Calculus: A new approach for schools that starts with simple algebra

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We outline a novel approach to tangents and derivatives that does not use any limits. Instead, it uses elementary algebraic concepts related to the quadratic equation, and therefore fits right into the school curriculum. Adding an elementary estimate to the central algebraic factorization naturally leads to the concept of continuity and thereby reveals that the algebraic derivative can also be captured by an approximation process. This turns out to be critical for handling non-algebraic functions, such as exponential functions. In order to capture the new elusive limits, students recognize that one needs to expand the familiar rational numbers to the much more intriguing real numbers. The solution of the tangent problem for exponential functions leads to the general notion of a differentiable function, in a formulation that is the natural generalization of the algebraic version, and which has been known for over 70 years. This approach gradually proceeds from most elementary concepts to the heart of analysis, making it clear to students along the way why more sophisticated tools are needed, and providing motivation for the more advanced concepts that are indispensable for a proper understanding of calculus.

1 The beginning

The standard introduction to calculus typically involves limits pretty much from the beginning. For example, the calculation of the slope of the tangent to $y = x^2$ at (a, a^2) leads us to consider $\lim_{h \to 0} \frac{(a+h)^2 - a^2}{h}$, whose intuitive answer $\frac{0}{0}$ is meaningless. However, as long as $h \neq 0$, it is easy to simplify the expression to $2a + h$, which makes it obvious that the answer should be $2a$. Yet there still remains the problem with the limit, something that prompted intensive discussions already in the early 18th century, led by the criticisms of the philosopher George Berkeley (1685–1753) (see [5, pp. 628–630]). Eventually, mathematicians in the 19th century placed limits on a solid foundation and since then students have had to learn about them first. But have you ever wondered if there shouldn't be an easier way that avoids limits until they really are needed?

Thanks to René Descartes (1596–1650), there actually is such an elementary way, and students in school can be introduced to it as soon as they have learned the basics of the quadratic function and equation. Tangents had been considered long before Descartes' days. Well over 2000 years ago, Greek geometers studied tangents to simple curves, such as circles, or more generally, conic sections. For them, a tangent to a curve at the point P is *a line that* ***touches*** *the curve at P, but does not* ***cut*** *it* (see [5, p. 120]). While this language is somewhat vague, its intuitive meaning is quite clear, and it turns out that it is very easy to give it a precise algebraic interpretation.

The crux of the matter is visible in Figure 1 that shows the graphs of two quadratic functions of the form $f(x) = (x - b)^2 + c$, where b, c are some fixed numbers.

The curve on the right corresponds to the case where $c < 0$: the x-axis intersects the graph of f in two *distinct* points, and surely it would seem appropriate to say that the axis "cuts" the curve in each of those two distinct points. The curve on the left corresponds to the case $c = 0$. In this case, the two points of intersection have blended together, one on top of the other, and it would be appropriate to say that the x-axis just "touches" the curve at that

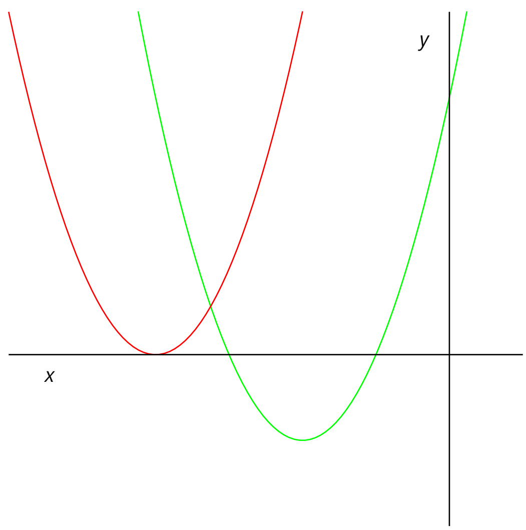

Figure 1

point, but does not "cut" it. So, according to Greek geometers, the x-axis surely is a tangent to the curve at that point. Algebraically, in this case, the point $(b, 0)$ of intersection is determined by the solution(s) of the equation $(x-b)^2 = 0$, which is a quadratic equation for which the two roots coincide; that is, we have a "double root (or zero)", or a root of "multiplicity 2". Geometrically, we have a "double point", that is, two points that just happen to lie on top of each other, so that only one point is visible. Of course, if the curve is just moved down a little bit, the two points of intersection separate and both become visible.

