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Bernstein Polynomials of a Smooth Function Restricted to an Isolated Hypersurface Singularity

By

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Abstract

Let f, g be two germs of holomorphic functions on $\mathbf{C}^n$ such that f is smooth at the origin and (f, g) defines an analytic complete intersection $(Z, 0)$ of codimension two. We study Bernstein polynomials of f associated with sections of the local cohomology module with support in $X = g^{-1}(0)$, and in particular some sections of its minimal extension. When $(X, 0)$ and $(Z, 0)$ have an isolated singularity, this may be reduced to the study of a minimal polynomial of an endomorphism on a finite dimensional vector space. As an application, we give an effective algorithm to compute those Bernstein polynomials when f is a coordinate and g is non-degenerate with respect to its Newton boundary.

§1. Introduction

Let $n \geq 2$ be an integer. Let us denote $\mathcal{O} = \mathbf{C}\{x_1, \ldots, x_n\}$ the ring of germs at 0 of complex holomorphic functions, and $\mathcal{D} = \mathcal{O}\langle \partial/\partial x_1, \ldots, \partial/\partial x_n \rangle$ the ring of linear differential operators with holomorphic coefficients.

Let $g \in \mathcal{O}$ be a nonzero germ such that $g(0) = 0$, and $\mathcal{R} = \mathcal{O}[1/g]/\mathcal{O}$ the local cohomology module with support in the hypersurface $(X, 0) \subset (\mathbf{C}^n, 0)$ defined by g. It is a regular holonomic $\mathcal{D}$-module such that its complex of holomorphic solutions is the perverse sheaf $\mathbf{C}_X[-1]$ (see [5], [6], [14]).

Given a germ of function $f \in \mathcal{O}$ nonzero on X, there are functional equations in $\mathcal{R}[1/f, s]f^s = \mathcal{R} \otimes_{\mathcal{O}} \mathcal{O}[1/f, s]f^s$ of the form:

$$b(s)\delta f^s = P \cdot \delta f^{s+1}$$

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for every $\delta \in \mathcal{R}$, with $b(s) \in \mathbf{C}[s]$ nonzero and $P \in \mathcal{D}[s] = \mathcal{D} \otimes \mathbf{C}[s]$ (see [6]). We call *Bernstein polynomial of* f associated with δ, and we denote $b(\delta f^s, s)$, the unitary generator of the ideal of polynomials $b(s)$ verifying such an identity. When f is not a unit, it is easy to check that $(s + r(\delta) + 1)$ is a factor of $b(\delta f^s, s)$, where $r(\delta) \in \mathbf{N}$ is such that $\delta \in f^{r(\delta)}\mathcal{R} - f^{r(\delta)+1}\mathcal{R}$; let us denote $\tilde{b}(\delta f^s, s) \in \mathbf{C}[s]$ the quotient of $b(\delta f^s, s)$ by $(s + r(\delta) + 1)$.

Because of the algebraic theory of vanishing cycles, roots of these polynomials determine the eigenvalues of the monodromy of $f|_X : (X, 0) \to (\mathbf{C}, 0)$ (see [7], [12], and [20] for examples). In particular, the singular monodromy theorem implies that their roots are rational numbers ([8], [10]).

The effective determination of these polynomials is a difficult question. Following ideas of B. Malgrange ([11], [2] part A), we have investigated this problem in [21] when X has an isolated singularity and (f, g) defines a germ of complete intersection isolated singularity $(Z, 0)$. First, for $\delta \in \mathcal{R}$ of the form $\dot{a}/g^\ell$ with $a \in \mathcal{O}$ nonzero on the components of Z, the holonomic $\mathcal{D}$-module:

$$\mathcal{N}_\delta = (s+1)\frac{\mathcal{D}[s]\delta f^s}{\mathcal{D}[s]\delta f^{s+1}}$$

is supported by 0. Then the minimal polynomial of the action of s on $\mathcal{N}_\delta$ - which is nothing else but $\tilde{b}(\delta f^s, s)$ - may be computed using its n^{th}-group of de Rham cohomology $H^n_{DR}(\mathcal{N}_\delta) = \mathcal{N}_\delta / \sum(\partial/\partial x_i)\mathcal{N}_\delta$. In order to do that, we need an explicit description of this group. So we imposed that the annihilator in $\mathcal{D}$ of δ is generated by operators of degree less or equal to one; but it is a very constraining condition, because this implies that g is weighted-homogeneous and that $a \in \mathcal{O}$ is a unit (see [21], [23]).

In this paper, we study the particular case where f is a germ of a smooth function. Let us recall that this contains the classical theory of the Bernstein polynomial of germs of holomorphic functions, because of the following relation:

$$b\Big(\frac{\dot{1}}{h-z}z^s, s\Big) = b(h^s, s)$$

for every $h \in \mathcal{O}$ nonzero, where $b(h^s, s)$ is the Bernstein polynomial of h and $\dot{1}/h - z \in \mathbf{C}\{x, z\}[1/h - z]/\mathbf{C}\{x, z\}$ (see Proposition 2.8 for example).

Without further condition on g, we prove in Theorem 2.1 that for some $\delta \in \mathcal{R}$, the $\mathcal{D}[s]$-module $\mathcal{N}_\delta$ coincides with:

$$\mathcal{N}_\ell = \frac{\mathcal{D}[s](\mathrm{jac}(g), g)\delta_\ell f^{s+1}}{\mathcal{D}[s]\mathcal{J}\delta_\ell f^{s+1}} \tag{1}$$

for an integer $\ell \in \mathbf{N}^*$, where $\mathrm{jac}(g) \subset \mathcal{O}$ is the jacobian ideal of g, $\mathcal{J} \subset \mathcal{O}$ is the ideal generated by g and by all the 2×2-minors of the jacobian matrix of

(f,g), and $\delta_\ell \in \mathcal{R}$ is defined by $(-1)^{\ell+1}(\ell-1)!/g^\ell \in \mathcal{O}[1/g]$. More precisely, $\mathcal{N}_\delta$ is equal to $\mathcal{N}_\ell$ (resp. $\mathcal{N}_{\ell+1}$) when $\delta = v(g)\delta_\ell$ (resp. $\delta = \delta_\ell$) for every generic regular vector field v such that $v(f) = 0$. This result enables us to treat in the same way the Bernstein polynomials of f associated with sections δ_ℓ, $\ell \in \mathbf{N}^*$, but also with certain generators of the minimal extension $\mathcal{L} \subset \mathcal{R}$ of the local algebraic cohomology with support in X (since D. Barlet and M. Kashiwara prove in [1] that $\mathcal{L}$ is generated by any nonzero section defined by $v(g)/g$, where $v \in \mathcal{D}$ is a vector field).

So we are interested in the determination of the minimal polynomial of the action of s on $\mathcal{N}_\ell$, denoted by $\tilde{b}_\ell(s)$, when f is smooth, X has an isolated singularity and (f,g) defines a germ of complete intersection isolated singularity. In the third part, we express $H^n_{DR}(\mathcal{N}_\ell)$ under these assumptions as a quotient of two finite dimensional vector spaces $\mathcal{Z}'_\ell$ and $\mathcal{Z}_\ell$ defined in section 3.2. Therefore:

Theorem 1.1. *For every $\ell \in \mathbf{N}^*$, $\tilde{b}_\ell(s)$ is the minimal polynomial of the action induced by s on $\mathcal{Z}'_\ell/\mathcal{Z}_\ell$.*

This needs the knowledge of the annihilator in $\mathcal{D}$ of $\delta_k f^s$, $\mathrm{Ann}_{\mathcal{D}}\, \delta_k f^s$, which authorizes the calculation of the n^{th}-group of the de Rham cohomology of the $\mathcal{D}$-module $\sum_{k\geq 1} \mathcal{D}\delta_k f^{s+1}$ (into which $\mathcal{D}[s](\mathrm{jac}(g), g)\delta_\ell f^{s+1}$ injects).

As an application, we develop in the last part an algorithm to compute $\tilde{b}_\ell(s)$ when $f = x_1$ and g is non-degenerate with respect to its Newton boundary in the sense of Kouchnirenko, which gives a generalization of [2]. Using the Newton function ρ on $\mathcal{O}$, we define a weight function ρ^* by $\rho^*(u\delta_k x_1^{s+1}) = \rho(ux_2\cdots x_n) - k$. Then Kouchnirenko division theorem makes it possible to establish that the filtration induced by ρ^* is suited to our construction of $H^n_{DR}(\sum_{k\geq 1}\mathcal{D}\delta_k f^{s+1})$. Moreover, the action of s respects the filtration induced by ρ^* on $\mathcal{Z}'_\ell/\mathcal{Z}_\ell$. Thus, if $\tilde{b}_{\ell,q}(s)$ is the minimal polynomial of the action of s on $\mathrm{gr}^*_q\mathcal{Z}'_\ell/\mathcal{Z}_\ell$, then the polynomial $\tilde{b}_\ell(s)$ is the l.c.m. of $\tilde{b}_{\ell,q}(s)$, $q \in \mathbf{Q}$ (Theorem 4.9). The technics 'rewriting by division' and 'increase in weight' allow us to give an explicit computation of the spaces $\mathcal{Z'}^*_{\ell,q}$, $\mathcal{Z}^*_{\ell,q}$ and of the action of s on $\mathcal{Z'}^*_{\ell,q}/\mathcal{Z}^*_{\ell,q}$, and thus to determine $\tilde{b}_\ell(s)$. In the particular case of semi-weighted-homogeneous germs, these computations are easier (Remark 4.12). On the way, we deduce from an algorithm for computing a multiple of the polynomials $\tilde{b}_{\ell,q}(s)$ that the multiplicities of the roots of $\tilde{b}_\ell(s)$ are strictly smaller than n (Theorem 4.10).

We end with the complete determination of the polynomials $\tilde{b}_\ell(s)$ when $g = x_1^d + x_2^d + x_3^d + (x_1x_2x_3)^2$, $d \geq 9$.

Finally, we point out that the methods at the root of the algorithm may be adapted to compute Bernstein functional equations associated with an analytic morphism - introduced by C. Sabbah ([15], [16]) - in the following case: $(g, x_1, \ldots, x_p) : (\mathbf{C}^n, 0) \to (\mathbf{C}^{p+1}, 0)$, $1 \leq p \leq n-1$. In particular, one can make explicit non trivial equations of the form:

$$\begin{aligned} d_0(\underline{s}) g^{s_0} x_1^{s_1} \cdots x_p^{s_p} &\in \mathcal{D}[\underline{s}] g^{s_0+1} x_1^{s_1} \cdots x_p^{s_p} \\ d_j(\underline{s}) g^{s_0} x_1^{s_1} \cdots x_p^{s_p} &\in \mathcal{D}[\underline{s}] x_j g^{s_0} x_1^{s_1} \cdots x_p^{s_p}, \ 1 \leq j \leq p \end{aligned}$$

where $d_0(\underline{s})$, $d_j(\underline{s}) \in \mathbf{C}[s_0, \ldots, s_p]$ and $\mathcal{D}[\underline{s}] = \mathcal{D} \otimes \mathbf{C}[s_0, \ldots, s_p]$. This completes H. Maynadier-Gervais results about these functional equations ([13]).

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§2. Some Equivalences of Functional Equations

In this part, we denote $f \in \mathcal{O}$ a germ of a smooth function and $g \in \mathcal{O}$ a germ which is not a unit and does not belong to $f\mathcal{O}$.

We first prove Theorem 2.1, where the $\mathcal{D}$-module $\mathcal{N}_\delta$ is identified to $\mathcal{N}_\ell$ for some $\delta \in \mathcal{R}$. Then we give relations between some Bernstein polynomials of f associated with sections of $\mathcal{R} = \mathcal{O}[1/g]/\mathcal{O}$.

§2.1. Some identifications of $\mathcal{N}_\delta$ with $\mathcal{N}_\ell$

Let us state the result at the root of this study.

