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Noncommutative Sobolev Spaces, C^∞ Algebras and Schwartz Distributions Associated with Semicircular Systems

By

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Abstract

When the C^*-algebra and the W^*-algebra generated by a semicircular system are viewed from the viewpoints of noncommutative topology and noncommutative probability theory, we may consider the C^*-algebra as a certain kind of a "noncommutative cubic space" and the W^*-algebra as a "noncommutative cubic measure space." In this paper we introduce the Sobolev spaces W_n^p associated with the W^*-algebra generated by a semicircular system, and the C^∞ algebra $\mathcal{S}$ is defined as the projective limit of W_n^p. The Schwartz distribution space is then defined as the dual space of $\mathcal{S}$ and the Fourier representation theorem is obtained for Schwartz distributions. We furthermore discuss vector fields on the C^∞ algebra $\mathcal{S}$. Appendix treats the K-theory of the noncommutative cubic space.

§1. Introduction

Let $(\mathcal{M}, \tau)$ be a W^*-probability space, i.e., $\mathcal{M}$ is a finite W^*-algebra with a faithful normal tracial state τ. Let $\{s_i : i \in I\}$ be a semicircular system in $\mathcal{M}$ with an index set I, i.e., s_i $(i \in I)$ are selfadjoint elements in $\mathcal{M}$ whose distributions with respect to τ are the standard semicircular distribution and they are free (or freely independent) in the sense of free probability. Then, any element in the closed real linear span of $\{s_i : i \in I\}$ in $L^2(\mathcal{M}, \tau)$ is a centered semicircular element, i.e., its distribution with respect to τ is a semicircular one with mean 0. Conversely, if V is a real subspace in $L^2(\mathcal{M}, \tau)$ consisting of only

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centered semicircular elements, then any orthonormal system of V becomes a semicircular system. This fact is easily proved by the use of the free cumulant (see also [14, Proposition 2.1]) and it is the free probability analog of the well-known fact on Gaussian systems and Gaussian spaces in classical probability theory. Assume now that $\mathcal{M}$ is generated by $\{s_i : i \in I\}$. Then,

$$(\mathcal{M},\tau) \cong (\star_{i\in I} W^*(\{s_i\}), \star_{i\in I}\tau|_{W^*(\{s_i\})}) \cong (\star_N \mathcal{L}(\mathbb{Z}), \star_N\langle\delta_e, \cdot\,\delta_e\rangle) \cong (\mathcal{L}(\mathbb{F}_N), \langle\delta_e, \cdot\,\delta_e\rangle), \tag{1}$$

where $\star$ denotes the W^*-free product, $\mathcal{L}(\,\cdot\,)$ is the group von Neumann algebra, $\mathbb{F}_N$ is the free group of N generators ($N \equiv \operatorname{card} I$) and $\langle\delta_e, \cdot\,\delta_e\rangle$ is the tracial vector state defined by the vector δ_e supported on the group unit e. Throughout the paper isomorphisms (of algebras, topological linear spaces or probability spaces) are denoted by the same symbol $\cong$. From now on we assume that $\mathcal{M} = W^*(\{s_i : i \in I\}) \cong \mathcal{L}(\mathbb{F}_N)$. Let $\mathcal{A}$ be the C^*-algebra generated by the set $\{1\} \cup \{s_i : i \in I\}$; then by the reduced C^*-free product version of (1),

$$(\mathcal{A},\tau) \cong (\star_{r,i\in I} C^*(\{1,s_i\}), \star_{r,i\in I}\tau|_{C^*(\{1,s_i\})}) \cong (\star_{r,N} C([-2,2]), \star_{r,N}\textstyle\int_{-2}^{2} \cdot\, d\mu(x)),$$

where $\star_r$ denotes the reduced C^*-free product, $C([-2,2])$ is the algebra of all continuous functions on the interval $[-2,2]$ and μ is the standard semicircular distribution. $\mathcal{A}$ is a simple C^*-algebra with a unique trace τ (see [6, Theorem 2]) and it is projectionless because it is a subalgebra of the reduced free group C^*-algebra $C^*_r(\mathbb{F}_N)$ which has no proper projections. Let $\star_{u,N}C([-2,2])$ be the universal free product C^*-algebra of N copies of $C([-2,2])$ and q be the natural quotient map from $\star_{u,N}C([-2,2])$ onto $\star_{r,N}C([-2,2])$. We may consider $\star_{u,N}C([-2,2])$ as the algebra of all continuous functions on the "noncommutative cubic space" $\star_N[-2,2]$ by analogy from the fact that $\otimes_N C([-2,2]) \cong C([-2,2]^N)$ is the algebra of all continuous functions on the usual cubic space. Also, the algebra $\mathcal{A}$ may be regarded as the algebra of all continuous functions on the "noncommutative topological support" in $\star_N[-2,2]$ of the "noncommutative measure" τq, and on the other hand $\mathcal{M}$ may be regarded as the algebra of all bounded measurable functions on $\star_N[-2,2]$ with respect to τq. In this paper we always write h_2h_1 to mean the composition of a map h_2 after a map h_1.

In this paper we construct the noncommutative Sobolev spaces, the C^∞ algebra and the space of Schwartz distributions on the "noncommutative cubic space" mentioned above. They have a certain aspect of the noncommutative geometry. They are also regarded as some ingredients of the "free Malliavin calculus"; indeed our constructions are the free probabilistic analog of those in Malliavin calculus (see [9] and [10]). The paper is organized as follows. Section 2 is for preliminaries from free probability theory. In Section 3 we introduce the Sobolev spaces W_n^p inside the noncommutative L^p-spaces and the C^∞ algebra $\mathcal{S}$ as the projective limit of the spaces W_n^p. It is shown that the abstractly constructed algebra $\mathcal{S}$ actually a subalgebra of the C^*-algebra $\mathcal{A}$. Some results in Section 3 are proved in Section 4, and for this sake we introduce the weak derivation of the free group factor. In Section 5 the space of Schwartz distributions is constructed as the dual space of the C^∞ algebra $\mathcal{S}$ and their Fourier representations are given. Finally in Section 6 we discuss vector fields on the free group factor, but our discussions there are not complete for our real motivation of this research. In addition, the K-theory of the noncommutative cubic space $\star_N[-2,2]$ is examined in the appendix to the paper.

§2. Preliminaries

Let $\mathcal{P}$ be the $*$-subalgebra algebraically generated by $\{1\} \cup \{s_i : i \in I\}$; then $\mathcal{P} \cong \mathbb{C}\langle X_i : i \in I\rangle$ because the free product by a faithful state is just the closure of the algebraic free product. Here $\mathbb{C}\langle X_i : i \in I\rangle$ is the noncommutative polynomial ring over indeterminates X_i $(i \in I)$, and s_i and X_i are associated under the above isomorphism. For $n \in \mathbb{N}_0$ $(\equiv \mathbb{N} \cup \{0\})$ let $T_n(X)$ be the nth Chebyshev polynomial of the second kind; $T_n(X)$'s are determined by the recursion formula $T_{n+1}(X) = XT_n(X) - T_{n-1}(X)$ $(n \in \mathbb{N})$ with $T_0(X) = 1$, $T_1(X) = X$, and they are the complete orthogonal polynomials for the standard semicircular distribution.

Definition 2.1. For $l \in \mathbb{N}_0$, $m_k \in \mathbb{N}$ and $j_k \in I$ $(1 \le k \le l)$ with $j_1 \neq j_2 \neq \cdots \neq j_l$, define

$$T_{\underbrace{j_1 j_1 \cdots j_1}_{m_1} \underbrace{j_2 j_2 \cdots j_2}_{m_2} \cdots \underbrace{j_l j_l \cdots j_l}_{m_l}} \equiv T_{m_1}(s_{j_1}) T_{m_2}(s_{j_2}) \cdots T_{m_l}(s_{j_l}),$$

and $T_\emptyset \equiv 1$ (for $l = 0$) by convention. We call $T_{i_1 i_2 \cdots i_n}$ $(n \in \mathbb{N}_0,\, i_1, i_2, \ldots, i_n \in I)$ *noncommutative Chebyshev polynomials.*

The set of all noncommutative Chebyshev polynomials $T_{i_1 i_2 \cdots i_n}$ is clearly a linear basis for $\mathcal{P}$, and moreover they form an orthonormal basis for $L^2(\mathcal{M}, \tau)$ (see [2, Proposition 2.7] and also the next paragraph). We call $x = \sum_{n=0}^\infty \sum_{i_1, i_2, \ldots, i_n \in I} \xi_{i_1 i_2 \cdots i_n} T_{i_1 i_2 \cdots i_n}$ the *Fourier expansion* of $x \in L^2(\mathcal{M}, \tau)$, where

$\xi_{i_1 i_2 \cdots i_n} = \tau(T^*_{i_1 i_2 \cdots i_n} x)$ are its Fourier coefficients. We also call $x = \sum_{n=0}^{\infty} x_n$ the Fourier expansion of x, where $x_n \equiv \sum_{i_1, i_2, \ldots, i_n \in I} \xi_{i_1 i_2 \cdots i_n} T_{i_1 i_2 \cdots i_n}$ $(n \in \mathbb{N}_0)$. When the Fourier expansion is referred to, we always consider it in $L^2(\mathcal{M}, \tau)$ norm, i.e.,

$$x = \lim_{m \to \infty} \sum_{n=0}^{m} x_n = \lim_{m \to \infty} \sum_{n=0}^{m} \left(\sum_{i_1, i_2, \ldots, i_n \in I} \xi_{i_1 i_2 \cdots i_n} T_{i_1 i_2 \cdots i_n} \right) \text{ in } L^2(\mathcal{M}, \tau) \text{ norm.}$$

Let $\mathcal{P}_n \equiv \operatorname{span}\{T_{i_1 i_2 \cdots i_n} : i_1, i_2, \ldots, i_n \in I\}$, the linear span, for $n \in \mathbb{N}_0$, which are of course mutually orthogonal. Define the linear operator L on $L^2(\mathcal{M}, \tau)$ with $\operatorname{dom} L = \mathcal{P}$ by $L(x) \equiv nx$ for $x \in \mathcal{P}_n$; then $-\overline{L}^{L^2(\mathcal{M}, \tau)}$ is non-positive and called the *Ornstein-Uhlenbeck Laplacian*. Here we use the notation $\overline{\cdot}^{L^2(\mathcal{M}, \tau)}$ to denote the closure in $L^2(\mathcal{M}, \tau)$ of an operator or a subspace. In this paper we also call L itself the Ornstein-Uhlenbeck Laplacian. Furthermore, it is known (see [2, Theorem 2.11] and also the discussions below) that the operators $\exp\big(-\overline{L}^{L^2(\mathcal{M}, \tau)} t\big)$ $(t \geq 0)$ leave $\mathcal{M}$ globally invariant and they form a σ-weakly continuous contraction semigroup of τ-preserving, unital, normal and completely positive maps on $\mathcal{M}$. This is called the *Ornstein-Uhlenbeck semigroup* and is the most natural diffusion process on $\mathcal{M}$.

We next explain Voiculescu's free functor and reintroduce the Ornstein-Uhlenbeck semigroup mentioned above in terms of this functor. Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space with dimension $N \equiv \operatorname{card} I$ and inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$. Let $\mathcal{H} \equiv \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification Hilbert space whose inner product is $\langle x \otimes_{\mathbb{R}} a, y \otimes_{\mathbb{R}} b \rangle \equiv \langle x, y \rangle_{\mathbb{R}} (\overline{a} b)$ for $x, y \in \mathcal{H}_{\mathbb{R}}$ and $a, b \in \mathbb{C}$. In this paper an inner product is assumed to be linear in the second variable. Let $\mathcal{F}(\mathcal{H}_{\mathbb{R}})$ be the full Fock space over $\mathcal{H}$, i.e.,

$$\mathcal{F}(\mathcal{H}_{\mathbb{R}}) \equiv \mathbb{C}\Phi \overline{\oplus} \overline{\bigoplus}_{n=1}^{\infty} \underbrace{\mathcal{H} \overline{\otimes} \mathcal{H} \overline{\otimes} \cdots \overline{\otimes} \mathcal{H}}_{n},$$

with inner product $\langle \cdot, \cdot \rangle_{\mathcal{F}(\mathcal{H}_{\mathbb{R}})}$, where Φ is the vacuum vector and $\overline{\oplus}$, $\overline{\otimes}$ are Hilbert space direct sum and Hilbert space tensor product. For $f \in \mathcal{H}$ define the *creation operator* $a^*(f)$ by

$$a^*(f)\Phi \equiv f,$$
$$a^*(f) g_1 \otimes g_2 \otimes \cdots \otimes g_n \equiv f \otimes g_1 \otimes g_2 \otimes \cdots \otimes g_n$$

for $n \in \mathbb{N}$, $g_j \in \mathcal{H}$ $(1 \leq j \leq n)$. The *annihilation operator* $a(f)$ is the adjoint operator $(a^*(f))^*$ so that

$$a(f)\Phi = 0,$$
$$a(f) g_1 \otimes g_2 \otimes \cdots \otimes g_n = \langle f, g_1 \rangle g_2 \otimes \cdots \otimes g_n.$$

Let $s(f)$ be the selfadjoint operator $a^*(f) + a(f) = 2\mathrm{Re}\, a^*(f)$, and for any orthonormal basis $\{e_i : i \in I\}$ of $\mathcal{H}_\mathbb{R}$ $(\subset \mathcal{H})$ let $\widehat{\mathcal{M}}$ $(= \widehat{\mathcal{M}}(\mathcal{H}_\mathbb{R}))$ denote the von Neumann algebra $\{s(e_i) : i \in I\}''$. It is known that $\hat{\tau} \equiv \langle \Phi, \cdot\,\Phi \rangle_{\mathcal{F}(\mathcal{H}_\mathbb{R})}$ is a faithful normal tracial state on $\widehat{\mathcal{M}}$ and $\{s(e_i) : i \in I\}$ is a semicircular system with respect to $\hat{\tau}$. So $\widehat{\mathcal{M}}$ is isomorphic to $\mathcal{M}$ by the isomorphism π given by $\pi(s_i) = s(e_i)$ $(i \in I)$, which will be sometimes used in the proofs below. Furthermore, thanks to the cyclicity of Φ for $\widehat{\mathcal{M}}$, $L^2(\widehat{\mathcal{M}}, \hat{\tau})$ and $\mathcal{F}(\mathcal{H}_\mathbb{R})$ are isomorphic by the isomorphism $\eta : L^2(\widehat{\mathcal{M}}, \hat{\tau}) \to \mathcal{F}(\mathcal{H}_\mathbb{R})$ given by $\eta(x) \equiv x\Phi$ for $x \in \widehat{\mathcal{M}}$. For $l \in \mathbb{N}$, $m_k \in \mathbb{N}$ and $j_k \in I$ $(1 \le k \le l)$ with $j_1 \ne j_2 \ne \cdots \ne j_l$ we then have

$$\begin{aligned}&\eta\big(T_{m_1}(s(e_{j_1}))T_{m_2}(s(e_{j_2}))\cdots T_{m_l}(s(e_{j_l}))\big)\\ &\quad = \underbrace{e_{j_1} \otimes e_{j_1} \otimes \cdots \otimes e_{j_1}}_{m_1} \otimes \underbrace{e_{j_2} \otimes e_{j_2} \otimes \cdots \otimes e_{j_2}}_{m_2} \otimes \cdots \otimes \underbrace{e_{j_l} \otimes e_{j_l} \otimes \cdots \otimes e_{j_l}}_{m_l},\end{aligned}$$

which shows that the above $T_{m_1}(s(e_{j_1}))T_{m_2}(s(e_{j_2}))\cdots T_{m_l}(s(e_{j_l}))$'s together with 1 form an orthonormal basis for $L^2(\widehat{\mathcal{M}}, \hat{\tau})$. Also, we notice $\eta\Big(\overline{\mathrm{span}\{s(e_i) : i \in I\}}^{L^2(\widehat{\mathcal{M}}, \hat{\tau})}\Big) = \mathcal{H}$ and $\widehat{\mathcal{A}} \equiv C^*(\{1\} \cup \{s(e_i) : i \in I\}) = C^*(\{1\} \cup \{s(f) : f \in \mathcal{H}_\mathbb{R}\})$ because $s(f)$ $(f \in \mathcal{H}_\mathbb{R})$ are centered semicircular with respect to $\hat{\tau}$ so that $\|s(f)\| = 2\|s(f)\|_{L^2(\widehat{\mathcal{M}}, \hat{\tau})}$. Via the above isomorphism the ultracontractivity (see [1, Proposition 2.1]) is formulated on $L^2(\mathcal{M}, \tau) = \bigoplus_{n=0}^\infty \overline{\mathcal{P}_n}^{L^2(\mathcal{M}, \tau)}$ as follows.