So, by just looking at this most simple example, we conclude that the ancient notion of a touching line is made precise geometrically by recognizing that the special point where the line touches the curve is really a double point, and furthermore, that this is made precise algebraically by saying that the relevant equation that determines the points of intersection has a double zero.

Given that the quadratic equation is a central and important topic in school algebra, which in particular includes the case of a double zero, this seems to be the perfect place to introduce the students to the basics of calculus. All that is needed is to appropriately interpret the picture above to conclude:

> A tangent to a curve at a point P is a line that intersects the curve at P with multiplicity 2 (or higher).

I can hear algebraic geometers say that they have known this for a long time. True, after all Descartes' idea did leave a trail. But have you ever used this idea when you are teaching a calculus class? If yes, great for you! But sadly, I have not yet found any calculus text that utilizes this simple idea to introduce students to tangents.

2 Historical remarks

As we just mentioned, the idea of finding tangents via double zeroes appears first in Descartes' work [3]. Actually, Descartes was more fascinated by *normals* to a curve rather than by tangents[1], but knowing either one determines the other. Perhaps Descartes had simply been thinking of generalizing what had long been known for a circle, namely that the tangent at a point P on a circle is perpendicular to the normal at P, which in this case is simply the line from the center of the circle to the point P. Therefore, in this case, the normal is the obvious tool to find the tangent. Descartes solved the problem for an ellipse and emphasized the generality of his method. Descartes' expositor Frans van Schooten applied the double point method to find the tangent to a parabola directly [9]. On the other hand, attempts to apply this method to more general curves led to formidable complications. The situation was captured by Howard Eves [4, pp. 284–285] as follows: "Here we have a general process which tells us exactly what to do to solve our problem, but it must be confessed that in the more complicated cases the required algebra may be quite forbidding." Thus Descartes' approach was eventually forgotten. On the other hand, as you will see shortly, the implementation of the double point method presented here, first published in [6], is completely elementary, and applies to all curves defined by algebraic expressions. So it is natural to wonder why mathematicians in the 17th century overlooked what today appears to be so obvious. One possible explanation is based on the fact that even though the concept of slope was certainly known at that time ("quotients of infinitesimals" were supposed to capture it), the *point-slope form* to describe lines – something every high school student today is familiar with – was not known or used explicitly at that time. In fact, it seems to have appeared first only more than 100 years later, in a 1784 paper by Gaspard Monge (1746–1818) (see [1, pp. 205–206]). Instead, Descartes, and everybody else around that time, described lines through the point P on the curve by using a second distant point Q on that line. One then changed the position of the line by moving that point Q along some other line, often a line of symmetry of the curve under consideration. But in case of more complicated curves, there was no obvious natural place to move that second point along. Perhaps philosophers and scientists in the 17th century were still so strongly under the spell of Euclid (approximately 325 BC – 265 BC), who had introduced the axiom that a line is defined by two (distinct) points, that they just couldn't conceive of describing a line by a single point and its slope.

3 Polynomials and the Chain Rule

This is not the place to present all the details of our proposed new approach to Calculus (see [8] for some more details). But I want to highlight two simple important results that can readily be presented to students pretty much at the beginning.

First, suppose P is a polynomial of degree n. We want to identify the tangent to its graph at the point $(a, P(a))$. An arbitrary (non-vertical) line through $(a, P(a))$ is described by an equation $y = P(a) + m(x-a)$, where m is the slope. Its points of intersection with our curve are given by the solutions of the equation

$$P(x) = P(a) + m(x-a).$$

Fortunately, we do not need to find all solutions of this equation. We only are interested in the *one* obvious zero, namely $x = a$, and we want to determine when this is a zero of multiplicity greater than 1. We rearrange the above equation in the form

$$P(x) - P(a) - m(x-a) = 0.$$

[1] Descartes wrote: "… when I have given a general method of drawing a straight line making right angles with the curve. And I dare say that this is not only the most useful and most general problem in geometry that I know, but even that I have ever desired to know." [3, p. 95]

A standard result gives the factorization

$$P(x) - P(a) = q(x)(x-a),$$

where q is a certain polynomial of degree $n-1$. While the existence of such a factorization is not trivial, it is an immediate consequence of the following fundamental result which should definitely be part of any discussion of polynomials in school.