Theorem 2.1. *Let $f \in \mathcal{O}$ be a germ of a smooth function at the origin, and $g \in \mathcal{O}$ a germ which is neither a unit nor a multiple of f. Let us denote $(Z, 0) \subset (\mathbf{C}^n, 0)$, the complete intersection defined by f and g.*

i) *For every non negative integer $\ell \in \mathbf{N}^*$, the $\mathcal{D}[s]$-module:*

$$(s+1)\frac{\mathcal{D}[s]\delta_\ell f^s}{\mathcal{D}[s]\delta_\ell f^{s+1}}$$

where $\delta_\ell = (-1)^{\ell+1}(\ell-1)!(\dot{1}/g^\ell) \in \mathcal{R}$, coincides with $\mathcal{N}_{\ell+1}$.

ii) *Let $v \in \mathcal{D}$ be a regular vector field such that $v(f) = 0$. Let us suppose that v is not tangent to $(Z, 0)$. Then, for every $\ell \in \mathbf{N}^*$, the $\mathcal{D}[s]$-module:*

$$(s+1)\frac{\mathcal{D}[s]v(g)\delta_\ell f^s}{\mathcal{D}[s]v(g)\delta_\ell f^{s+1}}$$

coincides with $\mathcal{N}_\ell$. Moreover, when $(Z, 0)$ does not have any irreducible smooth component, the equality is verified if v is not tangent to $(Sing(Z), 0)$.

iii) *Let us suppose that* $f = x_1$. *Let* $\tilde{v} \in \mathcal{D}$ *be a vector field of the form* $x_1(\partial/\partial x_1) + v$ *where* $v \in \mathbf{C}\{x_2, \dots, x_n\}\langle \partial/\partial x_2, \dots, \partial/\partial x_n\rangle$ *is a regular vector field. Let us suppose that* v *is not tangent to* $(Z, 0)$. *Then, for every* $\ell \in \mathbf{N}^*$, *the* $\mathcal{D}[s]$*-module*:

$$(s+1)\frac{\mathcal{D}[s]\tilde{v}(g)\delta_\ell f^s}{\mathcal{D}[s]\tilde{v}(g)\delta_\ell f^{s+1}}$$

coincides with $\mathcal{N}_\ell$. *Moreover, if* $(Z, 0)$ *does not have any irreducible smooth component, the equality is verified if* v *is not tangent to* $(Sing(Z), 0)$.

Given $\delta \in \mathcal{R}$, the $\mathcal{D}[s]$-module $\mathcal{N}_\delta$ coincides with $\mathcal{N}_\ell$, $\ell \in \mathbf{N}^*$, if and only if the following identities are verified:

$$(\dagger) \qquad \mathcal{D}[s]\delta f^{s+1} = \mathcal{D}[s]\mathcal{J}\delta_\ell f^{s+1}$$

$$(\ddagger) \qquad \mathcal{D}[s](s+1)\delta f^s + \mathcal{D}[s]\delta f^{s+1} = \mathcal{D}[s](\mathrm{jac}(g), g)\delta_\ell f^{s+1}$$

In order to prove the theorem, we will check that these identities are verified in any case.

Proof of Theorem 2.1, *case* i). The equality (†) results from the following identities:

$$(2) \qquad g\delta_{\ell+1} f^{s+1} = -\ell\delta_\ell f^{s+1}$$

$$(3) \qquad (f'_{x_j} g'_{x_i} - f'_{x_i} g'_{x_j})\delta_{\ell+1} f^{s+1} = \Big(f'_{x_j}\frac{\partial}{\partial x_i} - f'_{x_i}\frac{\partial}{\partial x_j}\Big)\delta_\ell f^{s+1}$$

So let r be an index such that f'_{x_r} is a unit. From the identities:

$$(s+1)\delta_\ell f^s = (f'_{x_r})^{-1}\frac{\partial}{\partial x_r}\delta_\ell f^{s+1} - (f'_{x_r})^{-1} g'_{x_r}\delta_{\ell+1} f^{s+1}$$

and (†), we deduce:

$$\mathcal{D}[s](s+1)\delta_\ell f^s + \mathcal{D}[s]\delta_\ell f^{s+1} = \mathcal{D}[s](g'_{x_r}, \mathcal{J})\delta_{\ell+1} f^{s+1}.$$

Thus (‡) is verified since the ideal $(g'_{x_r}, \{g'_{x_i} f'_{x_r} - g'_{x_r} f'_{x_i}\}_{i \neq r})\mathcal{O}$ coincides with $\mathrm{jac}(g)$. □

Proof of Theorem 2.1, *first part of* ii). Let $v \in \mathcal{D}$ be a regular vector field such that v annihilates f and is not tangent to $(Z, 0)$. Up to a change of coordinates, we may assume that $f = x_1$ and $v = \partial/\partial x_2$ (in particular $\mathcal{J} = (g'_{x_2}, \dots, g'_{x_n}, g)\mathcal{O}$). In algebraic terms, the geometrical assumption on v is: $g \notin (x_1, x_3, \dots, x_n)\mathcal{O}$. In other words, there exists $N \in \mathbf{N}^*$ such that $v^N(g)$ is a unit.

First we prove that the inclusion $\mathcal{D}[s]v(g)\delta_\ell {x_1}^{s+1} \subset \mathcal{D}[s]\mathcal{J}\delta_\ell {x_1}^{s+1}$ is an equality. It is enough to see that the ideal $I = \mathcal{D}[s]v(g) + \mathrm{Ann}_{\mathcal{D}[s]}\,\delta_\ell x_1^{s+1}$ contains $g'_{x_3},\dots,g'_{x_n}$ and g. Since the operators $(\partial/\partial x_i)v(g) - vg'_{x_i}$, $3 \le i \le n$, and $vg + (\ell-1)v(g)$ annihilate $\delta_\ell x_1^{s+1}$, then vg, $vg'_{x_3},\dots,vg'_{x_n} \in I$. So we have g, $g'_{x_3},\dots,g'_{x_n} \in I$ by using the following lemma. Thus (†) is true.

Lemma 2.2. *Let $\vartheta \in \mathcal{D}$ be a vector field and $h \in \mathcal{O}$ a nonzero germ such that $\vartheta^N(h)$ is a unit for a non negative integer $N \in \mathbf{N}^*$.*

Then, for every $a, c \in \mathcal{O}[s]$, the ideal $\mathcal{D}[s](\vartheta + c)a + \mathcal{D}[s]ha$ contains a.

Proof. It is enough to prove that $\vartheta^k(h)a$, $k \in \mathbf{N}^*$, belong to the given ideal. This may be done by induction, using the identities: $\vartheta a h - h(\vartheta + c)a = \vartheta(h)a - cah$ and $\vartheta\vartheta^k(h)a - \vartheta^k(h)(\vartheta + c)a = \vartheta^{k+1}(h)a - \vartheta^k(h)ca$, $k \in \mathbf{N}^*$. □

Let us prove (‡) for $\delta = v(g)\delta_\ell$. Since $\mathcal{D}[s]v(g)\delta_\ell x_1^{s+1}$ coincides with $\mathcal{D}[s]\mathcal{J}\delta_\ell x_1^{s+1}$, and using the equality:

$$(4)\quad (s+1)v(g)\delta_\ell x_1^s = \Big(v(g)\frac{\partial}{\partial x_1} - g'_{x_1}v\Big)\delta_\ell x_1^{s+1} = \Big(\frac{\partial}{\partial x_1}v(g) - vg'_{x_1}\Big)\delta_\ell x_1^{s+1}$$

it is enough to remark that g'_{x_1} belongs to $\mathcal{D}(v(g), vg'_{x_1})$. But this is a consequence of Lemma 2.2. Then (‡) is verified. □

Proof of Theorem 2.1, *first part of* iii). Let $\tilde{v}$ be the vector field $x_1(\partial/\partial x_1) + v$ where $v \in \mathbf{C}\{x_2,\dots,x_n\}\langle\partial/\partial x_2,\dots,\partial/\partial x_n\rangle$ is regular and such that $v^N(g)$ is a unit for a non negative integer $N \in \mathbf{N}^*$. From the case *ii)*, the $\mathcal{D}$-module $\mathcal{D}[s]v(g)\delta_\ell x_1^{s+1}$ coincides with $\mathcal{D}[s]\mathcal{J}\delta_\ell x_1^{s+1}$. So, to prove (†), we just have to remark that $x_1 g'_{x_1}\delta_\ell x_1^{s+1}$ belongs to $\mathcal{D}[s]\tilde{v}(g)\delta_\ell x_1^{s+1}$ and to $\mathcal{D}[s]\mathcal{J}\delta_\ell x_1^{s+1}$. First, it is easy to check that if $v^N(g)$ is a unit, then $\tilde{v}^N(g)$ is a unit too. Moreover, identity (4) implies that $(\tilde{v} - (s+1))x_1 g'_{x_1}$ (resp. $vx_1g'_{x_1}$) belongs to $\tilde{I} = \mathcal{D}[s]\tilde{v}(g) + \mathrm{Ann}_{\mathcal{D}[s]}\,\delta_\ell x_1^{s+1}$ (resp. $I = \mathcal{D}[s]v(g) + \mathrm{Ann}_{\mathcal{D}[s]}\,\delta_\ell x_1^{s+1}$). Thus the germ $x_1 g'_{x_1}$ belongs to I and to $\tilde{I}$ *i.e.* $x_1 g'_{x_1}\delta_\ell x_1^{s+1} \in \mathcal{D}[s]\tilde{v}(g)\delta_\ell f^{s+1}$ and $x_1 g'_{x_1}\delta_\ell x_1^{s+1} \in \mathcal{D}[s]\mathcal{J}\delta_\ell x_1^{s+1}$.

The proof of (‡) for $\delta = \tilde{v}(g)\delta_\ell x_1^{s+1}$ is similar to the one of the previous case, using the identity:

$$(s+1)\tilde{v}(g)\delta_\ell x_1^s = \Big(\frac{\partial}{\partial x_1}v(g) + (s+1-v)g'_{x_1}\Big)\delta_\ell x_1^{s+1}.$$

□

Remark 2.3. In the last case, we also prove that $\mathcal{D}[s](\mathrm{jac(g)}, g)\delta_\ell f^{s+2}$ is contained in $\mathcal{D}[s]\mathcal{J}\delta_\ell f^{s+1}$.

Proof of Theorem 2.1, *second part of* ii) *and* iii). We are going to prove that the equalities (†) and (‡) are true for every regular vector field v or $\tilde{v} = x_1(\partial/\partial x_1)+v$, where v is not tangent to the singular set of $(Z, 0)$ and fulfils the conditions of the exposition. Let us take some coordinates such that $f = x_1$ and $v = \partial/\partial x_2$. Thus the geometrical assumption on v means that there is at least one monomial x_2^N or $x_2^N x_i$, $i \geq 3$, in the Taylor expansion of $g|_{x_1=0} \in \mathbf{C}\{x_2, \dots, x_n\}$.

We start with the case $\delta = v(g)\delta_\ell f^{s+1}$. Under our assumption, there exists an integer $N \in \mathbf{N}^*$ such that $v^N(g) = l + h$ where l is a linear form, nonzero and not proportional to x_1, and $h \in (x_1, \dots, x_n)^2\mathcal{O}$. Let us remark that if l depends of the variable x_2, $v^{N+1}(g)$ is a unit and v is not tangent to $(Z, 0)$. Without loss of generality, we can also suppose that $n \geq 3$, $l = x_3$ and that there is no monomial of the form $x_2^{N'}$ in the Taylor expansion of h.

In order to get (†), we will prove that the ideal $I = \mathcal{D}[s]v(g) + \mathrm{Ann}_{\mathcal{D}[s]}\, \delta_\ell x_1^{s+1}$ contains $g'_{x_3}, \dots, g'_{x_n}$ and g (following the proof of the case 'v not tangent to $(Z, 0)$'). We start with the membership of I for g. As above, we have $vg, vg'_{x_3}, \dots, vg'_{x_n} \in I$; so $vgg'_{x_i} - v(g)g'_{x_i} \in I$ and then $vg'_{x_i}g$, $3 \leq i \leq n$, belong to I too. Using that $vg \in I$, we deduce: $v(g'_{x_3})g \in I$. Thus g belongs to the ideal I (Lemma 2.2).