Proposition 2.2. *For every $n \in \mathbb{N}_0$,*

$$\|x\|_{L^2(\mathcal{M},\tau)} \le \|x\| \le (n+1)\|x\|_{L^2(\mathcal{M},\tau)} \qquad \big(x \in \overline{\mathcal{P}_n}^{L^2(\mathcal{M},\tau)}\big).$$

This says that, for each $n \in \mathbb{N}_0$, $L^p(\mathcal{M}, \tau)$ norms $(2 \le p \le \infty)$ restricted on $\overline{\mathcal{P}_n}^{L^2(\mathcal{M},\tau)}$ are all equivalent. But in fact, due to the duality of noncommutative L^p-spaces, all $L^p(\mathcal{M}, \tau)$ norms $(1 \le p \le \infty)$ on each $\overline{\mathcal{P}_n}^{L^2(\mathcal{M},\tau)}$ are equivalent.

Remark 2.3. In the commutative case, if a linear space of measurable functions on a probability space is in $L^1 \cap L^\infty$ and closed with both L^1 and L^∞ norms, then it must be finite dimensional. But this is not the case in the noncommutative case; for example, $\overline{\mathcal{P}_n}^{L^2(\mathcal{M},\tau)}$ $(n \in \mathbb{N})$ are infinite dimensional when $N \equiv \mathrm{card}\, I$ is infinite.

We return to Fock space construction again. For a contraction $T : \mathcal{H}_\mathbb{R} \to \mathcal{H}^1_\mathbb{R}$ between real Hilbert spaces, let $T_\mathbb{C} : \mathcal{H} \to \mathcal{H}^1$ be its complexification and

$\mathcal{F}(T) : \mathcal{F}(\mathcal{H}) \to \mathcal{F}(\mathcal{H}^1)$ be defined by

$$\mathcal{F}(T) \equiv 1_{\mathbb{C}\Phi}\overline{\oplus}\overline{\bigoplus}_{n=1}^{\infty} T_{\mathbb{C}}\overline{\otimes}T_{\mathbb{C}}\overline{\otimes}\cdots\overline{\otimes}T_{\mathbb{C}}.$$

Then $\mathcal{F}(T)$ is obviously a contraction with $\mathcal{F}(1) = 1$, and $\mathcal{F}(T^1T) = \mathcal{F}(T^1)$ $\mathcal{F}(T)$ for another contraction $T^1 : \mathcal{H}^1_{\mathbb{R}} \to \mathcal{H}^2_{\mathbb{R}}$. Define the contraction operator $\Gamma(T) : L^2(\widehat{\mathcal{M}}, \hat{\tau}) \to L^2(\widehat{\mathcal{M}}^1, \hat{\tau}^1)$ ($\widehat{\mathcal{M}}^1 \equiv \widehat{\mathcal{M}}(\mathcal{H}^1_{\mathbb{R}})$) by the above isomorphism, i.e., $\Gamma(T)(x) \equiv \eta^{-1}(\mathcal{F}(T)(\eta(x)))$ for $x \in L^2(\widehat{\mathcal{M}}, \widehat{\tau})$. It is known (see [2]) that, for the general contraction T, $\Gamma(T)|_{\widehat{\mathcal{M}}}$ becomes a trace-preserving, unital, normal and completely positive map from $\widehat{\mathcal{M}}$ into $\widehat{\mathcal{M}}^1$. We write just $\Gamma(T)$ for the above $\Gamma(T)|_{\widehat{\mathcal{M}}}$ in the sequel. The functor from the category of $(\mathcal{H}_{\mathbb{R}}, T)$, real Hilbert spaces with contractions, to the category of $(\widehat{\mathcal{M}}, \Gamma(T))$, free group factors with trace-preserving unital complete positive maps, is so-called Voiculescu's free functor, and it is the free probabilistic analog of the classical Gaussian functor. For $T(t) = \exp(-t)1$ $(t \geq 0)$ on $\mathcal{H}_{\mathbb{R}}$, $\Gamma(T(t))$ is isomorphic to the Ornstein-Uhlenbeck semigroup via the isomorphism π, as easily checked on the noncommutative Chebyshev polynomials. So the Ornstein-Uhlenbeck semigroup is a σ-weakly continuous semigroup of τ-preserving, unital, normal and completely positive maps on $\mathcal{M}$.

For a finite ordered set of noncommutative Chebyshev polynomials $(T_{i_1 i_2 \cdots i_l}, T_{j_1 j_2 \cdots j_m}, \ldots, T_{k_1 k_2 \cdots k_n})$ where $l, m, \ldots, n \in \mathbb{N}$, $i_r, j_s, \ldots, k_t \in I$ $(1 \leq r \leq l,\ 1 \leq s \leq m,\ \ldots,\ 1 \leq t \leq n)$, we associate the following diagram with $l + m + \cdots + n$ vertices labeled by $i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_m, \ldots, k_1, k_2, \ldots, k_n$ and partitioned by the symbol $|$:

$$\begin{matrix} \bullet & \bullet & \cdots & \bullet & | & \bullet & \bullet & \cdots & \bullet & | & \cdots & | & \bullet & \bullet & \cdots & \bullet \\ i_1 & i_2 & \cdots & i_l & & j_1 & j_2 & \cdots & j_m & & \cdots & & k_1 & k_2 & \cdots & k_n \end{matrix}.$$

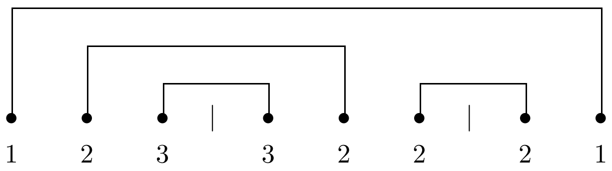
We call $(i_1, i_2, \ldots, i_l)$, $(j_1, j_2, \ldots, j_m)$, $\ldots$, $(k_1, k_2, \ldots, k_n)$ blocks of the diagram. A pair partition of the above $l + m + \cdots + n$ vertices is called a non-crossing pair partition of the diagram if the following conditions are satisfied:

1. the labels of two vertices paired in the partition are same,

2. the blocks of two vertices paired in the partition are different,

3. the pair partition is non-crossing.

A pair partition is presented by drawing the edges connecting paired vertices as follows:



The meaning of the word "non-crossing" is obvious from this presentation. The above is a unique non-crossing pair partition of the diagram associated with $(T_{123}, T_{322}, T_{21})$.

Lemma 2.4. *For $l, m, \ldots, n \in \mathbb{N}$, $i_r, j_s, \ldots, k_t \in I$ $(1 \leq r \leq l,\ 1 \leq s \leq m,\ \ldots,\ 1 \leq t \leq n)$, $\tau(T_{i_1 i_2 \cdots i_l} T_{j_1 j_2 \cdots j_m} \cdots T_{k_1 k_2 \cdots k_n})$ is equal to the number of non-crossing pair partitions of the diagram associated with $(T_{i_1 i_2 \cdots i_l}, T_{j_1 j_2 \cdots j_m}, \ldots, T_{k_1 k_2 \cdots k_n})$.*

Proof. *Step* 1. First, we see that $\pi(T_m(s_i)) = \sum_{n=0}^{m} a^*(e_i)^n a(e_i)^{m-n}$ for $i \in I$, $m \in \mathbb{N}_0$. The cases $m = 0, 1$ are trivial. Moreover, we have

$$\begin{aligned}
&(a^*(e_i) + a(e_i))\left(\sum_{n=0}^{m} a^*(e_i)^n a(e_i)^{m-n}\right) - \sum_{n=0}^{m-1} a^*(e_i)^n a(e_i)^{m-1-n} \\
&\quad = \sum_{n=0}^{m} a^*(e_i)^{n+1} a(e_i)^{m-n} + a(e_i)^{m+1} + \sum_{n=1}^{m} a^*(e_i)^{n-1} a(e_i)^{m-n} \\
&\qquad - \sum_{n=0}^{m-1} a^*(e_i)^n a(e_i)^{m-1-n} \\
&\quad = \sum_{n=0}^{m+1} a^*(e_i)^n a(e_i)^{m+1-n},
\end{aligned}$$

which is the same recursion formula as that for Chebyshev polynomials. So the claim is proved.

Step 2. Second, we prove the assertion of the lemma when $i_1 = \cdots = i_l = i$, $j_1 = \cdots = j_m = j$, $\ldots$, $k_1 = \cdots = k_n = k$ so that $T_{i_1 i_2 \cdots i_l} = T_l(s_i)$, $T_{j_1 j_2 \cdots j_m} = T_m(s_j)$, $\ldots$, $T_{k_1 k_2 \cdots k_n} = T_n(s_k)$. For this case we write $m(1), m(2), \ldots, m(r)$ for $l, m, \ldots, n$ and $i(1), i(2), \ldots, i(r)$ for $i, j, \ldots, k$, so our quantity in question is

$$\tau\big(T_{m(1)}(s_{i(1)}) T_{m(2)}(s_{i(2)}) \cdots T_{m(r)}(s_{i(r)})\big)$$

and the associated diagram is

(2)
$$\begin{array}{cccc|cccc|c|cccc} \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \cdots & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet \\ \underbrace{i(1) & i(1) & \cdots & i(1)}_{m(1)} & \underbrace{i(2) & i(2) & \cdots & i(2)}_{m(2)} & \cdots & \underbrace{i(r) & i(r) & \cdots & i(r)}_{m(r)}. \end{array}$$

By Step 1,

$$\begin{aligned}\tau\big(T_{m(1)}(s_{i(1)})T_{m(2)}(s_{i(2)})\cdots T_{m(r)}(s_{i(r)})\big) &= \sum_{n(1)=0}^{m(1)}\cdots\sum_{n(r)=0}^{m(r)}\big\langle\Phi, a^*(e_{i(1)})^{n(1)}a(e_{i(1)})^{m(1)-n(1)} \\ &\quad\times a^*(e_{i(2)})^{n(2)}a(e_{i(2)})^{m(2)-n(2)}\cdots a^*(e_{i(r)})^{n(r)}a(e_{i(r)})^{m(r)-n(r)}\Phi\big\rangle_{\mathcal{F}(\mathcal{H}_{\mathbb{R}})}.\end{aligned}$$

Note that each inner product in the above expression is either 0 or 1, and it is 1 if and only if

(3) $$a^*(e_{i(1)})^{n(1)}a(e_{i(1)})^{m(1)-n(1)}\cdots a^*(e_{i(r)})^{n(r)}a(e_{i(r)})^{m(r)-n(r)}\Phi = \Phi.$$

Hence it suffices to make a bijective correspondence between the set of non-crossing pair partitions of the diagram (2) and the set of terms $a^*(e_{i(1)})^{n(1)}$ $a(e_{i(1)})^{m(1)-n(1)}\cdots a^*(e_{i(r)})^{n(r)}a(e_{i(r)})^{m(r)-n(r)}$ satisfying (3). Assume that a term

$$\begin{aligned}X = a^*(e_{i(1)})^{n(1)}&a(e_{i(1)})^{m(1)-n(1)}a^*(e_{i(2)})^{n(2)}a(e_{i(2)})^{m(2)-n(2)}\cdots \\ &a^*(e_{i(r)})^{n(r)}a(e_{i(r)})^{m(r)-n(r)}\end{aligned}$$

satisfies $X\Phi = \Phi$. Let $I_T \equiv \{i(1), i(2), \ldots, i(r)\} \subset I$ and $M \equiv m(r) + m(r-1) + \cdots + m(1)$. Looking M factors of X from the right to the left we recursively define functions

$$h_i : \big\{0, 1, 2, \cdots, m(r) + m(r-1) + \cdots + m(1)\big\} \to \mathbb{N}_0 \qquad (i \in I_T)$$

as follows: $h_i(0) = 0$ $(i \in I_T)$ and for $1 \le k \le m(r) + m(r-1) + \cdots + m(1)$,

$$h_i(k) \equiv \begin{cases} h_i(k-1) + 1 & \text{if the } k\text{th factor of } X \text{ is } a^*(e_i), \\ h_i(k-1) - 1 & \text{if the } k\text{th factor of } X \text{ is } a(e_i), \\ h_i(k-1) & \text{otherwise.} \end{cases}$$

Furthermore, set $h \equiv \sum_{i\in I_T} h_i$. Since $X\Phi = \Phi$, for $1 \le j \le r$ we notice

$$a(e_{i(j)})^{k-(m(r)+m(r-1)+\cdots+m(j+1))}\cdots a^*(e_{i(r)})^{n(r)}a(e_{i(r)})^{m(r)-n(r)}\Phi \in \mathcal{P}_{h(k)}$$

if $m(r)+\cdots+m(j+1) < k \le m(r)+\cdots+m(j+1)+m(j)-n(j)$ (with obvious convention $m(r)+\cdots+m(j+1)=0$ for $j=r$), and

$$a^*(e_{i(j)})^{k-(m(r)+m(r-1)+\cdots+m(j)-n(j))}a(e_{i(j)})^{m(j)-n(j)}\cdots$$
$$a^*(e_{i(r)})^{n(r)}a(e_{i(r)})^{m(r)-n(r)}\Phi \in \mathcal{P}_{h(k)}$$

if $m(r)+\cdots+m(j)-n(j) < k \le m(r)+\cdots+m(j)$. So $h(k)$ is nonnegative for all k and it has a peak (since $h(M)=0$ due to the assumption $X\Phi=\Phi$). Let k be the far right peak point so that $h(k-1)+1 = h(k+1)+1 = h(k)$. There exists a unique $i \in I_T$ such that $h_i(k) = h_i(k-1)+1$ due to the definition of h_i's, and the kth factor must be $a^*(e_i)$. Then the $k+1$st factor must be $a(e_i)$ because otherwise $X\Phi = 0$. This shows that the kth and the $k+1$st vertices of (2) have the same label (counted from the right to the left) and these vertices belong to different blocks (say, jth and $j+1$st blocks). So we make a pair of these two vertices. Since $a(e_i)a^*(e_i) = 1$, we have $X'\Phi = \Phi$ for the shorter term

$$\begin{aligned} X' \equiv\, & a^*(e_{i(1)})^{n(1)}a(e_{i(1)})^{m(1)-n(1)}\cdots a^*(e_{i(j)})^{n(j)}a(e_{i(j)})^{m(j)-1-n(j)} \\ & \times a^*(e_{i(j+1)})^{n(j+1)-1}a(e_{i(j+1)})^{m(j+1)-n(j+1)}\cdots \\ & \quad a^*(e_{i(r)})^{n(r)}a(e_{i(r)})^{m(r)-n(r)}. \end{aligned}$$

Now the above procedure can be again applied to this new term so that we can choose two neighboring factors of X' having the same label and belonging to different blocks. This process can be repeated until all the factors are removed, and a pair partition of the vertices of (2) is obtained in this way. It is obvious from the procedure that the pair partition constructed is non-crossing.