Theorem. *If P is a polynomial of degree $n \geq 1$ that has a zero at the point a, i.e., $P(a) = 0$, then P has a linear factor of the form $x-a$; that is, there exists a polynomial q of degree $n-1$ such that $P(x) = q(x)(x-a)$.*

In our case, we simply apply this theorem to the polynomial $P(x) - P(a)$, which does indeed have a zero at a.

By using this result, we can now factor our equation $P(x) - P(a) - m(x-a) = 0$ and rearrange it in the form

$$[q(x) - m](x-a) = 0.$$

This clearly shows that a is a solution of the equation – after all we only consider lines that intersect the curve at $(a, P(a))$! Most significantly, this factorization shows that a is a zero of multiplicity greater than one if and only if the factor $[q(x) - m]$ also has a zero at a, and this occurs precisely if the slope $m = q(a)$.

Let us formalize our result in the following theorem that completely solves the tangent problem for any polynomial.

Theorem. *Let P be a polynomial of degree $n \geq 2$ and let $(a, P(a))$ be a point on its graph. Then there exists a unique line through $(a, P(a))$ that intersects the graph at that point with muliplicity greater than 1. The slope m of that line is given by $q(a)$, where q is the polynomial of degree $n-1$ in the factorization $P(x) - P(a) = q(x)(x-a)$.*

Of course, we call this unique line given by the theorem the *tangent to the graph of P at the point $(a, P(a))$*. Its slope $m = q(a)$ is called the *derivative of P at the point a*, and it is denoted by $D(P)(a)$ or also by $P'(a)$.

Example. If $g(x) = x^2$, then the relevant factorization is $x^2 - a^2 = (x+a)(x-a)$, so $q(x) = x + a$, and hence $D(g)(a) = q(a) = 2a$. So we have obtained the expected result without any limits whatsoever!

It is straightforward to generalize this to an arbitrary power function $P(x) = cx^n$, where c is a constant and n is a non-negative integer. Just as easily one can verify that if f and g are two polynomials, then $(f+g)'(x) = f'(x) + g'(x)$, and consequently one obtains the familiar formula for the derivative of a polynomial, all this without any limits!

Next we want to discuss the Chain Rule. This order also differs from the standard curriculum, which typically discusses product and quotient rules first. Perhaps this is due to the fact that multiplication and division are such standard operations in school algebra that it seems natural to extend these operations to polynomials or more general functions. Unfortunately, the relevant rules for differentiation look quite complicated and mysterious for the beginning student, adding more frustration. In contrast, as mathematicians know, the natural operation to consider for functions in the most general setting is *composition*, that is, apply one function after the other. It therefore should not come as a surprise that the rule for differentiation of compositions is as easy as it gets, with a completely transparent proof.

In detail, suppose f and g are polynomials, and consider the composition $(f \circ g)(x) = f(g(x))$, which is again a polynomial. We want to find its derivative at the point $x = a$. Let $b = g(a)$, and introduce the relevant factorizations

$$f(y) - f(b) = q_f(y)(y-b) \text{ and}$$
$$g(x) - g(a) = q_g(x)(x-a),$$

where q_f and q_g are polynomials. The obvious step is to replace $y = g(x)$ and $b = g(a)$ in the first equation, resulting in

$$f(g(x)) - f(g(a)) = q_f(g(x)) \cdot [g(x) - g(a)],$$

and then to introduce the second equation to obtain

$$\begin{aligned}(f \circ g)(x) - (f \circ g)(a) &= f(g(x)) - f(g(a)) \\ &= q_f(g(x)) \cdot [q_g(x)(x-a)] \\ &= [q_f(g(x)) \cdot q_g(x)](x-a).\end{aligned}$$

Clearly, this is the relevant factorization for $f \circ g$, and it follows that

$$D(f \circ g)(a) = q_f(g(a)) \cdot q_g(a) = D(f)(g(a)) \cdot D(g)(a).$$

We thus have established the Chain Rule.

