

Carleman Inequalities for Parabolic Equations in Sobolev Spaces of Negative Order and Exact Controllability for Semilinear Parabolic Equations

By

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Abstract

We prove Carleman inequalities for a second order parabolic equation when the coefficients are not bounded and norms of right hand sides are taken in the Sobolev space $L^2(0, T; W_2^{-\ell}(\Omega))$, $\ell \in [0, 1]$. Our Carleman inequality yields the unique continuation for L^2 -solutions. We further apply these inequalities to the global exact zero controllability of a semilinear parabolic equation whose semilinear term also contains derivatives of first order of solutions and is of sub-linear growth at the infinity.

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§1. Introduction

Since Carleman [3], there have been great concerns in Carleman inequalities. In particular, after the appearance of fundamental results by Hörmander [23], the theory is one of the most developing areas of linear partial differential equations. Among recent significant achievements, let us mention new unique continuation theorems for the hyperbolic operators in a spatial domain Ω and in a time interval $(0, T)$ (Hörmander [24], Robbiano [45], Ruiz [46], Tataru [52], [53], for example). Since [23], the theory has progressed in several directions, among which we mention the theory of Carleman inequalities in L^p -spaces with $p \neq 2$ (see Jerison and Kenig [31], Kim [32], [33], Sogge [49]) and the theory of Carleman inequalities with singular weight function. We can further refer to Jerison [30]. Note that these papers deal with either of the following “non-regular cases”

- (1) Coefficients of low order terms belong to the space $L^p(0, T; L^q(\Omega))$ for some $p, q \in [1, +\infty]$ and right hand sides are taken in some L^p -space.
- (2) Coefficients possess isolated singularities.

For Carleman inequalities for parabolic equations, see Isakov [28], [29], Kurata [34], Lavrent'ev, Romanov and Shishat-skiĭ [36], Lin [37], Mizohata [42], Poon [44], Saut and Scheurer [48], Sogge [50], for example. In their works, coefficients of first order terms are assumed to be at least bounded, and coefficients of zero order term are assumed to be from the space $L^p(0, T; L^q(\Omega))$. Such a boundedness assumption makes the proof simple, but prevents us from applying the inequalities to solutions of semilinear parabolic equations which are less regular. In particular, Fabre [9], [10], Fabre and Lebeau [11] establish Carleman inequalities with norm of right hand sides in negative order Sobolev spaces for the Laplace and heat operators and functions with compact supports.

The first purpose of this paper is to establish Carleman inequalities for linear parabolic equations where the coefficients of terms of lower order are not regular and the right hand sides are in Sobolev spaces of negative orders. More precisely, we consider parabolic equations of the second order in a bounded cylindrical domain $Q = (0, T) \times \Omega$ with the zero Dirichlet boundary conditions on $\partial\Omega$, where the coefficients of the zeroth order term in the equations are in $L^\infty(0, T; W_r^{-\kappa}(\Omega))$ and the right hand sides are in the spaces $L^2(0, T; W_2^{-\ell}(\Omega))$ for some $r > 1$ and $\ell \in [0, 1]$. However for the principal part, we have to assume that the coefficients are Lipschitz continuous (see Corollary 2.1). As for less regular coefficients of second-order terms, we can refer to Lu [41], Wolff [55] in the case of elliptic equations.

The technique in this paper is combinations of several methods. That is, the proof of the Carleman inequalities is based essentially on a duality argument, a theory of extremal problems, smoothing properties of parabolic and elliptic operators and L^2 -Carleman inequalities proved by Chae, Imanuvilov and Kim [4], Imanuvilov [25].

Here we state other characteristics of our Carleman inequalities.

- (1) Our Carleman inequalities hold over the whole domain Q , while classical Carleman inequalities are valid in subdomains bounded by level sets (e.g. Isakov [29], Lavrentiev, Romanov and Shishat'skiĭ [36]), or sufficiently small domains (e.g. Saut and Scheurer [48]).
- (2) For our Carleman inequalities, the solutions have to satisfy the boundary condition on the whole boundary over the time interval. On the other hand, for classical Carleman inequalities, we can discuss solutions locally in the spatial domain Ω by introducing appropriate cut-off functions.
- (3) Within solutions satisfying a boundary condition, our Carleman inequalities enable us to obtain unconditional global Lipschitz stability in a state estimation problem of determining a solution at a preceding time in terms of values of a solution in $(0, T) \times \omega$, where $\omega \subset \Omega$ is an arbitrary subdomain. On the other hand, classical Carleman inequalities cannot give such unconditional global Lipschitz stability estimates. Only Hölder stability can be proved (e.g. [29]).

Next we state the second purpose of this paper: applications of the Carleman inequalities. Firstly, with a suitable cut-off function, our Carleman inequalities imply the unique continuation theorem for parabolic operators within $L^2(Q)$ -solutions. That is, if any $L^2(Q)$ -solution of a parabolic equation with the zero right hand side vanishes in $(0, T) \times \mathcal{O} \subset Q$ where \mathcal{O} is an arbitrary open subset of Ω , then it identically vanishes in the whole domain Q (Theorem 2.2). Simultaneously by our Carleman inequalities, we can prove the above-mentioned unconditional and global Lipschitz stability for solutions at any intermediate time, provided that solutions satisfy the zero Dirichlet boundary condition.

Another important application of our Carleman inequalities is exact controllability of semilinear parabolic equations. In this paper we prove the exact zero controllability for semilinear parabolic equations of the second order where the nonlinear term depends on $(t, x) \in Q$, $y = y(t, x)$ and $\nabla y(t, x)$, and is of sub-linear growth at the infinity (see (3.7) and (3.8)). The methodology is same as in [20] and [25], but it relies on Carleman inequalities obtained in this paper.

We can further refer to Guo and Littman [22] for the exact zero controllability for a parabolic equation whose semilinear term contains ∇y and satisfies analyticity condition.

In relation with the other controllability, in the beginning of 1990's, Fabre, Puel and Zuazua in [12], [13] have proved the global approximate controllability for second order semilinear parabolic equations with nonlinear term $f(t, x, y)$ of sub-linear growth in the variable y at the infinity. Later the first named author of this paper proved the global exact controllability for the same equation ([25]). This result was improved by Fernández-Cara [15]. On the other hand, for the case of nonlinear term including ∇y , the approximate controllability was established only recently by Fernández and Zuazua [14] and Zuazua [56]. For other important results on boundary controllability of evolution equations of fluid mechanics, see Coron [5]–[7], Coron and Fursikov [8], Fabre [9], [10], Fabre and Lebeau [11], Fursikov and Imanuvilov [16]–[21], Imanuvilov [26].

We conclude this section with a remark on further applications of Carleman inequalities to inverse problems. As is seen in Isakov [29, Chapter 8] for example, Carleman inequalities are useful for proving the uniqueness and stability in inverse problems of determining spatially varying coefficients in partial differential equations by overdetermining data on lateral boundary. In particular, thanks to the above-mentioned global character of our Carleman inequalities, we can prove Lipschitz stability which is global in the whole domain for the inverse problems. In Imanuvilov and Yamamoto [27], we establish such stability within L^2 -coefficients for inverse parabolic problems, on the basis of the Carleman inequalities in usual L^2 -spaces in [20], [25]. The Carleman inequalities proved in this paper, enable us to extend the results in [27] to inverse problems of determining less regular coefficients (not in $L^2(\Omega)$) and in a forthcoming paper, we will give details.

§2. Carleman Inequalities

Let $(t, x) \in Q \equiv (0, T) \times \Omega$, $\Sigma \equiv (0, T) \times \partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is a connected bounded domain whose boundary $\partial\Omega$ is sufficiently smooth, $\nu(x)$ is the external unit normal to $\partial\Omega$, $T \in (0, +\infty)$ is an arbitrary moment of time, $D^\beta = D^{\beta_0} D^{\beta'}$ = $\frac{\partial^{\beta_0}}{\partial t^{\beta_0}} \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \dots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}$, $\beta = (\beta_0, \beta') = (\beta_0, \beta_1, \dots, \beta_n)$, $|\beta| = 2\beta_0 + \beta_1 + \dots + \beta_n$. Let $\omega \subset \Omega$ be an arbitrarily fixed subdomain and let us set $Q_\omega = (0, T) \times \omega$.

Throughout this paper, $W_p^\mu(\Omega) = W^{\mu,p}(\Omega)$, $\overset{\circ}{W}_p^\mu(\Omega) = W_0^{\mu,p}(\Omega)$, $p \geq 1$, $\mu \geq 0$ denote usual Sobolev spaces (e.g., Adams [1], Triebel [54]), and we set $L^2(\Omega) = W_2^0(\Omega)$. For non-integer $\ell = k + \gamma$, $k \in \mathbb{N} \cup \{0\}$, $\gamma \in (0, 1)$, we note

that

$$\|u\|_{W_p^\ell(\Omega)}^p = \|u\|_{W_p^k(\Omega)}^p + \int_{\Omega} \int_{\Omega} \sum_{|\beta|=k} \frac{|D^\beta u(y) - D^\beta u(x)|^p}{|y-x|^{n+\gamma p}} dx dy.$$

Moreover $W_p^{-\mu}(\Omega) = (\overset{\circ}{W}_p^\mu(\Omega))'$: the dual, where $\frac{1}{p} + \frac{1}{p'} = 1$. We set

$$W^{1,2}(Q) = \left\{ y(t,x) \mid \frac{\partial y}{\partial t} \in L^2(0,T;L^2(\Omega)), y \in L^2(0,T;W_2^2(\Omega)) \right\},$$

$$C^{1,2}(\overline{Q}) = \left\{ y = y(t,x) \mid y, \frac{\partial y}{\partial t}, \frac{\partial y}{\partial x_i}, \frac{\partial^2 y}{\partial x_i \partial x_j} \in C(\overline{Q}), 1 \leq i, j \leq n \right\}$$

and

$$C^{0,1}(\overline{Q}) = \left\{ y = y(t,x) \mid y, \frac{\partial y}{\partial x_i} \in C(\overline{Q}), 1 \leq i \leq n \right\}.$$

Henceforth $\mathcal{L}(X, Y)$ denotes the totality of bounded linear operators defined over a Banach space X with values in another Banach space Y .

Let us consider the initial boundary value problem

$$(2.1) \quad Ly \equiv \frac{\partial y}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(t,x) \frac{\partial y}{\partial x_j} \right) + \sum_{i=1}^n \frac{\partial}{\partial x_i} (b_i(t,x)y) + c(t,x)y = g \quad \text{in } Q,$$

$$(2.2) \quad y|_{\Sigma} = 0, \quad y(0, \cdot) = y_0.$$

Assume that

$$(2.3) \quad \left\{ \begin{array}{l} a_{ij} \in W_\infty^1(Q), \quad a_{ij} = a_{ji}, \quad 1 \leq i, j \leq n, \\ b_i \in L^\infty(0, T; L^r(\Omega)), \quad r > 2n, \quad 1 \leq i \leq n, \\ c \in L^\infty(0, T; W_{r_1}^{-\mu}(\Omega)), \quad 0 \leq \mu < \frac{1}{2}, \quad r_1 > \max \left\{ \frac{2n}{3-2\mu}, 1 \right\}, \end{array} \right\}$$

and the coefficients a_{ij} satisfy the uniform ellipticity: There exists $\beta > 0$ such that

$$(2.4) \quad a(t, x, \zeta) \equiv \sum_{i,j=1}^n a_{ij}(t, x) \zeta_i \zeta_j \geq \beta |\zeta|^2, \quad \zeta \in \mathbb{R}^n, \quad (t, x) \in Q.$$

To formulate our Carleman inequality we need a special weight function.

Lemma 2.1 ([4], [25]). *Let $\omega_0 \subset \omega$ be an arbitrary fixed subdomain of Ω such that $\overline{\omega_0} \subset \omega$. Then there exists a function $\psi \in C^2(\overline{\Omega})$ such that*

$$(2.5) \quad \left\{ \begin{array}{l} \psi(x) > 0 \quad \text{all } x \in \Omega, \quad \psi|_{\partial\Omega} = 0, \\ |\nabla\psi(x)| > 0 \quad \text{for all } x \in \overline{\Omega} \setminus \overline{\omega_0}. \end{array} \right\}$$

Now using the function ψ constructed in Lemma 2.1, we introduce weight functions:

$$(2.6) \quad \varphi(t, x) = e^{\lambda\psi(x)}/(t(T - t)), \quad \widehat{\varphi}(t) = 1/(t(T - t)),$$

$$(2.7) \quad \alpha(t, x) = (e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_{C(\overline{\Omega})}})/(t(T - t)),$$

where $\lambda > 0$ is a parameter. Moreover we set

$$(2.8) \quad \gamma = \sum_{i,j=1}^n \|a_{ij}\|_{W_\infty^1(Q)} + \sum_{i=1}^n \|b_i\|_{L^\infty(0,T;L^r(\Omega))} + \|c\|_{L^\infty(0,T;W_{r_1}^{-\mu}(\Omega))}.$$

Denote by L^* the operator formally adjoint to the operator L . Below we are dealing with weak L^2 -solutions to the problem (2.1)–(2.2). Since under assumption (2.3), the function $c(t, x)y(t, x)$ is a distribution, we have to introduce the notion of weak solution to this problem using the method of transposition.

Definition 2.1. We say that $y \in L^2(Q)$ is a (weak) solution to the problem (2.1)–(2.2) if for any $z \in L^2(0, T; W_2^1(\Omega))$ with $L^*z \in L^2(Q)$, $z|_{\partial\Omega} = 0$ and $z(T, \cdot) = 0$, the following equality holds true:

$$(y, L^*z)_{L^2(Q)} = (g, z)_{L^2(Q)} + (y_0, z(0, \cdot))_{L^2(\Omega)}.$$

We are ready to state our main result, which establishes Carleman inequalities in Sobolev spaces of negative orders.

Theorem 2.1. *Let (2.3)–(2.4) be fulfilled and the functions φ, α be defined by (2.6) and (2.7). Then there exists a number $\widehat{\lambda} > 0$ such that for an arbitrary $\lambda \geq \widehat{\lambda}$, we can choose $s_0(\lambda) > 0$ satisfying: there exists a constant $C_1 > 0$ such that for each $s \geq s_0(\lambda)$ the solution $y \in L^2(Q)$ to the problem (2.1) and (2.2) satisfies the following inequality:*

$$(2.9) \quad \int_Q ((s\varphi)^{1-2\ell} |\nabla y|^2 + (s\varphi)^{3-2\ell} y^2) e^{2s\alpha} dx dt \leq C_1 \left(\|ge^{s\alpha}\|_{L^2(0,T;W_2^{-\ell}(\Omega))}^2 + \int_{Q_\omega} (s\varphi)^{3-2\ell} y^2 e^{2s\alpha} dx dt \right),$$

for all $s \geq s_0(\lambda), \quad \ell \in [0, 1]$.

Here the constant C_1 is dependent continuously on γ, λ and independent of s .

Moreover if $g(t, x) = g_0(t, x) + \sum_{i=1}^n \frac{\partial g_i(t, x)}{\partial x_i}$ with $g_i \in L^2(Q)$, $1 \leq i \leq n$, then the following estimate holds true for any $d \in \mathbb{R}$:

$$(2.10) \quad \int_Q ((s\varphi)^{d-1} |\nabla y|^2 + (s\varphi)^{d+1} y^2) e^{2s\alpha} dx dt$$

$$\begin{aligned} &\leq C_2 \left(\|g_0(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(0,T;W_2^{-1}(\Omega))}^2 \right. \\ &\quad \left. + \sum_{i=1}^n \|g_i(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(Q)}^2 + \int_{Q_\omega} (s\varphi)^{1+d} y^2 e^{2s\alpha} dxdt \right), \\ &\quad \text{for all } s \geq s_0(\lambda, d), \end{aligned}$$

where the constant $C_2 > 0$ is dependent continuously on γ, λ, d and independent of s .

Corollary 2.1. *The statement of Theorem 2.1 holds true if we assume that the coefficients $a_{ij}, 1 \leq i, j \leq n$ of the principal part are just Lipschitz continuous on \overline{Q} .*

We postpone the proof of the corollary till the end of this section.

Carleman inequality (2.10) implies the following unique continuation result by a similar argument with using level sets of ψ (e.g. [23], [29, Chapter 3]).

Theorem 2.2. *Let the conditions in (2.3) hold for the coefficients $b_i, 1 \leq i \leq n, c$. Moreover let the coefficients a_{ij} with $a_{ij} = a_{ji}, 1 \leq i, j \leq n$ be Lipschitz continuous on \overline{Q} and let (2.4) hold. Suppose that $y \in L^2(Q)$ is a solution to equation (2.1) with the right hand side $g \equiv 0$. If y equals zero in $[0, T] \times \mathcal{O}$ where \mathcal{O} is some open set in Ω , then y identically equals zero over the whole Q .*

With more restrictive assumptions on regularity of coefficients of a parabolic operator, the Carleman inequality (2.9) with $\ell = 0$ was proved in Chae, Imanuvilov and Kim [4], Fursikov and Imanuvilov [20], Imanuvilov [25], for example. Theorem 2.1 generalizes such Carleman inequalities.

The rest part of this section is devoted to the proof of Theorem 2.1. First we show

Lemma 2.2. *Let $0 < \mu < \frac{1}{2}$ and $r_1 > \max\left\{\frac{2n}{3-2\mu}, 1\right\}, \frac{1}{r_1} + \frac{1}{r_1} = 1$. Then there exist constants $0 < \delta < \frac{1}{2}$ and $C > 0$ such that*

$$\|zv\|_{W_{r_1}^\mu(\Omega)} \leq C \|v\|_{W_2^1(\Omega)} \|z\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}$$

for all $v \in \overset{\circ}{W}_2^1(\Omega)$ and $z \in W_2^{\frac{1}{2}-\delta}(\Omega)$.