Next, we consider the process reverse to the above. Let a non-crossing pair partition of the diagram (2) be given. Define a product $\widehat{a}_M\widehat{a}_{M-1}\cdots\widehat{a}_1$ made from $a^*(e_i), a(e_i)$ $(i \in I_T)$ in the following way: If the kth and the lth vertices where $k > l$ (counted from the right to the left) are paired with the same label i, then $\widehat{a}_k \equiv a(e_i)$ and $\widehat{a}_l \equiv a^*(e_i)$. We then show that $\widehat{a}_M\widehat{a}_{M-1}\cdots\widehat{a}_1\Phi = \Phi$. Since the pair partition is non-crossing, there is a pair consisting of neighboring vertices with the same label i. Hence $\widehat{a}_{k+1} = a(e_i)$ and $\widehat{a}_k = a^*(e_i)$ for a certain k, so that $\widehat{a}_M\widehat{a}_{M-1}\cdots\widehat{a}_1 = \widehat{a}_M\cdots\widehat{a}_{k+2}\widehat{a}_{k-1}\cdots\widehat{a}_1$. Removing the kth and the $k+1$st vertices we have a non-crossing pair partition of the shortened diagram, and the product corresponding to the new pair partition is $\widehat{a}_M\cdots\widehat{a}_{k+2}\widehat{a}_{k-1}\cdots\widehat{a}_1$. This process can be continued until $\widehat{a}_M\cdots\widehat{a}_1$ is reduced to 1. So $\widehat{a}_M\cdots\widehat{a}_1\Phi = \Phi$ is shown. Moreover, it is immediate to see that the two maps given by the above procedures are the inverse of each other.

Step 3. For a general $(T_{i_1 i_2 \cdots i_l}, T_{j_1 j_2 \cdots j_m}, \dots, T_{k_1 k_2 \cdots k_n})$ we write

$$\begin{aligned}
T_{i_1 i_2 \cdots i_l} &= T_{m(1,1)}(s_{i(1,1)}) T_{m(1,2)}(s_{i(1,2)}) \cdots T_{m(1,r(1))}(s_{i(1,r(1))}),\\
T_{j_1 j_2 \cdots j_m} &= T_{m(2,1)}(s_{i(2,1)}) T_{m(2,2)}(s_{i(2,2)}) \cdots T_{m(2,r(2))}(s_{i(2,r(2))}),\\
&\vdots\\
T_{k_1 k_2 \cdots k_n} &= T_{m(p,1)}(s_{i(p,1)}) T_{m(p,2)}(s_{i(p,2)}) \cdots T_{m(p,r(p))}(s_{i(p,r(p))}),
\end{aligned}$$

where p is the number of noncommutative Chebyshev polynomials and $i(t,1) \neq i(t,2) \neq \cdots \neq i(t,r(t))$ for each $1 \leq t \leq p$. Let A and B denote the diagrams associated with $(T_{i_1 i_2 \cdots i_l}, T_{j_1 j_2 \cdots j_m}, \dots, T_{k_1 k_2 \cdots k_n})$ and

$$\Big(T_{m(1,1)}(s_{i(1,1)}), \dots, T_{m(1,r(1))}(s_{i(1,r(1))}), \dots,\\ T_{m(p,1)}(s_{i(p,1)}), \dots, T_{m(p,r(p))}(s_{i(p,r(p))})\Big),$$

respectively. By Step 2, $\tau(T_{i_1 i_2 \cdots i_l} T_{j_1 j_2 \cdots j_m} \cdots T_{k_1 k_2 \cdots k_n})$ is equal to the number of non-crossing pair partitions of B. So it remains to show that any non-crossing pair partition of B is also that of A, the converse being trivially true. Assume that two vertices paired by a non-crossing pair partition of B belong to the same block of A. From the non-crossingness we can choose such two neighboring vertices between the above two vertices. Then, by the construction of B, the two vertices cannot have the same label. This is a contradiction, and the proof is completed. □

Remarks 2.5.

(1) The above lemma extends the known formula for $\tau(s_{i_1} s_{i_2} \cdots s_{i_n})$ (see [7, Lemma 2.5.3]), which is the case where all noncommutative Chebyshev polynomials are the first T_1. In fact, the condition on blocks is irrelevant in this case.

(2) The assertion similar to Lemma 2.4 in classical probability is known (see [5]), where the non-crossing condition must be dropped.

(3) Lemma 2.4 is also used to prove the already mentioned fact that the set $\{T_{i_1 i_2 \cdots i_n} : n \in \mathbb{N}_0,\ i_1, \dots, i_n \in I\}$ is an orthonormal basis for $L^2(\mathcal{M}, \tau)$. In fact, for $n, m \in \mathbb{N}$ and $i_1, \dots, i_n, j_1, \dots, j_m \in I$,

$$\langle T_{i_1 i_2 \cdots i_n}, T_{j_1 j_2 \cdots j_m} \rangle_{L^2(\mathcal{M},\tau)} = \tau(T^*_{i_1 i_2 \cdots i_n} T_{j_1 j_2 \cdots j_m}) = \tau(T_{i_n i_{n-1} \cdots i_1} T_{j_1 j_2 \cdots j_m}),$$

which is nonzero (in fact, equal to 1) if and only if $n = m$ and $i_k = j_k$ $(1 \leq k \leq n)$.

(4) Here is another application of Lemma 2.4. For $n, m \in \mathbb{N}$ and $i_1, \ldots, i_n, j_1, \ldots, j_m \in I$, if $\tau(T_{i_1 i_2 \cdots i_n} T_{j_1} T_{j_2} \cdots T_{j_m})$ is nonzero, then n must be one of $m, m-2, m-4, \ldots$ and $\{i_1, i_2, \ldots, i_n\} \subset \{j_1, j_2, \ldots, j_m\}$. In fact, assume that there is a non-crossing pair partition of the diagram associated with $(T_{i_1 i_2 \cdots i_n}, T_{j_1}, T_{j_2}, \ldots, T_{j_m})$. If $n = m$, then $i_n = j_1, i_{n-1} = j_2, \cdots, i_1 = j_n$ and $\tau(T_{i_1 i_2 \cdots i_n} T_{j_1} T_{j_2} \cdots T_{j_n}) = 1$. If $n < m$, then there must be $1 \leq k(1) < k(2) < \cdots < k(n) \leq m$ such that the vertices in the first block with labels $i_n, i_{n-1}, \ldots, i_1$ are paired with the vertices with labels $j_{k(1)}, j_{k(2)}, \ldots, j_{k(n)}$, respectively.

§3. Sobolev Spaces and C^∞ Algebra

In this section we construct the Sobolev spaces W_n^p $(1 < p < \infty,\ n \in \mathbb{N}_0)$ and the C^∞ algebra $\mathcal{S}$. We begin by introducing the Sobolev norms. For $x = \sum_{i=0}^\infty x_i \in \mathcal{P}$ (a finite sum Fourier expansion), define $L^\alpha(x) \equiv \sum_{i=0}^\infty i^\alpha x_i$ for each $\alpha \in \mathbb{R}$. (This $L^\alpha(x)$ may be defined by the functional calculus of the selfadjoint operator $\overline{L}^{L^2(\mathcal{M},\tau)}$).

Definition 3.1. For every $x \in \mathcal{P}$, $1 < p \leq \infty$ and $n \in \mathbb{N}_0$, define the norm $\|x\|_{W_n^p}$ by

$$\|x\|_{W_n^p} \equiv \begin{cases} \left(\|x\|^p_{L^p(\mathcal{M},\tau)} + \|L^{\frac{1}{2}} x\|^p_{L^p(\mathcal{M},\tau)} + \cdots + \|L^{\frac{n}{2}} x\|^p_{L^p(\mathcal{M},\tau)}\right)^{\frac{1}{p}} & (1 < p < \infty), \\ \max\left\{\|x\|_{L^\infty(\mathcal{M},\tau)}, \|L^{\frac{1}{2}} x\|_{L^\infty(\mathcal{M},\tau)}, \ldots, \|L^{\frac{n}{2}} x\|_{L^\infty(\mathcal{M},\tau)}\right\} & (p = \infty), \end{cases}$$

where $\|\cdot\|_{L^\infty(\mathcal{M},\tau)}$ means C^*-norm of $\mathcal{M}$. We call the norm $\|\cdot\|_{W_n^p}$ on $\mathcal{P}$ the *Sobolev W_n^p norm.* (The W_n^∞ norm is included here for the convenience of later discussions.)

By definition, $\|\cdot\|_{W_0^p} = \|\cdot\|_{L^p(\mathcal{M},\tau)}$ for all $1 < p \leq \infty$. We also introduce the modified Sobolev W_n^p norm $(1 < p \leq \infty,\ n \in \mathbb{N}_0)$ for the convenience of the proofs bellow.

Definition 3.2. For every $x \in \mathcal{P}$, $1 < p \leq \infty$ and $n \in \mathbb{N}_0$, define the *modified Sobolev W_n^p norm* $\|x\|^*_{W_n^p}$ by

$$\|x\|^*_{W_n^p} \equiv \|x\|_{L^p(\mathcal{M},\tau)} + \|L^{\frac{1}{2}} x\|_{L^p(\mathcal{M},\tau)} + \cdots + \|L^{\frac{n}{2}} x\|_{L^p(\mathcal{M},\tau)}.$$

For each $1 < p \leq \infty$ and $n \in \mathbb{N}_0$,

$$n^{\frac{1}{p}-1} \|x\|^*_{W_n^p} \leq \|x\|_{W_n^p} \leq \|x\|^*_{W_n^p} \qquad (x \in \mathcal{P}).$$

Thus, the Sobolev norm and the modified one are equivalent, so we can use the modified Sobolev norm in proving continuity property, etc.

Definition 3.3. For $1 < p < \infty$ and $n \in \mathbb{N}_0$ we define the L^p *Sobolev space* W_n^p of order n as the (abstract) completion of $\mathcal{P}$ with respect to the norm $\|\cdot\|_{W_n^p}$.

By definition, for each $1 < p < \infty$, W_0^p is isometric to $L^p(\mathcal{M}, \tau)$ by the canonical isomorphism, so we just write $W_0^p = L^p(\mathcal{M}, \tau)$ in the sequel. For each $n \in \mathbb{N}_0$, W_n^2 is a Hilbert space with the following inner product

$$\begin{aligned}\langle x, y\rangle_{W_n^2} &\equiv \langle x, y\rangle_{L^2(\mathcal{M},\tau)} + \langle L^{\frac{1}{2}}(x), L^{\frac{1}{2}}(y)\rangle_{L^2(\mathcal{M},\tau)} + \cdots + \langle L^{\frac{n}{2}}(x), L^{\frac{n}{2}}(y)\rangle_{L^2(\mathcal{M},\tau)} \\ &= \langle x, (I + L^1 + \cdots + L^n)y\rangle_{L^2(\mathcal{M},\tau)}\end{aligned}$$

and the norm

$$\|x\|_{W_n^2} = \|(I + L^1 + \cdots + L^n)^{\frac{1}{2}} x\|_{L^2(\mathcal{M},\tau)}$$

for $x, y \in \mathcal{P}$. Assume that $1 < p < p' < \infty$ and $0 \le n < n'$. Since $\|\cdot\|^*_{W_n^p} \le \|\cdot\|^*_{W_n^{p'}}$ and $\|\cdot\|^*_{W_n^p} \le \|\cdot\|^*_{W_{n'}^p}$ on $\mathcal{P}$, we can extend the identity map on $\mathcal{P}$ as $\iota^n_{p',p} : W_n^{p'} \to W_n^p$ $(1 < p < p' < \infty, n \in \mathbb{N}_0)$ and $\iota^p_{n',n} : W^p_{n'} \to W^p_n$ $(1 < p < \infty, 0 \le n < n')$. Then the next proposition holds.

Theorem 3.4. *For each $1 < p < p' < \infty$ and $0 \le n < n'$, the maps $\iota^n_{p',p}$ and $\iota^p_{n',n}$ defined above are all injective. Consequently, all Sobolev spaces W_n^p can be regarded as linear subspaces of $L^1(\mathcal{M}, \tau)$.*

The proof of the theorem will be presented in the next section. In the rest of this section (also in Sections 5 and 6) we consider the abstract Sobolev spaces as subspaces of $L^1(\mathcal{M}, \tau)$. We now get the following commutative diagram. The commutativity is obvious from the fact that all the maps $\iota^n_{p',p}$ $(p < p')$ and $\iota^p_{n',n}$ $(n < n')$ are the closures of the identity map on $\mathcal{P}$. In the following the symbol $\Subset$ is used to mean continuous imbedding.

$$\begin{array}{ccccccc}
(L^\infty \equiv)\,\mathcal{M} \Subset & \begin{matrix} W_0^p = L^p \\ (2 < p < \infty)\end{matrix} & \Subset & W_0^2 = L^2 & \Subset & \begin{matrix} W_0^p = L^p \\ (1 < p < 2)\end{matrix} & \Subset L^1 \\
& \Cup & & \Cup & & \Cup & \\
& W_1^p & \Subset & W_1^2 & \Subset & W_1^p & \\
& \Cup & & \Cup & & \Cup & \\
& W_2^p & \Subset & W_2^2 & \Subset & W_2^p & \\
& \Cup & & \Cup & & \Cup & \\
& \vdots & & \vdots & & \vdots &
\end{array}$$

Definition 3.5. For each $n \in \mathbb{N}_0$ let $W_n^\cap$ be the projective limit of the Sobolev spaces W_n^p $(1 < p < \infty)$ (see the following diagram). Let $\mathcal{S}$ denote the projective limit of $W_n^{\cap}$ ($n \in \mathbb{N}_0$) (see the following diagram). We call $\mathcal{S}$ the C^∞ *algebra* or the *rapidly decreasing function algebra*. The justification of the terms "algebra" and "rapidly decreasing" are clarified below.

$$
\begin{array}{ccccccccc}
\mathcal{M} \Subset W_0^{\cap} & \Subset & \begin{array}{c} W_0^p = L^p \\ (2 < p < \infty) \end{array} & \Subset & W_0^2 = L^2 & \Subset & \begin{array}{c} W_0^p = L^p \\ (1 < p < 2) \end{array} & \Subset & L^1 \\
\Cup & & \Cup & & \Cup & & \Cup & & \\
W_1^{\cap} & \Subset & W_1^p & \Subset & W_1^2 & \Subset & W_1^p & & \\
\Cup & & \Cup & & \Cup & & \Cup & & \\
W_2^{\cap} & \Subset & W_2^p & \Subset & W_2^2 & \Subset & W_2^p & & \\
\Cup & & \Cup & & \Cup & & \Cup & & \\
\vdots & & \vdots & & \vdots & & \vdots & & \\
\Cup & & & & & & & & \\
\mathcal{S} & & & & & & & &
\end{array} \tag{4}
$$

Then we have the next theorem.