Yes, you might say that all this only handles the case of polynomials, while in the case of general differentiable functions things surely must be more complicated. Well, not really. As we will show later, the above proof, combined with natural properties of continuous functions, carries over as it stands.

It is natural to introduce inverse functions at this place, and then the relevant rule for derivatives. There is a new twist, as taking inverses where a function is one-to-one usually brings in a new kind of function, so that the relevant factorization needs to be established as well. But we shall skip these details and refer the interested reader to [6, 7]. Also, considering inverses will typically take us beyond the rational numbers, although this can be avoided by restricting the domain of the inverse of a function f defined on the interval I to the image $f(I)$. For example, if we just consider $\sqrt{x}$ on the domain $\{r^2 \mid r \in \mathbb{Q} \text{ and } r \geq 0\}$, everything can be done by just using rational numbers.

4 Rational functions

The next topic I propose for the school curriculum is to investigate the class of rational functions, defined by taking quotients of polynomials. There are two main reasons for this choice. First of all, it takes the student up one step on the ladder to mastering calculus. While no really new ideas are needed in order to handle tangents and derivatives, things do get more complicated algebraically. Also, this seems to be a good place to introduce the quotient rule for derivatives (and of course the product rule as well), since rational functions are the first non-trivial examples where quotients appear. In particular, the quotient rule shows that the derivative of a rational function is again a rational function, with the same domain. The other reason is that the study of the field $\mathbb{Q}(x)$ of rational functions is a good opportunity to strengthen the student's familiarity with the fundamental rules that she/he has learned while mastering the rational numbers $\mathbb{Q}$. And these rules (or axioms) will become even more critical when one must introduce the *real* numbers $\mathbb{R}$, which notoriously are very difficult to describe exactly, so that "following the rules" becomes even more important. Of course, mathematicians are used to dealing with abstract sets, whose elements are required to follow the rules that define the particular structure that is studied. But for students, this is a big leap, and I believe that it is important to help them to prepare to deal with the more axiomatic approach that will be needed for the field of real numbers.

The main lesson is that in order to learn about tangents for rational functions, no new ideas are needed beyond what we used for polynomials. In fact, the methods can be readily extended to all functions built up from polynomials by algebraic operations.

Furthermore, it is remarkable that everything we have covered so far does not involve any limits, and it can be handled by just using the rational numbers $\mathbb{Q}$. Students thus get introduced to tangents and the basic rules of differentiation within the familiar setting of rational numbers. Isn't it amazing how simple school algebra allows to solve a major problem – at least in the algebraic setting – that was the focus of fundamental investigations in the 17th century? Perhaps this will help students to better understand the importance of and to appreciate the basic algebra that they are learning in school.

5 Continuity and approximation of derivatives

At this point, students may wonder if this is it. Perhaps they have heard of exponential functions or trigonometric functions, and if they found the discussion of tangents for rational functions of some interest, they may ask whether all this works for these functions as well. Before we get into that in the next section, we want to discuss another remarkable consequence of the fundamental algebraic factorization that opens the door to the next stage, where we really will be entering a new world.

Consider a polynomial P and its standard factorization $P(x) - P(a) = q(x)(x-a)$ at a fixed point a, where q is also a polynomial. It is easy to see by standard estimations that any polynomial is bounded over any finite interval. In particular, there exists a constant $K > 0$, so that

$$|q(x)| \leq K \quad \text{for all } x \text{ with } |x-a| \leq 1.$$

The factorization then implies the estimate

$$|P(x) - P(a)| \leq K|x-a| \quad \text{for all } x \text{ with } |x-a| \leq 1.$$

Clearly, this estimate shows that as x gets closer to a, then $P(x)$ gets closer to $P(a)$ as well. In symbols, $P(x) \to P(a)$ as $x \to a$. In other words, we have discovered that polynomials enjoy a fundamental property that is known as *continuity* at each point a. In case you are hooked on ε and δ, just choose $\delta = \varepsilon/K$, and you are done. But there is no need to make things so complicated for the students at this point; the estimate speaks for itself, and it confirms what students can readily see by graphing polynomials with a graphing calculator.