It is more difficult to get the membership of I for $g'_{x_3}, \dots, g'_{x_n}$. Since vg'_{x_i}, $v(g)g'_{x_i} \in I$, we remark - with the help of technics of Lemma 2.2 - that $v^N(g)g'_{x_i}$, $3 \leq i \leq n$, belong to I. Multiplying the operators $(\partial/\partial x_3)g'_{x_i} - (\partial/\partial x_i)g'_{x_3} \in \mathrm{Ann}_\mathcal{D}\, \delta_\ell x_1^{s+1}$ by $v^N(g) = x_3 + h$, we deduce:

$$\text{for } i \neq 1, 3,\ (1 + h'_{x_3})g'_{x_i} - h'_{x_i}g'_{x_3} \text{ belongs to } I \tag{5}$$

Thus the operators $((\partial/\partial x_3)h'_{x_i}(1 + h'_{x_3})^{-1} - \partial/\partial x_i)g'_{x_3}$ belong to the ideal I. Dividing $h'_{x_i}(1 + h'_{x_3})^{-1}$ by $x_3 + h$, we get $((\partial/\partial x_3)\tilde{h}_i - \partial/\partial x_i)g'_{x_3} \in I$ where $\tilde{h}_i \in \mathcal{O}$ does not depend of x_3. Similarly, dividing g by $x_3 + h$, we have $g = q(x_3 + h) + \tilde{g}$, where $\tilde{g} \in \mathcal{O}$ does not depend of x_3, and is not proportional to x_1 because $(Z, 0)$ does not have any smooth irreducible component. Thus $\tilde{g}g'_{x_3}$ belongs to I. So the fact g'_{x_3} belongs to I comes from Lemma 2.2, taking $a = g'_{x_3}$, $h = \tilde{g}$ and $v = \sum_{i\neq 1,3} \lambda_i((\partial/\partial x_3)\tilde{h}_i - \partial/\partial x_i)$, $\lambda_i \in \mathbf{C}$ generic. From (5), we have then $g'_{x_4}, \dots, g'_{x_n} \in I$.

Now we consider (‡). Following the proof of the case *ii)* above, it is enough to remark that the ideal $I' = \mathcal{D}[s](vg'_{x_1}, g'_{x_2}, \dots, g'_{x_n}, g) + \mathrm{Ann}_{\mathcal{D}[s]}\, \delta_\ell x_1^{s+1}$ contains g'_{x_1}. Multiplying vg'_{x_1} by g'_{x_3}, we see that $v(g'_{x_3})g'_{x_1}$ belongs to I'. Then we conclude with Lemma 2.2 (with $h = v(g'_{x_3})$).

In the case $\delta = \tilde{v}(g)\delta_\ell f^s$, we can assume that $f = x_1$, $\tilde{v} = x_1(\partial/\partial x_1) + v$ where $v = \partial/\partial x_2$ and $\tilde{v}^N(g) = x_3 + h$, $h \in (x_1, \dots, x_n)^2\mathcal{O}$. Then the identities

(†) and (‡) may be got similarly, using that the operators $(\tilde{v}-(s+1))g$, $(\tilde{v}-(s+1))g'_{x_2},\dots,(\tilde{v}-(s+1))g'_{x_n}$ belong to the ideal $I=\mathcal{D}[s]\tilde{v}(g)+\mathrm{Ann}_{\mathcal{D}[s]}\,\delta_\ell x_1^{s+1}$. This comes from the identities:

$$(s+1)g\delta_\ell x_1^{s+1}=\Big[\Big(x_1\frac{\partial}{\partial x_1}+\vartheta\Big)g+(\ell-1)(x_1g'_{x_1}+\vartheta(g))\Big]\delta_\ell x_1^{s+1}$$
$$(s+1)\vartheta(g)\delta_\ell x_1^{s+1}=\Big[\Big(x_1\frac{\partial}{\partial x_1}+\vartheta\Big)g+\vartheta(x_1g'_{x_1}+\vartheta(g))\Big]\delta_\ell x_1^{s+1}$$

for every vector field $\vartheta\in\mathbf{C}\{x_2,\dots,x_n\}\langle\partial/\partial x_2,\dots,\partial/\partial x_n\rangle$.

Remark 2.4. From these identities, we deduce the following ones:

$$\mathcal{D}[s]_{\le d}\mathcal{J}\delta_\ell f^{s+1}=\mathcal{D}[s]_{\le d-1}fg'_{x_r}\delta_\ell f^{s+1}+\mathcal{D}\mathcal{J}\delta_\ell f^{s+1}$$
$$\mathcal{D}[s]_{\le d}(\mathrm{jac}(g),g)\delta_\ell f^{s+1}=\mathcal{D}[s]_{\le d}g'_{x_r}\delta_\ell f^{s+1}+\mathcal{D}\mathcal{J}\delta_\ell f^{s+1}$$

for every $d\in\mathbf{N}$, where r is an index such that f'_{x_r} is a unit and $\mathcal{D}[s]_{\le d}\subset\mathcal{D}[s]$ is the subspace of the operators which the degree in s is less or equal to d. This may be done by induction, and using that $fg'_{x_r}\delta_\ell f^{s+1}$ belongs to $\mathcal{D}[s]\mathcal{J}\delta_\ell f^{s+1}$ for every $\ell\in\mathbf{N}^*$ (Remark 2.3).

Remark 2.5. The identity (†) is not always true if $(Z,0)$ has an irreducible smooth component. For example, if $f=x_1$, $g=x_1^2+x_2x_3$, $v=\partial/\partial x_2$ and $\ell=1$, then $\mathcal{D}[s]v(g)+\mathrm{Ann}_{\mathcal{D}[s]}\,\delta_\ell x_1^{s+1}$ is equal to $\mathcal{D}[s](x_1^2,x_3,(\partial/\partial x_2)x_2,s+2-(\partial/\partial x_1)x_1)$, and then it is different from the ideal $\mathcal{D}[s]\mathcal{J}+\mathrm{Ann}_{\mathcal{D}[s]}\,\delta_\ell x_1^{s+1}=\mathcal{D}[s](x_1^2,x_2,x_3,s+2-(\partial/\partial x_1)x_1)$.

§2.2. Some relations between Bernstein polynomials

We start with some relations between the Bernstein polynomials of f associated with some elements of $\mathcal{R}$ and the polynomial $\tilde{b}_\ell(s)$, the minimal polynomial of the action of s on $\mathcal{N}_\ell$.

Corollary 2.6. *Let $f\in\mathcal{O}$ be a germ of a smooth function, and let $g\in\mathcal{O}$ be a germ which is neither a unit nor a multiple of f. Let us denote $(Z,0)\subset(\mathbf{C}^n,0)$, the complete intersection defined by (f,g). Let $\ell\in\mathbf{N}^*$ be a non negative integer.*

i) *The polynomial $\tilde{b}(\delta_\ell f^s,s)$ coincides with $\tilde{b}_{\ell+1}(s)$.*

ii) *Let v be a regular vector field v such that $v(f)=0$. If v is not tangent to $(Z,0)$, then $\tilde{b}(v(g)\delta_\ell f^s,s)$ coincides with $\tilde{b}_\ell(s)$. Moreover, when $(Z,0)$ does not have any irreducible smooth component, the equality is verified if v is not tangent to $(Sing(Z),0)$.*

iii) *Assume that* $f = x_1$. *Let* $v \in \mathbf{C}\{x_2, \ldots, x_n\}\langle \partial/\partial x_2, \ldots, \partial/\partial x_n\rangle$ *be a regular vector field. If* v *is not tangent to* $(Z,0)$, *then* $\tilde{b}((x_1 g'_{x_1} + v(g))\delta_\ell f^s, s)$ *coincides with* $\tilde{b}_\ell(s)$. *Moreover, when* $(Z,0)$ *does not have any smooth component, this equality is true if* v *is not tangent to* $(Sing(Z), 0)$.

iv) *Let* $u \in \mathrm{jac}(g) + g\mathcal{O}$ *be a generator of the* $\mathcal{O}$*-module* $(\mathrm{jac}(g)+g\mathcal{O})/\mathcal{J}$. *Then the polynomial* $b(u\delta_\ell f^s, s)$ *is a multiple of* $\tilde{b}_\ell(s-1)$.

Proof. The first 3 points are easy consequences of Theorem 2.1 and of the fact that $v(g)$ is not divisible by f for every v verifying the requisite conditions. The last point is a consequence of the surjectivity of the following $\mathcal{D}[s]$-linear morphism:

$$\frac{\mathcal{D}[s]u\delta_\ell f^{s+1}}{\mathcal{D}[s]u\delta_\ell f^{s+2}} \longrightarrow \frac{\mathcal{D}[s](\mathrm{jac}(g), g)\delta_\ell f^{s+1}}{\mathcal{D}[s]\mathcal{J}\delta_\ell f^{s+1}}$$

which is well defined from Remark 2.3. □

Hence, for every generic vector field v annihilating f, the polynomial $\tilde{b}(v(g)\delta_\ell, s)$ coincides with $\tilde{b}_\ell(s)$. However, because of *iv)*, this is not true for every regular vector field v.

The following corollary gives a similar result for the classical Bernstein polynomial of a germ of function.

Corollary 2.7. *Let* $h \in \mathcal{O}$ *be a germ neither zero nor a unit. Let us denote* $(\mathcal{H}, 0) \subset (\mathbf{C}^n, 0)$ *the hypersurface defined by* h *and* $\tilde{b}(s) \in \mathbf{C}[s]$ *its reduced Bernstein polynomial.*

Let $v \in \mathcal{D}$ *be a regular vector field. If* v *is not tangent to* $(\mathcal{H}, 0)$, *then the reduced Bernstein polynomial of* $v(h)h^s$ *is equal to* $\tilde{b}(s+1)$. *Moreover, when* $(\mathcal{H}, 0)$ *does not have any smooth component, the equality is true if* v *is not tangent to the singular set of* $(\mathcal{H}, 0)$.

This shifting in the roots of $\tilde{b}(s)$ is very clear in terms of poles of analytic continuation of distributions $\int_{\mathbf{C}^n} |h|^{2\lambda}\varphi$, where φ is a (n,n)-differential form with compact support around the origin, because:

$$\int_{\mathbf{C}^n} v(h)|h|^{2\lambda}\varphi = -\frac{1}{\lambda+1}\int_{\mathbf{C}^n} h|h|^{2\lambda}({}^t v.\varphi)$$

for every vector field v.

In order to prove this corollary, we will use the following result. This is the first explicit example of computation of the polynomials $\tilde{b}_\ell(s)$, $\ell \in \mathbf{N}^*$, and it generalizes a result of [19].

Proposition 2.8. *Let $h \in \mathcal{O}$ be a germ which is neither zero nor a unit. Let us denote $\tilde{b}(s)$ its reduced Bernstein polynomial. Let $N \in \mathbf{N}^*$ be a non negative integer and z a new variable.*

Up to a multiplicative constant, the polynomial $\tilde{b}_\ell(s)$, $\ell \in \mathbf{N}^*$, associated with $f = z$ and $g = h - z^N \in \mathbf{C}\{x, z\}$ is equal to $\tilde{b}(1 - \ell + (s+1)/N)$.

Proof. Without loss of generality, we will prove the result for $\tilde{h} = e^\tau h$, where τ is a new variable. In fact, it does not change the value of the studied Bernstein polynomials.