Theorem 3.6. *For each $n \in \mathbb{N}_0$, $W_n^{\cap}$ is a Fréchet $*$-algebra, i.e., a complete, locally convex and metrizable linear topological space with continuous $*$-operation and bicontinuous multiplication. Consequently, $\mathcal{S}$ is also a Fréchet $*$-algebra. Here, the $*$-operation and the multiplication are defined by continuously extending those of the algebra $\mathcal{P}$.*

This theorem as well as Theorem 3.4 will be proved in the next section.

Now, we show that the abstractly constructed C^∞ algebra $\mathcal{S}$ sits in the C^* algebra $\mathcal{A}$. The next lemma is regarded as the free analog of the *Sobolev lemma*.

Theorem 3.7. *Let $1 < p \leq \infty$ and $n \in \mathbb{N}_0$, and set $C \equiv \left(\sum_{i=1}^{\infty} (\frac{i+1}{i^2})^2\right)^{\frac{1}{2}}$. If $n' \geq 4$, then*

$$\|x\|^*_{W_n^p} \leq C\|x\|^*_{W^2_{n+n'}} \qquad (x \in \mathcal{P}). \tag{5}$$

Proof. Since $\|x\|_{L^p(\mathcal{M},\tau)} \leq \|x\|_{L^\infty(\mathcal{M},\tau)}$ and $\|x\|^*_{W^2_{n+4}} \leq \|x\|^*_{W^2_{n+n'}}$, we may prove that $\|x\|^*_{W_n^\infty} \leq C\|x\|^*_{W^2_{n+4}}$ ($n \in \mathbb{N}_0$) for $x = \sum_{i=0}^\infty x_i \in \mathcal{P}$ (a finite sum Fourier expansion). We estimate

$$
\begin{aligned}
\|x\|^*_{W_n^\infty} &= \|x\|_{L^\infty(\mathcal{M},\tau)} + \|L^{\frac{1}{2}}x\|_{L^\infty(\mathcal{M},\tau)} + \cdots + \|L^{\frac{n}{2}}x\|_{L^\infty(\mathcal{M},\tau)} \\
&\le \|x_0\|_{L^\infty(\mathcal{M},\tau)} + \sum_{i=1}^\infty \|x_i\|_{L^\infty(\mathcal{M},\tau)} \\
&\quad + \sum_{i=1}^\infty i^{\frac{1}{2}}\|x_i\|_{L^\infty(\mathcal{M},\tau)} + \cdots + \sum_{i=1}^\infty i^{\frac{n}{2}}\|x_i\|_{L^\infty(\mathcal{M},\tau)} \\
&\le \|x_0\|_{L^2(\mathcal{M},\tau)} + \sum_{i=1}^\infty (i+1)\|x_i\|_{L^2(\mathcal{M},\tau)} \\
&\quad + \sum_{i=1}^\infty (i+1)i^{\frac{1}{2}}\|x_i\|_{L^2(\mathcal{M},\tau)} + \cdots + \sum_{i=1}^\infty (i+1)i^{\frac{n}{2}}\|x_i\|_{L^2(\mathcal{M},\tau)} \\
&= \|x_0\|_{L^2(\mathcal{M},\tau)} + \sum_{i=1}^\infty \frac{(i+1)}{i^2} i^2\|x_i\|_{L^2(\mathcal{M},\tau)} \\
&\quad + \sum_{i=1}^\infty \frac{(i+1)}{i^2} i^{2+\frac{1}{2}}\|x_i\|_{L^2(\mathcal{M},\tau)} + \cdots + \sum_{i=1}^\infty \frac{(i+1)}{i^2} i^{2+\frac{n}{2}}\|x_i\|_{L^2(\mathcal{M},\tau)} \\
&\le \|x_0\|_2 + C\left(\sum_{i=1}^\infty i^4\|x_i\|^2_{L^2(\mathcal{M},\tau)}\right)^{\frac{1}{2}} \\
&\quad + C\left(\sum_{i=1}^\infty i^5\|x_i\|^2_{L^2(\mathcal{M},\tau)}\right)^{\frac{1}{2}} + \cdots + C\left(\sum_{i=1}^\infty i^{n+4}\|x_i\|^2_{L^2(\mathcal{M},\tau)}\right)^{\frac{1}{2}} \\
&\le C\|x\|^*_{W^2_{n+4}}.
\end{aligned}
$$

In the above, the second inequality is implied by Proposition 2.2 and the third is due to the Cauchy-Schwarz inequality. □

We have the free analog of the *Sobolev inclusion theorem*.

Corollary 3.8. *The space W_4^2 is continuously imbedded in $\mathcal{A}$, i.e., $W_4^2 \Subset \mathcal{A}$. Furthermore, the imbeddings $W_n^\cap \Subset \mathcal{A}$ for $n \ge 4$ and $\mathcal{S} \Subset \mathcal{A}$ are $*$-algebraic homomorphisms.*

Proof. This can be shown by using the commutative diagram (4). Let $\theta_{\infty,2}$ be the inclusion map from $\mathcal{A} = W_0^\infty$ into $L^2(\mathcal{M},\tau) = W_0^2$, and let $\iota_{(4,2),(0,\infty)} : W_4^2 \to W_0^\infty$ be the closure of the identity map on $\mathcal{P}$, whose existence is guaranteed by Theorem 3.7. Then $\iota^2_{4,0} = \theta_{\infty,2}\iota_{(4,2),(0,\infty)}$, so the map $\iota_{(4,2),(0,\infty)}$ is injective because so is $\iota^2_{4,0}$ by Theorem 3.4. The assertion that the inclusion map $W_4^{\cap} \Subset \mathcal{A}$ is a $*$-algebraic homomorphism will be proved in the proof of Theorem 3.6, which is in the next section. $\square$

Note that the proof of Theorem 3.7 shows a stronger result as follows: Let $x = \sum_{i=0}^{\infty} x_i$ be the Fourier expansion of $x \in L^2(\mathcal{M}, \tau)$. The inequality (5) with $n = 0$ and $n' = 4$ shows that if $\|x\|_{W_4^2} < \infty$, then the Fourier expansion $x = \sum_{i=0}^{\infty} x_i$ is an absolutely convergent series with respect to C^*-norm, i.e., $\sum_{i=0}^{\infty} \|x_i\|_{L^\infty(\mathcal{M},\tau)} < \infty$.

Corollary 3.9. *The Fréchet topology on the C^∞ algebra $\mathcal{S}$ is generated by the family of W_n^2 $(n \in \mathbb{N}_0)$ norms of Hilbert type.*

Proof. The norm families $\{\|\cdot\|_{W_n^p} : 1 < p < \infty,\ n \in \mathbb{N}_0\}$ and $\{\|\cdot\|_{W_n^2} : n \in \mathbb{N}_0\}$ are equivalent to $\{\|\cdot\|_{W_n^p}^* : 1 < p < \infty,\ n \in \mathbb{N}_0\}$ and $\{\|\cdot\|_{W_n^2}^* : n \in \mathbb{N}_0\}$, respectively. The latter two norm families are equivalent by Theorem 3.7. $\square$

Corollary 3.10. *The C^∞ algebra $\mathcal{S}$ is not nuclear.*

Proof. By the above corollary the Fréchet topology on $\mathcal{S}$ is defined by the norms $\|\cdot\|_{W_n^2}$ $(n \in \mathbb{N}_0)$ of Hilbert type. We say that a countable family $\{\|\cdot\|_n : n \in \mathbb{N}\}$ of Hilbert type seminorms on a linear space E has property A if it satisfies the condition that for any $n \in \mathbb{N}$ there exists $n' \in \mathbb{N}$ such that the identity map on E is a Hilbert-Schmidt operator from $(E, \|\cdot\|_{n'})$ to $(E, \|\cdot\|_n)$. A locally convex and metrizable linear topological space defined by a countable family of Hilbert type seminorms is said to be nuclear if the family of seminorms has property A (see [13, Definition 50.1]). Note that if a seminorm family $\{\|\cdot\|_{1,n} : n \in \mathbb{N}\}$ has the property A, then any other equivalent countable seminorm family $\{\|\cdot\|_{2,m} : m \in \mathbb{N}\}$ satisfies property A. In fact, for any $m \in \mathbb{N}$ there are $n, n', m' \in \mathbb{N}$ such that $\|\cdot\|_{2,m} \le C_1 \|\cdot\|_{1,n} \le_{HS} \|\cdot\|_{1,n'} \le C_2 \|\cdot\|_{2,m'}$ for some $C_1, C_2 > 0$, where $\le_{HS}$ means that the identity map is a Hilbert-Schmidt class operator. So, it suffices to show that for any $n' > n$, $\|\cdot\|_{W_{n'}^2}$ is never stronger than $\|\cdot\|_{W_n^2}$ in the Hilbert-Schmidt sense, i.e., $\iota_{n',n}^2$ is not a Hilbert-Schmidt operator.

The vectors $T_{i_1 i_2 \cdots i_k}$ $(k \in \mathbb{N}_0,\ i_1, i_2, \ldots, i_k \in I)$ form a countable complete orthogonal set of the space W_n^2 for each $n \in \mathbb{N}_0$ because they are a complete orthogonal eigenvectors for the Laplacian L. Let $N\,(\equiv \operatorname{card} I)$ be finite (the case of N being infinite is easier). Since $\|T_{i_1 i_2 \cdots i_k}\|_{W_n^2} = (1 + k + \cdots + k^n)^{\frac{1}{2}}$ and $\|T_{i_1 i_2 \cdots i_k}\|_{W_{n'}^2} = (1 + k + \cdots + k^{n'})^{\frac{1}{2}}$ for all $i_1, i_2, \ldots, i_k \in I$, we get

$$\sum_{k=1}^{\infty} \sum_{i_1, i_2, \ldots, i_k \in I} \left(\frac{\|T_{i_1 i_2 \cdots i_k}\|_{W_n^2}}{\|T_{i_1 i_2 \cdots i_k}\|_{W_{n'}^2}} \right)^2 = \sum_{k=1}^{\infty} N^k \frac{1 + k + \cdots + k^n}{1 + k + \cdots + k^{n'}} = \infty.$$

Hence $\iota^2_{n',n}$ is not a Hilbert Schmidt operator. □

We note that when N is finite and $n' > n$, the above $\iota^2_{n',n}$ is compact at least because

$$\lim_{k\to\infty} \frac{1+k+\cdots+k^n}{1+k+\cdots+k^{n'}} = 0.$$

§4. Weak Derivation

In this section we introduce the notion of weak derivation, and by making use of it we prove Theorems 3.4 and 3.6 stated in the previous section.

Let $\tilde{I}$ be a copy of I and $I^{(2)} \equiv I \cup \tilde{I}$. Let $\mathcal{M}^{(2)} \equiv W^*(\{s_i : i \in I^{(2)}\})$, which is regarded as an $\mathcal{M}$ bimodule naturally defined by the inclusion $\mathcal{M} \subset \mathcal{M}^{(2)}$. In the same way as before define $L^{(2)}$, $\mathcal{S}^{(2)}$, $\mathcal{P}^{(2)}$, $\tau^{(2)}$, etc. associated with $\mathcal{M}^{(2)}$. Also, let $i+N$ denote the element of $\tilde{I}$ corresponding to $i \in I$ (regardless of $N \equiv \operatorname{card} I$ being finite or not).

Definition 4.1. We define the *weak derivation* D from $\mathcal{M}$ to the $\mathcal{M}$ (or $\mathcal{P}$) bimodule $\mathcal{M}^{(2)}$ with $\operatorname{dom} D = \mathcal{P}$ by $D(s_i) = s_{i+N}$ for $i \in I$.

Note that the derivation D is well defined by the Leipniz rule because $\mathcal{P}$ is isomorphic to the noncommutative polynomial ring. We here use the full Fock space construction. Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space with $\dim \mathcal{H}_{\mathbb{R}} = N \equiv \operatorname{card} I$. Consider the one-parameter orthogonal group on $\mathcal{H}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{R}}$ defined by the following 2×2 matrix form:

$$O(t) \equiv \begin{pmatrix} \cos t \cdot 1_{\mathcal{H}_{\mathbb{R}}} & -\sin t \cdot 1_{\mathcal{H}_{\mathbb{R}}} \\ \sin t \cdot 1_{\mathcal{H}_{\mathbb{R}}} & \cos t \cdot 1_{\mathcal{H}_{\mathbb{R}}} \end{pmatrix} \qquad (t \in \mathbb{R}).$$

Then $\Gamma(O(t))$ $(t \in \mathbb{R})$ is a σ-weakly continuous one-parameter action on $\widehat{\mathcal{M}}$ $(\mathcal{H}_{\mathbb{R}} \otimes \mathbb{R}^2)$. Furthermore, let $\widehat{D}$ be the linear operator on $\mathcal{F}(\mathcal{H}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{R}})$ with

$$\operatorname{dom} \widehat{D} = \mathbb{C}\Phi \oplus_{\text{alg}} \bigoplus_{n=1}^{\infty} {}_{\text{alg}} \underbrace{(\mathcal{H} \oplus \mathcal{H}) \otimes_{\text{alg}} \cdots \otimes_{\text{alg}} (\mathcal{H} \oplus \mathcal{H})}_{n} \ \Big(\subset \mathcal{F}(\mathcal{H}_{\mathbb{R}} \oplus \mathcal{H}_{\mathbb{R}})\Big)$$

defined by

$$\widehat{D} \equiv 0 \oplus_{\text{alg}} \bigoplus_{n=1}^{\infty} {}_{\text{alg}} \left(\sum_{m=1}^{n} 1 \otimes_{\text{alg}} \cdots \otimes_{\text{alg}} \underbrace{A}_{m\text{th}} \otimes_{\text{alg}} \cdots \otimes_{\text{alg}} 1 \right),$$

where $\oplus_{\mathrm{alg}}$, $\otimes_{\mathrm{alg}}$ mean algebraic direct sum and algebraic tensor product and A is represented in the 2×2 matrix form

$$A \equiv \begin{pmatrix} 0_{\mathcal{H}} & -1_{\mathcal{H}} \\ 1_{\mathcal{H}} & 0_{\mathcal{H}} \end{pmatrix}.$$

We then have $\widehat{D} \subset \frac{d}{dt}\mathcal{F}(O(t))\big|_{t=0}$ and $\eta\big(\frac{d}{dt}\Gamma(O(t))\big|_{t=0}\big) \subset \frac{d}{dt}\mathcal{F}(O(t))\big|_{t=0}$. Note that $\frac{d}{dt}\Gamma(O(t))\big|_{t=0}$ is a derivation on $\widehat{\mathcal{M}}(\mathcal{H}_{\mathbb{R}} \otimes \mathbb{R}^2)$ because $\Gamma(O(t))$ is a one-parameter action. Let $J : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{R}^2$ be the isometry defined via the obvious identification $\mathcal{H}_{\mathbb{R}} \cong \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{R}(1,0) \subset \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{R}^2$. Then $\Gamma(J) : \widehat{\mathcal{M}}(\mathcal{H}_{\mathbb{R}}) \to \widehat{\mathcal{M}}(\mathcal{H}_{\mathbb{R}} \otimes \mathbb{R}^2)$ is an injective homomorphism. Now we obtain $D|_{\mathcal{P}} = \pi_2^{-1}\eta^{-1}\widehat{D}\eta\Gamma(J)\pi_1|_{\mathcal{P}}$ because both sides of this equality are derivations and

$$D(T_i) = D(s_i) = s_{i+N} = T_{i+N}, \quad \pi_2^{-1}\eta^{-1}\widehat{D}\eta\Gamma(J)\pi_1(T_i) = T_{i+N} \qquad (i \in I),$$

where $\pi_1 : \mathcal{M} \to \widehat{\mathcal{M}}(\mathcal{H}_{\mathbb{R}})$ and $\pi_2 : \mathcal{M}^{(2)} \to \widehat{\mathcal{M}}(\mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{R}^2)$ are isomorphisms explained in Section 2. An easy calculation gives

$$\pi_2^{-1}\eta^{-1}\widehat{D}\eta\Gamma(J)\pi_1(T_{i_1 i_2 \cdots i_n}) = \sum_{j=1}^{n} T_{i_1 i_2 \cdots (i_j+N) \cdots i_n} \qquad (n \in \mathbb{N},\ i_1, i_2, \ldots, i_n \in I)$$

so that we get the next lemma.