Remark. We want to emphasize that no limits are required yet. In fact, it is the above estimate that suggests the idea of *limit* in the simple case where the value of the limit is known and equals the value of the function. So we may introduce the notation

$$\lim_{x \to a} P(x) = P(a)$$

to summarize the statement that $P(x) \to P(a)$ as $x \to a$. Note how this approach is really the reverse of the standard one, which begins with limits, introduces continuity as a special case of limits, proves various limit theorems, and ultimately concludes that polynomials are continuous everywhere. I wonder which approach works better for our students?

Rational functions can essentially be handled by the same method, and similarly more general algebraic functions. It just takes a simple argument to show that every rational function is *locally* bounded near each point in its domain (i.e., where the denominator is *not* zero). So we have a rigorous proof that rational functions are continuous at every point where they are defined.

Let us now apply these new ideas to the polynomial (or rational) factor q in the standard factorization $P(x) - P(a) = q(x)(x-a)$. Using the new notation, we know that

$$\lim_{x \to a} q(x) = q(a) = D(P)(a).$$

This shows that the derivative of P at a is approximated by the values of $q(x)$ as $x \neq a$ gets closer and closer to a. So what are these values $q(x)$? As long as $x \neq a$, we can divide by $x - a \neq 0$, resulting in

$$q(x) = \frac{P(x) - P(a)}{x - a} \quad \text{for } x \neq a.$$

So we have reached the very beginning of the standard introduction to derivatives: the derivative is the limit of average rates of change, or, in geometric language, the slope of the tangent is the limit of the slopes of secants through the points $(a, P(a))$ and $(x, P(x))$ on the curve as $x \to a$. However, from the perspective of the student who is learning calculus as suggested in our approach, this is a new, *non-algebraic* technique to capture derivatives, which so far have been defined exactly by a simple algebraic method. It thus takes the student the first step beyond pure algebra further up the ladder towards mastering calculus. And as we start investigating non-algebraic functions, we will shortly discover how this latest insight will reveal amazing new phenomena that will force us to expand the foundations of the rational numbers $\mathbb{Q}$.

6 Exponential functions and mysterious limits

As we have reached the boundary of the algebraic techniques, the next step involves the study of other – non-algebraic – functions. Perhaps the most widely known such functions are the *exponential functions*. This past year just about everybody heard about the exponential growth of the spread of Covid-19, but other applications, such as compound interest, population growth, radioactive decay, and so on are also widely known. From the perspective of the experienced mathematician, exponential functions are the *eigenfunctions* of the differentiation operator, and therefore understanding them fully should reveal all there is to know about tangents and derivatives.

Exponential functions are easily defined for inputs that are natural numbers; most importantly, this basic case already reveals the fundamental rule

$$b^{m+n} = b^m \cdot b^n \quad \text{for } m, n \in \mathbb{N}.$$

While studying integers, rational numbers, and rational functions, students have learned that "following the rules" is a fundamental principle when one moves into new uncharted territory. So they should not be surprised that the rule we just recalled, that is, the functional equation for the exponential function, will guide us to properly define exponential functions for other numbers, such as negative integers, and then rational numbers. In particular, these rules, when extended to $1/n$ with $n \in \mathbb{N}$, require that the base b must be positive, and most importantly that

$$(b^{\frac{1}{n}})^n = b,$$

that is, $b^{1/n}$ must be the n-th root of b. While this forces us to go beyond the rational numbers, this is not much of a problem at the preliminary stage. After all, already Greek geometers recognized that there is no rational number $\frac{m}{n}$ that satisfies $(\frac{m}{n})^2 = 2$, and philosophers, scientists, and mathematicians lived for over 2000 years with this problem, and still were able to make amazing progress. So students, too, will just have to accept this reality as we move on with a preliminary investigation of exponential functions $E_b(x) = b^x$ with domain $\mathbb{Q}$, where $b > 0$.