The proof of the lemma is technical and so it is given in Appendix I.

Lemma 2.3. *Let (2.3)–(2.4) be fulfilled, $b_i \in C^{0,1}(\overline{Q})$, $c \in L^\infty(Q)$ and the functions φ , α be defined by (2.6) and (2.7). Let $d \in \mathbb{R}$. Then there exists $\widehat{\lambda} > 0$ such that for an arbitrary $\lambda \geq \widehat{\lambda}$, we can choose $s_0 = s_0(\lambda, d) > 0$ satisfying: there exists a constant $C_3 = C_3(\lambda, d) > 0$ such that a solution $y \in L^2(Q)$ to problem (2.1) and (2.2) satisfies the following inequality:*

$$(2.11) \quad \int_Q ((s\varphi)^{1+d} |\nabla y|^2 + (s\varphi)^{3+d} y^2) e^{2s\alpha} dx dt \\ \leq C_3 \left(\|(s\varphi)^{\frac{d}{2}} g e^{s\alpha}\|_{L^2(Q)}^2 + \int_{Q_\omega} (s\varphi)^{3+d} y^2 e^{2s\alpha} dx dt \right) \\ \text{for all } s \geq s_0(\lambda, d),$$

where the constant $C_3 > 0$ is independent of s .

Proof of Lemma 2.3. In the case of $d = 0$, inequality (2.11) with $C^{1,2}$ -coefficients a_{ij} is shown, for example, in [4], [25]. For completeness, in Appendix II, we will give the proof in the case of $d = 0$. Thus we have to prove (2.11) for $d \neq 0$. By taking a constant $C_3 > 0$ sufficiently large for λ if necessary, it suffices to prove (2.11) after the function $\varphi(t, x)$ is substituted by $\widehat{\varphi}(t)$ (see (2.6)). In fact, we can choose a constant $C'_3 > 0$ such that

$$\frac{1}{C'_3} \widehat{\varphi}(t) \leq \varphi(t, x) \leq C'_3 \widehat{\varphi}(t), \quad (t, x) \in Q.$$

Set $w(t, x) = y(t, x) \widehat{\varphi}(t)^{\frac{d}{2}}$. By (2.1), the function w satisfies

$$Lw = \frac{g}{(t(T-t))^{\frac{d}{2}}} - \frac{d}{2} \left(\frac{1}{t} - \frac{1}{T-t} \right) w = g \widehat{\varphi}^{\frac{d}{2}} - \frac{d}{2} (T-2t) \widehat{\varphi} w,$$

and

$$w|_\Sigma = 0.$$

Applying to this equation the Carleman estimate (2.11) with $d = 0$, we have

$$\int_Q (s\varphi |\nabla w|^2 + (s\varphi)^3 w^2) e^{2s\alpha} dx dt \\ \leq C''_3 \left(\|g \widehat{\varphi}^{\frac{d}{2}} e^{s\alpha}\|_{L^2(Q)}^2 + \int_Q \widehat{\varphi}^2 w^2 e^{2s\alpha} dx dt + \int_{Q_\omega} (s\varphi)^3 w^2 e^{2s\alpha} dx dt \right) \\ \leq C_3 \left(\|g \widehat{\varphi}^{\frac{d}{2}} e^{s\alpha}\|_{L^2(Q)}^2 + \int_Q \varphi^2 w^2 e^{2s\alpha} dx dt + \int_{Q_\omega} (s\varphi)^3 w^2 e^{2s\alpha} dx dt \right), \\ \forall s \geq s_0(\lambda).$$

Then, increasing the parameter s_0 if necessary, we obtain

$$\begin{aligned} & \int_Q (s\varphi|\nabla w|^2 + (s\varphi)^3 w^2) e^{2s\alpha} dx dt \\ & \leq C_3 \left(\|g\varphi^{\frac{d}{2}} e^{s\alpha}\|_{L^2(Q)}^2 + \int_{Q_\omega} (s\varphi)^3 w^2 e^{2s\alpha} dx dt \right), \\ & \text{for all } s \geq s_0(\lambda, d). \end{aligned}$$

Consequently, the change of $w(t, x) = y(t, x)\hat{\varphi}(t)^{\frac{d}{2}}$ yields (2.11). □

Lemma 2.4. *Let $y_0 \in L^2(\Omega)$, $g \in L^2(0, T; W_2^{-1}(\Omega))$ and conditions (2.3) be fulfilled for b_i , $1 \leq i \leq n$ and c . Moreover let $a_{ij} = a_{ji}$, $1 \leq i, j \leq n$ be Lipschitz continuous on \overline{Q} and let (2.4) be satisfied. Then there exists a solution $y \in L^2(0, T; \overset{\circ}{W}_2^1(\Omega))$ to (2.1) and (2.2) which is unique in $L^2(Q)$, and the estimate is true:*

$$(2.12) \quad \|y\|_{L^2(0, T; W_2^1(\Omega)) \cap C([0, T]; L^2(\Omega))} \leq C(\|y(0, \cdot)\|_{L^2(\Omega)} + \|g\|_{L^2(0, T; W_2^{-1}(\Omega))}),$$

where the constant $C > 0$ depends continuously only on parameter γ .

This lemma can be proved by a usual energy method (see, e.g., [35]) and, for completeness, we will give the proof in Appendix III.

Throughout this section, $C_k > 0$ and $C > 0$ denote generic constants which are independent of parameters s , λ and functions to be estimated.

Remark 2.1. To simplify the situation one can further assume that $y(t, x)$ equals zero in some neighbourhoods of $t = T$ and $t = 0$.

In fact, let us suppose that for such functions, estimate (2.9) is proved. Set

$$\tau_\varepsilon(t) = \begin{cases} 0, & t \in [0, \varepsilon] \cup [T - \varepsilon, T] \\ \frac{t-\varepsilon}{\varepsilon}, & t \in (\varepsilon, 2\varepsilon] \\ 1, & t \in (2\varepsilon, T - 2\varepsilon] \\ \frac{T-\varepsilon-t}{\varepsilon}, & t \in (T - 2\varepsilon, T - \varepsilon]. \end{cases}$$

By (2.9), the function $\tau_\varepsilon(t)y(t, x)$ satisfies the inequality

$$\int_Q ((s\varphi)^{1-2\ell} |\nabla y|^2 + (s\varphi)^{3-2\ell} y^2) \tau_\varepsilon^2 e^{2s\alpha} dx dt$$

$$\begin{aligned} &\leq C_1 \left(\|g\tau_\varepsilon e^{s\alpha}\|_{L^2(0,T;W_2^{-\ell}(\Omega))}^2 + \int_\varepsilon^{2\varepsilon} \int_\Omega \frac{1}{\varepsilon^2} y^2 e^{2s\alpha} dxdt \right. \\ &\quad \left. + \int_{T-2\varepsilon}^{T-\varepsilon} \int_\Omega \frac{1}{\varepsilon^2} y^2 e^{2s\alpha} dxdt + \int_{Q_\omega} (s\varphi)^{3-2\ell} y^2 \tau_\varepsilon^2 e^{2s\alpha} dxdt \right), \end{aligned}$$

for all $s \geq s_0(\lambda)$ and $\ell \in [0, 1]$. We note that there exists a constant $C_4 > 0$ independent of $\varepsilon > 0$ and $(t, x) \in Q$, such that

$$\int_\varepsilon^{2\varepsilon} \int_\Omega \frac{1}{\varepsilon^2} y^2 e^{2s\alpha} dxdt \leq C_4 \int_\varepsilon^{2\varepsilon} \int_\Omega \frac{1}{\varepsilon^2} y^2 \exp\left(-2s\frac{C_4}{\varepsilon}\right) dxdt,$$

because $\alpha(t, x) \leq -\frac{C_4}{\varepsilon}$, $\varepsilon < t < 2\varepsilon$, $x \in \Omega$ by (2.7). Moreover a similar estimate holds for the third integral at the right hand side. Therefore, passing to the limit in this inequality as $\varepsilon \rightarrow 0$ and keeping in mind that $y \in L^2(Q)$, we obtain (2.9). □

Henceforth we set

$$(2.13) \quad J(z, u) = \frac{1}{2} \int_Q (s\varphi)^{2-d} z^2 e^{-2s\alpha} dxdt + \frac{1}{2} \int_{Q_\omega} (s\varphi)^{-1-d} u^2 e^{-2s\alpha} dxdt.$$

Now let us consider the following extremal problem:

$$(2.14) \quad \inf_{(z,u) \in \mathcal{U}} J(z, u),$$

where \mathcal{U} is the totality of $(z, u) \in W^{1,2}(Q) \times L^2(Q)$ satisfying

$$(2.15) \quad L^* z = (s\varphi)^{1+d} y e^{2s\alpha} + \chi_\omega u, \quad z|_\Sigma = 0,$$

and

$$(2.16) \quad z(T, \cdot) = z(0, \cdot) = 0.$$

Here $\lambda > \widehat{\lambda}$, $s > s_0(\lambda, d)$, the parameters $\widehat{\lambda}$, $s_0(\lambda, d)$ are defined in the estimate (2.11) with d substituted by $d - 2$ and

$$L^* y \equiv -\frac{\partial y}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(t, x) \frac{\partial y}{\partial x_j} \right) - \sum_{i=1}^n b_i(t, x) \frac{\partial y}{\partial x_i} + c(t, x)y$$

is an operator formally adjoint to the operator L . Here and henceforth χ_ω denotes the characteristic function of ω .

Proposition 2.1. *Let $b_i \equiv c \equiv 0$, $1 \leq i \leq n$, and let conditions (2.3) and (2.4) hold true. Then there exists a unique solution $(\widehat{z}, \widehat{u}) \in W^{1,2}(Q) \times L^2(Q)$ to the extremal problem (2.14)–(2.16) and*

$$\begin{aligned}
 (2.17) \quad & \int_Q \sum_{|\beta| \leq 1} (s\varphi)^{2-2|\beta|-d} |D^\beta \widehat{z}|^2 e^{-2s\alpha} dxdt \\
 & + \int_0^T (s\widehat{\varphi})^{2-2\ell-d} \|\widehat{z} e^{-s\alpha}\|_{W_2^\ell(\Omega)}^2 dt \\
 & + \int_{Q_\omega} \frac{\widehat{u}^2}{(s\varphi)^{1+d}} e^{-2s\alpha} dxdt \leq C_5 \int_Q (s\varphi)^{1+d} y^2 e^{2s\alpha} dxdt \\
 & \forall s \geq s_0(\lambda, d), \ell \in [0, 1]
 \end{aligned}$$

Proof of Proposition 2.1. Since $y(t, x) \equiv 0$ in neighbourhoods of $t = T$ and $t = 0$, the existence of an admissible element for this problem was proved in [25]. Thus, by standard arguments (see Alekseev, Tikhomirov and Fomin [2], Lions [38], [39] for example), one can prove the existence of a unique solution $(\widehat{z}, \widehat{u}) \in W^{1,2}(Q) \times L^2(Q)$ to the problem (2.14)–(2.16).

We set

$$L^0 y \equiv \frac{\partial y}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(t, x) \frac{\partial y}{\partial x_j} \right) \quad \text{in } Q.$$

In other words, L^0 is the principal part of L defined by (2.1). We apply the Lagrange principle (see [2]) to the problem (2.14) where the admissible set \mathcal{U} of (z, u) is defined by (2.16) and

$$(2.15') \quad (L^0)^* z = (s\varphi)^{1+d} y e^{2s\alpha} + \chi_\omega u, \quad z|_\Sigma = 0.$$

Then we obtain the optimality system for this problem:

$$(2.18) \quad (L^0)^* \widehat{z} = (s\varphi)^{1+d} y e^{2s\alpha} + \chi_\omega \widehat{u}, \quad \widehat{z}|_\Sigma = 0, \quad \widehat{z}(T, \cdot) = \widehat{z}(0, \cdot) = 0,$$

$$(2.19) \quad L^0 p = -(s\varphi)^{2-d} \widehat{z} e^{-2s\alpha}, \quad p|_\Sigma = 0, \quad p|_{Q_\omega} = \frac{\widehat{u}}{(s\varphi)^{1+d}} e^{-2s\alpha}.$$

Applying to (2.19) Carleman estimate (2.11) with d substituted by $d - 2$, we have

$$\begin{aligned}
 (2.20) \quad & \int_Q (s\varphi)^{1+d} p^2 e^{2s\alpha} dxdt \leq C_6 \left(\int_Q (s\varphi)^{2-d} \widehat{z}^2 e^{-2s\alpha} dxdt \right. \\
 & \left. + \int_{Q_\omega} \frac{1}{(s\varphi)^{1+d}} \widehat{u}^2 e^{-2s\alpha} dxdt \right) = 2C_6 J(\widehat{z}, \widehat{u}) \quad \text{for all } s \geq s_0(\lambda, d).
 \end{aligned}$$

Then taking scalar products of (2.18) with p in $L^2(Q)$, integrating by parts and applying (2.19), we obtain

$$\begin{aligned} 0 &= \int_Q ((L^0)^* \widehat{z} - (s\varphi)^{1+d} y e^{2s\alpha} - \chi_\omega \widehat{u}) p dx dt \\ &= \int_Q L^0 p \widehat{z} dx dt - \int_{Q_\omega} \frac{\widehat{u}^2}{(s\varphi)^{1+d}} e^{-2s\alpha} dx dt - \int_Q (s\varphi)^{1+d} y p e^{2s\alpha} dx dt \\ &= -2J(\widehat{z}, \widehat{u}) - \int_Q (s\varphi)^{1+d} y p e^{2s\alpha} dx dt. \end{aligned}$$

Hence, by the Cauchy-Bunyakovskii inequality,

$$\begin{aligned} J(\widehat{z}, \widehat{u}) &= \frac{1}{2} \int_Q (s\varphi)^{1+d} y p e^{2s\alpha} dx dt \\ &\leq C_7 \left(\int_Q (s\varphi)^{1+d} y^2 e^{2s\alpha} dx dt \right)^{\frac{1}{2}} \left(\int_Q (s\varphi)^{1+d} p^2 e^{2s\alpha} dx dt \right)^{\frac{1}{2}}. \end{aligned}$$

Substitution of (2.20) into this inequality yields

$$\begin{aligned} (2.21) \quad J(\widehat{z}, \widehat{u}) &= \frac{1}{2} \int_Q (s\varphi)^{2-d} \widehat{z}^2 e^{-2s\alpha} dx dt + \frac{1}{2} \int_{Q_\omega} (s\varphi)^{-1-d} \widehat{u}^2 e^{-2s\alpha} dx dt \\ &\leq C_8 \int_Q (s\varphi)^{1+d} y^2 e^{2s\alpha} dx dt \quad \text{for all } s \geq s_0(\lambda, d). \end{aligned}$$

Multiplying (2.18) by $(s\varphi)^{-d} e^{-2s\alpha} \widehat{z}$ and integrating by parts, we obtain

$$\begin{aligned} (2.22) \quad &\int_Q |\nabla \widehat{z}|^2 (s\varphi)^{-d} e^{-2s\alpha} dx dt \\ &\leq C_9 \left(\int_Q (s\varphi)^{2-d} |\widehat{z}|^2 e^{-2s\alpha} dx dt + \int_Q (s\varphi)^{1+d} y^2 e^{2s\alpha} dx dt \right. \\ &\quad \left. + \int_Q \frac{\widehat{u}^2}{(s\varphi)^{1+d}} e^{-2s\alpha} dx dt \right) \\ &\leq C_{10} \int_Q (s\varphi)^{1+d} y^2 e^{2s\alpha} dx dt \quad \text{for all } s \geq s_0(\lambda, d). \end{aligned}$$

In fact, applying integration by parts in (2.18), we have

$$\begin{aligned} &-\sum_{i,j=1}^n \int_Q \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial \widehat{z}}{\partial x_j} \right) (s\varphi)^{-d} \widehat{z} e^{-2s\alpha} dx dt \\ &= \sum_{i,j=1}^n \int_Q a_{ij} \frac{\partial \widehat{z}}{\partial x_i} \frac{\partial \widehat{z}}{\partial x_j} (s\varphi)^{-d} e^{-2s\alpha} dx dt \end{aligned}$$

$$- \sum_{i,j=1}^n \int_Q a_{ij} \frac{\partial \widehat{z}}{\partial x_i} \widehat{z} (s\varphi)^{-d} (d\varphi^{-1}\varphi_{x_j} + 2s\alpha_{x_j}) e^{-2s\alpha} dx dt$$

and

$$\begin{aligned} & - \int_Q \frac{\partial \widehat{z}}{\partial t} (s\varphi)^{-d} \widehat{z} e^{-2s\alpha} dx dt \\ & = \int_Q \widehat{z} \frac{\partial \widehat{z}}{\partial t} (s\varphi)^{-d} e^{-2s\alpha} dx dt + \int_Q \widehat{z}^2 \frac{\partial}{\partial t} ((s\varphi)^{-d} e^{-2s\alpha}) dx dt, \end{aligned}$$

namely,

$$- \int_Q \frac{\partial \widehat{z}}{\partial t} (s\varphi)^{-d} \widehat{z} e^{-2s\alpha} dx dt = -\frac{1}{2} \int_Q \widehat{z}^2 (s\varphi)^{-d} (d\varphi^{-1}\varphi_t + 2s\alpha_t) e^{-2s\alpha} dx dt.$$