To prove that $\tilde{b}_\ell(s)$ is a multiple of $\tilde{b}(1 - \ell + (s+1)/N)$, we start with the 'Bernstein identity' of $\tilde{b}_\ell(s)$, *i.e.*:

$$\tilde{b}_\ell(s) z^{N-1} \in \mathcal{D}_{z,\tau}[s](\tilde{h}, \tilde{h}_{x_1}, \dots, \tilde{h}_{x_n}, \tilde{h} - z^N) + \mathrm{Ann}_{\mathcal{D}_{z,\tau}[s]}\, \delta_\ell z^{s+1}$$

where $\mathcal{D}_{z,\tau}$ is the ring of differential operators $\mathbf{C}\{x, z, \tau\}\langle \partial/\partial x, \partial/\partial z, \partial/\partial \tau \rangle$. As the operator $N(\partial/\partial\tau) + z(\partial/\partial z) - s - 1 + N\ell$ annihilates $\delta_\ell z^{s+1}$, this equation may be rewritten:

$$\tilde{b}_\ell\Big(N\frac{\partial}{\partial\tau} + z\frac{\partial}{\partial z} - N + N\ell\Big) z^{N-1} \in \mathcal{D}_{z,\tau}(\tilde{h}, \tilde{h}_{x_1}, \dots, \tilde{h}_{x_n}, z^N) + \mathrm{Ann}_{\mathcal{D}_{z,\tau}}\, \delta_\ell z^{s+1}$$

or:

$$\tilde{b}_\ell\Big(N\frac{\partial}{\partial\tau} - N - 1 + N\ell\Big) z^{N-1} \in \mathcal{D}_{z,\tau}(\tilde{h}, \tilde{h}_{x_1}, \dots, \tilde{h}_{x_n}, z^N) + \mathrm{Ann}_{\mathcal{D}_{z,\tau}}\, \delta_\ell z^{s+1}.$$

Then we remark that $\mathrm{Ann}_{\mathcal{D}_{z,\tau}}\, \delta_\ell z^{s+1}$ is generated by its operators which are not dependant of $\partial/\partial z$. Indeed, if $P = \sum_{i=0}^{d} (\partial/\partial z)^i P_i$ with $P_i \in \widetilde{\mathcal{D}}_{z,\tau} = \mathbf{C}\{x, z, \tau\}\langle \partial/\partial x, \partial/\partial\tau \rangle$ annihilates $\delta_\ell z^{s+1}$, so does $[P, z] = \sum_{i=1}^{d} i(\partial/\partial z)^{i-1} P_i$. So we prove by induction that the operators $P_0, \dots, P_d$ annihilate $\delta_\ell z^{s+1}$. The identity becomes:

$$(6) \quad \tilde{b}_\ell\Big(N\frac{\partial}{\partial\tau} - N - 1 + N\ell\Big) z^{N-1} \in \widetilde{\mathcal{D}}_{z,\tau}(\tilde{h}, \tilde{h}_{x_1}, \dots, \tilde{h}_{x_n}, z^N) + \mathrm{Ann}_{\widetilde{\mathcal{D}}_{z,\tau}}\, \delta_\ell\ .$$

By division, an operator $P \in \mathrm{Ann}_{\widetilde{\mathcal{D}}_{z,\tau}}\, \delta_\ell$ may be written:

$$P = \widetilde{Q}\Big(\frac{\partial}{\partial\tau}(\tilde{h} - z^N) + (\ell - 1)\tilde{h}\Big) + \sum_{i=1}^{n} Q_i \Big(\frac{\partial}{\partial x_i}(\tilde{h} - z^N) + (\ell - 1)\tilde{h}'_{x_i}\Big)$$
$$+\, q(\tilde{h} - z^N)^\ell + \underbrace{R' + \overbrace{\sum_{i=1}^{\ell} r_i (\tilde{h} - z^N)^{\ell - i}}^{r}}_{R}$$

where $R' \in (\partial/\partial x, \partial/\partial\tau)\mathbf{C}\{x,\tau\}\langle\partial/\partial x, \partial/\partial\tau\rangle[z]$ and $r_1, \dots, r_\ell \in \mathbf{C}\{x,\tau\}[z]$ have a degree in z strictly less than N, and $\widetilde{Q}$, $Q_i \in \widetilde{\mathcal{D}}_{z,\tau}$, $q \in \mathbf{C}\{x,z,\tau\}$. So we have:

$$R\frac{1}{(\tilde{h}-z^N)^\ell} = \sum_{i=1}^{d}(-1)^i\frac{(\ell+i-1)!}{(\ell-1)!}\frac{r_i'}{(\tilde{h}-z^N)^{\ell+i}} + \sum_{i=1}^{\ell}\frac{r_i}{(\tilde{h}-z^N)^i}$$

and

$$R\tilde{h}^s = \sum_{i=1}^{d} s(s-1)\cdots(s-i+1)\frac{r_i'}{h^i}h^s + rh^s$$

where $d = \deg R$ and $r_i' \in \mathbf{C}\{x,\tau\}[z]$ has a degree in z strictly less than N. As R annihilates δ_ℓ, all the germs r_i and r_i' are necessarily equal to zero, and then R annihilates $\tilde{h}^s$. Hence (6) implies that:

$$\tilde{b}_\ell\Big(N\frac{\partial}{\partial\tau} - N - 1 + N\ell\Big)z^{N-1} \in \widetilde{\mathcal{D}}_{z,\tau}(\tilde{h}, \tilde{h}_{x_1}, \dots, \tilde{h}_{x_n}, z^N) + \widetilde{\mathcal{D}}_{z,\tau}\mathrm{Ann}_{\mathcal{D}_\tau}\tilde{h}^s$$

where $\mathcal{D}_\tau = \mathbf{C}\{x,\tau\}\langle\partial/\partial x, \partial/\partial\tau\rangle$. Consequently, $\tilde{b}_\ell(N(\partial/\partial\tau) - N - 1 + N\ell)$ belongs to the ideal $\mathcal{D}_\tau(\tilde{h}, \tilde{h}_{x_1}, \dots, \tilde{h}_{x_n}) + \mathrm{Ann}_{\mathcal{D}_\tau}\tilde{h}^s$ *i.e.* $\tilde{b}_\ell(Ns - N - 1 + N\ell)$ is definitely a multiple of $\tilde{b}(s)$.

The proof of the converse relation is similar (see [19]). □

Proof of Corollary 2.7. By similar computations, we prove easily that the polynomial $b(\dot{a}/(h-z)z^s, s)$ coincides with the Bernstein polynomial of ah^s. So the assertion is a direct consequence of Corollary 2.6 and Proposition 2.8. □

We end with a relation between the Bernstein polynomial of f associated with some particular element of $\mathcal{O}[1/g]$ and of $\mathcal{R} = \mathcal{O}[1/g]/\mathcal{O}$. From the point of view of the monodromy, it is very clear (because $\Phi_f(\mathcal{O})$ is zero when f is smooth).

Proposition 2.9. *Let $f \in \mathcal{O}$ be a germ of a smooth function, and $g \in \mathcal{O}$ a germ which is neither a unit nor a multiple of f.*

For every $\ell \in \mathbf{N}^*$, the Bernstein polynomial of $(1/g^\ell)f^s$ coincides with $b(\delta_\ell f^s, s)$.

Proof. We just prove that the Bernstein polynomial of $(1/g^\ell)f^s \in \mathcal{O}[1/fg, s]f^s$, denoted by $b((1/g^\ell)f^s, s)$, is a factor of $b(\delta_\ell f^s, s)$ (the converse relation is evident). Let $R \in \mathcal{D}[s]$ be an operator realizing the functional equation of $\delta_\ell f^s$: $b(\delta_\ell f^s, s)\delta_\ell f^s = R\delta_\ell f^{s+1}$. So there are an integer $d \in \mathbf{Z}$ and

$a \in \mathcal{O}[s]$, $a \notin f\mathcal{O}[s] - \{0\}$, such that:

$$b(\delta f^s, s)\frac{1}{g^\ell}f^s = R\frac{1}{g^\ell}f^{s+1} + af^{s+d} \tag{7}$$

in $\mathcal{O}[1/fg, s]f^s$. If a is zero, $b((1/g^\ell)f^s, s)$ divides definitely $b(\delta_\ell f^s, s)$. Otherwise, let us prove that af^{s+d} belongs to $\mathcal{D}[s]f^{s+1}$. If $d \geq 1$, it is trivial. So we suppose that $d \leq 0$. By specializations of s in $-1, 0, \ldots, -d-1$, we remark that $(s+1)s\cdots(s+d+1)$ is a factor of a. Hence af^{s+d} belongs to $\mathcal{D}[s]f^{s+1}$, because:

$$\left[(f'_{x_r})^{-1}\left(\frac{\partial}{\partial x_r}\right)\right]^{-d+1} f^{s+1} = (s+1)\cdots(s+d+1)f^{s+d}$$

where r is an index such that f'_{x_r} is a unit. So the equation (7) implies that $b(\delta_\ell f^s, s)(1/g^\ell)f^s \in \mathcal{D}[s](1/g^\ell)f^{s+1}$, and our assertion is proved. □

§3. The Case of Isolated Singularities

In this part, the germ $g \in \mathcal{O}$ defines an isolated singularity, and $f \in \mathcal{O}$ is a germ of smooth function such that $f(0) = 0$ and (f, g) defines a complete intersection isolated singularity.

Following [2], [21], we give an explicit description of $H^n_{DR}(\mathcal{N}_\ell)$ in order to study the polynomials $\tilde{b}_\ell(s)$ (Theorem 1.1). So we introduce the $\mathcal{D}$-module $\sum_{k\geq 1}\mathcal{D}\delta_k f^{s+1}$.

§3.1. A suitable $\mathcal{D}$-module

First, we remark that for every $\ell \in \mathbf{N}^*$, the $\mathcal{D}[s]$-module $\mathcal{D}[s]\delta_\ell f^{s+1}$ is a submodule of $\sum_{k\geq 1}\mathcal{D}\delta_k f^{s+1}$. This comes from the identities:

$$(s+2)\delta_k f^{s+1} = (f'_{x_r})^{-1}\frac{\partial}{\partial x_r}f\delta_k f^{s+1} - (f'_{x_r})^{-1}g'_{x_r}f\delta_{k+1}f^{s+1},\ k \in \mathbf{N}^* \tag{8}$$

where r is an index such that the germ f'_{x_r} is a unit. Indeed, the $\mathcal{D}$-module $\sum_{k\geq 1}\mathcal{D}\delta_k f^{s+1}$ coincides with $\sum_{k\geq 1}\sum_{i\geq 0}\mathcal{D}\delta_k\xi_i \subset \mathcal{R}[1/f, s]f^{s+1}$, where $\delta_k\xi_i$ is the element $(s-i+2)\cdots(s+1)\delta_k f^{s-i+1}$, because:

$$\delta_k\xi_i = (f'_{x_r})^{-1}\frac{\partial}{\partial x_r}\delta_k\xi_{i-1} - (f'_{x_r})^{-1}g'_{x_r}\delta_{k+1}\xi_{i-1},\ k \in \mathbf{N}^*$$

for $i \in \mathbf{N}$.

We give now some results about the $\mathcal{D}$-module $\sum_{k\geq 1}\mathcal{D}\delta_k f^{s+1}$.

Lemma 3.1. *For every non negative integer $\ell \in \mathbf{N}^*$, the $\mathcal{D}$-module*:

$$\frac{\sum_{k\geq 1} \mathcal{D}\delta_k f^{s+1}}{\mathcal{D}\mathcal{J}\delta_\ell f^{s+1}}$$

is supported by the origin.

Proof. Under our assumptions, the ideal $\mathcal{J}$ defines zero (see its definition page 798). So we have to prove that for every $P \in \mathcal{D}$ and every non negative integer $k \geq \ell$, there is an integer $m \in \mathbf{N}^*$ such that $hP\delta_k f^{s+1}$ belongs to $\mathcal{D}\mathcal{J}\delta_\ell f^{s+1}$ for every $h \in \mathcal{J}^m$. This may be done by induction on $k - \ell \in \mathbf{N}$ and on the degree d of the operator P, using that $hP \in \mathcal{D}\mathcal{J}$ for $h \in \mathcal{J}^{d+1}$ and that $u\delta_k f^{s+1} \in \mathcal{D}\delta_{k-1} f^{s+1}$ for $u \in \mathcal{J}$ (with the help of identities (2) & (3), page 801). □

Let E be a $\mathbf{C}$-vector subspace of $\mathcal{O}$ isomorphic to $\mathcal{O}/\mathcal{J}$ by projection, $D \subset \mathcal{D}$ the ring of differential operators with constant coefficients, $DE \subset \mathcal{D}$ the subspace generated by $\partial^\beta e$, $e \in E$, and $\mathcal{D}\mathcal{J} \subset \mathcal{D}$ the left ideal generated by $\mathcal{J}$.