Lemma 4.2. *For every $n \in \mathbb{N}$ and $i_1, i_2, \ldots, i_n \in I$,*

$$D(T_{i_1 i_2 \cdots i_n}) = \sum_{j=1}^{n} T_{i_1 i_2 \cdots (i_j+N) \cdots i_n}.$$

For $i \in \tilde{I}$ $(\subset I^{(2)} = I \cup \tilde{I})$ so that $i = j + N$ with $j \in I$, let $i - N \equiv j$. Let D_{dual} be the linear operator from $\mathcal{M}^{(2)}$ to $\mathcal{M}$ with $\operatorname{dom} D_{\mathrm{dual}} = \mathcal{P}^{(2)}$ defined by

$$D_{\mathrm{dual}}(T_{i_1 i_2 \cdots i_n}) \equiv \begin{cases} T_{i_1 i_2 \cdots i_j - N \cdots i_n} & \text{if } i_j \in \tilde{I} \text{ and } i_k \in I\ (k \neq j) \text{ for some } j, \\ 0 & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}_0$ and $i_1, \ldots, i_n \in I^{(2)}$.

Lemma 4.3. *If $x \in \mathcal{P}$ and $y \in \mathcal{P}^{(2)}$, then*

$$\tau^{(2)}(y^* D(x)) = \tau(D_{\mathrm{dual}}(y^*)x).$$

Proof. We may assume that $x = T_{i_1 i_2 \cdots i_n}$ and $y = T_{j_1 j_2 \cdots j_m}$ where $n, m \in \mathbb{N}_0$, $i_1, \ldots, i_n \in I$ and $j_1, \ldots, j_m \in I^{(2)}$. But the desired equality for this case is directly checked by Lemma 4.2 and the definition of D_{dual}. □

Based on the familiar duality between the spaces $L^p(\mathcal{M}, \tau)$ and $L^q(\mathcal{M}, \tau)$ where $1 < p < \infty$ and $q = \frac{p}{p-1}$, we get the next corollary since $\mathcal{P}^{(2)}$ is dense in $L^p(\mathcal{M}^{(2)}, \tau^{(2)})$.

Corollary 4.4. *The operators D and D_{dual} are closable as operators from $L^p(\mathcal{M}, \tau)$ to $L^p(\mathcal{M}^{(2)}, \tau^{(2)})$ and from $L^q(\mathcal{M}^{(2)}, \tau^{(2)})$ to $L^q(\mathcal{M}, \tau)$, respectively.*

For any $\alpha \in \mathbb{R}$ the operator L^α on $\mathcal{P}$ is naturally defined as mentioned in Section 3, and its kernel is $\mathcal{P}_0 = \mathbb{C}1$. The next corollary is also an easy consequence of the Lemma 4.2.

Corollary 4.5. *For every $x \in \mathcal{P}$ and $\alpha \in \mathbb{R}$,*

$$D_{\mathrm{dual}} D x = Lx, \quad D L^\alpha x = (L^{(2)})^\alpha D x.$$

Proof. Direct computations using Lemma 4.2 give the above equalities for $x = T_{i_1 i_2 \cdots i_n}$ $(n \in \mathbb{N}_0,\ i_1, i_2, \ldots, i_n \in I)$. □

It is easy to see that $DL^{-\frac{1}{2}}$ is isometric on $\bigoplus_{n=1}^\infty \mathcal{P}_n$ $(\subset \mathcal{P}_0^\perp$ in $L^2(\mathcal{M}, \tau))$ with respect to $L^2(\mathcal{M}, \tau)$ and $L^2(\mathcal{M}^{(2)}, \tau^{(2)})$ norms. Hence we have an isometry $DL^{-\frac{1}{2}} : L^2(\mathcal{M}, \tau) \ominus \mathcal{P}_0 \to L^2(\mathcal{M}^{(2)}, \tau^{(2)})$. Furthermore, we can see that $DL^{-\frac{1}{2}}$ is a bounded operator from $L^p(\mathcal{M}, \tau)$ to $L^p(\mathcal{M}^{(2)}, \tau^{(2)})$. This can be proved by the same technique as in Pisier [11] in classical probability, and in fact the next proposition as well as Lemma 4.2 above was given in [8] in the setting of more general Fock spaces. Note that $N \equiv \operatorname{card} I < \infty$ was assumed in [8], but the cardinality of I is irrelevant to the proposition in the case of free probability.

Proposition 4.6. *For $1 < p < \infty$ the operator $DL^{-\frac{1}{2}}$ on $\bigoplus_{n=1}^\infty \mathcal{P}_n$ is bounded above and below with respect to $L^p(\mathcal{M}, \tau)$ and $L^p(\mathcal{M}^{(2)}, \tau^{(2)})$ norms. More precisely, there exist constants $C_{0,p}, C_{1,p} > 0$, independent of $N \equiv \operatorname{card} I$, such that*

$$C_{0,p} \|x\|_{L^p(\mathcal{M}, \tau)} \le \|DL^{-\frac{1}{2}} x\|_{L^p(\mathcal{M}^{(2)}, \tau^{(2)})} \le C_{1,p} \|x\|_{L^p(\mathcal{M}, \tau)}$$

for all $x \in \bigoplus_{n=1}^\infty \mathcal{P}_n$.

The following three corollaries are easy consequences of the above proposition.

Corollary 4.7. *For every $n \in \mathbb{N}$, $\|L^{\frac{n}{2}} \cdot \|_{L^p(\mathcal{M},\tau)} \simeq \|(L^{(2)})^{\frac{n-1}{2}} D \cdot \|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})}$ on $\mathcal{P}$, where $\simeq$ means the equivalence of two norms.*

Proof. For $n \in \mathbb{N}$ and $x \in \mathcal{P}$, by Corollary 4.5 and Proposition 4.6 we have

$$\begin{aligned}\|(L^{(2)})^{\frac{n-1}{2}} Dx\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})} &= \|(L^{(2)})^{\frac{n-1}{2}} D(x-\tau(x))\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})} \\ &= \|DL^{-\frac{1}{2}} L^{\frac{n}{2}}(x-\tau(x))\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})} \\ &\leq C_{1,p}\|L^{\frac{n}{2}}(x-\tau(x))\|_{L^p(\mathcal{M},\tau)} \\ &= C_{1,p}\|L^{\frac{n}{2}} x\|_{L^p(\mathcal{M},\tau)}.\end{aligned}$$

The proof of the inverse inequality is similar. □

Corollary 4.8. *D extends to a continuous operator from $\mathcal{S}$ to $\mathcal{S}^{(2)}$.*

Corollary 4.9. *On $\mathcal{P}$ the norm $\|\cdot\|^*_{W_1^p}$ is equivalent to $\|\cdot\|_{L^p(\mathcal{M},\tau)} + \|D\cdot\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})}$, the graph norm of $\overline{D}^p$.*

Proof. This is immediate since

$$\begin{aligned}\|\cdot\|^*_{W_1^p} &= \|\cdot\|_{L^p(\mathcal{M},\tau)} + \|L^{\frac{1}{2}}\cdot\|_{L^p(\mathcal{M},\tau)} \\ &\simeq \|\cdot\|_{L^p(\mathcal{M},\tau)} + \|D\cdot\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})}.\end{aligned}$$

□

Let $\mathcal{M}^{(1)} \equiv \mathcal{M}$, $\mathcal{P}^{(1)} \equiv \mathcal{P}$ and $D^{(1)} \equiv D : \mathcal{P}^{(1)} \to \mathcal{P}^{(2)}$. In the same way as above we recursively define $I^{(3)} \equiv I^{(2)} \cup \tilde{I}^{(2)}$ ($\tilde{I}^{(2)}$ being a copy of $I^{(2)}$), $\mathcal{M}^{(3)}$, $D^{(2)} : \mathcal{P}^{(2)} \to \mathcal{P}^{(3)}$ ($\subset \mathcal{M}^{(3)}$), $I^{(4)}$, $\mathcal{M}^{(4)}$, $D^{(3)}$, and so on. Now, an induction argument proves the following proposition.

Proposition 4.10. *Let $1 < p < \infty$ and $n \in \mathbb{N}$. On $\mathcal{P}$ the norm $\|\cdot\|^*_{W_n^p}$ is equivalent to $\|\cdot\|_{L^p(\mathcal{M}^{(1)},\tau^{(1)})} + \|D^{(1)}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})} + \cdots + \|D^{(n)}D^{(n-1)}\cdots D^{(1)}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(n+1)},\tau^{(n+1)})}$. Furthermore, the linear operator $D^{(n)}D^{(n-1)}\cdots D^{(1)} : \mathcal{P}^{(1)} \to \mathcal{P}^{(n+1)}$ is closable with respect to the norms $\|\cdot\|_{L^p(\mathcal{M}^{(1)},\tau^{(1)})} + \|D^{(1)}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})} + \cdots + \|D^{(n-1)}D^{(n-2)}\cdots D^{(1)}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(n)},\tau^{(n)})}$ on $\mathcal{P}^{(1)}$ and $\|\cdot\|_{L^p(\mathcal{M}^{(n+1)},\tau^{(n+1)})}$ on $\mathcal{P}^{(n+1)}$.*

Proof. By Corollary 4.5 and Proposition 4.6 repeatedly applied to $D^{(1)}$,

$D^{(2)}, \dots, D^{(n)}$, the first assertion is seen as follows:

$$\begin{aligned}\|\cdot\|^{*}_{W^p_n} &= \|\cdot\|_{L^p(\mathcal{M}^{(1)},\tau^{(1)})} + \|(L^{(1)})^{\frac{1}{2}}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(1)},\tau^{(1)})}\\ &\quad + \|(L^{(1)})^{\frac{2}{2}}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(1)},\tau^{(1)})}\\ &\quad + \cdots + \|(L^{(1)})^{\frac{n}{2}}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(1)},\tau^{(1)})}\\ &\simeq \|\cdot\|_{L^p(\mathcal{M}^{(1)},\tau^{(1)})} + \|D^{(1)}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})}\\ &\quad + \|(L^{(2)})^{\frac{1}{2}}D^{(1)}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})}\\ &\quad + \cdots + \|(L^{(2)})^{\frac{n-1}{2}}D^{(1)}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})}\\ &\ \ \vdots\\ &\simeq \|\cdot\|_{L^p(\mathcal{M}^{(1)},\tau^{(1)})} + \|D^{(1)}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})} + \|D^{(2)}D^{(1)}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(3)},\tau^{(3)})}\\ &\quad + \cdots + \|D^{(n)}D^{(n-1)}\cdots D^{(1)}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(n+1)},\tau^{(n+1)})}.\end{aligned}$$

The latter assertion is obvious because Corollary 4.4 says that $D^{(n)} : \mathcal{P}^{(n)} \to \mathcal{P}^{(n+1)}$ is closable with respect to $L^p(\mathcal{M}^{(n)}, \tau^{(n)})$ and $L^p(\mathcal{M}^{(n+1)}, \tau^{(n+1)})$ norms. □

We denote by $\overline{D^{(n)}D^{(n-1)}\cdots D^{(1)}}^{p}$ the closure of $D^{(n)}D^{(n-1)}\cdots D^{(1)}$ with respect to the norms stated in the above proposition.

In the rest of this section we prove Theorems 3.4 and 3.6 in the previous section. To do so we need the following easy lemma. For each $m \in \mathbb{N}$ let $\iota^{(m)}$ be the natural inclusion of $\mathcal{P}^{(m)}$ into $\mathcal{P}^{(m+1)}$, which obviously extends to an injective homomorphism from $\mathcal{M}^{(m)}$ into $\mathcal{M}^{(m+1)}$ with $\tau^{(m+1)}\iota^{(m)} = \tau^{(m)}$.

Lemma 4.11. *For every $m \in \mathbb{N}$ and $x \in \mathcal{P}^{(m)}$,*

$$\|x\|^{*}_{(W^p_n)^{(m)}} = \|\iota^{(m)}(x)\|^{*}_{(W^p_n)^{(m+1)}} \qquad (1 < p < \infty,\ n \in \mathbb{N}_0),$$

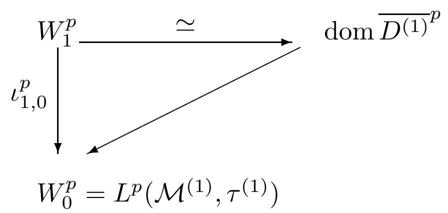
where $\|x\|^{*}_{(W^p_n)^{(m)}}$ denotes the modified Sobolev W^p_n norm associated with $\mathcal{M}^{(m)}$ (see Definition 3.2). Consequently, $\iota^{(m)}$ extends to a continuous injection from the C^∞ algebra $\mathcal{S}^{(m)}$ (associated with $\mathcal{M}^{(m)}$) into $\mathcal{S}^{(m+1)}$. Furthermore, if $1 \le m < m'$ and $x \in \mathcal{P}^{(m)}$, then

$$\begin{aligned}&\|D^{(m')}\iota^{(m'-1)}\cdots\iota^{(m)}(x)\|^{*}_{(W^p_n)^{(m'+1)}}\\ &\qquad = \|D^{(m)}(x)\|^{*}_{(W^p_n)^{(m+1)}} \qquad (1 < p < \infty,\ n \in \mathbb{N}_0).\end{aligned}$$

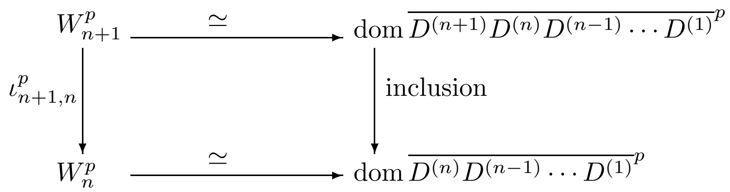
Proof. Since $\iota^{(m)}$ is a trace-preserving homomorphism as mentioned above, we notice $\|x\|_{L^p(\mathcal{M}^{(m)},\tau^{(m)})} = \|\iota^{(m)}(x)\|_{L^p(\mathcal{M}^{(m+1)},\tau^{(m+1)})}$ for $x \in \mathcal{P}^{(m)}$.