Therefore the first equation in (2.18) implies

$$\begin{aligned} & \int_Q (L^0)^* \widehat{z} (s\varphi)^{-d} \widehat{z} e^{-2s\alpha} dx dt = -\frac{1}{2} \int_Q \widehat{z}^2 (s\varphi)^{-d} (d\varphi^{-1}\varphi_t + 2s\alpha_t) e^{-2s\alpha} dx dt \\ & + \sum_{i,j=1}^n \int_Q a_{ij} \frac{\partial \widehat{z}}{\partial x_i} \frac{\partial \widehat{z}}{\partial x_j} (s\varphi)^{-d} e^{-2s\alpha} dx dt \\ & - \sum_{i,j=1}^n \int_Q a_{ij} \frac{\partial \widehat{z}}{\partial x_i} \widehat{z} (s\varphi)^{-d} (d\varphi^{-1}\varphi_{x_j} + 2s\alpha_{x_j}) e^{-2s\alpha} dx dt \\ & = \int_Q s\varphi y \widehat{z} dx dt + \int_{Q_\omega} (s\varphi)^{-d} \widehat{u} \widehat{z} e^{-2s\alpha} dx dt. \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{i,j=1}^n \int_Q a_{ij} \frac{\partial \widehat{z}}{\partial x_i} \frac{\partial \widehat{z}}{\partial x_j} (s\varphi)^{-d} e^{-2s\alpha} dx dt \\ & = \int_Q s\varphi y \widehat{z} dx dt + \int_{Q_\omega} (s\varphi)^{-d} \widehat{u} \widehat{z} e^{-2s\alpha} dx dt \\ & + \frac{1}{2} \int_Q \widehat{z}^2 (s\varphi)^{-d} (d\varphi^{-1}\varphi_t + 2s\alpha_t) e^{-2s\alpha} dx dt \\ & + \sum_{i,j=1}^n \int_Q a_{ij} \frac{\partial \widehat{z}}{\partial x_i} \widehat{z} (s\varphi)^{-d} (d\varphi^{-1}\varphi_{x_j} + 2s\alpha_{x_j}) e^{-2s\alpha} dx dt. \end{aligned}$$

On the other hand, we can see

$$(2.23) \quad \left\{ \begin{array}{l} 0 < C_{11}^{-1} \leq \varphi \\ |\varphi_t| \leq C_{12}\varphi^2, \quad |\varphi_{x_j}| \leq C_{12}\varphi \\ |\alpha_t| \leq C_{12}\varphi^2, \quad |\alpha_{x_j}| \leq C_{12}\varphi \quad \text{in } Q \end{array} \right\}.$$

Therefore by the uniform ellipticity (2.4) and (2.23), for a small constant $\varepsilon > 0$, we have

$$\begin{aligned}
& C_{13} \int_Q |\nabla \widehat{z}|^2 (s\varphi)^{-d} e^{-2s\alpha} dxdt \\
& \leq \int_Q \left| (s\varphi)^{\frac{1+d}{2}} y e^{s\alpha} \right| \left| (s\varphi)^{\frac{2-d}{2}} \widehat{z} e^{-s\alpha} \right| dxdt \\
& \quad + \int_{Q_\omega} \left| (s\varphi)^{-\frac{1+d}{2}} \widehat{u} e^{-s\alpha} \right| \left| (s\varphi)^{\frac{2-d}{2}} \widehat{z} e^{-s\alpha} \right| dxdt + \int_Q (s\varphi)^{2-d} |\widehat{z}|^2 e^{-2s\alpha} dxdt \\
& \quad + \sum_{i,j=1}^n \int_Q \left| \varepsilon \frac{\partial \widehat{z}}{\partial x_i} (s\varphi)^{-\frac{d}{2}} e^{-s\alpha} \right| \left| \frac{1}{\varepsilon} \widehat{z} (s\varphi)^{-\frac{d-2}{2}} e^{-s\alpha} \right| dxdt \\
& \leq \left(\int_Q (s\varphi)^{1+d} y^2 e^{2s\alpha} dxdt + \int_Q (s\varphi)^{2-d} |\widehat{z}|^2 e^{-2s\alpha} dxdt \right) \\
& \quad + \left(\int_{Q_\omega} \frac{\widehat{u}^2}{(s\varphi)^{1+d}} e^{-2s\alpha} dxdt + \int_Q (s\varphi)^{2-d} |\widehat{z}|^2 e^{-2s\alpha} dxdt \right) \\
& \quad + \int_Q (s\varphi)^{2-d} |\widehat{z}|^2 e^{-2s\alpha} dxdt \\
& \quad + \varepsilon^2 \sum_{i,j=1}^n \int_Q \left| \frac{\partial \widehat{z}}{\partial x_i} \right|^2 (s\varphi)^{-d} e^{-2s\alpha} dxdt + \frac{1}{\varepsilon^2} \int_Q (s\varphi)^{2-d} |\widehat{z}|^2 e^{-2s\alpha} dxdt.
\end{aligned}$$

Here we have used $|ab| \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$. Taking $\varepsilon > 0$ sufficiently small, we obtain the first inequality in (2.22). The second inequality in (2.22) follows from (2.21). Consequently we have proved (2.17) for $|\beta| \leq 1$. Thus the proof of Proposition 2.1 is complete. \square

Proof of Theorem 2.1. Let \widehat{z} and \widehat{u} be the pair constructed in Proposition 2.1. Then by definition of weak solution, we have

$$\begin{aligned}
0 &= \int_Q y(L^* \widehat{z} - g\widehat{z}) dxdt = \int_Q (y(L^0)^* \widehat{z} - \sum_{i=1}^n b_i y \frac{\partial \widehat{z}}{\partial x_i} + (cy - g)\widehat{z}) dxdt \\
&= \int_Q (s\varphi)^{1+d} y^2 e^{2s\alpha} dxdt + \int_{Q_\omega} \widehat{u} y dxdt - \int_Q \left(\sum_{i=1}^n b_i y \frac{\partial \widehat{z}}{\partial x_i} + (g - cy)\widehat{z} \right) dxdt.
\end{aligned}$$

Hence

$$(2.24) \quad \int_Q (s\varphi)^{1+d} y^2 e^{2s\alpha} dxdt = \int_Q \left(\sum_{i=1}^n b_i y \frac{\partial \widehat{z}}{\partial x_i} + (g - cy)\widehat{z} \right) dxdt - \int_{Q_\omega} \widehat{u} y dxdt.$$

We can directly prove that

$$\begin{aligned}
 (2.25) \quad & \left| \int_Q \left(\sum_{i=1}^n b_i y \frac{\partial \widehat{z}}{\partial x_i} - cy \widehat{z} \right) dx dt \right| \\
 & \leq C(\varepsilon) \left(\sum_{i=1}^n \|b_i y(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(Q)}^2 + \|cy(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(0,T;W_2^{-1}(\Omega))}^2 \right) \\
 & \quad + \varepsilon (\|(\nabla \widehat{z}) e^{-s\alpha} (s\varphi)^{-\frac{d}{2}}\|_{L^2(Q)}^2 + \|\widehat{z} e^{-s\alpha} (s\varphi)^{\frac{1-d}{2}}\|_{L^2(Q)}^2).
 \end{aligned}$$

Let us estimate the first two terms at the right hand side of (2.25). Since $r > 2n$ and the Hölder inequality, we have

$$\begin{aligned}
 (2.26) \quad & \sum_{i=1}^n \|b_i y(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(Q)}^2 \\
 & \leq C'_{14} \int_0^T \sum_{i=1}^n \|b_i(t, \cdot)\|_{L^r(\Omega)}^2 \|y(t, \cdot)(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^{\frac{2r}{r-2}}(\Omega)}^2 dt \\
 & \leq C_{14} \int_0^T \|y(t, \cdot)(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}^2 dt
 \end{aligned}$$

with some $0 < \delta < \frac{r-2n}{2r}$.

By the Sobolev embedding theorem, we have

$$\begin{aligned}
 & \|cy(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(0,T;W_2^{-1}(\Omega))}^2 \leq C'_{15} \int_0^T \sup_{\|v(t, \cdot)\|_{\overset{\circ}{W}_2^1(\Omega)} = 1} \left| \int_{\Omega} cy(s\varphi)^{\frac{d}{2}} e^{s\alpha} v dx \right|^2 dt \\
 & \leq C''_{15} \int_0^T \|c\|_{L^\infty(0,T;W_{r_1}^{-\mu}(\Omega))}^2 \sup_{\|v(t, \cdot)\|_{\overset{\circ}{W}_2^1(\Omega)} = 1} \|y(s\varphi)^{\frac{d}{2}} e^{s\alpha} v\|_{W_{r_1}^{\mu}(\Omega)}^2 dt \\
 & \leq C_{15} \int_0^T \sup_{\|v(t, \cdot)\|_{\overset{\circ}{W}_2^1(\Omega)} = 1} \|y(t, \cdot)v(t, \cdot)(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{W_{r_1}^{\mu}(\Omega)}^2 dt.
 \end{aligned}$$

Henceforth we take the 0-extension of $y(s\varphi)^{\frac{d}{2}} e^{s\alpha}$ and v outside Ω . Then by Lemma 2.2, we obtain

$$\|y(t, \cdot)v(t, \cdot)(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{W_{r_1}^{\mu}(\Omega)}^2 \leq C'_{16} \|v(t, \cdot)\|_{\overset{\circ}{W}_2^1(\Omega)} \|y(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}$$

and so we see

$$(2.27) \quad \|cy(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(0,T;W_2^{-1}(\Omega))}^2 \leq C_{16} \int_0^T \|y(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}^2 dt.$$

From (2.25)–(2.27), we obtain

$$(2.28) \quad \left| \int_Q \left(\sum_{i=1}^n b_i y \frac{\partial \widehat{z}}{\partial x_i} - cy\widehat{z} \right) dx dt \right| \leq C(\varepsilon) \int_0^T \|y(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}^2 dt \\ + \varepsilon \left(\|(\nabla \widehat{z})e^{-s\alpha}(s\varphi)^{-\frac{d}{2}}\|_{L^2(Q)}^2 + \|\widehat{z}e^{-s\alpha}(s\varphi)^{\frac{1-d}{2}}\|_{L^2(Q)}^2 \right).$$

Thanks to (2.17), noting that $|ab| \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$, we have

$$(2.29) \quad \left| \int_{Q_\omega} \widehat{u} y dx dt \right| = \left| \int_{Q_\omega} \widehat{u}(s\varphi)^{-\frac{1+d}{2}} e^{-s\alpha} y(s\varphi)^{\frac{1+d}{2}} e^{s\alpha} dx dt \right| \\ \leq \frac{1}{2} \int_{Q_\omega} \frac{\widehat{u}^2}{(s\varphi)^{1+d}} e^{-2s\alpha} dx dt + \frac{1}{2} \int_{Q_\omega} y^2 (s\varphi)^{1+d} e^{2s\alpha} dx dt \\ \leq \frac{C_5}{2} \int_{Q_\omega} y^2 (s\varphi)^{1+d} e^{2s\alpha} dx dt + \frac{1}{2} \int_{Q_\omega} y^2 (s\varphi)^{1+d} e^{2s\alpha} dx dt.$$

By (2.17), (2.28) and (2.29), we obtain from (2.24)

$$(2.30) \quad \int_Q (s\varphi)^{1+d} y^2 e^{2s\alpha} dx dt \\ \leq C_{17} \left(\left| \int_Q g \widehat{z} dx dt \right| + \int_0^T \|y(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}^2 dt \right. \\ \left. + \int_{Q_\omega} (s\varphi)^{1+d} y^2 e^{2s\alpha} dx dt \right) \quad \text{for all } s \geq s_0(\lambda, d).$$

Taking scalar products of (2.1) with $(s\varphi)^{d-1} y e^{2s\alpha}$ in $L^2(Q)$, we obtain

$$\int_\Omega \left(\int_0^T \frac{\partial y}{\partial t} y e^{2s\alpha} (s\varphi)^{d-1} dt \right) dx \\ - \int_0^T \left(\int_\Omega \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial y}{\partial x_j} \right) (s\varphi)^{d-1} e^{2s\alpha} y dx \right) dt \\ = \int_Q \left(\sum_{i=1}^n b_i y \frac{\partial}{\partial x_i} ((s\varphi)^{d-1} y e^{2s\alpha}) - cy^2 (s\varphi)^{d-1} e^{2s\alpha} \right) dx dt \\ + \int_Q g y e^{2s\alpha} (s\varphi)^{d-1} dx dt.$$

By integration by parts, $\lim_{t \uparrow T} \alpha(t, x) = \lim_{t \downarrow 0} \alpha(t, x) = -\infty$ and $y|_\Sigma = 0$, we have

$$\int_Q y \frac{\partial y}{\partial t} e^{2s\alpha} (s\varphi)^{d-1} dx dt = -\frac{1}{2} \int_Q y^2 \frac{\partial}{\partial t} ((s\varphi)^{d-1} e^{2s\alpha}) dt dx$$

and

$$\begin{aligned}
 & - \int_0^T \left(\int_{\Omega} \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial y}{\partial x_j} \right) y (s\varphi)^{d-1} e^{2s\alpha} dx \right) dt \\
 & = \int_Q \sum_{i,j=1}^n a_{ij} \frac{\partial y}{\partial x_j} \frac{\partial y}{\partial x_i} (s\varphi)^{d-1} e^{2s\alpha} dx dt \\
 & \quad + \int_Q \sum_{i,j=1}^n a_{ij} \frac{\partial y}{\partial x_j} y \frac{\partial}{\partial x_i} ((s\varphi)^{d-1} e^{2s\alpha}) dx dt.
 \end{aligned}$$

Hence

$$\begin{aligned}
 (2.31) \quad & \int_Q \sum_{i,j=1}^n a_{ij} \frac{\partial y}{\partial x_j} \frac{\partial y}{\partial x_i} (s\varphi)^{d-1} e^{2s\alpha} dx dt = \frac{1}{2} \int_Q y^2 \frac{\partial}{\partial t} ((s\varphi)^{d-1} e^{2s\alpha}) dx dt \\
 & - \int_Q \sum_{i,j=1}^n a_{ij} \frac{\partial y}{\partial x_j} y \frac{\partial}{\partial x_i} ((s\varphi)^{d-1} e^{2s\alpha}) dx dt \\
 & + \int_Q \left(\sum_{i=1}^n b_{iy} \frac{\partial}{\partial x_i} ((s\varphi)^{d-1} y e^{2s\alpha}) - cy^2 (s\varphi)^{d-1} e^{2s\alpha} \right) dx dt \\
 & + \int_Q gy (s\varphi)^{d-1} e^{2s\alpha} dx dt.
 \end{aligned}$$

By (2.26) and (2.27), in terms of the Cauchy-Bunyakovkii inequality, we have

$$\begin{aligned}
 (2.32) \quad & \left| \int_Q \left\{ \sum_{i=1}^n b_{iy} \frac{\partial}{\partial x_i} ((s\varphi)^{d-1} y e^{2s\alpha}) - cy^2 (s\varphi)^{d-1} e^{2s\alpha} \right\} dx dt \right| \\
 & \leq C(\varepsilon) \left(\sum_{i=1}^n \|b_{iy} (s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(Q)}^2 + \|cy (s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(0,T;W_2^{-1}(\Omega))}^2 \right) \\
 & \quad + \varepsilon \int_Q (s\varphi)^{-d} e^{-2s\alpha} \left| \frac{\partial}{\partial x_i} ((s\varphi)^{d-1} y e^{2s\alpha}) \right|^2 dx dt \\
 & \quad + \varepsilon \sum_{i=1}^n \int_Q \left| \frac{\partial}{\partial x_i} (y e^{s\alpha} (s\varphi)^{\frac{d-2}{2}}) \right|^2 dx dt \\
 & \leq C(\varepsilon) \int_0^T \|y (s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}^2 dt \\
 & \quad + \varepsilon \int_Q (|\nabla y|^2 (s\varphi)^{d-1} e^{2s\alpha} + (s\varphi)^{1+d} y^2 e^{2s\alpha}) dx dt.
 \end{aligned}$$

By (2.23), we have

$$(2.33) \quad \left| \frac{\partial}{\partial t} ((s\varphi)^{d-1} e^{2s\alpha}) \right| \leq C_{18} (s\varphi)^{d+1} e^{2s\alpha}$$

and

$$(2.34) \quad \left| \frac{\partial}{\partial x_i} ((s\varphi)^{d-1} e^{2s\alpha}) \right| \leq C_{18} (s\varphi)^d e^{2s\alpha}, \quad 1 \leq i \leq n$$

for large $s > 0$. Therefore, in terms of (2.4), using (2.31)–(2.34), we obtain

$$(2.35) \quad \begin{aligned} \int_Q |\nabla y|^2 (s\varphi)^{d-1} e^{2s\alpha} dx dt &\leq C_{19} \int_Q y^2 (s\varphi)^{d+1} e^{2s\alpha} dx dt \\ &+ C_{19} \int_Q \left((s\varphi)^{\frac{d-1}{2}} |\nabla y| e^{s\alpha} \right) \left((s\varphi)^{\frac{d+1}{2}} y e^{s\alpha} \right) dx dt \\ &+ C_{19} \left| \int_Q g e^{s\alpha} (s\varphi)^{d-1} y e^{s\alpha} dx dt \right| \\ &+ C(\varepsilon) \int_0^T \|y (s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}^2 dt \\ &+ \varepsilon \int_Q (|\nabla y|^2 (s\varphi)^{d-1} e^{2s\alpha} + (s\varphi)^{1+d} y^2 e^{2s\alpha}) dx dt. \end{aligned}$$

In (2.35), we note that

$$\begin{aligned} &\left| \int_Q g e^{s\alpha} (s\varphi)^{d-1} y e^{s\alpha} dx dt \right| \\ &\leq C(\varepsilon) \|g (s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(0,T;W_2^{-1}(\Omega))}^2 + \varepsilon \|\nabla((s\varphi)^{\frac{d}{2}-1} y e^{s\alpha})\|_{L^2(0,T;L^2(\Omega))}^2 \\ &\leq C(\varepsilon) \|g (s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(0,T;W_2^{-1}(\Omega))}^2 \\ &\quad + \varepsilon \int_Q (s\varphi)^{d-1} |\nabla y|^2 e^{2s\alpha} dx dt + C\varepsilon \int_Q (s\varphi)^{d+1} y^2 e^{2s\alpha} dx dt. \end{aligned}$$

Consequently, combining (2.30) and (2.35) and taking a sufficiently small $\varepsilon > 0$, in terms of the Cauchy-Bunyakovskii inequality, we obtain

$$(2.36) \quad \begin{aligned} I(s) &\equiv \int_Q (|\nabla y|^2 (s\varphi)^{d-1} + (s\varphi)^{1+d} y^2) e^{2s\alpha} dx dt \\ &\leq C_{20} \left(\left| \int_Q g \widehat{z} dx dt \right| + \|g (s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(0,T;W_2^{-1}(\Omega))}^2 \right. \\ &\quad \left. + \int_0^T \|y (s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}^2 dt + \int_{Q_\omega} (s\varphi)^{1+d} y^2 e^{2s\alpha} dx dt \right) \\ &\quad \text{for all } s \geq s_0(\lambda, d). \end{aligned}$$

We can prove an interpolation inequality:

$$s^{2\delta} \|y(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(0,T;W_2^{\frac{1}{2}-\delta}(\Omega))}^2 \leq C_{21} I(s) \quad \text{whenever } \delta \in \left(0, \frac{1}{2}\right),$$

so that we obtain from (2.36)

$$(2.37) \quad \begin{aligned} I(s) &= \int_Q (|\nabla y|^2 (s\varphi)^{d-1} + (s\varphi)^{1+d} y^2) e^{2s\alpha} dx dt \\ &\leq C_{22} \left(\left| \int_Q g \widehat{z} dx dt \right| + \|g(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(0,T;W_2^{-1}(\Omega))}^2 + \int_{Q_\omega} (s\varphi)^{1+d} y^2 e^{2s\alpha} dx dt \right) \end{aligned}$$

for all $s \geq s_0(\lambda, d)$.