Proposition 3.2. *For every $\ell \in \mathbf{N}^*$, there is a decomposition*:

$$\sum_{k\geq 1} \mathcal{D}\delta_k f^{s+1} = \mathcal{D}\mathcal{J}\delta_\ell f^{s+1} \oplus \Big(\bigoplus_{k\geq \ell} DE\delta_k f^{s+1}\Big)$$

Proof. First remark that the $\mathcal{D}$-modules $\mathcal{D}\delta_k f^{s+1}$, $1 \leq k \leq \ell - 1$, are contained in $\mathcal{D}\mathcal{J}\delta_\ell f^{s+1}$ (since $g \in \mathcal{J}$). So, to get the existence of the decomposition, it is enough to prove it only for the elements $u\delta_k f^{s+1}$, $u \in \mathcal{O}$, $k \geq \ell$. By division by $\mathcal{J}$, there exists a uniquely defined element $e \in E$, and $h, \lambda_{i,j} \in \mathcal{O}$, $1 \leq i < j \leq n$ such that $u = e + hg + \sum_{i<j} \lambda_{i,j}(f'_{x_j} g'_{x_i} - f'_{x_i} g'_{x_j})$. Hence we have:

$$\begin{aligned} u\delta_k f^{s+1} &= e\delta_k f^{s+1} - (k-1)h\delta_{k-1} f^{s+1} \\ &\quad + \left[\sum_{i<j} \Big(\frac{\partial}{\partial x_i} f'_{x_j} - \frac{\partial}{\partial x_j} f'_{x_i}\Big)\lambda_{i,j} - \Big(f'_{x_j}\frac{\partial \lambda_{i,j}}{\partial x_i} - f'_{x_i}\frac{\partial \lambda_{i,j}}{\partial x_j}\Big)\right] \delta_{k-1} f^{s+1} \end{aligned}$$

for $k \geq \ell + 1$. So, by induction on k, every element of $\sum_{k\geq 1} \mathcal{D}\delta_k f^{s+1}$ may be decomposed in $\mathcal{D}\mathcal{J}\delta_\ell f^{s+1} \oplus (\bigoplus_{k\geq\ell} DE\delta_k f^{s+1})$.

The proof of the uniqueness uses that the ideals $\mathrm{Ann}_{\mathcal{D}}\, \delta_k f^{s+1}$, $k \in \mathbf{N}^*$, are contained in $\mathcal{D}\mathcal{J}$ (see [19], [21]). Suppose that $V\delta_\ell f^{s+1} + \sum_{k=\ell}^{L} U_k \delta_k f^{s+1} = 0$

with $V \in \mathcal{DJ}$ and $U_k \in DE$. This may be written:

$$\left[(-1)^{L+\ell}\frac{(\ell-1)!}{(L-1)!}Vg^{L-\ell}+U_L+\sum_{k=\ell}^{L-1}(-1)^{L+k}\frac{(k-1)!}{(L-1)!}U_kg^{L-k}\right]\delta_Lf^{s+1}=0$$

As $\mathrm{Ann}_{\mathcal{D}}\,\delta_Lf^{s+1}\subset\mathcal{DJ}$, the operator U_L belongs to DE and to $\mathcal{DJ}$ in the same time, and so it is zero. By induction, we prove that U_k, $\ell\le k\le L-1$, are zero too, and then $V\delta_\ell f^{s+1}=0$. Consequently, we get the assertion. □

Let $D'\subset D$ be the ideal of operators without nonzero constant term. Given $\kappa\in\mathbf{N}^*$, we consider the linear morphism:

$$c_\kappa:\sum_{k\ge1}\mathcal{D}\delta_kf^{s+1}=\mathcal{DJ}\delta_\kappa f^{s+1}\oplus\Big(\bigoplus_{k\ge\kappa}DE\delta_kf^{s+1}\Big)\longrightarrow\bigoplus_{k\ge\kappa}E\delta_kf^{s+1}$$

defined by $c_\kappa(\mathcal{DJ}\delta_\kappa f^{s+1})=0$ and if $Q=Q'+e$ with $Q'\in D'E$, $e\in E$, then $c_\kappa(Q\delta_kf^{s+1})=e\delta_kf^{s+1}$ for every $k\ge\kappa$. Its kernel is $\mathcal{DJ}\delta_\kappa f^{s+1}\oplus(\bigoplus_{k\ge\kappa}D'E\delta_kf^{s+1})$. So we have the inclusion: $\bigoplus_{k\ge1}D'\mathcal{O}\delta_kf^{s+1}\subset\ker c_\kappa$. Hence c_κ induces an isomorphism:

$$\bar{c}_\kappa:H^n_{DR}\Big(\frac{\sum_{k\ge1}\mathcal{D}\delta_kf^{s+1}}{\mathcal{DJ}\delta_\kappa f^{s+1}}\Big)\longrightarrow\bigoplus_{k\ge\kappa}E\delta_kf^{s+1}. \tag{9}$$

§3.2. The spaces $\mathcal{Z}_\ell$, $\mathcal{Z}'_\ell$ and the polynomial $\tilde{b}_\ell(s)$

Given $\ell\in\mathbf{N}^*$, let us denote $\mathcal{Z}'_\ell=c_\ell(\mathcal{D}[s](\mathrm{jac}(g),g)\delta_\ell f^{s+1})$ and $\mathcal{Z}_\ell=c_\ell(\mathcal{D}[s]\mathcal{J}\delta_\ell f^{s+1})\subset\mathcal{Z}'_\ell$. Now we give some general results on these $\mathbf{C}$-vector spaces.

Lemma 3.3. *For every $\ell\in\mathbf{N}^*$, there are the following identifications*:

$$\mathcal{Z}_\ell{}'=c_\ell(\mathcal{D}[s]g'_{x_r}\delta_\ell f^{s+1}),\ \mathcal{Z}_\ell=c_\ell(\mathcal{D}[s]fg'_{x_r}\delta_\ell f^{s+1})$$

where r is an index such that f'_{x_r} is a unit.

It is a consequence of Remark 2.4.

Proposition 3.4. *For every $\ell\in\mathbf{N}^*$, the dimensions of the spaces $\mathcal{Z}_\ell$ and $\mathcal{Z}'_\ell$ are finite.*

Proof. From regularity of the holonomic $\mathcal{D}$-module $\mathcal{R}$, there exist good operators in s in the annihilator of δf^s, $\delta\in\mathcal{R}$, *i.e.* of the form $s^N+P_1s^{N-1}$

$+\cdots+P_N \in \mathcal{D}[s]$ where the degree of $P_i \in \mathcal{D}$ is less or equal to i (see [4], [18]). If N is the degree of such an operator annihilating $\delta_\ell f^{s+1}$, then:

$$\mathcal{D}[s]\delta_\ell f^{s+1} = \sum_{i=0}^{N-1} s^i \mathcal{D}\delta_\ell f^{s+1} \subset \sum_{k=1}^{N+\ell-1} \mathcal{D}\delta_k f^{s+1}$$

(see identity (8)). In particular, the dimension of $c_\ell(\mathcal{D}[s]\delta_\ell f^{s+1})$ is finite, and the one of $\mathcal{Z}'_\ell$, $\mathcal{Z}_\ell$ are finite too. □

Remark that the dimension of $\mathcal{Z}_\ell$, $\mathcal{Z}'_\ell$ and $\mathcal{Z}'_\ell/\mathcal{Z}_\ell$ depends on the integer ℓ (see the example studied in the last part).

Given $\ell \in \mathbf{N}^*$, we define the action of s on $\bigoplus_{k\geq\ell} E\delta_k f^{s+1}$ by $s.U = c_\ell(sU)$. Remark that $c_\ell(sU) \in \mathcal{Z}_\ell$ when $U \in \ker c_\ell$. Indeed, $s\bigoplus_{k\geq\ell} D'E\delta_k f^{s+1}$ is contained in the kernel of c_ℓ. Hence, the action of s on $\bigoplus_{k\geq\ell} E\delta_k f^{s+1}$ is well defined on $\mathcal{Z}_\ell$, $\mathcal{Z}'_\ell$, and then on $\mathcal{Z}'_\ell/\mathcal{Z}_\ell$.

The proof of Theorem 1.1 is the very same as the one of [21], Theorem 1.1. It uses Lemma 3.1, the identification (9) and the fact that the functor H^n_{DR}, from the category of $\mathcal{D}$-modules supported by zero to the category of $\mathbf{C}$-vector spaces, is an exact and faithful functor ([11]).

§4. The Computational Algorithm for Non Degenerate Hypersurfaces

Here we adapt to the case of polynomials $\tilde{b}_\ell(s)$ the algorithm of computation of Bernstein polynomial of a non-degenerate convenient germ with respect to its Newton boundary in the sense of Kouchnirenko (see [2]). We invite the reader to see [2] for the proof of some results which may be easily extended.

§4.1. Division by $\mathcal{J}$ and increase in weight

Let $g \in \mathcal{O}$ be a nonzero germ of an holomorphic function with $g(0) = 0$. Its Taylor expansion is written $\sum_{A\in\mathbf{N}^n} g_A x^A$ where $g_A \in \mathbf{C}$ and $x^A = x_1^{a_1}\cdots x_n^{a_n}$ for $A = (a_1,\dots,a_n) \in \mathbf{N}^n$.

Let $N(g) = \{A \in \mathbf{N}^n \mid g_A \neq 0\}$ be the Newton cloud of g and $\Gamma(g) \subset (\mathbf{R}^+)^n$ its Newton boundary, the union of compact faces of the convex hull of $N(g) + \mathbf{N}^n$. For every face $\Delta \subset \Gamma(g)$ and every $u = \sum_{A\in\mathbf{N}^n} u_A x^A \in \mathcal{O}$, we denote $u|_\Delta = \sum_{A\in\Delta} u_A x^A$ the restriction of u to Δ.

We make the following assumptions on g:

- g is *convenient*: each coordinate line has a point contained in $\Gamma(g)$.
- g is *non-degenerate with respect to its Newton boundary*: for every face $\Delta \subset \Gamma(g)$, the system:

$$\Big(x_1 \frac{\partial g}{\partial x_1}\Big)\Big|_\Delta = \cdots = \Big(x_n \frac{\partial g}{\partial x_n}\Big)\Big|_\Delta = 0$$

does not have any solution in $(\mathbf{C}^*)^n$.

Under these conditions, g defines an isolated singularity. We will suppose that $f = x_1$. In particular, the ideal $\mathcal{J}$ is $(g, g_{x_2}, \ldots, g_{x_n})\mathcal{O}$. Moreover the morphism (x_1, g) defines a isolated singularity too, because the restriction of g to $x_1 = 0$ is also convenient and non-degenerate.

Remark that the system of equations in the definition of the non-degeneracy condition is equivalent to the following one:

$$g|_\Delta = \Big(x_2 \frac{\partial g}{\partial x_2}\Big)\Big|_\Delta = \cdots = \Big(x_n \frac{\partial g}{\partial x_n}\Big)\Big|_\Delta = 0$$

because $g|_\Delta$ is a weighted-homogeneous polynomial in restriction to every face $\Delta \subset \Gamma(g)$. Let us recall that a nonzero polynomial is *weighted-homogeneous* of weight $d \in \mathbf{Q}^+$ for a system $\alpha \in (\mathbf{Q}^{*+})^n$ if it is a $\mathbf{C}$-linear combination of monomials x^A with $\langle \alpha, A\rangle = d$.

Now we introduce some notations before giving the division theorem by the ideal $\mathcal{J}$ which is adapted to our situation.