Hence the first assertion follows from $L^{(m+1)}\iota^{(m)} = \iota^{(m+1)}L^{(m)}$. By a direct computation for noncommutative Chebyshev polynomials, we can see that $\|D^{(m+1)}\iota^{(m)}x\| = \|\iota^{(m+1)}D^{(m)}x\|$ for all $x \in \mathcal{P}^{(m)}$. This and the first assertion imply the second. $\square$

Proof of Theorem 3.4. First, we prove that $\iota^p_{n+1,n}$ is injective for $1 < p < \infty$ and $n \in \mathbb{N}_0$. Thanks to Corollary 4.9 the modified Sobolev norm $\|\cdot\|^*_{W^p_1}$ and the graph norm $\|x\|_{L^p(\mathcal{M}^{(1)},\tau^{(1)})} + \|D(x)\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})}$ of the closable operator $D^{(1)}$ are equivalent, so the Sobolev space W^p_1 is isomorphic to the Banach space $\operatorname{dom}\overline{D^{(1)}}^p$. Since the following diagram is commutative, $\iota^p_{1,0}$ is injective.



Similarly, Proposition 4.10 says that the norm $\|\cdot\|^*_{W^p_{n+1}}$ is equivalent to the graph norm $\|\cdot\|_{L^p(\mathcal{M}^{(1)},\tau^{(1)})} + \|D^{(1)}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})} + \cdots + \|D^{(n+1)}D^{(n)}\cdots D^{(1)}(\,\cdot\,)\|_{L^p(\mathcal{M}^{(n+2)},\tau^{(n+2)})}$ of the closable operator $D^{(n+1)}D^{(n)}\cdots D^{(1)}$. Hence the following commutative diagram implies the injectivity of $\iota^p_{n+1,n}$.



Now, the theorem is easily proved as follows. Let $1 < p < p' < \infty$ and $0 \leq n < n'$. Then $\iota^p_{n',n} = \iota^p_{n',n'-1}\iota^p_{n'-1,n'-2}\cdots\iota^p_{n+1,n}$ is injective. Furthermore, since $\iota^p_{n,0}\iota^n_{p',p} = \iota^0_{p',p}\iota^{p'}_{n,0}$ is injective thanks to the compatibility of $L^p(\mathcal{M},\tau)$ norms, $\iota^n_{p',p}$ is also injective. $\square$

Proof of Theorem 3.6. First, for each $n \in \mathbb{N}_0$ we show that the $*$-operation on $\mathcal{P}$ is continuous in $W^\cap_n$ topology and that the multiplication on $\mathcal{P}$ is bicontinuous in $W^\cap_n$ topology. The first claim is seen because

$$\|L^{\frac{n}{2}}(x^*)\|_{L^p(\mathcal{M},\tau)} = \|(L^{\frac{n}{2}}x)^*\|_{L^p(\mathcal{M},\tau)} = \|L^{\frac{n}{2}}x\|_{L^p(\mathcal{M},\tau)} \qquad (x \in \mathcal{P}).$$

For the second claim, it suffices to prove that for every $0 < p < \infty$ there exists a constant $C_{n,p}$ such that $\|xy\|^*_{W^p_n} \le C_{n,p}\|x\|^*_{W^{2p}_n}\|y\|^*_{W^{2p}_n}$ for all $x, y \in \mathcal{P}$. We estimate

$$
\begin{aligned}
\|xy\|^*_{W^p_n} \le\, & C\Big\{\|xy\|_{L^p(\mathcal{M}^{(1)},\tau^{(1)})} + \|D^{(1)}(xy)\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})} \\
& \qquad + \cdots + \|D^{(n)}D^{(n-1)}\cdots D^{(1)}(xy)\|_{L^p(\mathcal{M}^{(n+1)},\tau^{(n+1)})}\Big\} \\
\le\, & C\Bigg\{\|xy\|_{L^p(\mathcal{M}^{(1)},\tau^{(1)})} + \|D^{(1)}(x)\iota^{(1)}(y)\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})} \\
& + \|\iota^{(1)}(x)D^{(1)}(y)\|_{L^p(\mathcal{M}^{(2)},\tau^{(2)})} \\
& + \cdots + \sum_{J\subset\{1,2,\dots,n\}} \|A^{(n)}_J A^{(n-1)}_J \cdots A^{(1)}_J(x) A^{(n)}_{J^c} A^{(n-1)}_{J^c} \\
& \qquad \cdots A^{(1)}_{J^c}(y)\|_{L^p(\mathcal{M}^{(n+1)},\tau^{(n+1)})}\Bigg\} \\
\le\, & 2^n C\Big\{\|x\|_{L^{2p}(\mathcal{M}^{(1)},\tau^{(1)})} + \|D^{(1)}(x)\|_{L^{2p}(\mathcal{M}^{(2)},\tau^{(2)})} \\
& \qquad + \cdots + \|D^{(n)}D^{(n-1)}D^{(1)}(x)\|_{L^{2p}(\mathcal{M}^{(n+1)},\tau^{(n+1)})}\Big\} \\
& \times \Big\{\|y\|_{L^{2p}(\mathcal{M}^{(1)},\tau^{(1)})} + \|D^{(1)}(y)\|_{L^{2p}(\mathcal{M}^{(2)},\tau^{(2)})} \\
& \qquad + \cdots + \|D^{(n)}D^{(n-1)}D^{(1)}(y)\|_{L^{2p}(\mathcal{M}^{(n+1)},\tau^{(n+1)})}\Big\} \\
\le\, & 2^n CC'\|x\|^*_{W^{2p}_n}\|y\|^*_{W^{2p}_n}.
\end{aligned}
$$

In the above, $\sum_{J\subset\{1,2,\dots,n\}}$ is the summation over all subsets J of $\{1, 2, \dots, n\}$, $J^c \equiv \{1, 2, \dots, n\} \setminus J$ and

$$
A^{(k)}_J \equiv \begin{cases} D^{(k)} & \text{if } k \in J, \\ \iota^{(k)} & \text{if } k \notin J. \end{cases}
$$

The first and last inequalities in the above follow from Proposition 4.10, and the third one is due to Lemma 4.11.

Now, the $*$-operation and the multiplication on $\mathcal{P}$ can be canonically extended on $W^{\cap}_n$ by continuity so that $W^{\cap}_n$ and also $\mathcal{S}$ become Fréchet $*$-algebras. Finally, it is easily verified (see the proof of Theorem 3.7) that the $*$-operation and the multiplication on $W^{\cap}_n$ where $n \ge 4$ are equal to those of the C^*-algebra $\mathcal{A}$, that is, the inclusion $W^{\cap}_n \Subset \mathcal{A}$ for $n \ge 4$ is a $*$-algebraic homomorphism. The same is true for $\mathcal{S}$ too, and the proof of Corollary 3.8 is also completed. □

§5. Schwartz Distribution Space

The C^∞ algebra $\mathcal{S}$ constructed in Section 3 may be regarded as the free analog of the rapidly decreasing C^∞ function algebra. So it is natural to introduce Schwartz distributions in free probability as continuous linear functionals on $\mathcal{S}$.

Definition 5.1. Let $\mathcal{S}'$ denote the dual space of $\mathcal{S}$, i.e., the space of all continuous linear functionals on $\mathcal{S}$ equipped with weak* topology. We call $\mathcal{S}'$ the *Schwartz distribution space.*

For each $1 < p < \infty$ and $n \in \mathbb{N}_0$, the dual space of W^p_n is denoted by $\widehat{W}^q_{-n}$ with $q = \frac{p}{p-1}$, which we are going to formulate as the Sobolev space with minus index. In the following we use the symbol $(\,\cdot\,)'$ to mean the dual of a space or a map. Let $1 < p < p' < \infty$, $q = \frac{p}{p-1}$, $q' = \frac{p'}{p'-1}$ $(1 < q' < q < \infty)$ and $0 \le n < n'$. Since $\iota^n_{p',p}$ and $\iota^p_{n',n}$ have dense ranges, the dual maps $\hat{\iota}^{-n}_{q,q'} \equiv \big(\iota^n_{p',p}\big)' : \widehat{W}^q_{-n} \to \widehat{W}^{q'}_{-n}$ and $\hat{\iota}^q_{-n,-n'} \equiv \big(\iota^p_{n',n}\big)' : \widehat{W}^q_{-n} \to \widehat{W}^q_{-n'}$ are injective, and we have

$$\begin{aligned}\hat{\iota}^{-n'}_{q,q'}\hat{\iota}^q_{-n,-n'} &= \big(\iota^{n'}_{p',p}\big)'\big(\iota^p_{n',n}\big)' = \big(\iota^p_{n',n}\iota^{n'}_{p',p}\big)' \\ &= \big(\iota^n_{p',p}\iota^{p'}_{n',n}\big)' = \hat{\iota}^{q'}_{-n,-n'}\hat{\iota}^{-n}_{q,q'}.\end{aligned}$$

Thus, we obtain the following commuting diagram:

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \widehat{W}^q_{-2} & \Subset & \widehat{W}^2_{-2} & \Subset & \widehat{W}^q_{-2} \\ \Cup & & \Cup & & \Cup \\ \widehat{W}^q_{-1} & \Subset & \widehat{W}^2_{-1} & \Subset & \widehat{W}^q_{-1} \\ \Cup & & \Cup & & \Cup \\ \widehat{W}^q_{-0} & \Subset & \widehat{W}^2_{-0} & \Subset & \widehat{W}^q_{-0} \\ (2 < q < \infty) & & & & (1 < q < 2) \end{array} \tag{6}$$

Due to Theorem 3.7 (and Definition 3.5) we see that $\mathcal{S}'$ is the inductive limit space of $\big(\mathcal{S}, \|\cdot\|_{W^2_n}\big)'$ that is the inductive limit of $\widehat{W}^2_{-n}$ as $n \to \infty$. The duality for noncommutative L^p spaces says that the Banach spaces $W^p_0 = L^p(\mathcal{M}, \tau)$ $(1 < p < \infty)$ and $\widehat{W}^p_{-0} = \big(W^q_0\big)' = \big(L^q(\mathcal{M}, \tau)\big)'$ are identified via the linear isometry $\lambda_p : W^p_0 \to \widehat{W}^p_{-0}$ given by

$$\lambda_p(x) = \tau(x\,\cdot\,) \in \widehat{W}^p_{-0} = \big(L^q\big(\mathcal{M}, \tau)\big)' \qquad \big(x \in W^p_0 = L^p(\mathcal{M}, \tau)\big).$$

By combining the diagrams (4) and (6) with identification $W^p_0 = \widehat{W}^p_{-0}$ via λ_p, we finally obtain the diagram of all the Sobolev spaces with positive and negative indices as follows. In the sequel we omit the symbol $\widehat{\ }$ for Sobolev spaces $\widehat{W}^p_{-0}$.

$$
\begin{array}{ccccccccccc}
 & & & & & & & & & & \mathcal{S}' \\
 & & & & \vdots & & \vdots & & \vdots & & \Cup \\
 & & & & W^p_{-2} & \Subset & W^2_{-2} & \Subset & W^p_{-2} & \Subset & W^{\cup}_{-2} \\
 & & & & \Cup & & \Cup & & \Cup & & \Cup \\
 & & & & W^p_{-1} & \Subset & W^2_{-1} & \Subset & W^p_{-1} & \Subset & W^{\cup}_{-1} \\
 & & & & \Cup & & \Cup & & \Cup & & \\
\mathcal{M} & \Subset & W^{\cap}_0 & \Subset & W^p_0 = L^p & \Subset & W^2_0 = L^2 & \Subset & W^p_0 = L^p & \Subset & L^1 \\
 & & & & (2 < p < \infty) & & & & (1 < p < 2) & & \\
 & & \Cup & & \Cup & & \Cup & & \Cup & & \\
 & & W^{\cap}_1 & \Subset & W^p_1 & \Subset & W^2_1 & \Subset & W^p_1 & & \\
 & & \Cup & & \Cup & & \Cup & & \Cup & & \\
 & & W^{\cap}_2 & \Subset & W^p_2 & \Subset & W^2_2 & \Subset & W^p_2 & & \\
 & & \Cup & & \Cup & & \Cup & & \Cup & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \Cup & & & & & & & & \\
 & & \mathcal{S} & & & & & & & &
\end{array}
\tag{7}
$$

In the above diagram, for each $n \in \mathbb{N}$ let $W^{\cup}_{-n}$ be the inductive limit space of W^p_{-n} $(1 < p < \infty)$, then $\mathcal{S}'$ is the inductive limit of $W^{\cup}_{-n}$ $(n \in \mathbb{N})$. We may and do regard $\mathcal{S}$, $\mathcal{A}$ and $\mathcal{M}$ as subspaces of $\mathcal{S}'$ as described in the above diagram.

Let $\mathcal{K}$ be the direct product linear space of $\overline{\mathcal{P}_k}^{L^2(\mathcal{M},\tau)}$ $(k \in \mathbb{N}_0)$, i.e., the linear space of all sequences $(x_0, x_1, \ldots, x_k, \ldots)$ such that $x_k \in \overline{\mathcal{P}}_k^{L^2(\mathcal{M},\tau)}$ for all $k \in \mathbb{N}_0$. For each $n \in \mathbb{Z}$ define the subspace

$$H^2_n \equiv \left\{ x = (x_0, x_1, \ldots, x_k, \ldots) \in \mathcal{K} : \sum_{k=0}^{\infty} (1+k)^n \|x_k\|^2_{L^2(\mathcal{M},\tau)} < \infty \right\}.$$

Then it is easy to see that H^2_n is a Hilbert space with the inner product

$$\langle x, x' \rangle_{H^2_n} \equiv \sum_{k=0}^{\infty} (1+k)^n \langle x_k, x'_k \rangle_{L^2(\mathcal{M},\tau)} \qquad (x, x' \in H^2_n),$$

and it is naturally isomorphic to W^2_n. For instance, when $n < 0$, the isomorphism $W^2_n \simeq H^2_n$ is given as $x(y) \equiv \sum_{k=0}^{\infty} \langle x^*_k, y_k \rangle$ for $x \in H^2_n$ and $y \in \mathcal{P} \subset W^2_n$. In fact, this defines a bounded linear functional $x(\,\cdot\,)$ on W^2_n, whose norm is computed as

$$\|x(\,\cdot\,)\| = \left\{ \sum_{k=0}^{\infty} (1 + k + \cdots + k^n)^{-2} \|x_k\|^2_{L^2(\mathcal{M},\tau)} \right\}^{\frac{1}{2}} \simeq \|x\|_{H^2_n}.$$

If $n < n'$, then $H^2_{n'}$ is a subspace of H^2_n, so we denote its inclusion map by $\kappa_{n',n}$. On the other hand, the inclusion map of $W^2_{n'} \Subset W^2_n$ in the diagram (7) is denoted by $\iota_{n',n}$. Then, $\kappa_{n',n} : H^2_{n'} \to H^2_n$ and $\iota_{n',n} : W^2_{n'} \to W^2_n$ are conjugate via $W^2_n \simeq H^2_n$ and $W^2_{n'} \simeq H^2_{n'}$; namely we obtain the commuting diagram

$$\begin{array}{ccc} W^2_n & \simeq & H^2_n \\ \Cup & & \Cup \\ W^2_{n'} & \simeq & H^2_{n'} \end{array}.$$

Now, it is straightforward to verify the next representation theorem for $\mathcal{S}$ and $\mathcal{S}'$ based on the facts explained so far; we omit the detailed proof.