Let $g = g_0 + \sum_{i=1}^n \frac{\partial}{\partial x_i} g_i$. Then $\int_Q g \widehat{z} dx dt = \int_Q \left(g_0 \widehat{z} - \sum_{i=1}^n g_i \frac{\partial \widehat{z}}{\partial x_i} \right) dx dt$.

Consequently, by (2.17) and the Cauchy-Bunyakovskii inequality, for any $\varepsilon > 0$, there exists a constant $C_{23}(\varepsilon) > 0$ such that

$$(2.38) \quad \begin{aligned} \left| \int_Q g \widehat{z} dx dt \right| &\leq \|g_0(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(0,T;W_2^{-1}(\Omega))} \| (s\varphi)^{-\frac{d}{2}} e^{-s\alpha} \widehat{z} \|_{L^2(0,T;W_2^1(\Omega))} \\ &\quad + \sum_{i=1}^n \|g_i(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(Q)} \| \nabla \widehat{z} (s\varphi)^{-\frac{d}{2}} e^{-s\alpha} \|_{L^2(Q)} \\ &\leq C_{23}(\varepsilon) \|g_0(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(0,T;W_2^{-1}(\Omega))}^2 \\ &\quad + C_{23}(\varepsilon) \sum_{i=1}^n \|g_i(s\varphi)^{\frac{d}{2}} e^{s\alpha}\|_{L^2(Q)}^2 + \varepsilon \int_Q (s\varphi)^{1+d} y^2 e^{2s\alpha} dx dt. \end{aligned}$$

Thus (2.10) follows from (2.37) and (2.38).

Let $d = 2 - 2\ell$. We note that $\widehat{\varphi}(t)$ is defined by (2.6). Then the duality, the Hölder inequality and the interpolation inequality (e.g. [1]), yield

$$\begin{aligned} &\left| \int_Q g e^{s\alpha} (s\varphi)^{1-2\ell} y e^{s\alpha} dx dt \right| \\ &\leq \frac{1}{\varepsilon} \|g e^{s\alpha}\|_{L^2(0,T;W_2^{-\ell}(\Omega))} \times \varepsilon \| (s\varphi)^{1-2\ell} y e^{s\alpha} \|_{L^2(0,T;W_2^\ell(\Omega))} \end{aligned}$$

and

$$\begin{aligned} &\varepsilon \| (s\varphi)^{1-2\ell} y e^{s\alpha}(t, \cdot) \|_{W_2^\ell(\Omega)} \\ &\leq C_{24} \varepsilon \left\{ (s\widehat{\varphi})^{\frac{\ell-2\ell^2}{2}} \|y e^{s\alpha}(t, \cdot)\|_{W_2^1(\Omega)}^\ell \right\} \left\{ (s\widehat{\varphi})^{1-2\ell-\frac{\ell-2\ell^2}{2}} \|y e^{s\alpha}(t, \cdot)\|_{L^2(\Omega)}^{1-\ell} \right\} \end{aligned}$$

$$\begin{aligned} &\leq C_{24}\varepsilon(s\widehat{\varphi})^{\frac{1-2\ell}{2}}\|ye^{s\alpha}(t, \cdot)\|_{W_2^1(\Omega)} + C_{24}\varepsilon(s\widehat{\varphi})^{\frac{2\ell^2-5\ell+2}{2(1-\ell)}}\|ye^{s\alpha}(t, \cdot)\|_{L^2(\Omega)} \\ &\leq C_{25}\varepsilon\|(s\varphi)^{\frac{1-2\ell}{2}}ye^{s\alpha}(t, \cdot)\|_{W_2^1(\Omega)} + C_{25}\varepsilon\|(s\varphi)^{\frac{3-2\ell}{2}}ye^{s\alpha}(t, \cdot)\|_{L^2(\Omega)}. \end{aligned}$$

Here and henceforth, we have also used $C_3'\widehat{\varphi}(t) \leq \varphi(t, x) \leq C_3''\widehat{\varphi}(t)$ for $(t, x) \in \overline{Q}$ and $\frac{2\ell^2-5\ell+2}{2(1-\ell)} \leq \frac{3-2\ell}{2}$.

Therefore we have

$$\begin{aligned} (2.39) \quad &\left| \int_Q ge^{s\alpha}(s\varphi)^{1-2\ell}ye^{s\alpha} dx dt \right| \\ &\leq \frac{1}{\varepsilon^2}\|ge^{s\alpha}\|_{L^2(0,T;W_2^{-\ell}(\Omega))} + C_{26}\varepsilon^2 \int_Q ((s\varphi)^{1-2\ell}|\nabla y|^2 + (s\varphi)^{3-2\ell}y^2)e^{2s\alpha} dx dt. \end{aligned}$$

Then, similarly to (2.37), we apply (2.35) and (2.39) where we choose $\varepsilon > 0$ sufficiently small, so that we obtain

$$\begin{aligned} &\int_Q ((s\varphi)^{1-2\ell}|\nabla y|^2 + (s\varphi)^{3-2\ell}y^2)e^{2s\alpha} dx dt \\ &\leq C_{27} \left(\int_0^T \|\widehat{z}e^{-s\alpha}\|_{W_2^\ell(\Omega)}^2 dt \right)^{\frac{1}{2}} \left(\int_0^T \|ge^{s\alpha}\|_{W_2^{-\ell}(\Omega)}^2 dt \right)^{\frac{1}{2}} \\ &\quad + C_{27} \int_{Q_\omega} (s\varphi)^{3-2\ell}y^2e^{2s\alpha} dx dt \\ &\leq C_{27}\|ge^{s\alpha}\|_{L^2(0,T;W_2^{-\ell}(\Omega))}^2 + C_{27} \int_{Q_\omega} (s\varphi)^{3-2\ell}y^2e^{2s\alpha} dx dt. \end{aligned}$$

This inequality implies (2.9). The proof of theorem is complete. □

Proof of Corollary 2.1. We will approximate a_{ij} by W_∞^1 -functions with the aid of the mollifiers (e.g. Adams [1]). Let $\kappa \in C^\infty(\mathbb{R}^{n+1})$, $\int_{\mathbb{R}^{n+1}} \kappa(t, x) dt dx = 1$, $\kappa(t, x) \geq 0$ for all $(t, x) \in \mathbb{R}^{n+1}$ and $\text{supp } \kappa \subset \{(t, x) \mid |(t, x)| \leq 1\}$. Set $a_{ij}^\varepsilon(t, x) = \frac{1}{\varepsilon^{n+1}} \int_{\mathbb{R}^{n+1}} \kappa\left(\frac{t-t'}{\varepsilon}, \frac{x-x'}{\varepsilon}\right) a_{ij}(t', x') dt' dx'$.

Then, since a_{ij} are Lipschitz continuous on \overline{Q} , we can see that

$$(2.40) \quad a_{ij}^\varepsilon \rightarrow a_{ij} \quad \text{in } C(\overline{Q}), \quad \|a_{ij}^\varepsilon\|_{W_\infty^1(Q)} \leq C, \quad 1 \leq i, j \leq n, \varepsilon > 0.$$

Here $C > 0$ is independent of $\varepsilon > 0$. Therefore for $\{a_{ij}^\varepsilon\}_{1 \leq i, j \leq n}$ with any $\varepsilon > 0$, the constant γ in (2.8) is bounded and (2.4) is true with the same $\beta > 0$. Let L^ε be the linear parabolic operator obtained from L after change of the coefficients a_{ij} by a_{ij}^ε . Let us consider the boundary value problem

$$(2.41) \quad L^\varepsilon y_\varepsilon = g \quad \text{in } Q, \quad y_\varepsilon|_\Sigma = 0, \quad y_\varepsilon(0, \cdot) = 0.$$

By (2.40) and Lemma 2.4, we can prove that

$$(2.42) \quad y_\varepsilon \rightarrow y \quad \text{in } L^2(0, T; W_2^1(\Omega)) \cap C([0, T]; L^2(\Omega)),$$

as $\varepsilon \rightarrow 0$, where $y(t, x)$ is a solution to (2.1) and (2.2) with $y_0 = 0$. Moreover by Theorem 2.1 for a solution to (2.41), inequalities (2.9) and (2.10) hold true with the constants C_1 and C_2 independent of ε . Passing to the limit in these inequalities and keeping in mind (2.42), we complete the proof of corollary. \square

§3. Exact Controllability of Semilinear Parabolic Equations

Henceforth $a_{ij}, b_i, 1 \leq i, j \leq n$ and c are assumed to satisfy (2.3). We consider the semilinear parabolic equation

$$(3.1) \quad G(y) = \frac{\partial y}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(t, x) \frac{\partial y}{\partial x_j} \right) + \sum_{i=1}^n b_i(t, x) \frac{\partial y}{\partial x_i}(t, x) + c(t, x)y + f(t, x, \nabla y, y) = u + g \quad \text{in } Q \text{ with } u \in \mathcal{U}(\omega),$$

and

$$(3.2) \quad y|_\Sigma = 0, \quad y(0, x) = v_0(x),$$

where v_0 and g are given, and $u(t, x)$ is a locally distributed control in the space

$$(3.3) \quad \mathcal{U}(\omega) = \{u(t, x) \in L^2(Q) \mid \text{supp } u \subset \overline{Q_\omega}\}.$$

By the exact controllability, we mean a problem of finding a control $u \in \mathcal{U}(\omega)$ such that

$$(3.4) \quad y(T, x) = v_1(x), \quad x \in \Omega,$$

where $v_1(x)$ is a given function.

In this paper we also consider the exact boundary controllability, by which we mean a problem of finding a boundary control $u(t, x)$ such that

$$(3.5) \quad G(y) = g \quad \text{in } Q, \quad y(0, x) = v_0(x), \quad y(T, x) = v_1(x),$$

$$(3.6) \quad y|_{]0, T[\times \Gamma_0} = u, \quad y|_{]0, T[\times (\partial\Omega \setminus \Gamma_0)} = 0,$$

where $\Gamma_0 \subset \partial\Omega$ is an arbitrary fixed subboundary, and v_0, v_1, g are given functions.

For a semilinear term f , let us assume that

$$(3.7) \quad f(t, x, \zeta', \zeta_0) \in C^1(\overline{Q} \times \mathbb{R}^{n+1}), \quad f(t, x, 0, 0) = 0, \quad \forall (t, x) \in Q,$$

and

$$(3.8) \quad \left| \frac{\partial f(t, x, \zeta', \zeta_0)}{\partial \zeta_i} \right| \leq K, \quad \forall (t, x) \in Q, \quad \forall \zeta \equiv (\zeta', \zeta_0) = (\zeta_1, \dots, \zeta_n, \zeta_0) \in \mathbb{R}^{n+1}$$

for $0 \leq i \leq n$. Set

$$(3.9) \quad \eta(t, x) = (-e^{\lambda\psi(x)} + e^{2\lambda\|\psi\|_{C(\bar{\Omega})}})/((T-t)\ell(t)),$$

where

$$(3.10) \quad \ell \in C^\infty[0, T], \quad \ell(t) > 0, \quad \ell(t) \geq t, \quad \forall t \in [0, T] \text{ and } \ell(t) = t, \quad \forall t \in \left[\frac{T}{2}, T\right].$$

We set

$$(3.11) \quad L_0 y = \frac{\partial y}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(t, x) \frac{\partial y}{\partial x_j} \right) + \sum_{i=1}^n b_i(t, x) \frac{\partial y}{\partial x_i}(t, x) + c(t, x)y, \quad (t, x) \in Q.$$

Henceforth we define a weighted L^2 -space with a weight function $\kappa(t, x) > 0$ for almost all $(t, x) \in Q$:

$$L^2(Q, \kappa) = \left\{ y \mid \int_Q |y(t, x)|^2 \kappa(t, x) dx dt < \infty \right\}$$

with the norm

$$\|y\|_{L^2(Q, \kappa)} = \left(\int_Q |y(t, x)|^2 \kappa(t, x) dx dt \right)^{\frac{1}{2}}.$$

Now, in order to formulate our results, we introduce the function spaces

$$(3.12) \quad X_s^\lambda(Q) = L^2(Q, (T-t)e^{2s\eta}),$$

$$(3.13) \quad Z_s^\lambda(Q) = \{y \mid y|_\Sigma = 0, y, \nabla y \in L^2(Q, e^{2s\eta}), L_0 y \in X_s^\lambda(Q)\}$$

with the norm

$$(3.14) \quad \|y\|_{Z_s^\lambda(Q)}^2 = \|L_0 y\|_{X_s^\lambda(Q)}^2 + \|y\|_{L^2(Q, e^{2s\eta})}^2 + \|\nabla y\|_{(L^2(Q, e^{2s\eta}))^n}^2,$$

and

$$(3.15) \quad Y(Q) = \{y(t, x) \mid L_0 y \in L^2(0, T; L^2(\Omega)), y|_\Sigma = 0, y(0, \cdot) \in W_2^1(\Omega)\}$$

with the norm

$$(3.16) \quad \|y\|_{Y(Q)}^2 = \|L_0y\|_{L^2(0,T;L^2(\Omega))}^2 + \|y(0, \cdot)\|_{W_2^1(\Omega)}^2.$$

We state our main result which is the global exact zero controllability for semi-linear parabolic equation (3.1).

Theorem 3.1. *Let $v_0 \in \overset{\circ}{W}_2^1(\Omega)$, $v_1 \equiv 0$, and let conditions (2.3)–(2.4), (3.7) and (3.8) be fulfilled. Then there exists $\widehat{\lambda} > 0$ such that for $\lambda \geq \widehat{\lambda}$ there exists a constant $s_0(\lambda)$ such that for $g \in X_s^\lambda(Q)$, with $\lambda \geq \widehat{\lambda}$, $s \geq s_0(\lambda)$ there exists a solution pair $(y, u) \in Y(Q) \times \mathcal{U}(\omega)$ to (3.1), (3.2) and (3.4).*

First we prove the existence of solution for a controllability problem in the case of linear parabolic equation.

$$(3.17) \quad L_0y = g + u, \quad u \in \mathcal{U}(\omega), \quad y|_\Sigma = 0, \quad y(0, x) = v_0, \quad y(T, x) = 0.$$

We have

Lemma 3.1. *Let $\lambda \geq \widehat{\lambda}$ and $v_0 \in \overset{\circ}{W}_2^1(\Omega)$, $v_1 \equiv 0$, and let conditions (2.3)–(2.4) be fulfilled. Then there exists a constant $s_0(\lambda) > 0$ such that if $g \in X_s^\lambda(Q)$ with $s \geq s_0(\lambda)$, then the problem (3.17) has a solution $(y, u) \in (Y(Q) \cap Z_s^\lambda(Q)) \times (\mathcal{U}(\omega) \cap X_s^\lambda(Q))$ which satisfies the following estimate:*

$$(3.18) \quad \|(y, u)\|_{(Y(Q) \cap Z_s^\lambda(Q)) \times (\mathcal{U}(\omega) \cap X_s^\lambda(Q))} \leq C(\lambda, s, \gamma)(\|v_0\|_{W_2^1(\Omega)} + \|g\|_{X_s^\lambda(Q)}).$$

Proof. We recall that the parameters $\widehat{\lambda}$ and $s_0(\lambda)$ were defined in Theorem 2.1. For $k \in \mathbb{N}$, let us consider the extremal problem

$$(3.19) \quad \begin{aligned} \mathcal{J}_k(y, u) &= \frac{1}{2} \int_Q \rho_k (|\nabla y|^2 + y^2) dxdt \\ &\quad + \frac{1}{2} \int_Q (T-t)e^{2s\eta(t,x)} m_k u^2 dxdt \rightarrow \inf, \end{aligned}$$

$$(3.20) \quad L_0y = g + u \quad \text{in } Q, \quad y|_\Sigma = 0, \quad y(0, x) = v_0, \quad y(T, x) = 0,$$

where

$$(3.21) \quad \rho_k(t, x) = \exp\left(\frac{2s\eta(t, x)(T-t)}{T-t+1/k}\right), \quad m_k(x) = \begin{cases} 1, & x \in \overline{\omega}, \\ k, & x \in \Omega \setminus \overline{\omega}, \end{cases}$$

and the parameters $s \geq s_0(\lambda)$, $\lambda \geq \hat{\lambda}$ are fixed.