Notation 4.1. Let $\mathcal{F}$ be the set of $n-1$ dimensional faces of $\Gamma(g)$. Given $F \in \mathcal{F}$, we consider the vector $\alpha_F = (\alpha_{F,1}, \ldots, \alpha_{F,n}) \in (\mathbf{Q}^{*+})^n$ such that $\langle \alpha_F, A\rangle = 1$ for every $A \in F$. The weight $\rho_F(u)$ in relation to the face $F \in \mathcal{F}$ of a nonzero germ $u = \sum_{A\in\mathbf{N}^n} u_A x^A \in \mathcal{O}$ is also defined by $\rho_F(u) = \inf\{\langle \alpha_F, A\rangle \,|\, u_A \neq 0\} \in \mathbf{Q}^+$. By agreement, we fix $\rho_F(0) = +\infty$. Then we define the weight of a germ $u \in \mathcal{O}$ in relation to $\Gamma(g)$ by $\rho(u) = \inf_{F\in\mathcal{F}} \rho_F(u)$.

For every rational $q \in \mathbf{Q}$, let us denote $\mathcal{O}_{\geq q} = \{u \in \mathcal{O} \,|\, \rho(u) \geq q\}$, $\mathcal{O}_{>q} = \{u \in \mathcal{O} \,|\, \rho(u) > q\}$ and $\operatorname{gr}\mathcal{O} = \bigoplus_{q\in\mathbf{Q}^+} \mathcal{O}_{\geq q}/\mathcal{O}_{>q}$.

We define another weight function, $\rho^* : \mathcal{O} \to \mathbf{Q}^+ \cup \{+\infty\}$, by $\rho^*(u) = \inf_{F\in\mathcal{F}} \rho^*_F(u)$ where $\rho^*_F(u) = \rho_F(ux_2\cdots x_n)$ for every $u \in \mathcal{O}$. As above, we have the spaces $\mathcal{O}^*_{>q}$, $\mathcal{O}^*_{\geq q}$, $q \in \mathbf{Q}$. If $\mathcal{O}^*_q$ is the set of germs $u \in \mathcal{O}$ such that $ux_2\cdots x_n$ is a polynomial supported by $q\Gamma(g)$, then $\operatorname{gr}^*\mathcal{O} = \bigoplus_q \mathcal{O}^*_{\geq q}/\mathcal{O}^*_{>q}$ may be identified to $\bigoplus_q \mathcal{O}^*_q$.

For every $u \in \mathcal{O}$ nonzero, let $\operatorname{in}^*(u)$ be the coset of u in $\mathcal{O}^*_{\geq\rho^*(u)}/\mathcal{O}^*_{>\rho^*(u)}$ identified to $\mathcal{O}^*_{\rho^*(u)}$. For every $q \in \mathbf{Q}^+$, let $E^*_q \subset \mathcal{O}^*_q$ be a supplementary of

$\mathcal{O}_q^* \cap \mathrm{in}^*(\mathcal{J})$ in $\mathcal{O}_q^*$, where $\mathrm{in}^*(\mathcal{J}) \subset \mathbf{C}[x]$ is the ideal generated by the initial parts of the elements of $\mathcal{J}$. Finally, let $E_{\geq q}^* \subset E$ be the space $\bigoplus_{q' \geq q} E_{q'}^*$.

Theorem 4.2. ([2], [9]) *For every $u \in \mathcal{O}$, there exists a unique element $v \in E = \bigoplus_q E_q^*$ and $\lambda_1, \ldots, \lambda_n \in \mathcal{O}$ such that:*

$$u = v + \lambda_1 g + \sum_{i=2}^{n} \lambda_i g'_{x_i}$$

where $\rho^*(v) \geq \rho^*(u)$, $\rho^*(\lambda_1) \geq \rho^*(u) - 1$, and for $2 \leq i \leq n$: $\rho^*(\lambda_i g'_{x_i}) \geq \rho^*(u)$, $\rho^*(\lambda_i) \geq \rho^*(u) - 1 + \rho(x_i)$, $\rho^*(\partial\lambda_i/\partial x_i) \geq \rho^*(u) - 1$.

The proof is a direct adaptation of the one of Proposition B.1.2.2, B.1.2.3, B.1.2.6 of [2], which need Theorems 2.8 and 4.1 of [9]. In particular, the multiplication by $x_2 \cdots x_n$ induces a strict isomorphism λ from $(\mathcal{O}/\mathcal{J}, \rho^*)$ to $(\mathcal{O}x_2 \cdots x_n/\mathcal{O}x_2 \cdots x_n \cap I(g), \rho)$ where $I(g) = (g, x_2 g'_{x_2}, \ldots, x_n g'_{x_n})\mathcal{O}$.

Indeed, these Kouchnirenko results are true for every non-degenerate family $h_1, \ldots, h_n \in \mathcal{O}$, i.e. satisfying the non-degeneracy condition and such that $\rho(h_i) = 1$ for $1 \leq i \leq n$. In particular, the family $\{g, x_2 g'_{x_2}, \ldots, x_n g'_{x_n}\}$ is non-degenerate.

Let us denote $\Pi^* = \{q \in \mathbf{Q}^+ \mid E_q^* \neq 0\}$ and $\sigma^* = \sup\{q \mid E_q^* \neq 0\}$. Rewriting [2, p. 566], we get:

$$n - \sup_{F \in \mathcal{F}} \rho_F(x_1 \cdots x_n) \leq \sigma^* < n$$

The estimation is obtained by using the Rees function $\overline{\nu}_{I(g)}$, which coincides with the weight function ρ under our assumptions ([3], [17]).

We end by giving the technical lemmas at the root of the algorithm. First we give a filtered version of Proposition 3.2.

Lemma 4.3. *Given $N, \ell \in \mathbf{N}^*$, $q \in \mathbf{Q}$, there is the following identity in $\sum_{k \geq 1} \mathcal{D}\delta_k x_1^{s+1}$:*

$$\sum_{k=1}^{N} \mathcal{D}\mathcal{O}_{\geq q+k}^* \delta_k x_1^{s+1} = \mathcal{D}\mathcal{J}_{\geq q+\ell} \delta_\ell x_1^{s+1} \oplus \bigoplus_{k=\ell}^{N} DE_{\geq q+k}^* \delta_k x_1^{s+1}$$

where $\mathcal{J}_{\geq q+\ell} = \mathcal{J} \cap \mathcal{O}_{\geq q+\ell}^*$.

For every face $F \in \mathcal{F}$, let us denote $|\alpha_F| \in \mathbf{Q}^{*+}$ the sum $\sum_{i=1}^n \alpha_{F,i}$, $\chi_F = \sum_{i=1}^n \alpha_{F,i} x_i (\partial/\partial x_i)$ the Euler vector field associated with F, $\overline{\chi}_F = \sum_{i=1}^n \alpha_{F,i} (\partial/\partial x_i) x_i = \chi_F + |\alpha_F|$ and $h_F = \chi_F(g) - g \in \mathcal{O}$.

Lemma 4.4. *Given $w \in \mathbf{C}$, $F \in \mathcal{F}$, $u \in \mathcal{O}$ and $k \in \mathbf{N}^*$, there is an identity*:

$$(\alpha_{F,1}(s+1) + |\alpha_F| + w)u\delta_k x_1^{s+1} = [\overline{\chi}_F u + [(w+k)u - \chi_F(u)]] \cdot \delta_k x_1^{s+1} - uh_F\delta_{k+1}x_1^{s+1}$$

and the following identities, for every $F' \in \mathcal{F}$:

$$\rho^*_{F'}(x_j u) > \rho^*(u),\ \rho^*_{F'}((w+k)u - \chi_F(u)) \geq \rho^*(u),\ \rho^*_{F'}(uh_F) \geq \rho^*(u) + 1$$

If $F' = F$, then $\rho^*_F(uh_F) > \rho^*(u) + 1$. Moreover, if $\rho^*_F(u) > \rho^*(u)$ or $\rho^*_F(u) = \rho^*(u) = w + k + |\alpha_F| - \alpha_{F,1}$, then $\rho^*_F((w+k)u - \chi_F(u)) > \rho^*(u)$.

For every monomial u, let $\mathcal{F}^*(u) \subset \mathcal{F}$ be the set of the faces F with $\rho^*_F(u) = \rho^*(u)$; if $u \in \mathcal{O}$ is nonzero, then $\mathcal{F}^*(u) \subset \mathcal{F}$ is the set of $F \in \mathcal{F}$ such that there exists a monomial v in $\mathrm{in}^*(u)$ with $\rho^*_F(v) = \rho^*(u)$. Using Lemma 4.4, we get the following formula:

Lemma 4.5. *For every $u \in \mathcal{O}$ nonzero and $k \in \mathbf{N}^*$*:

$$\left[\prod_{F \in \mathcal{F}^*(u)} (\alpha_{F,1}(s+2) + \rho^*(u) - k)\right] u\delta_k x_1^{s+1} \in \sum_{i=0}^{\#\mathcal{F}^*(u)} \mathcal{D}\mathcal{O}^*_{>\rho^*(u)+i}\delta_{k+i}x_1^{s+1}$$

Remark that the multiplicity of a factor $(\alpha_{F,1}(s+2) + \rho^*(u) - k)$ in the given polynomial may be arbitrarily high. The next result states the existence of a polynomial such that the multiplicities are strictly smaller than n.

Proposition 4.6. *Let $u \in \mathcal{O}$ nonzero and $k \in \mathbf{N}^*$. Let $\mathcal{A}^*(u) \subset \mathbf{Q}^{*+}$ be the set of $\alpha_{F,1}$ with $F \in \mathcal{F}^*(u)$. Then*:

$$\left[\prod_{a \in \mathcal{A}^*(u)} (a(s+2) + \rho^*(u) - k)\right]^{n-1} u\delta_k x_1^{s+1} \in \sum_{i=0}^{(n-1)\times\#\mathcal{A}^*(u)} \mathcal{D}\mathcal{O}^*_{>\rho^*(u)+i}\delta_{k+i}x_1^{s+1}$$

We prove this result in the next paragraph.

§4.2. Proof of Proposition 4.6

We need some additional notations.

Let us attach to any face $F \in \mathcal{F}$ the closed cone $C(F) \subset (\mathbf{R}^+)^n$, the union of linear half-lines going through F. In particular, $A \in (\mathbf{R}^+)^n$ belongs to $C(F)$ if and only if $\inf_{F' \in \mathcal{F}} \langle \alpha_{F'}, A \rangle = \langle \alpha_F, A \rangle$. Let us denote $\mathcal{C}$ the fan with support in $(\mathbf{R}^+)^n$ associated with the Newton boundary $\Gamma(g)$. We recall that it is the smallest family of convex polyhedral rational convex cones of $(\mathbf{R}^+)^n$ which contains the cones $C(F)$, $F \in \mathcal{F}$, and verifies the conditions:

- if C is a facet of a cone of $\mathcal{C}$ then $C \in \mathcal{C}$;
- if $C_1, C_2 \in \mathcal{C}$, then $C_1 \cap C_2$ is a facet of C_1 and C_2.

For every $A \in (\mathbf{R}^+)^n$ nonzero, we note $C(A) \in \mathcal{C}$ the cone of smallest dimension which contains A, and $d(A) \in \mathbf{N}$ its dimension. In particular, we have $1 \leq d(A) \leq n$ and $d(A) = n$ if and only if A belongs to the interior of a cone $C(F)$.

The proof of the proposition uses the following elementary results.

Lemma 4.7. *Let $F \in \mathcal{F}$ and let $A, A' \in C(F)$ be two nonzero vectors such that $A' \notin C(A)$. Then $A, A' \in C(A + A')$ and so $d(A + A') \geq d(A) + 1$.*

Lemma 4.8. *Let $F_1, \ldots, F_m \in \mathcal{F}$ be faces such that $\alpha_{F_1,1}, \ldots, \alpha_{F_m,1}$ are equal. Let $A \in (\mathbf{R}^+)^n$ be a vector belonging to the cone $C(F_1, \ldots, F_m) = C(F_1) \cap \cdots \cap C(F_m)$ and such that $\inf_{F \mid A \in C(F)} \alpha_{F,1} = \alpha_{F_1,1}$. Then, for every $\epsilon \in \mathbf{R}^{*+}$ small enough, the vector $A + \epsilon(1, 0, \ldots, 0)$ belongs to $C(F_1, \ldots, F_m)$.*

Proof of Proposition 4.6. Without loss of generality, we assume that u is a monomial; we denote $A \in \mathbf{N} \times (\mathbf{N}^*)^{n-1}$ the n-uplet such that $ux_2 \cdots x_n$ is $\mathbf{C}$-proportional to x^A.