So let us look at the tangent problem for the concrete case $b = 2$ at the point $(0, 1)$. Guided by the algebraic approach, we look for an analogous factorization

$$2^x - 2^0 = q(x)(x - 0),$$

and we immediately realize that there is no formula for q that would unambiguously tell us the value of $q(0)$. Just as in the algebraic case, that value would give us the slope of that line through $(0, 1)$ that intersects the graph of E_2 with multiplicity greater than one. Since we do know that $q(x) = \frac{2^x - 1}{x}$ for $x \neq 0$, and since we just learned that in the algebraic case the values $q(x)$ approximate $q(0)$ as $x \to 0$, it thus seems reasonable to study

$$q(x) = \frac{2^x - 1}{x} \quad \text{as } x \to 0.$$

Unfortunately, we have no tools available yet to determine whether such a limit actually exists. And if it does, what is it? If we look at numerical approximations for $q(0)$ by considering inputs $x_k = 10^{-k}$ for $k = 1, 2, 3, \dots$, we see that the numbers $q(x_k)$ approach a strange number whose decimal expansion begins with 0.6931471805… as k gets larger, so that $10^{-k} \to 0$. To summarize: the student recognizes that we are faced with fundamentally new phenomena that force us to take a deeper look at the numbers that we are using, and to thoroughly study the mysterious limit process that is central to understanding the exponential function.

7 Analysis

It is now time to bring in the real numbers $\mathbb{R}$ by adding one more axiom beyond those that the students have been familiar with all along from the rational numbers $\mathbb{Q}$. We introduce the *completeness* axiom by requiring that the field of real numbers satisfies the *Least Upper Bound Property*, and hence also the analogous *Greatest Lower Bound Property*. This version makes precise something that is intuitively quite obvious, and it provides easy proofs of the existence of the relevant limits that are needed to rigorously complete the discussion of the exponential function. Once students are well familiar with the real numbers $\mathbb{R}$, the domain of exponential functions is extended to all real numbers, and one verifies that the functional equation continues to hold in this more general setting. The tangent problem is solved by verifying in detail the existence in $\mathbb{R}$ of the limit

$$\lim_{x \to 0} \frac{2^x - 1}{x} = \inf\left\{\frac{2^x - 1}{x} \;\middle|\; x > 0\right\} = c_2,$$

where the expression on the right is the Greatest Lower Bound of the relevant set, whose existence is guaranteed by the completeness axiom. The subscript 2 in the notation c_2 relates to the base 2 of the exponential function E_2 that is considered. We then define

$$q(x) = \begin{cases} \frac{2^x - 1}{x} & \text{for } x \neq 0, \\ c_2 & \text{for } x = 0, \end{cases}$$

so the function q is continuous at 0, since $\lim_{x \to 0} q(x) = q(0)$.

We therefore conclude that the function $E_2(x) = 2^x$ has a factorization

$$E_2(x) - E_2(0) = q(x) \cdot (x - 0), \quad \text{with } q \text{ continuous at } 0,$$

and we define its derivative $D(E_2)(0) = q(0)$. Of course, the same results hold for exponential functions E_b with arbitrary base $b > 0$, with

$$D(E_b)(0) = c_b = \lim_{x \to 0} \frac{b^x - 1}{x}.$$

The functional equation readily implies that analogous results hold at every point $a \in \mathbb{R}$, leading to the differentiation formula

$$D(E_b)(a) = E_b(a) \cdot D(E_b)(0).$$

Furthermore, the number $e = 2^{1/c_2}$, where c_2, as introduced above, equals the derivative $D(E_2)(0)$, is the unique number for which $D(E_e)(0) = 1$, so that the *natural* exponential function $E(x) = E_e(x) = e^x$ satisfies

$$D(e^x) = e^x \quad \text{for all } x \in \mathbb{R}.$$

Once we have identified the relevant property for the exponential function that identifies the slope of the tangent, it is natural to use it as the defining property for general *differentiable* functions, as follows.

Definition. The function f defined in an interval surrounding the point a is said to be differentiable at a if there exists a factorization

$$f(x) - f(a) = q(x) \cdot (x - a),$$

where the function q is continuous at a. The value $q(a)$ is called the derivative of f at a and it is denoted by $D(f)(a)$ or $f'(a)$.