It is easy to prove (see Lions [38], [39]) that the problem (3.19)–(3.20) has a unique solution, which we denote by $(\hat{y}_k, \hat{u}_k) \in Y(Q) \times L^2(Q)$.

Applying the Lagrange principle to the problem (3.19)–(3.20) (see [2], [38]), we obtain

$$(3.22) \quad L_0 \hat{y}_k = g + \hat{u}_k \quad \text{in } Q, \quad \hat{y}_k|_{\Sigma} = 0, \quad \hat{y}_k(T, \cdot) \equiv 0, \quad \hat{y}_k(0, \cdot) = v_0,$$

$$(3.23) \quad L_0^* p_k = \nabla \cdot (\rho_k \nabla \hat{y}_k) - \rho_k \hat{y}_k \quad \text{in } Q, \\ p_k|_{\Sigma} = 0, \quad p_k - (T - t)e^{2s\eta(t,x)} m_k \hat{u}_k = 0 \quad \text{in } Q.$$

Henceforth $C = C(\lambda, s) > 0$ denotes a generic constant which is dependent on λ and s , but independent of k .

For the proof of Lemma 3.1, we will prove

Lemma 3.2.

(3.24)

$$\int_Q e^{-2s\eta} \frac{|p_k|^2}{T-t} dxdt + \int_{\Omega} |p_k(0, x)|^2 dx \\ \leq C(\lambda, s) \left(\int_Q \rho_k^2 e^{-2s\eta} (|\nabla \hat{y}_k|^2 + |\hat{y}_k|^2) dxdt + \int_{Q_\omega} e^{-2s\eta} \frac{p_k^2}{T-t} dxdt \right).$$

Proof of Lemma 3.2. First applying to (3.23) estimate (2.10) with $d = 0$, we have

$$\int_Q s\varphi |p_k|^2 e^{2s\alpha} dxdt \leq C \int_Q \rho_k^2 e^{2s\alpha} (|\nabla \hat{y}_k|^2 + |\hat{y}_k|^2) dxdt + C \int_{Q_\omega} s\varphi |p_k|^2 e^{2s\alpha} dxdt.$$

Hence

$$\int_Q \frac{1}{t(T-t)} |p_k|^2 e^{2s\alpha} dxdt \\ \leq C \int_Q \rho_k^2 e^{2s\alpha} (|\nabla \hat{y}_k|^2 + |\hat{y}_k|^2) dxdt + C \int_{Q_\omega} \frac{1}{t(T-t)} |p_k|^2 e^{2s\alpha} dxdt.$$

By definition (3.9), we have

$$-\eta(t, x) = \alpha(t, x), \quad \frac{T}{2} \leq t \leq T$$

and

$$-\eta(t, x) \geq \alpha(t, x), \quad (t, x) \in Q.$$

Moreover we have

$$\frac{1}{t}e^{2s\alpha} \leq C(s)e^{-2s\eta}, \quad 0 < t < T,$$

where $C(s) > 0$ depends on $s > 0$. In fact, this is equivalent to

$$\mu_s(x, t) \equiv \frac{1}{t} \exp \left(2s \left(\frac{1}{t(T-t)} - \frac{1}{(T-t)\ell(t)} \right) (e^{\lambda\psi(t,x)} - e^{2\lambda\|\psi\|_{C(\bar{\Omega})}}) \right) \leq C(s)$$

for $0 \leq t \leq T$ and $x \in \bar{\Omega}$. For this, it is sufficient to verify $\lim_{t \rightarrow 0} \mu_s(x, t) < \infty$ for $s > 0$ and $x \in \bar{\Omega}$. By $\ell(t) \geq t$ for $0 \leq t \leq T$ and $\ell(0) > 0$ from (3.10), we can directly see that

$$\lim_{t \rightarrow 0} \mu_s(x, t) = 0$$

for $s > 0$ and $x \in \bar{\Omega}$.

Therefore

$$(3.25) \quad \int_{\frac{T}{2}}^T \int_{\Omega} \frac{1}{t(T-t)} |p_k|^2 e^{-2s\eta} dx dt \\ \leq C \int_Q \rho_k^2 e^{-2s\eta} (|\nabla \hat{y}_k|^2 + |\hat{y}_k|^2) dx dt + C \int_{Q_\omega} \frac{1}{T-t} |p_k|^2 e^{-2s\eta} dx dt.$$

Let $\chi = \chi(t) \in C^\infty[0, T]$ be a function such that $1 \geq \chi(t) \geq 0$ for $t \in [0, T]$, $\chi(t) = 1$ for $t \in [0, \frac{T}{2}]$, $\chi(t) = 0$ for $t \geq \frac{3T}{4}$. Multiplying (3.23) by $p_k \chi$ and taking scalar products in $L^2(\Omega)$ and integrating by parts, we obtain

$$(3.26) \quad -\frac{1}{2} \frac{d}{dt} \int_{\Omega} \chi p_k^2 dx + \int_{\Omega} \sum_{i,j=1}^n a_{ij} \frac{\partial p_k}{\partial x_i} \frac{\partial p_k}{\partial x_j} \chi(t) dx = -\frac{1}{2} \int_{\Omega} \frac{d\chi}{dt} p_k^2 dx \\ - \int_{\Omega} \left(\sum_{i=1}^n b_i p_k \chi \frac{\partial p_k}{\partial x_i} + c p_k^2 \chi \right) dx - \int_{\Omega} (\rho_k \chi \nabla \hat{y}_k \cdot \nabla p_k + \rho_k \hat{y}_k p_k \chi(t)) dx.$$

Here, similarly to (2.26), we use the interpolation inequality (e.g., [1]) to obtain

$$(3.27) \quad \left| \int_{\Omega} b_i p_k \chi \frac{\partial p_k}{\partial x_i} dx \right| \leq \|b_i p_k \sqrt{\chi}\|_{L^2(\Omega)} \left\| \frac{\partial p_k}{\partial x_i} \sqrt{\chi} \right\|_{L^2(\Omega)} \\ \leq C \|p_k \sqrt{\chi}\|_{W_2^{\frac{1}{2}-\delta}(\Omega)} \left\| \frac{\partial p_k}{\partial x_i} \sqrt{\chi} \right\|_{L^2(\Omega)} \\ \leq C \left(\varepsilon \|p_k \sqrt{\chi}\|_{W_2^1(\Omega)} + C(\varepsilon) \|p_k \sqrt{\chi}\|_{L^2(\Omega)} \right) \times \|p_k \sqrt{\chi}\|_{W_2^1(\Omega)}$$

$$\leq C\varepsilon\|p_k\sqrt{\chi}\|_{W_2^1(\Omega)}^2 + C(\varepsilon)\|p_k\sqrt{\chi}\|_{L^2(\Omega)}^2.$$

Moreover, by (2.3), Lemma 2.2 and the interpolation inequality, we see that

$$\begin{aligned} (3.28) \quad \left| \int_{\Omega} c\chi p_k^2 dx \right| &\leq \|c\|_{W_{r_1}^{-\mu}(\Omega)} \|\sqrt{\chi}p_k\sqrt{\chi}p_k\|_{W_{r_1'}^{\mu}(\Omega)} \\ &\leq C\|\sqrt{\chi}p_k\|_{W_2^1(\Omega)} \|\sqrt{\chi}p_k\|_{W_2^{\frac{1}{2}-\delta}(\Omega)} \\ &\leq C\|\sqrt{\chi}p_k\|_{W_2^1(\Omega)} \left(\varepsilon\|\sqrt{\chi}p_k\|_{W_2^1(\Omega)} + C(\varepsilon)\|\sqrt{\chi}p_k\|_{L^2(\Omega)} \right) \\ &\leq \varepsilon\|\sqrt{\chi}p_k\|_{W_2^1(\Omega)}^2 + C(\varepsilon)\|\sqrt{\chi}p_k\|_{L^2(\Omega)}^2. \end{aligned}$$

Noting (2.4) and

$$2|\rho_k\chi\nabla\hat{y}_k \cdot \nabla p_k| \leq 2|\rho_k\sqrt{\chi}\nabla\hat{y}_k \cdot \nabla p_k| \leq \varepsilon\chi|\nabla p_k|^2 + \frac{1}{\varepsilon}\rho_k^2|\nabla\hat{y}_k|^2,$$

and taking sufficiently small $\varepsilon > 0$, we apply (2.4), (3.27) and (3.28) to (3.26), so that we obtain

$$\begin{aligned} -\frac{1}{2}\frac{d}{dt}\int_{\Omega}\chi p_k^2 dx + \int_{\Omega}|\nabla p_k|^2\chi(t) dx \\ \leq C\int_{\Omega}\left\{\left(\left|\frac{d\chi}{dt}\right| + \chi\right)p_k^2 + \rho_k^2(|\nabla\hat{y}_k|^2 + |\hat{y}_k|^2)\right\} dx, \end{aligned}$$

that is,

$$(3.29) \quad \frac{df}{dt}(t) \leq Cf(t) + C\int_{\Omega}\left|\frac{d\chi}{dt}\right|p_k^2 dx + C\int_{\Omega}\rho_k^2(|\nabla\hat{y}_k|^2 + |\hat{y}_k|^2) dx, \quad 0 \leq t \leq T,$$

where $f(t) = \int_{\Omega}\chi p_k^2 dx$. Let us consider inequality (3.29) on the time interval $[0, \frac{3T}{4}]$. Note that $\chi(\frac{3T}{4})p_k(\frac{3T}{4}, \cdot) = 0$. Applying the Gronwall inequality to (3.29) and taking into account that $\frac{d\chi}{dt}(t) = 0, 0 \leq t \leq \frac{T}{2}$, we obtain

$$\begin{aligned} f(t) &\leq C\int_0^{\frac{3T}{4}}\int_{\Omega}\left|\frac{d\chi}{dt}\right|p_k^2 dx dt + C\int_0^{\frac{3T}{4}}\int_{\Omega}\rho_k^2(|\nabla\hat{y}_k|^2 + |\hat{y}_k|^2) dx dt \\ &\leq C\int_{\frac{T}{2}}^{\frac{3T}{4}}\int_{\Omega}p_k^2 dx + C\int_0^{\frac{3T}{4}}\int_{\Omega}\rho_k^2(|\nabla\hat{y}_k|^2 + |\hat{y}_k|^2) dx dt, \end{aligned}$$

that is,

$$(3.30) \quad \int_0^{\frac{T}{2}}\int_{\Omega}p_k^2 dx dt + \|p_k(0, \cdot)\|_{L^2(\Omega)}^2$$

$$\begin{aligned} &\leq C \int_{\frac{T}{2}}^{\frac{3T}{4}} \int_{\Omega} p_k^2 dxdt + C \int_0^{\frac{3T}{4}} \int_{\Omega} \rho_k^2 (|\nabla \widehat{y}_k|^2 + |\widehat{y}_k|^2) dxdt \\ &\leq C \int_{\frac{T}{2}}^{\frac{3T}{4}} \int_{\Omega} p_k^2 dxdt + C \int_0^{\frac{3T}{4}} \int_{\Omega} e^{-2s\eta} \rho_k^2 (|\nabla \widehat{y}_k|^2 + |\widehat{y}_k|^2) dxdt. \end{aligned}$$

Now inequalities (3.25) and (3.30) imply

$$\begin{aligned} &\int_0^{\frac{T}{2}} \int_{\Omega} p_k^2 dxdt + \|p_k(0, \cdot)\|_{L^2(\Omega)}^2 \\ &\leq C(s) \left(\int_Q e^{-2s\eta} \rho_k^2 (|\nabla \widehat{y}_k|^2 + |\widehat{y}_k|^2) dxdt + \int_{Q_\omega} \frac{1}{T-t} p_k^2 e^{-2s\eta} dxdt \right). \end{aligned}$$

Hence

$$\begin{aligned} (3.31) \quad &\int_0^{\frac{T}{2}} \int_{\Omega} \frac{1}{T-t} p_k^2 e^{-2s\eta} dxdt + \|p_k(0, \cdot)\|_{L^2(\Omega)}^2 \\ &\leq C(\lambda, s) \left(\int_Q e^{-2s\eta} \rho_k^2 (|\nabla \widehat{y}_k|^2 + |\widehat{y}_k|^2) dxdt + \int_{Q_\omega} \frac{1}{T-t} p_k^2 e^{-2s\eta} dxdt \right). \end{aligned}$$

On the other hand, it follows from (3.25) that

$$\begin{aligned} &\int_{\frac{T}{2}}^T \int_{\Omega} \frac{1}{T-t} p_k^2 e^{-2s\eta} dxdt \\ &\leq C(\lambda, s) \left(\int_Q e^{-2s\eta} \rho_k^2 (|\nabla \widehat{y}_k|^2 + |\widehat{y}_k|^2) dxdt + \int_{Q_\omega} \frac{1}{T-t} p_k^2 e^{-2s\eta} dxdt \right). \end{aligned}$$

This inequality and (3.31) complete the proof of Lemma 3.2. \square

We observe that $|\rho_k(t, x)e^{-2s\eta(t, x)}| \leq 1, (t, x) \in Q$. Thus, by (3.24) and (3.23), we have

$$\begin{aligned} (3.32) \quad &\int_{\Omega} |p_k(0, x)|^2 dx + \int_Q \frac{|p_k|^2}{T-t} e^{-2s\eta} dxdt \\ &\leq C(\lambda, s) \left(\int_Q \rho_k (|\nabla \widehat{y}_k|^2 + \widehat{y}_k^2) dxdt + \int_{Q_\omega} e^{2s\eta} (T-t) \widehat{u}_k^2 dxdt \right). \end{aligned}$$

Multiplying (3.23) by \widehat{y}_k , taking scalar products in $L^2(Q)$ and integrating by parts with respect to t and x , we have

$$0 = (L_0^* p_k - \nabla \cdot (\rho_k \nabla \widehat{y}_k) + \rho_k \widehat{y}_k, \widehat{y}_k)_{L^2(Q)} = \int_Q \rho_k (|\nabla \widehat{y}_k|^2 + \widehat{y}_k^2) dxdt$$

$$\begin{aligned}
 &+ (p_k, L_0 \widehat{y}_k)_{L^2(Q)} + (p_k(0, \cdot), \widehat{y}_k(0, \cdot))_{L^2(\Omega)} = \int_Q \rho_k (|\nabla \widehat{y}_k|^2 + \widehat{y}_k^2) dxdt \\
 &+ \int_Q (T-t)e^{2s\eta(t,x)} m_k \widehat{u}_k^2 dxdt + \int_Q gp_k dxdt + (p_k(0, \cdot), v_0)_{L^2(\Omega)}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 (3.33) \quad \mathcal{J}_k(\widehat{y}_k, \widehat{u}_k) &= \frac{1}{2} \int_Q \left(\rho_k (|\nabla \widehat{y}_k|^2 + \widehat{y}_k^2) + (T-t)e^{2s\eta(t,x)} m_k \widehat{u}_k^2 \right) dxdt \\
 &= \frac{1}{2} \left(- \int_Q gp_k dxdt - (p_k(0, \cdot), v_0)_{L^2(Q)} \right).
 \end{aligned}$$

By (3.32) and (3.33) we obtain

$$\mathcal{J}_k(\widehat{y}_k, \widehat{u}_k) \leq C(\|g\|_{X_s^\lambda(Q)} + \|v_0\|_{L^2(\Omega)}) \sqrt{\mathcal{J}_k(\widehat{y}_k, \widehat{u}_k)}.$$

It follows that

$$(3.34) \quad \mathcal{J}_k(\widehat{y}_k, \widehat{u}_k) \leq C^2(\|g\|_{X_s^\lambda(Q)} + \|v_0\|_{L^2(\Omega)})^2.$$

By virtue of (3.34), we have a subsequence $\{(\widehat{y}_k, \widehat{u}_k)\}_{k=1}^\infty$ such that

$$\begin{aligned}
 (3.35) \quad (\widehat{y}_k, \widehat{u}_k) &\rightarrow (y, u) \text{ weakly in } Y(Q) \times L^2(Q), \\
 \widehat{u}_k &\rightarrow 0 \text{ in } L^2((0, T) \times (\Omega \setminus \omega)), \\
 e^{s\eta} \widehat{u}_k &\rightarrow e^{s\eta} u \text{ weakly in } L^2(Q_\omega), \\
 \left(\sqrt{\rho_k} \frac{\partial \widehat{y}_k}{\partial x_i}, \sqrt{\rho_k} \widehat{y}_k \right) &\rightarrow \left(e^{s\eta} \frac{\partial y}{\partial x_i}, e^{s\eta} y \right) \text{ weakly in } L^2((0, T - \varepsilon) \times \Omega) \quad \forall \varepsilon > 0.
 \end{aligned}$$

Using (3.35), we pass to the limit in (3.22) and obtain that pair (y, u) is a solution to the problem (3.17). Estimate (3.18) follows from (3.34), (3.35) and Fatou’s theorem. □

Lemma 3.3. *The imbedding $Y(Q) \subset L^2(0, T; W_2^1(\Omega))$ is compact.*

Proof of Lemma 3.3. Let

$$\|y_k\|_{Y(Q)} \equiv (\|L_0 y_k\|_{L^2(0, T; L^2(\Omega))}^2 + \|y(0, \cdot)\|_{W_2^1(\Omega)}^2)^{\frac{1}{2}} \leq C,$$

for $k \in \mathbb{N}$. Henceforth a generic constant $C > 0$ is independent of $k \in \mathbb{N}$. Then we have to prove that $\{y_k\}_{k=1}^\infty$ contains a subsequence which is convergent in $L^2(0, T; W_2^1(\Omega))$. Application of Lemma 2.4 to $L_0 y_k = g_k$ yields

$$\|y_k\|_{L^2(0, T; W_2^1(\Omega))} \leq C, \quad k \in \mathbb{N}.$$

Therefore the sequence $\{y_k\}_{k=1}^\infty$ contains a subsequence which converges weakly in $L^2(0, T; W_2^1(\Omega))$ to some function y . Without loss of generality, we can assume that $y \equiv 0$.