Let $F_1 \in \mathcal{F}^*(u)$. Using Lemma 4.4, we have:

$$(\alpha_{F_1,1}(s+2) + \rho^*(u) - k) u \delta_k x_1^{s+1} = \overline{\chi}_{F_1} \cdot u \delta_k x_1^{s+1} - u h_{F_1} \delta_{k+1} x_1^{s+1}$$

where $\overline{\chi}_{F_1} \cdot u \delta_k x_1^{s+1} \in \mathcal{D}\mathcal{O}^*_{>\rho^*(u)} \delta_k x_1^{s+1}$. If $w_1 = x^{A_1'}$ is a monomial of the Taylor expansion of h_{F_1}, then two cases are possible:

- First case: $\rho^*(uw_1) > \rho^*(u) + 1$. Then $uw_1 \delta_{k+1} x_1^{s+1} \in \mathcal{O}^*_{>\rho^*(u)+1} \delta_{k+1} x_1^{s+1}$.
- Second case: $\rho^*(uw_1) = \rho^*(u) + 1$. As $\rho_F(h_{F_1}) \geq 1$ with an equality if and only if $F \neq F_1$, we have also $\mathcal{F}^*(uw_1) = \{F \in \mathcal{F}^*(u) \mid A_1' \in F\}$ and this set does not contain F_1. From Lemma 4.7 applied with $A \in C(F_1) \cap C(F_2)$, $A' = A_1' \in C(F_2) - C(F_1)$ for $F_2 \in \mathcal{F}^*(uw_1)$, we get $d(A + A_1') \geq d(A) + 1$.

Hence, up to an element of the $\mathcal{D}$-module $\sum_{i=0}^{1}\mathcal{D}\mathcal{O}^*_{>\rho^*(u)+i}\delta_{k+i}x_1^{s+1}$, the element $(\alpha_{F_1,1}(s+2)+\rho^*(u)-k)u\delta_k x_1^{s+1}$ is equal to a $\mathbf{C}$-linear finite combination of terms $uw_1\delta_{k+1}x_1^{s+1}$ with weight $\rho^*(u)-k$ such that $\mathcal{F}^*(uw_1)\subset\mathcal{F}^*(u)-\{F_1\}$ and $d(A+A_1')\geq 2$ if $w_1ux_2\cdots x_n = x^{A+A_1'}$.

Remark that if $d(A+A')=n$ then $\mathcal{F}^*(uw)$ has necessarily one element. So, when a polynomial $c(s)\in\mathbf{C}[s]$ allows to use n times this process, we prove that $c(s)u\delta_k x_1^{s+1}$ belongs to $\mathcal{D}[s]_{\leq \deg c(s)-n}\sum_{i=0}^{n}\mathcal{D}\mathcal{O}^*_{>\rho^*(u)+i}\delta_{k+i}x_1^{s+1}$ then to $\sum_{i=0}^{\deg c(s)}\mathcal{D}\mathcal{O}^*_{>\rho^*(u)+i}\delta_{k+i}x_1^{s+1}$ (Lemma 4.4). In particular, the polynomial $\left[\prod_{a\in\mathcal{A}^*(u)}(a(s+2)+\rho^*(u)-k)\right]^n$ is suitable. We will prove that the power $n-1$ is sufficient.

It is easy to see that it is true if $d(A)\geq 2$. Remark that it is again true when there exists $a\in\mathcal{A}^*(u)$ such that $\alpha_{F,1}=a$ for at most $n-1$ faces $F\in\mathcal{F}^*(u)$ (this is true if $n=2$). Indeed, by taking such a face $F_1\in\mathcal{F}^*(u)$, the polynomials of degree less or equal to n so used to get terms $uw_1\cdots w_i\delta_{k+i}x_1^{s+1}$, $i\leq n$, with a weight strictly greater than $\rho^*(u)-k$, are multiples of $(a(s+2)+\rho^*(u)-k)$, but they can not be equal to $(a(s+2)+\rho^*(u)-k)^n$. A similar argument allow us to conclude when there exists $F_1\in\mathcal{A}^*(u)$ such that, for every monomial w_1 of the Taylor expansion of h_{F_1} with $\rho^*(uw_1)=\rho^*(u)+1$, the set $\mathcal{A}^*(uw_1)$ is not reduced to $\{\alpha_{F_1,1}\}$.

So we have just to consider the following case: $n\geq 3$, $d(A)=1$, and, for every $F\in\mathcal{F}^*(u)$, there exists at least one monomial $w=x^{A'}$ in the Taylor expansion of h_F such that $\rho^*(uw)=\rho^*(u)+1$, $d(A+A')=2$, $\mathcal{A}^*(uw)=\{\alpha_{F,1}\}$ and the set $\mathcal{F}^*(uw)$ has at least $n-1$ elements. We will prove that after at least $n-1$ iterations of the general process given above, we get a sum of terms $uw_1\cdots w_i\delta_{k+i}x_1^{s+1}$, $i\leq n-1$ with a weight strictly bigger than $\rho^*(u)-k$.

Let $F_1\in\mathcal{F}^*(u)$ such that $\alpha_{F_1,1}$ is the smallest element of $\mathcal{A}^*(u)$. Let $w_1=x^{A_1'}$ be a monomial in the Taylor expansion of h_{F_1} which verifies the requisite conditions, and let $\mathcal{F}^*(uw_1)=\{F_2,\ldots,F_m\}$. Let us prove that $A+A_1'$ is necessarily in the cone $\{0\}\times(\mathbf{R}^+)^{n-1}$. Otherwise the vector $A+A_1'\in(\mathbf{N}^*)^n$ is in the interior of the cone $C(F_2,\ldots,F_m)=C(F_2)\cap\cdots\cap C(F_m)\in\mathcal{C}$, i.e. $C(A+A_1')=C(F_2,\ldots,F_m)$. As $A\in C(A+A_1')\cap C(F_1)$ and $A_1'\neq C(F_1)$, the cone $C(F_1,F_2,\ldots,F_m)$ is contained in a facet of $C(A+A_1')$. Then for a dimensional argument, it coincides with $C(A)$. But, from Lemma 4.8, this is not possible because $d(A)=1$ and $A\in\mathbf{N}\times(\mathbf{N}^*)^{n-1}$. So the assertion is proved.

Now we apply this process for the face F_2. If $d(A+A_1'+A_2')\geq 4$, at least $n-3$ additional iterations are enough for ending. So we can assume that $d(A+A_1'+A_2')=3$. But $d(A+A_1')=2$ and $C(A+A_1')\subset\{0\}\times(\mathbf{R}^+)^{n-1}$.

So, using again the above argument, we obtain also that $A'_2 \in \{0\} \times (\mathbf{R}^+)^{n-1}$ necessarily, and then $C(A + A'_1 + A'_2) \subset \{0\} \times (\mathbf{R}^+)^{n-1}$. Iterating again at least $n-4$ times this process and the argument, if it is not finished, then $C(A+A'_1+\cdots+A'_{n-2})$ is a cone in $\{0\}\times(\mathbf{R}^+)^{n-1}$ of dimension $n-1$. But also $\mathcal{F}^*(uw_1\cdots w_{n-2})$ is reduced to $\{F\}$ and after a last iteration, $\rho^*(uw_1\cdots w_{n-2} h_F\delta_{k+n-1}x_1^{s+1})$ is strictly greater than $\rho^*(u)-k$. This ends the proof. □

§4.3. Filtrations and roots of $\tilde{b}_\ell(s)$

For every $\ell \in \mathbf{N}^*$, the weight function ρ^* may be extend to $\bigoplus_{k\geq\ell} E\delta_k x_1^{s+1}$ by $\rho^*(\sum_k u_k\delta_k x_1^{s+1}) = \min_k\{\rho^*(u_k)-k\}$. It induces the decreasing filtration $(\bigoplus_{k\geq\ell} E\delta_k x_1^{s+1})_{\geq q} = \bigoplus_{k\geq\ell} E^*_{\geq q+k}\delta_k x_1^{s+1}$, $q\in\mathbf{Q}$. Then the spaces $\mathcal{Z}_\ell$, $\mathcal{Z}'_\ell$ and $\mathcal{Z}'_\ell/\mathcal{Z}_\ell$ get the induced filtrations and we have:

$$\mathrm{gr}^*\mathcal{Z}_\ell \hookrightarrow \mathrm{gr}^*\mathcal{Z}'_\ell \hookrightarrow \mathrm{gr}^*\Big(\bigoplus_{k\geq\ell} E\delta_k x_1^{s+1}\Big) \cong \bigoplus_q \Big(\bigoplus_{k\geq\ell} E^*_{q+k}\delta_k x_1^{s+1}\Big)$$

For every $U=\sum_k u_k\delta_k x_1^{s+1} \in \bigoplus_{k\geq\ell} E\delta_k x_1^{s+1}$ nonzero, the *initial part* of U is the element $\mathrm{in}^*(U)\in\bigoplus_{k\geq\ell} E^*_{\rho^*(U)+k}\delta_k x_1^{s+1}$ defined by:

$$\mathrm{in}^*(U) = \sum_{\rho^*(u_k)-k=\rho^*(U)} \mathrm{in}^*(u_k)\delta_k x_1^{s+1}$$

If $G\subset\bigoplus_{k\geq\ell} E\delta_k x_1^{s+1}$ is a nonzero subspace, we will denote $\mathrm{in}^*(G)$ the subspace of $\bigoplus_q(\bigoplus_{k\geq\ell} E^*_{q+k}\delta_k x_1^{s+1})$ generated by the initial parts of the nonzero vectors of G. For $q\in\mathbf{Q}$, let us denote $\mathcal{Z}^*_{\ell,q} = \mathrm{in}^*(\mathcal{Z}_\ell)\cap\bigoplus_{k\geq\ell} E^*_{q+k}\delta_k x_1^{s+1}$, and ${\mathcal{Z}'}^*_{\ell,q} = \mathrm{in}^*(\mathcal{Z}'_\ell)\cap\bigoplus_{k\geq\ell} E^*_{q+k}\delta_k x_1^{s+1}$. In particular, the rational numbers q with ${\mathcal{Z}'}^*_{\ell,q}\neq 0$ are contained in $\{q\in\mathbf{Q}\mid \exists k\in\mathbf{N},\ q+k\in\Pi^*\}$.

Using (8) and Lemma 4.3, we prove that the action of s on $\mathcal{Z}'_\ell/\mathcal{Z}_\ell$ respects the filtration by ρ^* and induces an action of degree zero on $\mathrm{gr}^*(\mathcal{Z}'_\ell/\mathcal{Z}_\ell)$. For every $q\in\mathbf{Q}$, let us denote $\tilde{b}_{\ell,q}(s)$ the minimal polynomial of s on $\mathrm{gr}^*_q(\mathcal{Z}'_\ell/\mathcal{Z}_\ell)$. So, from Theorem 1.1, we have:

Theorem 4.9. *The polynomial $\tilde{b}_\ell(s)$ is the l.c.m. of the polynomials $\tilde{b}_{\ell,q}(s)$:*

$$\tilde{b}_\ell(s) = \mathrm{l.c.m.}_{\mathcal{Z}^*_{\ell,q}\subsetneq{\mathcal{Z}'}^*_{\ell,q}}\tilde{b}_{\ell,q}(s)$$

Remark that, contrary to the classical case, the polynomials $\tilde{b}_{\ell,q}(s)$ are not a power of an affine form (see Lemma 4.5). In Proposition 4.6, we have proved that the multiplicities of their roots are strictly smaller than n. Thus:

Theorem 4.10. *The multiplicity of a root of* $\tilde{b}_\ell(s)$ *is at most* $n-1$.

Remark 4.11. Up to a change of notations, the first part of the proof of Proposition 4.6 allows to prove in the case of a non-degenerate convenient germ that the multiplicities of its reduced Bernstein polynomial are raised by n.