Theorem 5.2. *The Schwartz distribution space $\mathcal{S}'$ is represented as the subspace of $\mathcal{K}$ consisting of all slowly increasing sequences, i.e., $x = (x_0, x_1, \ldots, x_k, \ldots) \in \mathcal{K}$ such that*

$$\sum_{k=0}^{\infty} (1+k)^n \|x_k\|^2_{L^2(\mathcal{M},\tau)} < \infty \qquad \textit{for some} \quad n \in \mathbb{Z}.$$

On the other hand, the C^∞ algebra $\mathcal{S}$ is represented as the subspace of $\mathcal{K}$ consisting of all rapidly decreeing sequences, i.e., $y = (y_0, y_1, \ldots, y_k, \ldots) \in \mathcal{K}$ such that

$$\sum_{k=0}^{\infty} (1+k)^n \|y_k\|^2_{L^2(\mathcal{M},\tau)} < \infty \qquad \textit{for all} \quad n \in \mathbb{Z}.$$

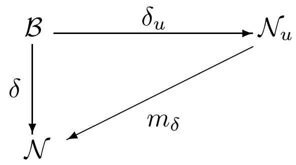
Under these representations, the duality is given by

$$x(y) \equiv \sum_{k=0}^{\infty} \langle x_k^*, y_k \rangle_{L^2(\mathcal{M},\tau)} \qquad (x \in \mathcal{S}',\ y \in \mathcal{S}).$$

Furthermore, for every $x \in \mathcal{S}'$, $\sum_{k=0}^{m} x_k$ converges to x as $m \to \infty$ in $\mathcal{S}'$ topology, where $x_k \equiv \sum_{i_1,i_2,\ldots,i_k \in I} \xi_{i_1 i_2 \cdots i_k} \widehat{T}_{i_1 i_2 \cdots i_k}$ (in $\mathcal{S}'$ topology) and $\xi_{i_1 i_2 \cdots i_k} \equiv x(T_{i_1 i_2 \cdots i_k})$. (Here, $\widehat{T}_{i_1 i_2 \cdots i_k}$ and $x(\,\cdot\,)$ are regarded as linear functionals on $\mathcal{S}$.) Similarly, for every $y \in \mathcal{S}$, $\sum_{k=0}^{m} y_k$ converges to y as $m \to \infty$ in $\mathcal{S}$ topology, where $y_k \equiv \sum_{i_1,i_2,\ldots,i_k \in I} \eta_{i_1 i_2 \cdots i_k} T_{i_1 i_2 \cdots i_k}$ (in $\mathcal{S}$ topology) and $\eta_{i_1 i_2 \cdots i_n} \equiv \tau(T^*_{i_1 i_2 \cdots i_k} y)$.

§6. C^∞ Vector Fields

Let $\mathcal{S}^I$ be the direct product of $\mathcal{S}$ over I which is a linear space. For any $k = (k_i)_{i \in I} \in \mathcal{S}^I$ the derivation δ_k on $\mathcal{S}$ with $\operatorname{dom} \delta_k = \mathcal{P}$ is uniquely well defined by the condition $\delta_k(s_i) = k_i$ $(i \in I)$, because $\mathcal{P}$ is isomorphic to the noncommutative polynomial ring $\mathbb{C}\langle X_i : i \in I \rangle$. Then, it is obvious that δ_k is $*$-derivation if and only if $k = (k_i)_{i \in I} \in (\mathcal{S}^{sa})^I \subset \mathcal{S}^I$, where $\mathcal{S}^{sa}$ is the real linear space consisting of selfadjoint elements of $\mathcal{S}$. We denote by $\mathcal{D}$ denote the linear span of $T_{i_1 i_2 \cdots i_n} \in \mathcal{S}^{(2)}$ for $n \in \mathbb{N}_0$ and $i_1, \ldots, i_n \in I^{(2)}$ such that $i_j \in \tilde{I}$ (a copy of I) and $i_k \in I$ $(k \neq j)$ for some $1 \leq j \leq n$. Then $\mathcal{D}$ becomes a $\mathcal{P}$ subbimodule of $\mathcal{S}^{(2)}$. Let $\mathcal{B}$ be an algebra, $\mathcal{N}_u$ a $\mathcal{B}$ bimodule and $\delta_u : \mathcal{B} \to \mathcal{N}_u$ a derivation. Then (δ_u, N_u) is called a *universal differential bimodule* for $\mathcal{B}$ if, for any derivation $\delta : \mathcal{B} \to \mathcal{N}$ with a $\mathcal{B}$ bimodule $\mathcal{N}$, there exists a unique bimodule map $m_\delta : \mathcal{N}_u \to \mathcal{N}$ such that $\delta = m_\delta \delta_u$.



Due to the universality definition, a universal differential bimodule for a given algebra is unique up to isomorphism. For $\mathcal{P}$ let $\mathcal{N}_\mathcal{P}$ be the algebraic direct sum $\bigoplus_{\mathrm{alg},\, i \in I} (\mathcal{P} \otimes \mathcal{P})$, a $\mathcal{P}$ bimodule, and define the derivation $\delta_\mathcal{P} : \mathcal{P} \to \mathcal{N}_\mathcal{P}$ by

$$\delta_\mathcal{P}(s_i) \equiv (0, \cdots, 0, \underbrace{1 \otimes 1}_{i\text{th}}, 0, \cdots) \in \mathcal{N}_\mathcal{P}.$$

Then $(\mathcal{N}_\mathcal{P}, \delta_\mathcal{P})$ is the universal differential bimodule for $\mathcal{P}$ being isomorphic to the noncommutative polynomial ring.

Lemma 6.1. *Consider the weak derivation D given in Definition* 4.1 *as a derivation $D : \mathcal{P} \to \mathcal{D}$. Then $(\mathcal{N}_\mathcal{P}, \delta_\mathcal{P})$ and $(\mathcal{D}, D)$ are isomorphic, so $(\mathcal{D}, D)$ is the universal differential bimodule for $\mathcal{P}$.*

Proof. The elements $T_{i_1 i_2 \cdots i_n} \otimes T_{j_1 j_2 \cdots j_l}$ where $n, l \in \mathbb{N}_0$ and $i_1, i_2, \ldots, i_n$, $j_1, j_2, \ldots, j_l \in I$ form a linear basis of $\mathcal{P} \otimes \mathcal{P}$. For each $i \in I$ define the linear map

$$m_i : 0 \oplus \cdots \oplus 0 \oplus \underbrace{\mathcal{P} \otimes \mathcal{P}}_{i\text{th}} \oplus 0 \cdots \big(\subset \mathcal{N}_\mathcal{P}\big) \to \mathcal{D}$$

by

$$m_i\Big(0\oplus\cdots\oplus 0\oplus \underbrace{T_{i_1 i_2\cdots i_n}\otimes T_{j_1 j_2\cdots j_l}}_{i\text{th}}\oplus 0\cdots\Big)\equiv T_{i_1 i_2\cdots i_n (i+N) j_1 j_2\cdots j_l},$$

and consider $m\equiv\bigoplus_{i\in I} m_i : \mathcal{N}_{\mathcal{P}}\to\mathcal{D}$. Then it is immediate to see that $\bigoplus_{i\in I} m_i$ is an injective $\mathcal{P}$ bimodule map from $\mathcal{N}_{\mathcal{P}}$ onto $\mathcal{D}$. Since

$$m\delta_{\mathcal{P}}(s_i)=m(0,\dots,0,1\otimes 1,0,\dots)=T_{i+N}=D(s_i)\qquad (i\in I),$$

we have $m\delta_{\mathcal{P}}=D$. □

Thank to the above lemma, for any $k=(k_i)_{i\in I}\in\mathcal{S}^I$ there exists a unique $\mathcal{P}$ bimodule map $m_k:\mathcal{D}\to\mathcal{S}$ such that $m_k(T_{i+N})=k_i$ $(i\in I)$; it satisfies $\delta_k=m_k D$. For each $i\in I$ the partial weak derivation $\frac{\partial}{\partial s_i}:\mathcal{P}\to\mathcal{D}$ and the $\mathcal{P}$ bimodule map $m_{k_i}:\mathcal{D}\to\mathcal{P}$ are well defined by

$$\frac{\partial}{\partial s_i}(T_j)=\begin{cases}T_{i+N} & \text{if } j=i,\\ 0 & \text{otherwise},\end{cases}$$

$$m_{k_i}(T_{j+N})=\begin{cases}k_i & \text{if } j=i,\\ 0 & \text{otherwise}\end{cases}$$

for all $j\in I$. Then we obtain

$$D=\sum_{i\in I}\frac{\partial}{\partial s_i},$$

$$m_k=\sum_{i\in I} m_{k_i},$$

$$\delta_k=m_k D=\sum_{i\in I} m_{k_i}\frac{\partial}{\partial s_i}.$$

We call the derivation $\delta_k:\mathcal{P}\to\mathcal{D}$ defined as above a C^∞ *vector field* because it is a differential operator of the first order with C^∞ coefficients. The linear space of all C^∞ vector fields can be identified with the linear space $\mathcal{S}^I$.

In the next theorem, we find a necessary and sufficient condition for a C^∞ vector field δ_k to have divergence zero, that is, for the τ-preservation of δ_k in the sense that $\tau(\delta_k(x))=0$ for all $x\in\mathcal{P}$. The condition is presented as a simple cocyclic condition for the Fourier coefficients of the vector field. Note that the same is obtained in [15, Corollary 7.6] but our derivation is more direct than that in [15].

Theorem 6.2. *Let δ_k be a C^∞ vector field, and assume that the Fourier expansions of its coefficients are*

$$k_i = \sum_{n=0}^{\infty} \sum_{i_1,\dots,i_n \in I} c_{i,i_1 i_2 \cdots i_n} T_{i_1 i_2 \cdots i_n} \qquad (i \in I).$$

Then, $\tau(\delta_k(x)) = 0$ holds for all $x \in \mathcal{P}$ if and only if the conditions

$$c_{i_1,i_2\cdots i_n} + c_{i_2,i_3\cdots i_n i_1} + c_{i_3,i_4\cdots i_1 i_2} + \cdots + c_{i_n,i_1\cdots i_{n-1}} = 0$$

are satisfied for all $n \in \mathbb{N}$ and $i_1, i_2, \dots, i_n \in I$.

Proof. *Sufficiency.* Assume the cocyclic conditions stated above. We may prove that $\tau\big(\delta_k(s_{j_1} s_{j_2} \cdots s_{j_l})\big) = 0$ for all $l \in \mathbb{N}$ and $j_1, j_2, \dots, j_l \in I$ because the set $\{1\} \cup \{s_{j_1} s_{j_2} \cdots s_{j_l} : l \in \mathbb{N},\ j_1, j_2, \dots, j_l \in I\}$ is a linear basis of $\operatorname{dom} \delta_k = \mathcal{P}$. We get

$$\begin{aligned}
\delta_k(s_{j_1} s_{j_2} \cdots s_{j_l}) &= \delta_k(T_{j_1} T_{j_2} \cdots T_{j_l}) \\
&= \delta_k(T_{j_1}) T_{j_2} \cdots T_{j_l} + T_{j_1} \delta_k(T_{j_2}) T_{j_3} \cdots T_{j_l} \\
&\quad + \cdots + T_{j_1} \cdots T_{j_{l-1}} \delta_k(T_{j_l}) \\
&= \left(\sum_{n=0}^{\infty} \sum_{i_1,i_2,\dots,i_n \in I} c_{j_1,i_1 i_2\cdots i_n} T_{i_1 i_2 \cdots i_n} \right) T_{j_2} \cdots T_{j_l} \\
&\quad + T_{j_1} \left(\sum_{n=0}^{\infty} \sum_{i_1,i_2,\dots,i_n \in I} c_{j_2,i_1 i_2\cdots i_n} T_{i_1 i_2 \cdots i_n} \right) T_{j_3} \cdots T_{j_l} \\
&\quad + \cdots + T_{j_1} \cdots T_{j_{l-1}} \left(\sum_{n=0}^{\infty} \sum_{i_1,i_2,\dots,i_n \in I} c_{j_l,i_1 i_2\cdots i_n} T_{i_1 i_2 \cdots i_n} \right).
\end{aligned}$$

Hence, by the norm convergence of Fourier expansions (as noted after Corollary 3.8) we compute

(8)

$$\begin{aligned}
&\tau\big(\delta_k(s_{j_1} s_{j_2} \cdots s_{j_l})\big) \\
&\quad = \sum_{n=0}^{\infty} \sum_{i_1,i_2,\dots,i_n \in I} c_{j_1,i_1 i_2\cdots i_n} \tau(T_{i_1 i_2 \cdots i_n} T_{j_2} T_{j_3} \cdots T_{j_l}) \\
&\qquad + \sum_{n=0}^{\infty} \sum_{i_1,i_2,\dots,i_n \in I} c_{j_2,i_1 i_2\cdots i_n} \tau(T_{j_1} T_{i_1 i_2 \cdots i_n} T_{j_3} \cdots T_{j_l}) \\
&\qquad + \cdots + \sum_{n=0}^{\infty} \sum_{i_1,i_2,\dots,i_n \in I} c_{j_l,i_1 i_2\cdots i_n} \tau(T_{j_1} T_{j_2} \cdots T_{j_{l-1}} T_{i_1 i_2 \cdots i_n})
\end{aligned}$$

$$= \sum_{n\in\{l-1,l-3,\dots\}} \Biggl\{ \sum_{\{i_1,i_2,\dots,i_n\}\subset\{j_2,j_3,\dots,j_l\}} c_{j_1,i_1i_2\cdots i_n}\tau(T_{i_1i_2\cdots i_n}T_{j_2}T_{j_3}\cdots T_{j_l})$$
$$+ \sum_{\{i_1,i_2,\dots,i_n\}\subset\{j_1,j_3,\dots,j_l\}} c_{j_2,i_1i_2\cdots i_n}\tau(T_{i_1i_2\cdots i_n}T_{j_3}\cdots T_{j_l}T_{j_1})$$
$$+\cdots+ \sum_{\{i_1,i_2,\dots,i_n\}\subset\{j_1,j_2,\dots,j_{l-1}\}} c_{j_l,i_1i_2\cdots i_n}\tau(T_{i_1i_2\cdots i_n}T_{j_1}T_{j_2}\cdots T_{j_{l-1}}) \Biggr\}$$

thanks to Remark 2.5 (4). We now proceed to compute based on Lemma 2.4 and Remark 2.5 (4). Let $n \in \{l-1, l-3, \dots\}$ be fixed and $(r_1, r_2, \dots, r_{l-1})$ be one of $(j_2, j_3, \dots, j_l)$, $(j_3, \dots, j_l, j_1)$, $\dots$, $(j_1, j_2, \dots, j_{l-1})$. Consider the following diagram associated with $(T_{i_1i_2\cdots i_n}, T_{r_1}, T_{r_2}, \dots, T_{r_{l-1}})$:

$$\begin{array}{ccccccccccc} \bullet & \bullet & \cdots & \bullet & | & \bullet & | & \bullet & | \cdots | & \bullet \\ i_1 & i_2 & \cdots & i_n & & r_1 & & r_2 & \cdots & r_{l-1} \end{array}. \tag{9}$$