Moreover we have

$$\|\Delta y_k\|_{L^2(0, T; W_2^{-1}(\Omega))} \leq C, \quad k \in \mathbb{N}.$$

First $\|g_k\|_{L^2(Q)} \leq C$ and $\|y_k(0, \cdot)\|_{W_2^1(\Omega)} \leq C$ for $k \in \mathbb{N}$. Therefore

(3.36) $\{g_k\}_{k=1}^\infty$ contains a subsequence which is weakly convergent to 0 in $L^2(Q)$ and $\{y_k(0, \cdot)\}_{k=1}^\infty$ contains a subsequence which is strongly convergent to 0 in $L^2(\Omega)$.

Let p_k be a solution to the problem

(3.37) $L_0^* p_k = -\Delta y_k$ in Q , $p_k|_\Sigma = 0$, $p_k(T, \cdot) = 0$.

By Lemma 2.4, the sequence $\{p_k\}_{k=1}^\infty$ is uniformly bounded in $L^2(0, T; W_2^1(\Omega)) \cap C([0, T]; L^2(\Omega))$.

Similarly to (2.26) and (2.27), we can prove

$$\left\| \sum_{i=1}^n \frac{\partial}{\partial x_i} (b_i p_k) \right\|_{L^2(0, T; W_2^{-1}(\Omega))} \leq C \sum_{i=1}^n \|b_i p_k\|_{L^2(Q)} \leq C \|p_k\|_{L^2(0, T; W_2^1(\Omega))}$$

and

$$\|c p_k\|_{L^2(0, T; W_2^{-1}(\Omega))} \leq C \|p_k\|_{L^2(0, T; W_2^1(\Omega))}$$

for all $k \in \mathbb{N}$. Therefore we see

$$\left\| - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial p_k}{\partial x_j} \right) - \sum_{i=1}^n \frac{\partial}{\partial x_i} (b_i p_k) + c p_k \right\|_{L^2(0, T; W_2^{-1}(\Omega))} \leq C \|p_k\|_{L^2(0, T; W_2^1(\Omega))},$$

which implies that $\{\frac{\partial p_k}{\partial t}\}_{k=1}^\infty$ is uniformly bounded in $L^2(0, T; W_2^{-1}(\Omega))$ by means of

$$\|L_0^* p_k\|_{L^2(0, T; W_2^{-1}(\Omega))} \leq C, \quad k \in \mathbb{N}.$$

Thus, by a theorem on compactness, we can extract a subsequence $\{p_{k'}\}_{k'=1}^\infty$ such that

(3.38) $p_{k'} \longrightarrow p_0$ in $L^2(Q)$.

Furthermore, by the uniform boundedness of $\|p_k\|_{C([0,T];L^2(\Omega))}$, $k \in \mathbb{N}$, we see that

$$(3.39) \quad \{p_k(0, \cdot)\}_{k=1}^\infty \text{ contains a weakly convergent subsequence in } L^2(\Omega).$$

Multiplying (3.37) by y_k , taking scalar products in $L^2(Q)$ and integrating by parts, we obtain

$$(3.40) \quad \int_Q |\nabla y_k|^2 dx dt = \int_Q g_k p_k dx dt + \int_\Omega y_k(0, \cdot) p_k(0, \cdot) dx.$$

Applying (3.36), (3.38) and (3.39) at the right side of (3.40), we complete the proof of the lemma. \square

Proof of Theorem 3.1. By (3.7), in terms of the mean value theorem, we can choose continuous functions f_0, f_1, \dots, f_n such that

$$(3.41) \quad f(t, x, \zeta', \zeta_0) = \sum_{i=0}^n f_i(t, x, \zeta', \zeta_0) \zeta_i \quad (t, x) \in Q.$$

Moreover, by (3.8), we have

$$(3.42) \quad |f_i(t, x, \zeta', \zeta_0)| \leq K, \quad \forall (t, x, \zeta) \in Q \times \mathbb{R}^{n+1}, 0 \leq i \leq n.$$

For the linear parabolic operator

$$R(y)z = L_0 z + \sum_{i=1}^n f_i(t, x, \nabla y, y) \frac{\partial z}{\partial x_i} + f_0(t, x, \nabla y, y)z,$$

we define the parameter $\gamma(y)$ by

$$\begin{aligned} \gamma(y) = & \sum_{i,j=1}^n \|a_{ij}\|_{W_\infty^1(Q)} + \sum_{i=1}^n (\|b_i\|_{L^\infty(0,T;L^r(\Omega))} + \|f_i(\cdot, \cdot, \nabla y, y)\|_{L^\infty(0,T;L^r(\Omega))}) \\ & + \|c\|_{L^\infty(0,T;W_{r_1}^{-\mu}(\Omega))} + \|f_0(\cdot, \cdot, \nabla y, y)\|_{L^\infty(0,T;W_{r_1}^{-\mu}(\Omega))} \end{aligned}$$

for every $y \in L^2(0, T; W_2^1(\Omega))$. Then by (3.8), (3.41) and (3.42), we obtain

$$(3.43) \quad \gamma(y) \leq C,$$

where $C > 0$ is a constant independent of y .

Let us consider the problem of exact controllability of parabolic equations

$$(3.44) \quad R(y)z = u + g \quad \text{in } Q, \quad u \in \mathcal{U}(\omega),$$

$$z|_{\Sigma} = 0, \quad z(0, x) = v_0(x), \quad z(T, x) = 0.$$

By (3.43) and Lemma 3.1, we can choose $\widehat{\lambda} > 0$ such that for $\lambda \geq \widehat{\lambda}$, there exists $s_0(\lambda)$ that if $\lambda \geq \widehat{\lambda}$ and $s \geq s_0(\lambda)$, then the problem of exact controllability (3.44) has solutions in the space $(Y(Q) \cap Z_s^\lambda(Q)) \times (\mathcal{U}(\omega) \cap X_s^\lambda(Q))$ for all initial data $(v_0, g) \in \overset{\circ}{W}_2^1(\Omega) \times X_s^\lambda(Q)$. Moreover these solutions satisfy (3.18) where $C(\lambda, s, \gamma) > 0$ is independent of $y \in L^2(0, T; W_2^1(\Omega))$.

Let us introduce mappings $\Psi : y \rightarrow \widehat{z}$ and $\Psi_1 : y \rightarrow (\widehat{z}, \widehat{u})$ as follows: For $y \in L^2(0, T; W_2^1(\Omega))$, a pair $(\widehat{z}, \widehat{u})$ is the solution to the extremal problem:

(3.45)

$$\mathcal{J}(z, u) = \int_Q e^{2s\eta(t,x)} (|\nabla z|^2 + z^2) dxdt + \int_Q (T-t)e^{2s\eta(t,x)} u^2 dxdt \rightarrow \inf,$$

(3.46)

$$R(y)z = g + u \quad \text{in } Q, \quad u \in \mathcal{U}(\omega), \\ z|_{\Sigma} = 0, \quad z(0, x) = v_0(x), \quad z(T, x) = 0.$$

By Lemma 3.1, for all $y \in L^2(0, T; W_2^1(\Omega))$, there exists a unique solution $(\widehat{z}, \widehat{u}) \in (Y(Q) \cap Z_s^\lambda(Q)) \times (\mathcal{U}(\omega) \cap X_s^\lambda(Q))$ to the problem (3.45)–(3.46). Consequently the mappings Ψ and Ψ_1 are well defined on the whole space $L^2(0, T; W_2^1(\Omega))$.

Let us prove that $\Psi : L^2(0, T; W_2^1(\Omega)) \rightarrow Y(Q) \cap Z_s^\lambda(Q)$ is a continuous mapping. Assume the contrary. Then there exist functions $y_k \in L^2(0, T; W_2^1(\Omega))$ and a sequence $\{(y_k, \widehat{z}_k, \widehat{u}_k)\}_{k=1}^\infty$ satisfying (3.47)–(3.49):

(3.47)

$$y_k \rightarrow y \quad \text{in } L^2(0, T; W_2^1(\Omega)), \quad \Psi(y_k) = \widehat{z}_k \rightarrow z \quad \text{weakly in } Y(Q) \cap Z_s^\lambda(Q), \\ \widehat{u}_k \rightarrow u \quad \text{weakly in } \mathcal{U}(\omega) \cap X_s^\lambda(Q).$$

(3.48)

$$\Psi(y) = (\widehat{z}, \widehat{u}) \neq (z, u), \quad \widehat{z} \in Z_s^\lambda(Q).$$

(3.49)

$$\text{the triple } (y_k, \widehat{z}_k, \widehat{u}_k) \text{ satisfies (3.46) and} \\ \mathcal{J}(\widehat{z}, \widehat{u}) < \mu_0 < \mathcal{J}(\widehat{z}_k, \widehat{u}_k), \quad k \in \mathbb{N}.$$

with some $\mu_0 > 0$.

By (3.42), (3.47) and (3.48)

$$(3.50) \quad \widehat{z}(f_0(t, x, \nabla y_k, y_k) - f_0(t, x, \nabla y, y)) + \sum_{i=1}^n (f_i(t, x, \nabla y_k, y_k) - f_i(t, x, \nabla y, y)) \frac{\partial \widehat{z}}{\partial x_i} \rightarrow 0$$

in $X_s^\lambda(Q)$ as $k \rightarrow \infty$.

By (3.50) and Lemma 3.1, there exists a subsequence $\{(\delta_k, q_k)\}_{k=1}^\infty \subset (Y(Q) \cap Z_s^\lambda(Q)) \times (\mathcal{U}(\omega) \cap X_s^\lambda(Q))$ such that

$$\begin{aligned}
 (3.51) \quad & L_0 \delta_k + \sum_{i=1}^n f_i(t, x, \nabla y, y) \frac{\partial \delta_k}{\partial x_i} + f_0(t, x, \nabla y, y) \delta_k \\
 &= \widehat{z}(f_0(t, x, \nabla y_k, y_k) - f_0(t, x, \nabla y, y)) \\
 &\quad + \sum_{i=1}^n (f_i(t, x, \nabla y_k, y_k) - f_i(t, x, \nabla y, y)) \frac{\partial \widehat{z}}{\partial x_i} + q_k \quad \text{in } Q,
 \end{aligned}$$

$$(3.52) \quad \delta_k|_\Sigma = 0, \quad \delta_k(0, x) = \delta_k(T, x) = 0, \quad q_k \in U(\omega),$$

$$(3.53) \quad \|\delta_k\|_{Y(Q) \cap Z_s^\lambda(Q)} + \|q_k\|_{X_s^\lambda(Q)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

We set

$$(3.54) \quad \tilde{z}_k = \widehat{z} - \delta_k, \quad \tilde{u}_k = \widehat{u} - q_k.$$

By (3.51) and (3.52), the following holds:

$$\begin{aligned}
 (3.55) \quad & L_0 \tilde{z}_k + \sum_{i=1}^n f_i(t, x, \nabla y_k, y_k) \frac{\partial \tilde{z}_k}{\partial x_i} + f_0(t, x, \nabla y_k, y_k) \tilde{z}_k \\
 &= g + \tilde{u}_k \quad \text{in } Q, \quad \tilde{u}_k \in \mathcal{U}(\omega),
 \end{aligned}$$

$$(3.56) \quad \tilde{z}_k|_\Sigma = 0, \quad \tilde{z}_k(0, x) = v_0(x), \quad \tilde{z}_k(T, x) = 0.$$

Moreover, by (3.53),

$$(3.57) \quad \lim_{k \rightarrow \infty} \mathcal{J}(\tilde{z}_k, \tilde{u}_k) = \mathcal{J}(\widehat{z}, \widehat{u}).$$

By (3.55) and (3.56), the pair $(\tilde{z}_k, \tilde{u}_k)$ is an admissible element of the extremal problem (3.45)–(3.46). Therefore by the definition of the mapping Ψ_1 , we obtain

$$(3.58) \quad \mathcal{J}(\widehat{z}_k, \widehat{u}_k) \leq \mathcal{J}(\tilde{z}_k, \tilde{u}_k), \quad k \in \mathbb{N}.$$

Now (3.57) and (3.58) contradict (3.49). Thus the continuity of Ψ is proved.

Denote by B_r the ball in $L^2(0, T; W_2^1(\Omega))$ with the radius r and the centre at zero. By (3.18) and (3.43), if $s > 0$ is sufficiently large, then, for all sufficiently large r , we obtain

$$(3.59) \quad \Psi(B_r) \subset B_r.$$

Moreover, if \mathfrak{S} is a bounded set in $L^2(0, T; W_2^1(\Omega))$, then by (3.18) the set $\Psi\mathfrak{S}$ is bounded in $Y(Q)$. Since, by Lemma 3.3, the imbedding $Y(Q) \subset$

$L^2(0, T; W_2^1(\Omega))$ is compact, the mapping Ψ from $L^2(0, T; W_2^1(\Omega))$ to itself is compact.

Applying the Schauder fixed point theorem, we find that there exists a fixed point y of the mapping Ψ :

$$\Psi(y) = y.$$

Obviously a pair $\Psi_1(y) = (y, u)$ is a solution to (3.1)–(3.2) with $v(T, x) = 0, x \in \Omega$. □

Finally we state the global exact zero controllability by boundary control.

Theorem 3.2. *Let $v_0 \in W_2^1(\Omega), v_1 \equiv 0$, and let conditions (2.3)–(2.4), (3.7) and (3.8) be fulfilled. Then there exists $\widehat{\lambda} > 0$ such that for $\lambda \geq \widehat{\lambda}$, there exists a constant $s_0(\lambda)$ such that if $g \in X_s^\lambda(Q)$ with $\lambda \geq \widehat{\lambda}$ and $s \geq s_0(\lambda)$, then there exists a solution pair $(y, u) \in Y(Q) \times L^2(0, T; H^{\frac{1}{2}}(\partial\Omega))$ of the problem (3.5)–(3.6).*

The proof of Theorem 3.2 is done by applying the argument in the proof of Theorem 3.3 from [25] on the basis of Theorem 3.1. We omit the details.

Appendix I

Proof of Lemma 2.2. The proof for $n = 1, 2$ is similar to the case of $n \geq 3$, and we give the proof only for the case of $n \geq 3$. Henceforth we set $\Delta_h u = u(x + h) - u(x), x \in \mathbb{R}^n$. Then we have

$$\|u\|_{W_p^\mu(\mathbb{R}^n)} = \left\{ \|u\|_{L^p(\mathbb{R}^n)}^p + \int_{\mathbb{R}^n} \frac{1}{|h|^{n+\mu p}} \|\Delta_h u\|_{L^p(\mathbb{R}^n)}^p dh \right\}^{\frac{1}{p}},$$

where $0 < \mu < 1/2, 1 < p < \infty$. By the smoothness of $\partial\Omega$, for the proof, instead of a function v , we can consider its extension in \mathbb{R}^n such that

$$\|v\|_{W_2^l(\mathbb{R}^n)} \leq C\|v\|_{W_2^l(\Omega)}, \quad \forall l \in \left[0, \frac{1}{2}\right]; \quad \|v\|_{W_2^\ell(\mathbb{R}^n)} \leq C\|v\|_{W_2^\ell(\Omega)}, \quad \forall \ell \in [0, 1].$$

By $B_{p,q}^\mu(\mathbb{R}^n)$, we denote the Besov space (e.g., Triebel [54]). Henceforth, taking these extensions of functions under consideration, we identify $\|\cdot\|_{B_{p,q}^\mu(\mathbb{R}^n)}$ with $\|\cdot\|_{B_{p,q}^\mu(\Omega)}$ (e.g., Theorem 4.2.2 in [54]). Since $v \in W_2^1(\Omega)$ and $z \in W_2^{\frac{1}{2}-\delta}(\Omega)$, we take the 0-extensions, so that we regard $v \in W_2^1(\mathbb{R}^n)$ and $z \in W_2^{\frac{1}{2}-\delta}(\mathbb{R}^n)$. By the definition, we have

$$\|zv\|_{W_{r_1}^\mu(\Omega)}^2 \leq 2\|zv\|_{L^{r_1}(\Omega)}^2 + 2 \left(\int_{\mathbb{R}^n} \frac{\|\Delta_h(zv)\|_{L^{r_1}(\mathbb{R}^n)}^{r_1'}}{|h|^{n+\mu r_1'}} dh \right)^{\frac{2}{r_1}}$$

$$\leq 2\|zv\|_{L^{r'_1}(\Omega)}^2 + C \left(\int_{\mathbb{R}^n} \frac{\|v\Delta_h z\|_{L^{r'_1}(\mathbb{R}^n)}^{r'_1} + \|z\Delta_h v\|_{L^{r'_1}(\mathbb{R}^n)}^{r'_1}}{|h|^{n+\mu r'_1}} dh \right)^{\frac{2}{r'_1}}.$$

We set

$$\kappa = \frac{2nr'_1}{2n - nr'_1 + 2r'_1}, \quad \kappa_1 = \frac{2nr'_1}{2n - nr'_1 + (1 - 2\delta)r'_1}.$$

