A remark on Piron's paper

By

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Abstract.

The following statement (Piron's Theorem 22) is proved: The lattice $L(V)$ of all subspaces of a prehilbert space V is orthomodular if and only if V is complete (i. e. a Hilbert space).

§1. Introduction.

In an attempt to formulate the postulate of quantum theory, Piron [1] has studied an irreducible complete orthomodular OAC-lattice. (Piron's irreducible system of propositions. See §2 for a definition.) He has shown that any such lattice of dimension larger than 3 can be realized as a lattice $L(V)$ of subspaces of a vector space V in the following manner:

Let K be a field with an involutive antiautomorphism $*$ and V be a vector space over the field K equipped with a definite hermitian form $f(x,y)$. For any subset S of V, $S^{\perp}$ denotes the set of all x such that $f(x,y)=0$ for all $y\in S$. We shall call S a subspace of V if $(S^{\perp})^{\perp}=S$. [If K is the field of complex numbers and V is a Hilbert space, this definition of a subspace coincides with that of a closed linear subset.] Then, $L(V)$ is the lattice of all subspaces of V with the join, meet and orthocomplementation defined by

$$S_1\vee S_2=[(S_1\cup S_2)^{\perp}]^{\perp}, \tag{1.1}$$

$$S_1\wedge S_2=S_1\cap S_2, \tag{1.2}$$

$$S\rightarrow S^{\perp}. \tag{1.3}$$

The whole space V and the trivial subspace 0 are 1 and 0 in this lat-

Received September 1, 1966.
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tice.

Piron's theorem has been sharpened in the following way [2]: $L(V)$ is always an irreducible complete OAC-lattice. (See §2 for a definition.) Conversely, any irreducible complete OAC-lattice of more than 3 dimensions is isomorphic to an $L(V)$ for some field K with an involution $*$ and some vector space V over K with a definite hermitian form f. However the necessary and sufficient condition for $(K, *, V, \mathrm{f})$ such that $L(V)$ is orthomodular is not known.

If K is the field of complex numbers, V is a prehilbert space with a positive definite inner product $\mathrm{f}(x, y)$. Piron states in his Theorem 22 that $L(V)$ for a complex field K is orthomodular if and only if V is complete (i. e. it is a Hilbert space). Unfortunately his proof is incomplete. In this note, we shall give a complete proof of Piron's Theorem 22.

§2. Preliminaries.

An orthocomplemented lattice L is called an OAC-lattice if

(A) L is relatively atomic, namely $a<b$ implies the existence of an atom p such that $p\leqq b$ holds and $p\leqq a$ does not hold where p is an atom if $c<p$ implies $c=0$.

(C) L has the covering property, namely if a is an arbitrary element of L and p is an atom, then there is no element b such that $a<b<a\vee p$.

A lattice L is said to be complete if a family of elements S_α in L always has a l. u. b $\bigvee_\alpha S_\alpha$ and a g. l. b $\bigwedge_\alpha S_\alpha$.

A lattice L is said to be irreducible if it is never isomorphic to a direct product $L_1\times L_2$ of two nontrivial lattices.

An orthocomplemented lattice is said to be orthomodular [3] if $a\leqq b$ implies $b=a\vee(b\wedge a^\perp)$. (There are many other equivalent conditions.)

A mapping f from $V\times V$ into K is called a definite hermitian form if

$$\mathrm{f}(x+\lambda y, z)=\mathrm{f}(x, z)+\lambda\mathrm{f}(y, z), \tag{2.1}$$

$$f(x,y)^*=f(y,x), \tag{2.2}$$

$$f(x,x)=0 \quad \text{implies} \quad x=0. \tag{2.3}$$

We consider the case where K is the complex field and V is a prehilbert space with the inner product f.

We need the following lemma which holds for a general K.

Lemma. Any finite demensional linear subset S of V is a subspace.

Proof. Let S be generated by n independent vectors $Q_1 \cdots Q_n$. Suppose Q is an arbitrary vector in $(S^{\perp})^{\perp}$. We can write

$$Q=\sum_{i=1}^{n} c_i Q_i + Q'$$

where $c_i \in K$ and $Q' \in V$ can be made orthogonal to all Q_i. Since $Q \in (S^{\perp})^{\perp}$ and $Q_i \in S$, Q' also belongs to $(S^{\perp})^{\perp}$. At the same time $Q' \in S^{\perp}$, by construction. Since $S^{\perp} \cap (S^{\perp})^{\perp}=0$ for any S, Q' must be 0 and we have $Q \in S$.

Corollary. A subspace p is an atom of $L(V)$ if and only if it is one dimensional.

§3. Main theorem and proof.

Theorem. Let V be a prehilbert space. $L(V)$ is orthomodular if and only if V is complete.

Proof. The "if" part is obvious and we concentrate on the proof of "only if" part.