Lemma 2.4 implies that $\tau(T_{i_1i_2\cdots i_n}T_{r_1}T_{r_2}\cdots T_{r_{l-1}})$ is equal to the number of non-crossing pair partitions of the diagram (9). In a non-crossing pair partition of (9), let $1 \le k(1) < k(2) < \cdots < k(n) \le l-1$ be the indices such that the n vertices in the first block with labels $i_n, i_{n-1}, \dots, i_1$ are paired with the vertices with labels $r_{k(1)}, r_{k(2)}, \dots, r_{k(n)}$, respectively. For each $1 \le p(0) < p(1) < \cdots < p(n) \le l$ consider the case

$$(r_1, r_2, \dots, r_{l-1}) = (j_{p(t)+1}, j_{p(t)+2}, \dots, j_{p(t)+l-1}),$$

where $t = 0, 1, \dots, n$ and $p(t) + q$ $(1 \le q \le l-1)$ are understood under *mod* l (so $j_{p(t)+l-1} = j_{p(t)-1}$ for instance). For this case the number of pair partitions of (9) such that

$$(k(1), k(2), \dots, k(n)) = (p(t+1), p(t+2), \dots, p(t-1)) \qquad \text{(in the cyclic order)}$$

is equal to

$$M(p(0), p(1), \dots, p(n)) \equiv M_0M_1\cdots M_n,$$

where M_u is the number of pair partitions of the subdiagram (interval) of (9):

$$\begin{array}{ccccccc} \bullet & | & \bullet & | \cdots | & \bullet \\ p(u)+1 & & p(u)+2 & \cdots & p(u+1)-1 \end{array}$$

for $u = 0, 1, \dots, n$. Therefore, the contribution of these pair partitions to $\tau\bigl(\delta_k(s_{j_1}s_{j_2}\cdots s_{j_l})\bigr)$ is

$$M(p(0), p(1), \dots, p(n))\, c_{j_{p(t)},\, j_{p(t+1)}j_{p(t+2)}\cdots j_{p(t-1)}},$$

whose sum over $t = 0, 1, \ldots, n$ is

$$
\begin{aligned}
&M(p(0), p(1), \ldots, p(n)) \\
&\quad \times \Big\{ c_{j_{p(0)},\, j_{p(1)} j_{p(2)} \cdots j_{p(n)}} + c_{j_{p(1)},\, j_{p(2)} \cdots j_{p(n)} j_{p(0)}} + \cdots + c_{j_{p(n)},\, j_{p(0)} j_{p(1)} \cdots j_{p(n-1)}} \Big\} \\
&\quad = 0
\end{aligned}
$$

by assumption. Finally, notice that every non-crossing pair partition of one of the diagrams associated with the terms in (8) is counted just once in the above computation. Summing up zeros over all choices of $1 \leq p(0) < p(1) < \cdots < p(n) \leq l$ for $n \in \{l-1, l-3, \ldots\}$ we obtain $\tau\big(\delta_k(s_{j_1} s_{j_2} \cdots s_{j_l})\big) = 0$, as required.

Necessity. Assume that there exist $i_1, i_2, \ldots, i_n$ such that

$$
c_{i_1, i_2 \cdots i_n} + c_{i_2, i_3 \cdots i_n i_1} + c_{i_3, i_4 \cdots i_1 i_2} + \cdots + c_{i_n, i_1 \cdots i_{n-1}} \neq 0.
$$

Let $i_1, i_2, \ldots, i_n$ be one of such strings with least n. Then, from the above proof of sufficiency we see that

$$
\tau\big(\delta_k(s_{i_1} s_{i_2} \cdots s_{i_n})\big) = c_{i_1, i_2 \cdots i_n} + c_{i_2, i_3 \cdots i_n i_1} + c_{i_3, i_4 \cdots i_1 i_2} + \cdots + c_{i_n, i_1 \cdots i_{n-1}} \neq 0.
$$

□

Although it cannot be proved at the moment, we expect that, for any C^∞ vector field δ_k, the bimodule map $m_k : \mathcal{D} \to \mathcal{S}$ is continuous when $\mathcal{D}$ is equipped with $\mathcal{S}^{(2)}$ topology and $\mathcal{S}$ with $\mathcal{S}$ topology. If this is the case, then any C^∞ vector field δ_k is continuous on $\mathcal{S}$ thanks to $\delta_k = m_k D$, and so the (continuous) dual map $(\delta_k)'$ is defined on the space $\mathcal{S}'$ of Schwartz distributions. Furthermore, in this situation, the next proposition says that any τ-preserving C^∞ vector field δ_k is continuous with respect to $\mathcal{S}'$ topology and it has the closure defined on the whole space $\mathcal{S}'$.

Proposition 6.3. *If a C^∞ vector field δ_k is τ-preserving and continuous on $\mathcal{S}$, then*

$$
-(\delta_k)'|_{\mathcal{S}} = \overline{\delta_k}^{\mathcal{S}}.
$$

Consequently, δ_k is continuous with respect to $\mathcal{S}'$ topology and $\overline{\delta_k}^{\mathcal{S}'}$ is defined on the whole space $\mathcal{S}'$ of Schwartz distributions

Proof. First, we prove that $-(\delta_k)'|_{\mathcal{P}} = \delta_k|_{\mathcal{P}}$. To do so, we may prove that

$$
-(\delta_k)'(T_{i_1 i_2 \cdots i_n}) = \delta_k(T_{i_1 i_2 \cdots i_n})
$$

for all $n \in \mathbb{N}$ and $i_1, i_2, \dots, i_n \in I$. For every $l \in \mathbb{N}$ and $j_1, j_2, \dots, j_l \in I$, using the derivation formula and the τ-preservation, we get

$$\begin{aligned}\tau\big(\delta_k(T_{i_1 i_2 \cdots i_n})T_{j_1 j_2 \cdots j_l}\big) &= -\tau\big(T_{i_1 i_2 \cdots i_n}\delta_k(T_{j_1 j_2 \cdots j_l})\big) + \tau\big(\delta_k(T_{i_1 i_2 \cdots i_n}T_{j_1 j_2 \cdots j_l})\big) \\ &= -\tau\big(T_{i_1 i_2 \cdots i_n}\delta_k(T_{j_1 j_2 \cdots j_l})\big) \\ &= -(\delta_k)'(T_{i_1 i_2 \cdots i_n})(T_{j_1 j_2 \cdots j_l}).\end{aligned}$$

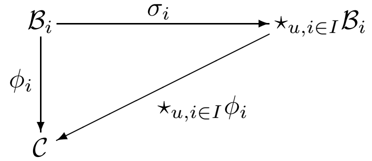
Hence we have $-(\delta_k)'|_{\mathcal{P}} = \delta_k|_{\mathcal{P}}$. Since $\mathcal{P}$ is dense in $\mathcal{S}$, this implies that $-(\delta_k)'|_{\mathcal{S}} = \overline{\delta_k}^{\mathcal{S}}$ so that δ_k is continuous with respect to $\mathcal{S}'$ topology. The last assertion is obvious because $\mathcal{P}$ is dense in $\mathcal{S}'$ by Theorem 5.2. $\square$

Finally, we would like to mention that our final goal would be attained by analyzing the actions on the free group factors by the pseudo-differential technique from the PDE theory in contrast to the traditional method ([3] and [12]). This viewpoint looks most attractive to the author and is the reason why we have constructed the free analog of the Schwartz distribution space.

§7. Appendix: Noncommutative Cubic Space

We here discuss a bit C^*-algebras that appeared in this paper. The K-theory of the C^*-algebra $\star_{u,N}C([-2,2])$ can be obtained in the same way as in [4], but we here get it in a simpler way.

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be two unital C^*-algebras and ψ_0, ψ_1 be two unital homomorphisms from $\mathcal{B}_1$ to $\mathcal{B}_2$. We say that ψ_0 and ψ_1 are *unitally homotopic* and write $\psi_0 \sim_h \psi_1$ if there exists a point-norm-continuous map $\widetilde{\psi}(t)$ on $t \in [0,1]$ into the unital homomorphisms from $\mathcal{B}_1$ to $\mathcal{B}_2$ such that $\widetilde{\psi}(0) = \psi_0$ and $\widetilde{\psi}(1) = \psi_1$. Moreover, we say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are unitally homotopic and write $\mathcal{B}_1 \sim_h \mathcal{B}_2$ if there exist two unital homomorphism $\psi_1 : \mathcal{B}_1 \to \mathcal{B}_2$ and $\psi_2 : \mathcal{B}_2 \to \mathcal{B}_1$ such that $\psi_2\psi_1 \sim_h \mathrm{id}_{\mathcal{B}_1}$ and $\psi_1\psi_2 \sim_h \mathrm{id}_{\mathcal{B}_2}$. Let $\mathcal{B}_i$ $(i \in I)$ be unital C^*-algebras indexed by I. Then the universal C^*-free product $\star_{u,i\in I}\mathcal{B}_i$ is a unique C^*-algebra with unital homomorphisms $\sigma_i : \mathcal{B}_i \to \star_{u,i\in I}\mathcal{B}_i$ $(i \in I)$ satisfying the following universality property: for any C^*-algebra $\mathcal{C}$ and for any family of unital homomorphisms $\phi_i : \mathcal{B}_i \to \mathcal{C}$ $(i \in I)$, there exists a unique unital homomorphism $\star_{u,i\in I}\phi_i : \star_{u,i\in I}\mathcal{B}_i \to \mathcal{C}$ such that the following diagram is commuting for each $i \in I$:



Proposition 7.1. *Let $\mathcal{B}_{1,i}$ and $\mathcal{B}_{2,i}$ $(i \in I)$ be two families of unital C^*-algebras indexed by I. If $\mathcal{B}_{1,i} \sim_h \mathcal{B}_{2,i}$ for all $i \in I$, then $\star_{u,i\in I}\mathcal{B}_{1,i} \sim_h \star_{u,i\in I}\mathcal{B}_{2,i}$.*

Proof. By assumption there exist C^*-homomorphisms $\psi_{1,i} : \mathcal{B}_{1,i} \to \mathcal{B}_{2,i}$ and $\psi_{2,i} : \mathcal{B}_{2,i} \to \mathcal{B}_{1,i}$ $(i \in I)$ such that $\psi_{2,i}\psi_{1,i} \sim_h \mathrm{id}_{\mathcal{B}_{1,i}}$ and $\psi_{1,i}\psi_{2,i} \sim_h \mathrm{id}_{\mathcal{B}_{2,i}}$. Then we have the homomorphisms $\star_{u,i\in I}\psi_{1,i} : \star_{u,i\in I}\mathcal{B}_{1,i} \to \star_{u,i\in I}\mathcal{B}_{2,i}$ and $\star_{u,i\in I}\psi_{2,i} : \star_{u,i\in I}\mathcal{B}_{2,i} \to \star_{u,i\in I}\mathcal{B}_{1,i}$. We need to prove that

$$\begin{aligned}\big(\star_{u,i\in I}\psi_{2,i}\big)\big(\star_{u,i\in I}\psi_{1,i}\big) &\sim_h \mathrm{id}_{\star_{u,i\in I}\mathcal{B}_{1,i}},\\ \big(\star_{u,i\in I}\psi_{1,i}\big)\big(\star_{u,i\in I}\psi_{2,i}\big) &\sim_h \mathrm{id}_{\star_{u,i\in I}\mathcal{B}_{2,i}}.\end{aligned}$$

By symmetry it is enough to prove the first assertion. For each $i \in I$ let $\widetilde{\psi}_i(t)$ $(t \in [0,1])$ be a homotopy map connecting $\psi_{2,i}\psi_{1,i}$ and $\mathrm{id}_{\mathcal{B}_{1,i}}$ so that $\widetilde{\psi}_i(0) = \psi_{2,i}\psi_{1,i}$ and $\widetilde{\psi}_i(1) = \mathrm{id}_{\mathcal{B}_{1,i}}$. Then, by construction we have

$$\star_{u,i\in I}\widetilde{\psi}_i(0) = \star_{u,i\in I}(\psi_{2,i}\psi_{1,i}) = \big(\star_{u,i\in I}\psi_{2,i}\big)\big(\star_{u,i\in I}\psi_{1,i}\big)$$

and

$$\star_{u,i\in I}\widetilde{\psi}_i(1) = \star_{u,i\in I}\mathrm{id}_{\mathcal{B}_{1,i}} = \mathrm{id}_{\star_{u,i\in I}\mathcal{B}_{1,i}}.$$

So it suffices to show the point-norm-continuity of the one-parameter unital homomorphisms $\star_{u,i\in I}\widetilde{\psi}_i(t) : \star_{u,i\in I}\mathcal{B}_{1,i} \to \star_{u,i\in I}\mathcal{B}_{1,i}$. But this is immediately verified by applying to elements in the algebraic free product of $\mathcal{B}_{1,i}$, a dense subalgebra of $\star_{u,i\in I}\mathcal{B}_{1,i}$. □

Corollary 7.2. *The C^*-algebra $\star_{u,N}C([-2,2])$ are contractible to one point for any $N \equiv \operatorname{card} I$, i.e., $\star_{u,N}C([-2,2]) \sim_h \mathbb{C}$. Consequently, $\star_{u,N}C$ $([-2,2])$ are projectionless and*

$$K_0\big(\star_{u,N}C([-2,2])\big) = \mathbb{Z}, \quad K_1\big(\star_{u,N}C([-2,2])\big) = 0.$$

In the K_0 isomorphism, $[1_{\star_{u,N}C([-2,2])}]$ corresponds to $1 \in \mathbb{Z}$.

Proof. Since $C([-2,2]) \sim_h \mathbb{C}$, the above proposition implies that $\star_{u,N}C^0$ $([-2,2]) \sim_h \star_{u,N}\mathbb{C} = \mathbb{C}$. This shows the assertions on the K-theory. There is a homotopy map $\widetilde{\phi}(t)$ on $t \in [0,1]$ into the unital homomorphisms of $C([-2,2])$ into itself such that $\widetilde{\phi}(0) = \mathrm{id}_{C([-2,2])}$ and $\widetilde{\phi}(1) = \delta_0$ where $\delta_0(f) = f(0)$ for $f \in C([-2,2])$. Set $\widetilde{\psi}(t) \equiv \star_{u,N}\widetilde{\phi}(t)$ for $t \in [0,1]$, which is a homotopy map connecting $\mathrm{id}_{\star_{u,N}C([-2,2])}$ and $\star_{u,N}\delta_0 : \star_{u,N}C([-2,2]) \to \mathbb{C}$ $(\subset \star_{u,N}C([-2,2]))$. Assume that a nontrivial projection p exists in $\star_{u,N}C([-2,2])$. Then $\widetilde{\psi}(t)(p)$ is a continuous path from p to a projection in $\mathbb{C}$. This gives a contradiction. □

Let $q : \star_{u,N}C([-2,2]) \to \mathcal{A} \equiv \star_{r,N}C([-2,2])$ be the natural surjective homomorphism. Then $K_0(q)$ is an injective map because $\mathcal{A}$ has a faithful tracial state. An interesting problem is to decide whether it is surjective.

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