We fix $\delta \in (0, \mu)$. Here and henceforth, $C > 0$ denotes a generic constant which is independent of z and v . Obviously $2n/(n - 1 + 2\delta)r' > 1$. Using Theorem 2.5.1 in [54] and the Hölder inequality, we obtain

$$\begin{aligned} \|zv\|_{W_{r'_1}^\mu(\Omega)}^2 &\leq C \|z\|_{L^\kappa(\Omega)}^2 \|v\|_{L^{\frac{2n}{n-2}}(\Omega)}^2 \\ &\quad + C \left(\int_{\mathbb{R}^n} \frac{\|\Delta_h z\|_{L^\kappa(\Omega)}^{r'_1} \|v\|_{L^{\frac{2n}{n-2}}(\Omega)}^{r'_1}}{|h|^{n+\mu r'_1}} dh \right)^{\frac{2}{r'_1}} \\ &\quad + C \left(\int_{\mathbb{R}^n} \frac{\|\Delta_h v\|_{L^{\kappa_1}(\Omega)}^{r'_1} \|z\|_{L^{\frac{2n}{n-1+2\delta}}(\Omega)}^{r'_1}}{|h|^{n+\mu r'_1}} dh \right)^{\frac{2}{r'_1}} \\ &\leq C \left(\|z\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}^2 \|v\|_{W_2^1(\Omega)}^2 \right. \\ &\quad \left. + \|z\|_{B_{\kappa, r'_1}^\mu(\Omega)}^2 \|v\|_{W_2^1(\Omega)}^2 + \|z\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}^2 \|v\|_{B_{\kappa_1, r'_1}^\mu(\Omega)}^2 \right). \end{aligned}$$

Here we have also used

$$W_2^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega), \quad W_2^{\frac{1}{2}-\delta}(\Omega) \subset L^{\frac{2n}{n-1+2\delta}}(\Omega)$$

and

$$W_2^{\frac{1}{2}-\delta}(\Omega) \subset L^\kappa(\Omega)$$

which are true by the Sobolev imbedding (e.g., [1]) and $0 < \delta < \mu$,

$$(1) \quad r'_1 < \frac{2n}{2n - 3 + 2\mu}.$$

Henceforth, by (1), we can set

$$(2) \quad \frac{n}{r'_1} = n - \frac{3}{2} + \mu + \varepsilon$$

with some $\varepsilon > 0$. On the other hand, we can prove

$$(3) \quad W_2^{\frac{1}{2}-\delta}(\Omega) \subset B_{\kappa, r'_1}^\mu(\Omega)$$

and

$$(4) \quad W_2^1(\Omega) \subset B_{\kappa_1, r'_1}^\mu(\Omega).$$

Proof of (3). We can take sufficiently small $\delta > 0$. By the imbedding of Besov spaces (e.g., Triebel [54, Theorem 2.3.2 (c)]), noting $r_1 > 1$, we have

$$(5) \quad W_2^{\frac{1}{2}-\delta}(\mathbb{R}^n) \subset B_{2,2}^{\frac{1}{2}-\delta}(\mathbb{R}^n) \subset B_{2,1}^{\frac{1}{2}-2\delta}(\mathbb{R}^n) \subset B_{2,r'_1}^{\frac{1}{2}-2\delta}(\mathbb{R}^n).$$

By (2), we can easily verify that $\frac{1}{2} - 2\delta - \frac{n}{2} \geq \mu - \frac{n}{\kappa}$ and $\mu < \frac{1}{2} - 2\delta$ if we take sufficiently small $\delta > 0$ such that $0 < \delta < \min\{\frac{\varepsilon}{2}, \frac{1}{2}(\frac{1}{2} - \mu)\}$. Therefore, by the imbedding of Besov spaces (e.g., Triebel [54, Theorem 4.6.2]), we can see

$$B_{2,r'_1}^{\frac{1}{2}-2\delta}(\Omega) \subset B_{\kappa, r'_1}^\mu(\Omega).$$

Thus (5) implies (3). □

Proof of (4). Similarly we can see

$$(6) \quad W_2^1(\mathbb{R}^n) \subset B_{2,2}^1(\mathbb{R}^n) \subset B_{2,1}^{1-\delta_1}(\mathbb{R}^n) \subset B_{2,r'_1}^{1-\delta_1}(\mathbb{R}^n)$$

for any small $\delta_1 > 0$ (e.g., [54, Theorem 2.3.2 (c)]). By (2), we can see $1 - \delta_1 - \frac{n}{2} \geq \mu - \frac{n}{\kappa_1}$ and $\mu < 1 - \delta_1$ if $0 < \delta < \varepsilon$ and $0 < \delta_1 < \min\{\varepsilon - \delta, 1 - \mu\}$. Then we obtain

$$B_{2,r'_1}^{1-\delta_1}(\Omega) \subset B_{\kappa_1, r'_1}^\mu(\Omega)$$

with which we combine (6) to obtain (4).

In view of (3) and (4), we have

$$\|z\|_{B_{\kappa, r'_1}^\mu(\Omega)} \leq C \|z\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}$$

and

$$\|v\|_{B_{\kappa_1, r'_1}^\mu(\Omega)} \leq C \|v\|_{W_2^1(\Omega)}$$

where $C = C(\Omega, \mu, \kappa, \kappa_1, r'_1, \delta) > 0$ is independent of z . Thus the proof of Lemma 2.2 is complete. □

Remark. In terms of the Triebel-Lizorkin space, we can give the following concise proof: First we extend the functions z, v by zero on \mathbb{R}^n . We introduce the Triebel-Lizorkin space $F_{p,q}^s$ (e.g., p. 8 in Runst and Sickel [47]). Then we note that $W_p^s(\mathbb{R}^n) = F_{p,p}^s$. Now we show that the statement of this lemma follows from the general embedding theorem proved in [47, p. 189]. Let

us check the conditions of that theorem in the case of $n \geq 3$. For the case of $n = 1, 2$, the proof is similar and omitted. We set $s = \mu$, $p = q = r'_1$, $p_1 = p_2 = q_1 = q_2 = 2$. Let $s_1 = \frac{1}{2} - \delta$ and $s_2 = 1$. There we choose the parameter $\delta \in (0, \frac{1}{2})$ such that $s < s_1 < s_2$. Obviously $\frac{1}{p} = 1 - \frac{1}{r_1} \leq \sum_{j=1}^2 \frac{1}{p_j} = 1$ and $n > \sum_{j=1}^2 (\frac{n}{p_j} - s_j) = n - \frac{3}{2} + \delta$. Finally, by the condition on r_1 , there exists $\varepsilon > 0$ such that $\frac{n}{p} - s = n - \frac{n}{r_1} - \mu = n - \frac{3}{2} + \varepsilon > \sum_{j=1}^2 (\frac{n}{p_j} - s_j) = n - \frac{3}{2} + \delta$ provided that $\delta < \varepsilon$. Hence we have $F_{p_1, q_1}^{s_1} \cdot F_{p_2, q_2}^{s_2} \subset F_{p, q}^s$, that is, $F_{2, 2}^{\frac{1}{2} - \delta} \cdot F_{2, 2}^1 \subset F_{r'_1, r'_1}^\mu$. The proof of this lemma is complete. \square

Appendix II

Proof of Lemma 2.3 in the case of $d = 0$. The proof is similar to the proof in [4], [25] where $a_{ij} \in C^{1,2}(\bar{\Omega})$, $1 \leq i, j \leq n$. Let us consider the operator

$$(1) \quad \widehat{L}y = \frac{\partial y}{\partial t} - \sum_{i,j=1}^n a_{ij}(t, x) \frac{\partial^2 y}{\partial x_i \partial x_j}.$$

We set

$$\widehat{c}(t, x) = c(t, x) + \sum_{i=1}^n \frac{\partial b_i}{\partial x_i}(t, x)$$

and

$$(2) \quad \widetilde{g}(t, x) = \left(g(t, x) - \sum_{i=1}^n b_i(t, x) \frac{\partial y}{\partial x_i} - \widehat{c}(t, x)y + \sum_{i,j=1}^n \frac{\partial a_{ij}(t, x)}{\partial x_i} \frac{\partial y}{\partial x_j} \right).$$

We denote $w(t, x) = e^{s\alpha}y(t, x)$. By (2.7), we have

$$(3) \quad w(T, \cdot) = w(0, \cdot) = 0 \quad \text{in } \Omega.$$

We define an operator P by

$$(4) \quad Pw = e^{s\alpha} \widehat{L}(e^{-s\alpha}w).$$

It follows from (2.1) and (1), (2) that

$$(5) \quad Pw = e^{s\alpha} \widetilde{g} \quad \text{in } Q.$$

We notice that the operator P can be written explicitly as follows

$$(6) \quad Pw = \frac{\partial w}{\partial t} - \sum_{i,j=1}^n a_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} + 2s\lambda\varphi \sum_{i,j=1}^n a_{ij}\psi_{x_i} \frac{\partial w}{\partial x_j} + s\lambda^2\varphi a(t, x, \nabla\psi, \nabla\psi)w$$

$$-s^2\lambda^2\varphi^2a(t,x,\nabla\psi,\nabla\psi)w+s\lambda\varphi w\sum_{i,j=1}^na_{ij}\psi_{x_ix_j}-s\frac{\partial\alpha}{\partial t}w.$$

Here and henceforth, we set $\psi_{x_i} = \frac{\partial\psi}{\partial x_i}$, $\psi_{x_ix_j} = \frac{\partial^2\psi}{\partial x_i\partial x_j}$, $1 \leq i, j \leq n$. We recall that the quadratic form $a(t, x, \xi, \eta)$ was defined in (2.4). We further introduce the operators L_1, L_2 as follows:

$$(7) \quad L_1w = -\sum_{i,j=1}^na_{ij}\frac{\partial^2w}{\partial x_i\partial x_j} - \lambda^2s^2\varphi^2a(t,x,\nabla\psi,\nabla\psi)w - s\frac{\partial\alpha}{\partial t}w,$$

$$(8) \quad L_2w = \frac{\partial w}{\partial t} + 2s\lambda\varphi\sum_{i,j=1}^na_{ij}\psi_{x_i}\frac{\partial w}{\partial x_j} + 2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)w.$$

It follows from (2), (6), (7) and (8) that

$$(9) \quad L_1w + L_2w = f_s \quad \text{in } Q,$$

where

$$f_s(t,x) = \tilde{g}e^{s\alpha} - s\lambda\varphi w\sum_{i,j=1}^na_{ij}\psi_{x_ix_j} + s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)w.$$

Taking L_2 -norms of the both sides of (9), we obtain

$$(10) \quad \|f_s\|_{L^2(Q)}^2 = \|L_1w\|_{L^2(Q)}^2 + \|L_2w\|_{L^2(Q)}^2 + 2(L_1w, L_2w)_{L^2(Q)}.$$

By (7) and (8), we have the following equality:

$$(11) \quad \begin{aligned} &(L_1w, L_2w)_{L^2(Q)} \\ &= \left(-\sum_{i,j=1}^na_{ij}\frac{\partial^2w}{\partial x_i\partial x_j} - \lambda^2s^2\varphi^2a(t,x,\nabla\psi,\nabla\psi)w \right. \\ &\quad \left. - s\frac{\partial\alpha}{\partial t}w, \frac{\partial w}{\partial t} + 2s\lambda\varphi a(t,x,\nabla\psi,\nabla\psi)w \right)_{L^2(Q)} \\ &\quad - \int_Q \left(2\lambda^3s^3\varphi^3a(t,x,\nabla\psi,\nabla\psi)w + 2s^2\lambda\varphi\frac{\partial\alpha}{\partial t}w \right) a(t,x,\nabla\psi,\nabla w) dxdt \\ &\quad - \int_Q \left(\sum_{i,j=1}^na_{ij}\frac{\partial^2w}{\partial x_i\partial x_j} \right) 2s\lambda\varphi a(t,x,\nabla\psi,\nabla w) dxdt. \end{aligned}$$

Integrating by parts in the first term of the right-hand side of (11), we obtain

$$\begin{aligned}
 (12) \quad A_0 &= \left(- \sum_{i,j=1}^n a_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} - \lambda^2 s^2 \varphi^2 a(t, x, \nabla \psi, \nabla \psi) w \right. \\
 &\quad \left. - s w \frac{\partial \alpha}{\partial t}, \frac{\partial w}{\partial t} + 2s \lambda^2 \varphi a(t, x, \nabla \psi, \nabla \psi) w \right)_{L^2(Q)} \\
 &= \int_Q \left(\frac{\partial w}{\partial t} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j} \frac{\partial w}{\partial x_i} + \sum_{i,j=1}^n a_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w_t}{\partial x_j} \right. \\
 &\quad - \frac{\lambda^2 s^2 \varphi^2}{2} a(t, x, \nabla \psi, \nabla \psi) \frac{\partial w^2}{\partial t} - \frac{s}{2} \frac{\partial \alpha}{\partial t} \frac{\partial w^2}{\partial t} - 2s^3 \varphi^3 \lambda^4 a(t, x, \nabla \psi, \nabla \psi)^2 w^2 \\
 &\quad - 2s^2 \lambda^2 \frac{\partial \alpha}{\partial t} \varphi a(t, x, \nabla \psi, \nabla \psi) w^2 + 2\lambda^2 s \varphi a(t, x, \nabla \psi, \nabla \psi) w \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j} \frac{\partial w}{\partial x_i} \\
 &\quad + 2s \lambda^2 \varphi a(t, x, \nabla \psi, \nabla \psi) a(t, x, \nabla w, \nabla w) \\
 &\quad \left. + 2s \lambda^2 w \sum_{i,j=1}^n a_{ij} \frac{\partial w}{\partial x_j} \frac{\partial}{\partial x_i} (\varphi a(t, x, \nabla \psi, \nabla \psi)) \right) dx dt.
 \end{aligned}$$

Integrating by parts in the second term of the right-hand side of (11), we have

$$\begin{aligned}
 (13) \quad & - \int_Q (2\lambda^3 s^3 w \varphi^3 a(t, x, \nabla \psi, \nabla \psi) a(t, x, \nabla \psi, \nabla w) \\
 & \quad + 2s^2 \lambda \frac{\partial \alpha}{\partial t} w \varphi a(t, x, \nabla \psi, \nabla w)) dx dt \\
 &= - \int_Q (\lambda^3 s^3 \varphi^3 a(t, x, \nabla \psi, \nabla \psi) a(t, x, \nabla \psi, \nabla w^2) \\
 & \quad + s^2 \frac{\partial \alpha}{\partial t} \varphi \lambda a(t, x, \nabla \psi, \nabla w^2)) dx dt \\
 &= \int_Q \left\{ (3\lambda^4 s^3 \varphi^3 a(t, x, \nabla \psi, \nabla \psi))^2 w^2 \right. \\
 & \quad + w^2 \varphi^3 \lambda^3 s^3 \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij} \psi_{x_j} a(t, x, \nabla \psi, \nabla \psi)) \\
 & \quad \left. + \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(\frac{s^2 \lambda^2 \varphi}{2} a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial \alpha}{\partial t} \right) w^2 \right\} dx dt.
 \end{aligned}$$

Finally, integrating by parts the third term of the right-hand side of (11) and taking into account (2.5), we have

$$\begin{aligned}
 (14) \quad A_1 &= \int_Q - \left(\sum_{i,j=1}^n a_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} \right) \left(2s\lambda\varphi \sum_{k,\ell=1}^n a_{k\ell} \psi_{x_k} \frac{\partial w}{\partial x_\ell} \right) dxdt \\
 &= \int_Q \left(\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j} \frac{\partial w}{\partial x_i} 2s\lambda\varphi \sum_{k,\ell=1}^n a_{k\ell} \psi_{x_k} \frac{\partial w}{\partial x_\ell} + 2s\lambda^2 \varphi a(t, x, \nabla\psi, \nabla w)^2 \right. \\
 &\quad + 2s\lambda\varphi \sum_{i,j=1}^n \left(a_{ij} \frac{\partial w}{\partial x_i} \sum_{k,\ell=1}^n \frac{\partial}{\partial x_j} (a_{k\ell} \psi_{x_k}) \frac{\partial w}{\partial x_\ell} \right) \\
 &\quad + 2s\lambda\varphi \sum_{i,j=1}^n a_{ij} \frac{\partial w}{\partial x_i} \sum_{k,\ell=1}^n a_{k\ell} \psi_{x_k} \frac{\partial^2 w}{\partial x_j \partial x_\ell} \Big) dxdt \\
 &\quad + \int_\Sigma 2s\lambda\varphi |\nabla\psi| \left| \frac{\partial w}{\partial \nu_A} \right|^2 d\Sigma \\
 &= \int_Q \left\{ \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j} \frac{\partial w}{\partial x_i} 2s\lambda\varphi \sum_{k,\ell=1}^n a_{k\ell} \psi_{x_k} \frac{\partial w}{\partial x_\ell} + 2s\lambda^2 \varphi a(t, x, \nabla\psi, \nabla w)^2 \right. \\
 &\quad + 2s\lambda\varphi \sum_{i,j=1}^n \left(a_{ij} \frac{\partial w}{\partial x_i} \sum_{k,\ell=1}^n \frac{\partial}{\partial x_j} (a_{k\ell} \psi_{x_k}) \frac{\partial w}{\partial x_\ell} \right) \\
 &\quad - s\lambda\varphi \sum_{k,\ell=1}^n a_{k\ell} \psi_{x_k} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_\ell} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \\
 &\quad + s\lambda\varphi \sum_{k,\ell=1}^n a_{k\ell} \psi_{x_k} \frac{\partial}{\partial x_\ell} \sum_{i,j=1}^n a_{ij} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \Big\} dxdt \\
 &\quad + \int_\Sigma 2s\lambda\varphi |\nabla\psi| \left| \frac{\partial w}{\partial \nu_A} \right|^2 d\Sigma.
 \end{aligned}$$

Integrating by parts once again, we obtain

$$\begin{aligned}
 (15) \quad A_1 &= \int_Q \left\{ \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j} \frac{\partial w}{\partial x_i} 2s\lambda\varphi \sum_{k,\ell=1}^n a_{k\ell} \psi_{x_k} \frac{\partial w}{\partial x_\ell} + 2s\lambda^2 \varphi a(t, x, \nabla\psi, \nabla w)^2 \right. \\
 &\quad + 2s\lambda\varphi \sum_{i,j=1}^n \left(a_{ij} \frac{\partial w}{\partial x_i} \sum_{k,\ell=1}^n \frac{\partial}{\partial x_j} (a_{k\ell} \psi_{x_k}) \frac{\partial w}{\partial x_\ell} \right)
 \end{aligned}$$

$$\begin{aligned}
& -s\lambda\varphi \sum_{k,\ell=1}^n a_{k\ell}\psi_{x_k} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_\ell} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \\
& -s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)a(t,x,\nabla w,\nabla w) \\
& -a(t,x,\nabla w,\nabla w)s\lambda\varphi \sum_{k,\ell=1}^n \frac{\partial}{\partial x_\ell}(a_{k\ell}\psi_{x_k}) \Big\} dxdt \\
& + \int_{\Sigma} s\lambda\varphi |\nabla\psi| \left| \frac{\partial w}{\partial \nu_A} \right|^2 d\Sigma,
\end{aligned}$$

where we used the fact: $\nu = -\nabla\psi/|\nabla\psi|$ which is seen from $\psi|_{\partial\Omega} = 0$.