Note how this definition naturally generalizes what has been central for polynomials and rational functions. Since in those cases the factor q is of the same algebraic type and known to be continuous, rational functions are trivially differentiable according to this definition at every point where defined. And of course, we just saw that exponential functions are differentiable everywhere.

It is obvious that the definition above is equivalent to the standard definition of differentiability.

This formulation of differentiability has been around for quite a while, but unfortunately it is not widely known. To my knowledge, it was first introduced by Constantin Carathéodory (1873–1950) at least as early as 1950, in his classic text *Funktionentheorie* [2]. It has been used in Germany in courses and books since the early 1960s, and it slowly has appeared in US textbooks since the late 1990s (see [6] and [7, pp. xxvi–xxvii] for more details).

It should be mentioned that Carathéodory's formulation of differentiability naturally suggests the basic relationship $(df)_a = f'(a)\,dx$ between "differentials" that is widely used in applications, namely, that a small change dx in the input leads to a small change $df(dx)$ in the output, which is well approximated by the product $f'(a) \cdot dx$. In fact, the error between $\Delta f = q(x)\Delta x$ and $(df)_a(\Delta x)$ is given by $[q(x) - q(a)]\Delta x$, which, because of the continuity of q at $x = a$, is negligible compared to Δx as long as x is very close to a. Of course, working with differentials df and dx is loaded with much historical baggage, given their classical interpretation as "infinitesimals", while for differential geometers it is standard to view the differential $(df)_a$ at a as a linear function defined on the tangent space at that point. It is hard enough for mathematicians to keep all this straight, and for students this is bound to be very confusing. Clearly, we need to find a way to help our students to sort this out. Perhaps it is time to introduce *different* notations for infinitesimals, small changes in input/output, and the underlying linear function, instead of just using df.

The Chain Rule, revisited. We already introduced and proved this rule for polynomials above. The proof for general differentiable functions follows exactly the same structure as the proof in that case, and is completed by just using natural properties of continuous functions, such as the fact that compositions and products of continuous functions are continuous. Again, the reader should compare this with the various proofs of this important result found in standard textbooks, and decide for her/himself which proof we should teach our students.

Aside from allowing the simplest proof of the Chain Rule known to me, Carathéodory's formulation has other advantages. In particular, it naturally extends to functions and maps of several variables once the appropriate linear algebra is introduced, and it allows simple proofs of the Chain Rule and Inverse Function Rule in the higher dimensional setting.

8 Concluding remarks

This concludes our outline of how we propose to introduce students to differential calculus, beginning with rational numbers and the quadratic equation. Once we have reached the heart of Analysis and in particular the general notion of differentiable functions, the story continues, more or less, along more traditional lines. We hope that this brief outline has convinced the reader that there is an alternative to the traditional approach, a new approach that should make things easier for students. It gradually builds up from simple algebra to lead them to the point where they will recognize the true necessity of deep new ideas involving more advanced mathematical tools that are the heart of analysis. The author is currently working on a detailed textbook that implements this outline and which may be viewed as a substantial expansion of the first half of his earlier book [7].

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R. Michael Range earned his *Diplom in Mathematik* at the University of Göttingen in 1968, where lectures of Hans Grauert got him hooked on multidimensional complex analysis. A Fulbright Fellowship brought him to the United States, where he earned a Ph.D. at UCLA in 1971. He has held academic positions at Yale University, the University of Washington, and at the State University of New York in Albany, as well as research positions at institutes in Bonn, Stockholm, Barcelona, and Berkeley. Range has published numerous research articles, and is the author of *Holomorphic Functions and Integral Representations in Several Complex Variables* (Springer 1986 and 1998), a widely used introduction to the field. He has lectured in many countries, including mini-courses at the Chinese Academy of Sciences in Beijing, at POSTECH in South Korea, and in Trondheim, Norway. More recently he has written historical and expository articles on multidimensional complex analysis, one of which was recognized with the Lester R. Ford award of the Mathematical Association of America in 2004. He also has spent much time thinking about the calculus curriculum, with the goal of making things easier for the ever increasing number of students. Range loves mountains and enjoys hiking, biking, and especially downhill skiing. In 2015 he retired to Park City, high in the beautiful mountains near Salt Lake City in Utah.

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