Assume that $L(V)$ is orthomodular.

Step 1. $V=S+S^{\perp}$ for any subspace S: Take an atom p not contained in S and $S^{\perp}$. We first show that $q=(S \vee p) \wedge S^{\perp}$ and $r=(S^{\perp} \vee p) \wedge S$ are atoms. By orthomodularity for the pair S and $p \vee S$, $p \vee S = q \vee S$, which excludes the possibility $q=0$. If $q>a$, we have $a \vee S \leqq p \vee S$ which implies $a \vee S = S$ or $a \vee S = p \vee S$ due to covering property. The former and $a<S^{\perp}$ implies $a=0$. The latter implies $a=S^{\perp} \wedge (a \vee S)=q$ due to $a<S^{\perp}$ and the orthomodularity. In the same

manner, r is also an atom. For these two atoms, we have

$$q \vee r \geqq p \tag{3.1}$$

due to the orthomodularity. By the Lemma in §2, a vector P in p is a linear combination of vectors in $q \leqq S^{\perp}$ and $r \leqq S$.

Step 2. Let H be the f-completion of V. If H_1 is a closed linear subset of H with finite dimensional $H_1^{\perp}$ ($\perp$ taken in H), then $V \cap H_1$ is dense in H_1: Let $g_1 \cdots g_n$ be a basis of $H_1^{\perp}$ and let $e_1 \cdots e_n$ be elements in V such that, when, considered as dual elements of H_1, span the dual of $H_1^{\perp}$. Let $q \in H_1$ and $q_m \in V$ such that $\lim q_m = q$. Then let c_i^m be the solution of $\mathrm{f}(g_j, q_m) = \sum_{i=1}^{n} c_i^m \mathrm{f}(g_j, e_i)$, $j = 1 \cdots n$. Since $\mathrm{f}(q, g_j) = 0$, $\lim_{m \to \infty} c_i^m = 0$ and hence

$$q'_m = q_m - \sum_{i=1}^{n} c_i^m e_i \in V \cap H_1$$

satisfies $\lim q'_m = q$.

Step 3. If $P, Q \in H$ and $P \perp Q$, there exists sequence $\{u_n\}$, $\{v_n\}$, both in V such that (1) $u_l \perp v_m$, $u_l \perp Q$, $P \perp v_m$ for all n and m, (2) $u_l \to P$, $v_m \to Q$.

To construct u_n and v_n by mathematical induction, first choose $\varepsilon_n > 0$ such that $\varepsilon_n \to 0$. Start from $u_0 = v_0 = 0$. Assume that u_m and v_m for $m < n$ has been constructed in such a way that the condition (1) is satisfied for l, $m < n$, $u_m \in V$, $v_m \in V$, $\|u_m - P\| < \varepsilon_m$ and $\|v_m - Q\| < \varepsilon_m$ for all $m < n$. We then want to construct u_n and v_n both in V such that (1) is satisfied for l, $m \leqq n$, $\|u_n - P\| < \varepsilon_n$ and $\|v_n - Q\| < \varepsilon_n$. This is easily achieved due to Step 2. Take the linear space spanned by $v_1 \cdots v_{n-1}$, Q as $H_1^{\perp}$ and find u_n in $V \cap H_1$ such that $\|u_n - P\| < \varepsilon_n$. Then take the linear space spanned by $u_1 \cdots u_n$, P as $H_1^{\perp}$ and find v_n in $V \cap H_1$ such that $\|v_n - Q\| < \varepsilon_n$.

Step 4. For any $P \in H$, there exists $R \in V$ such that $P \perp \mathrm{R} - P$: Use R' in V with $\mathrm{f}(P, R') \neq 0$ to construct $R = \mathrm{f}(P, P) R' / \mathrm{f}(P, R')$.

Now we are ready for the proof of Theorem. Take P and $Q \equiv R - P$ from Step 4. Construct $\{u_n\}$ and $\{v_n\}$ of Step 3. Let S be

$\{v_1 v_2 \cdots\}^{\perp}$ ($\perp$ taken in V) and E be the projection on the closure of S in H. Since $v_n \perp S$, we have $Q \perp S$. Since $u_n \in S$, P belongs to the closure of S in H. Thus $R = Q + P$, $R \in V$, $Q \perp S$, $P \perp S^{\perp}$ ($\perp$ of $S^{\perp}$ taken in V) and hence $P = ER$.

On the other hand, Step 1 implies $R = u + v$, $u \in S$, $v \in S^{\perp}$. Hence we must have $u = ER = P$ and $P \in V$. This proves $V = H$.

Acknowlegement.

The second named author (H. A) would like to thank Professor M. H. Stone and Dr. C. Piron for bringing his attention to Piron's Theorem 22. Authors would like to thank Professor S. Maeda for mediating our cooperations.

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