By virtue of (12), (13) and (15), one can rewrite (11) as follows.

$$\begin{aligned}
(16) \quad & (L_1w, L_2w)_{L^2(Q)} \\
& = \int_Q \left\{ \lambda^4 s^3 \varphi^3 a(t,x,\nabla\psi,\nabla\psi)^2 w^2 \right. \\
& \quad + s\lambda^2\varphi a(t,x,\nabla\psi,\nabla\psi)a(t,x,\nabla w,\nabla w) + L_2w \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j} \frac{\partial w}{\partial x_i} \\
& \quad \left. + 2s\lambda^2\varphi a(t,x,\nabla\psi,\nabla w)^2 \right\} dxdt \\
& + \int_{\Sigma} s\lambda\varphi |\nabla\psi| \left| \frac{\partial w}{\partial \nu_A} \right|^2 d\Sigma + X_1,
\end{aligned}$$

where we put

$$\begin{aligned}
X_1 = & \int_Q \left\{ 2s\lambda^2w \sum_{i,j=1}^n a_{ij} \frac{\partial w}{\partial x_j} \frac{\partial}{\partial x_i}(\varphi a(t,x,\nabla\psi,\nabla\psi)) \right. \\
& + \frac{1}{2} \frac{\partial}{\partial t} (\lambda^2 s^2 \varphi^2 a(t,x,\nabla\psi,\nabla\psi)) w^2 - \frac{sw^2}{2} \frac{\partial^2 \alpha}{\partial t^2} \\
& + 2s\lambda\varphi \sum_{i,j=1}^n \left(a_{ij} \frac{\partial w}{\partial x_i} \sum_{k,\ell=1}^n \frac{\partial(a_{k\ell}\psi_{x_k})}{\partial x_j} \frac{\partial w}{\partial x_\ell} \right) \\
& - s\lambda\varphi \sum_{k,\ell=1}^n a_{k\ell}\psi_{x_k} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_\ell} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} \\
& \left. - a(t,x,\nabla w,\nabla w)s\lambda\varphi \sum_{k,\ell=1}^n \frac{\partial}{\partial x_\ell}(a_{k\ell}\psi_{x_k}) \right\}
\end{aligned}$$

$$\begin{aligned}
 & - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial t} \frac{\partial w}{\partial x_i} \frac{\partial w}{\partial x_j} + w^2 \varphi^3 \lambda^3 s^3 \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij} \psi_{x_j} a(t, x, \nabla \psi, \nabla \psi)) \\
 & - \left. \sum_{i,j=1}^n \frac{\partial}{\partial x_j} \left(\frac{s^2 \varphi \lambda^2}{2} a_{ij} \frac{\partial \psi}{\partial x_i} \frac{\partial \alpha}{\partial t} \right) w^2 \right\} dxdt.
 \end{aligned}$$

Hence we can easily prove

$$\begin{aligned}
 (17) \quad |X_1| & \leq C \int_Q ((s^3 \lambda^3 \varphi^3 + s^2 \lambda^4 \varphi^3) w^2 + (s \lambda \varphi + 1) |\nabla w|^2) dxdt \\
 & \quad s \geq 1, \quad \lambda \geq 1.
 \end{aligned}$$

Here and henceforth $C > 0$ denotes a generic constant which is independent of s and λ .

Therefore, by virtue of (10) and (16), we have

$$\begin{aligned}
 (18) \quad \|f_s\|_{L^2(Q)}^2 & = \|L_1 w\|_{L^2(Q)}^2 + \|L_2 w\|_{L^2(Q)}^2 + 2 \int_Q (\lambda^4 s^3 \varphi^3 a(t, x, \nabla \psi, \nabla \psi)^2 w^2 \\
 & \quad + s \lambda^2 \varphi a(t, x, \nabla \psi, \nabla \psi) a(t, x, \nabla w, \nabla w) \\
 & \quad + (L_2 w) \left(\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j} \frac{\partial w}{\partial x_i} \right) + 2s \lambda^2 \varphi a(t, x, \nabla \psi, \nabla \psi)^2 dxdt \\
 & \quad + \int_{\Sigma} s \lambda |\nabla \psi|^2 \varphi \left| \frac{\partial w}{\partial \nu_A} \right|^2 d\Sigma + X_1.
 \end{aligned}$$

Applying the Cauchy-Bunyakovskii inequality in (18), we obtain

$$\begin{aligned}
 (19) \quad \|L_1 w\|_{L^2(Q)}^2 & + \frac{1}{2} \|L_2 w\|_{L^2(Q)}^2 \\
 & + 2 \int_Q \left(\lambda^4 s^3 \varphi^3 a(t, x, \nabla \psi, \nabla \psi)^2 w^2 \right. \\
 & \quad + s \lambda^2 \varphi a(t, x, \nabla \psi, \nabla \psi) a(t, x, \nabla w, \nabla w) \\
 & \quad \left. - 4 \left(\sum_{i,j=1}^n \frac{\partial a_{ij}}{\partial x_j} \frac{\partial w}{\partial x_i} \right)^2 \right) dxdt + X_1 \leq \|f_s\|_{L^2(Q)}^2.
 \end{aligned}$$

We recall that by Lemma 2.1

$$|\nabla \psi(x)| > \kappa > 0, \quad \forall x \in \overline{\Omega \setminus \omega_0}.$$

Hence, taking a parameter $\lambda > \hat{\lambda} > 0$ sufficiently large in (19), by virtue of (17), we obtain: There exists $s_0(\lambda) > 0$ such that

$$(20) \quad \|L_1 w\|_{L^2(Q)}^2 + \frac{1}{2} \|L_2 w\|_{L^2(Q)}^2 + \int_Q (\lambda^4 s^3 \varphi^3 w^2 + s \lambda^2 \varphi |\nabla w|^2) dxdt \leq C \left(\int_{Q_\omega} (\lambda^4 s^3 \varphi^3 w^2 + s \lambda^2 \varphi |\nabla w|^2) dxdt + \|\tilde{g}e^{s\alpha}\|_{L^2(Q)}^2 \right), \quad \forall s \geq s_0.$$

Thus, from (7), (8) and (20), we have

$$(21) \quad \int_Q \left\{ \frac{1}{s\varphi} \left(\frac{\partial w}{\partial t} \right)^2 + \frac{1}{s\varphi} \sum_{i,j=1}^n \left(\frac{\partial^2 w}{\partial x_i \partial x_j} \right)^2 + s \lambda^2 \varphi |\nabla w|^2 + \lambda^4 s^3 \varphi^3 w^2 \right\} dxdt \leq C \left(\int_{Q_\omega} (\lambda^4 s^3 \varphi^3 w^2 + s \lambda^2 \varphi |\nabla w|^2) dxdt + \|\tilde{g}e^{s\alpha}\|_{L^2(Q)}^2 \right), \quad \forall s \geq s_0.$$

Replacing w by $e^{s\alpha}y$ in (21), we obtain

$$(22) \quad \int_Q \left(\frac{1}{s\varphi} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{s\varphi} \sum_{i,j=1}^n \left(\frac{\partial^2 y}{\partial x_i \partial x_j} \right)^2 + s \lambda^2 \varphi |\nabla y|^2 + s^3 \lambda^4 \varphi^3 y^2 \right) e^{2s\alpha} dxdt \leq C_1(\lambda) \left(\int_{Q_\omega} (\lambda^4 s^3 \varphi^3 y^2 + s \lambda^2 \varphi |\nabla y|^2) e^{2s\alpha} dxdt + \|ge^{s\alpha}\|_{L^2(Q)}^2 \right), \quad \forall s \geq s_1.$$

Let us consider a function $\rho \in C_0^\infty(\omega)$, $\rho(x) \equiv 1$ in ω_0 . We multiply equation (6) by $s\varphi\lambda^2 y \rho e^{2s\alpha}$ and take scalar products in $L^2(Q)$. Integrating by parts with respect to t and x and applying the Cauchy-Bunyakovskii inequality, we obtain

$$(23) \quad \int_{(0,T) \times \omega_0} \lambda^2 s \varphi |\nabla y|^2 e^{2s\alpha} dxdt \leq C \left(\int_{Q_\omega} \lambda^4 (s\varphi)^3 y^2 e^{2s\alpha} dxdt + \|\tilde{g}e^{s\alpha}\|_{L^2(Q)}^2 \right).$$

By virtue of (22) and (23), we have

$$\begin{aligned}
 (24) \quad & \int_Q \left(\frac{1}{s\varphi} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{s\varphi} \left(\sum_{i,j=1}^n \frac{\partial^2 y}{\partial x_i \partial x_j} \right)^2 + s\lambda^2 \varphi |\nabla y|^2 + s^3 \lambda^4 \varphi^3 y^2 \right) e^{2s\alpha} dx dt \\
 & \leq C \left\{ \int_{Q_\omega} \lambda^4 (s\varphi)^3 y^2 e^{2s\alpha} dx dt + \|ge^{s\alpha}\|_{L^2(Q)}^2 \right\}, \quad \forall s \geq s_0.
 \end{aligned}$$

By (24), we finally obtain (2.11) with $d = 0$. □

Appendix III

Proof of Lemma 2.4. Since $a_{ij}, 1 \leq i, j \leq n$, are Lipschitz continuous on \overline{Q} , the unique existence of the solution in $L^2(0, T; \overset{\circ}{W}_2^1(\Omega)) \cap C([0, T]; L^2(\Omega))$ is seen in the case of $b_i = 0, 1 \leq i \leq n$ and $c = 0$, for example, by Ladyzenskaja, Solonnikov and Ural'ceva [35], Lions and Magenes [40, Chapter 3, Section 4.7], Pazy [43], Tanabe [51]. To prove the uniqueness of weak solutions to the problem (2.1)–(2.2) and a priori estimate (2.12), it suffices to prove that the problem

$$(1) \quad L^* z = f, \quad z|_\Sigma = 0, \quad z(T, \cdot) = 0$$

has a solution $z \in L^2(0, T; \overset{\circ}{W}_2^1(\Omega)) \cap C([0, T]; L^2(\Omega))$ for any $f \in L^2(0, T; W_2^{-1}(\Omega))$. To prove the solvability of problem (1), it is sufficient to prove the analogue of (2.12) for this problem.

Henceforth $C > 0$ denotes a generic constant which is independent of functions to be estimated. Multiplication of (1) with z and integration by parts in x yield

$$\begin{aligned}
 & -\frac{1}{2} \frac{d}{dt} \|z(t, \cdot)\|_{L^2(\Omega)}^2 + \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial z}{\partial x_j} \frac{\partial z}{\partial x_i} dx \\
 & = \int_\Omega \sum_{i=1}^n b_i z \frac{\partial z}{\partial x_i} dx - \int_\Omega cz^2 dx + \int_\Omega f z dx.
 \end{aligned}$$

By the uniform ellipticity, we see

$$\begin{aligned}
 (2) \quad & -\frac{d}{dt} \|z(t, \cdot)\|_{L^2(\Omega)}^2 + \|\nabla z(t, \cdot)\|_{L^2(\Omega)}^2 \\
 & \leq \sum_{i=1}^n \left| \int_\Omega b_i z \frac{\partial z}{\partial x_i} dx \right| + \left| \int_\Omega cz^2 dx \right| + \left| \int_\Omega \left(\frac{1}{\varepsilon} f \right) (\varepsilon z) dx \right|.
 \end{aligned}$$

Here $\varepsilon > 0$ is a sufficiently small parameter which is fixed later, and we use $2|ab| \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2$.

First we estimate $\sum_{i=1}^n \left| \int_{\Omega} b_i z \frac{\partial z}{\partial x_i} dx \right|$. We take $r > 2n$. By the Hölder inequality, we have

$$\left| \int_{\Omega} b_i z \frac{\partial z}{\partial x_i} dx \right| \leq \|b_i(t, \cdot)\|_{L^r(\Omega)} \|z(t, \cdot)\|_{L^{\frac{2r}{r-2}}(\Omega)} \left\| \frac{\partial z}{\partial x_i}(t, \cdot) \right\|_{L^2(\Omega)}.$$

Since $r > 2n$, the Sobolev imbedding theorem implies $W_2^{\frac{1}{2}-\delta}(\Omega) \subset L^{\frac{2r}{r-2}}(\Omega)$ for sufficiently small $\delta > 0$. Hence, with small $\varepsilon > 0$, we have

$$\begin{aligned} \left| \int_{\Omega} b_i z \frac{\partial z}{\partial x_i} dx \right| &\leq \|z(t, \cdot)\|_{W_2^{\frac{1}{2}-\delta}(\Omega)} \|z(t, \cdot)\|_{W_2^1(\Omega)} \\ &\leq \varepsilon \|z(t, \cdot)\|_{W_2^1(\Omega)}^2 + \frac{C}{\varepsilon} \|z(t, \cdot)\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}^2. \end{aligned}$$

By the interpolation inequality, we see

$$(3) \quad \|z(t, \cdot)\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}^2 \leq \delta \|z(t, \cdot)\|_{W_2^1(\Omega)}^2 + C(\delta) \|z(t, \cdot)\|_{L^2(\Omega)}^2$$

for small $\delta > 0$. We choose sufficiently small $\varepsilon > 0$ and $\delta > 0$ such that $\frac{\delta}{\varepsilon}$ is also small, so that

$$(4) \quad \sum_{i=1}^n \left| \int_{\Omega} b_i z \frac{\partial z}{\partial x_i} dx \right| \leq C\varepsilon \|z(t, \cdot)\|_{W_2^1(\Omega)}^2 + C(\varepsilon) \|z(t, \cdot)\|_{L^2(\Omega)}^2.$$

Now, by Lemma 2.2, we have

$$\begin{aligned} \left| \int_{\Omega} cz^2 dx \right| &\leq \|c(t, \cdot)\|_{W_{r_1}^{-\mu}(\Omega)} \|z^2(t, \cdot)\|_{W_{r_1}^{\mu}(\Omega)} \\ &\leq C \|z(t, \cdot)\|_{W_2^{\frac{1}{2}-\delta}(\Omega)} \|z(t, \cdot)\|_{W_2^1(\Omega)} \\ &\leq C\varepsilon \|\nabla z(t, \cdot)\|_{L^2(\Omega)}^2 + \frac{C}{\varepsilon} \|z(t, \cdot)\|_{W_2^{\frac{1}{2}-\delta}(\Omega)}^2 \end{aligned}$$

with $0 < \delta < \frac{1}{2}$. In view of interpolation inequality (3), taking $\varepsilon > 0$ and $\delta > 0$ so small that $\frac{\delta}{\varepsilon}$ is also small, we obtain

$$(5) \quad \left| \int_{\Omega} cz^2 dx \right| \leq \varepsilon \|\nabla z(t, \cdot)\|_{L^2(\Omega)}^2 + C(\varepsilon) \|z(t, \cdot)\|_{L^2(\Omega)}^2.$$

On the other hand, we have

$$(6) \quad \left| \int_{\Omega} f z dx \right| = \left| \int_{\Omega} (\varepsilon f) \left(\frac{1}{\varepsilon} z \right) dx \right| \leq C(\varepsilon) \|f(t, \cdot)\|_{W_2^{-1}(\Omega)}^2 + \varepsilon \|\nabla z(t, \cdot)\|_{L^2(\Omega)}^2.$$

Applying (4)–(6) in (2), we have

$$(7) \quad -\frac{d}{dt} \int_{\Omega} z(t, \cdot)^2 dx + \|\nabla z(t, \cdot)\|_{L^2(\Omega)}^2 \leq C \int_{\Omega} z(t, \cdot)^2 dx + C \|f(t, \cdot)\|_{W_2^{-1}(\Omega)}^2, \quad t \geq 0.$$

In particular,

$$-\frac{d}{dt} \int_{\Omega} z(t, \cdot)^2 dx \leq C \int_{\Omega} z(t, \cdot)^2 dx + C \|f(t, \cdot)\|_{W_2^{-1}(\Omega)}^2, \quad t \geq 0.$$

Hence by $z(T, \cdot) = 0$, the Gronwall inequality implies

$$(8) \quad \|z(t, \cdot)\|_{L^2(\Omega)}^2 \leq C \|f\|_{L^2(0,T;W_2^{-1}(\Omega))}^2, \quad 0 \leq t \leq T.$$

Integrating (7) in t from 0 to T , we obtain

$$(9) \quad \|z(0, \cdot)\|_{L^2(\Omega)}^2 + \|\nabla z(t, \cdot)\|_{L^2(Q)}^2 \leq C \|z\|_{L^2(Q)}^2 + C \|f\|_{L^2(0,T;W_2^{-1}(\Omega))}^2.$$

By (8) and (9), we have

$$\|z\|_{L^2(0,T;W_2^1(\Omega))}^2 \leq C \|f\|_{L^2(0,T;W_2^{-1}(\Omega))}^2.$$

The proof of Lemma 2.4 is complete. □

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