§4.4. The effective computation

Thus the determination of $\tilde{b}_\ell(s)$ needs the one of spaces $\mathcal{Z}^*_{\ell,q}$ and $\mathcal{Z}'^*_{\ell,q}$, $q \in \mathbf{Q}$. Here we adapt the method given in [2], and we apply it on an example.

Using the following formula:

$$\begin{aligned}&(\alpha_{F,1}(s+1) + w - \langle \alpha_F, \beta\rangle - \overline{\chi}_F)\partial^\beta u \delta_k x_1^{s+1}\\ &= \partial^\beta[(w + k - |\alpha_F|)u - \chi_F(u)]\delta_k x_1^{s+1} - \partial^\beta u h_F \delta_{k+1} x_1^{s+1}\end{aligned}$$

for $u \in \mathcal{O}$, $k \in \mathbf{N}^*$, $w \in \mathbf{C}$, $\beta \in \mathbf{N}^n$, and Lemma 4.3, we construct a sequence $(S_{\ell,m})_{1\le m\le M_\ell}$ of good operators $S_{\ell,m}$ in s of degree m, a creasing sequence of rational numbers $(q_{\ell,m})_{1\le m\le M_\ell-1}$ with $q_{\ell,1} \ge \rho^*(x_1 g'_{x_1})$ and a sequence $(H_{\ell,m})_{1\le m\le M_\ell-1}$ of elements of $\bigoplus_{k\ge\ell} DE\delta_k x_1^{s+1}$ such that:

- $S_{\ell,m} x_1 g'_{x_1} \delta_\ell x_1^{s+1} - H_{\ell,m} \in \mathcal{D}\mathcal{J}\delta_\ell x_1^{s+1}$ for $1 \le m \le M_\ell - 1$;
- $S_{\ell,M_\ell} x_1 g'_{x_1} \delta_\ell x_1^{s+1} \in \mathcal{D}\mathcal{J}\delta_\ell x_1^{s+1}$;
- $H_{\ell,m} = \sum_{\ell\le k\le \ell+n-2} H_{\ell,m,k}\delta_k x_1^{s+1}$ with $H_{\ell,m,k} \in DE^*_{\ge q_{\ell,m}+k-\ell}$ of degree at least $m+\ell-k-1$.

Then this sequence $(H_{\ell,m})$ determines $\mathcal{Z}_\ell$:

$$\mathcal{Z}_\ell = \left\{ \sum_{m=1}^{M_\ell-1} c_\ell(a_m H_{\ell,m}) + c_\ell(a_0 x_1 g'_{x_1}\delta_\ell x_1^{s+1}) \,|\, a_m \in \mathcal{O} \right\} \tag{10}$$

because $\mathcal{Z}_\ell$ coincides with $c_\ell(\mathcal{D}[s] x_1 g'_{x_1}\delta_\ell x_1^{s+1})$ (Lemma 3.3) and, for every $P(s) \in \mathcal{D}[s]$:

$$P(s) x_1 g'_{x_1}\delta_\ell x_1^{s+1} \in \sum_{m=1}^{M_\ell-1} \mathcal{D}S_{\ell,m} x_1 g'_{x_1}\delta_\ell x_1^{s+1} + \mathcal{D}x_1 g'_{x_1}\delta_\ell x_1^{s+1} + \mathcal{D}\mathcal{J}\delta_\ell x_1^{s+1}$$

Indeed, by division we have: $P(s) = P_{M_\ell}(s) S_{\ell,M_\ell} + \sum_{m=1}^{M_\ell-1} P_m S_{\ell,m} + P_0$ where $P_m \in \mathcal{D}$, $0 \le m \le M_\ell - 1$, and $P_{M_\ell}(s) \in \mathcal{D}[s]_{\le d-M_\ell}$ if $d \in \mathbf{N}$ is the degree in s of $P(s)$. An induction on d allows us to conclude, using Remark 2.4 and that $S_{\ell,M_\ell} x_1 g'_{x_1}\delta_\ell x_1^{s+1} \in \mathcal{D}\mathcal{J}\delta_\ell x_1^{s+1}$.

The determination of $\mathcal{Z}'_\ell = c_\ell(\mathcal{D}[s] g'_{x_1}\delta_\ell x_1^{s+1})$ is similar, using sequences $(S'_{\ell,m})_{1\le m\le M'_\ell}$, $(q'_{\ell,m})_{1\le m\le M'_\ell-1}$ with $q'_{\ell,1} \ge \rho^*(g'_{x_1})$, and $(H'_{\ell,m})_{1\le m\le M'_\ell-1}$.

Remark 4.12. If the Newton polyhedron of g has only one $(n-1)$-dimensional face F - with normal vector $\alpha \in (\mathbf{Q}^{*+})^n$ -, the algorithm is very simple, exactly as in [2], part 2. In fact, it is enough to suppose that $g|_F$ and $(g|_F, x_1)$ define some isolated singularities, *i.e.* g, (g, x_1) are *semi-weighted-homogeneous* morphism. Then the division theorem used in [2], p. 593, is sufficient, and so the weight function $\rho = \rho_F$ is enough. Moreover, Π is also the set of the weights of a weighted-homogeneous co-basis of the ideal $\mathrm{in}(\mathcal{J}) = (\mathrm{in}(g), \mathrm{in}(g_{x_2}), \ldots, \mathrm{in}\ (g_{x_n}))\mathbf{C}[x]$, with $\sigma = n - 2|\alpha| + \alpha_1$, and the formula given in Lemma 4.4 ends in one time:

$$\begin{aligned}&(\alpha_1(s+1) + |\alpha| + \rho(u) - k)u\delta_k x_1^{s+1}\\ &\quad \in \mathcal{DO}_{>\rho(u)}\delta_k x_1^{s+1} + \ \mathcal{DO}_{\geq \rho(u)+\rho(h)}\delta_{k+1}x_1^{s+1}\end{aligned}$$

where $h = \chi(g) - g$. Hence $(\alpha_1(s+1) + |\alpha| + q)$ annihilates $\mathrm{gr}_q \mathcal{Z}'_\ell / \mathcal{Z}_\ell$, and the polynomial $\tilde{b}_\ell(s)$ is given by:

$$\tilde{b}_\ell(s) = \prod_{\mathcal{Z}_{\ell,q} \subsetneq \mathcal{Z}'_{\ell,q}} \left(s + 1 + \frac{|\alpha| + q}{\alpha_1}\right)$$

When g is in fact a weighted-homogeneous polynomial, we easily get:

$$\tilde{b}_\ell(s) = \prod_{p \in \Pi'} \left(s + \frac{|\alpha| + 1 + p - \ell}{\alpha_1}\right)$$

where $\Pi' \subset \mathbf{Q}^+$ is the set of the weights of a weighted homogeneous cobasis of $(x_1, g_{x_2}, \ldots, g_{x_n})\mathcal{O}$ (see [22]).

Example. Let g be the germ $x_1^d + x_2^d + x_3^d + x_1^2 x_2^2 x_3^2$ with $d \geq 9$, and $f = x_1$. The computation of the Bernstein polynomial of g is done in [2]. Here we determinate the polynomials $\tilde{b}_\ell(s)$, $\ell \in \mathbf{N}^*$.

The Newton polyhedron of g has exactly three 2-dimensional faces F_1, F_2, F_3, with normal vectors associated:

$$\alpha_{F_1} = \left(\frac{1}{2} - \frac{2}{d}, \frac{1}{d}, \frac{1}{d}\right),\ \alpha_{F_2} = \left(\frac{1}{d}, \frac{1}{2} - \frac{2}{d}, \frac{1}{d}\right),\ \alpha_{F_3} = \left(\frac{1}{d}, \frac{1}{d}, \frac{1}{2} - \frac{2}{d}\right)$$

So $|\alpha_{F_i}| = 1/2$ and $h_{F_i} = (d/2 - 3)x_i^d$, $1 \leq i \leq 3$.

The ideal $\mathcal{J}$ is generated by g, $g'_{x_2} = dx_2^{d-1} + 2x_1^2 x_2 x_3^2$ and $g'_{x_3} = dx_3^{d-1} + 2x_1^2 x_2^2 x_3$. By taking away the non multiple of $x_2 x_3$ monomials from the monomial basis of $I(g) = (g, x_2 g'_{x_2}, x_3 g'_{x_3})\mathcal{O}$ given in [2], B.4.2.2.3, we obtain (using the isomorphism λ) the following monomials:

u	$\rho^*(u)$	
$(x_1x_2x_3)^\varepsilon x_1$	$(\varepsilon+1)/2$	$0 \le \varepsilon \le 4$
$(x_1x_2x_3)^\varepsilon x_1x_\theta^i$	$(\varepsilon+1)/2+i/d$	$0 \le \varepsilon \le 2,\ 1 \le i \le d-1,\ 1 \le \theta \le 3$
$(x_1x_2x_3)^\varepsilon x_2^ix_3^j$	$\varepsilon/2+(i+j+2)/d$	$0 \le \varepsilon \le 1,\ 0 \le i,j \le d-2$
$x_1^{i+1}x_\theta^j$	$1/2+(i+j)/d$	$1 \le i,j \le d-1,\ \theta=2,3$

So this gives a basis of a supplementary $E \subset \mathcal{O}$ of the ideal $\mathcal{J}$. Thus $\sigma^* = 5/2$, and $\Pi^* = \{1/2+k/d \,|\, 0 \le k \le 2d\} \cup \{k/d \mid 2 \le k \le 2d\}$.

Now we determinate the space $\mathcal{Z}_\ell = c_\ell(\mathcal{D}[s]x_1g'_{x_1}\delta_\ell x_1^{s+1})$. First we remark that the division of $x_1g'_{x_1}$ by $\mathcal{J}$ is given by:

$$x_1g'_{x_1} = dx_1^d + \frac{2}{d-4}(dg - x_2g'_{x_2} - x_3g'_{x_3})$$

Without loss of generality, it is also enough to find the sequence $(H_{\ell,m})$ associated with $x_1^d\delta_\ell x_1^{s+1}$. We have the identities:

$$\Big(\frac{1}{d}(s+1)+\frac{3}{2}-\ell-\overline{\chi}_{F_2}\Big)x_1^d\delta_\ell x_1^{s+1} = \Big(\frac{6-d}{2}\Big)x_1^dx_2^d\delta_{\ell+1}x_1^{s+1}$$
$$\Big(\frac{1}{d}(s+1)+\frac{3}{2}-\ell-\overline{\chi}_{F_3}\Big)x_1^dx_2^d\delta_{\ell+1}x_1^{s+1} = \Big(\frac{6-d}{2}\Big)x_1^dx_2^dx_3^d\delta_{\ell+2}x_1^{s+1}$$

where $\rho^*((x_1x_2x_3)^d) = d/2+2/d > \sigma^*+2$ because $d \ge 9$. Hence the term $(x_1x_2x_3)^d\delta_{\ell+2}x_1^{s+1}$ belongs to $\mathcal{D}\mathcal{J}\delta_\ell x_1^{s+1}$ and so $M_\ell = 2$. We get $H_{\ell,1}$ by rewriting $(d(6-d)/2)x_1^dx_2^d\delta_{\ell+1}x_1^{s+1}$. As $dx_1^dx_2^d = x_1^dx_2g'_{x_2} - 2(x_1x_2x_3)^2x_1^d$, we obtain:

$$H_{\ell,1} = (d-6)(x_1x_2x_3)^2x_1^d\delta_{\ell+1}x_1^{s+1} + d\Big(\frac{d-6}{2}\Big)\Big[x_1^d - \frac{\partial}{\partial x_2}x_1^dx_2\Big]\delta_\ell x_1^{s+1}$$

Consequently, $\mathcal{Z}_\ell$ is equal to $c_\ell(\mathcal{O}x_1^d\delta_\ell x_1^{s+1} + \mathcal{O}(x_1x_2x_3)^2x_1^d\delta_{\ell+1}x_1^{s+1})$. So we find:

$$\mathcal{Z}_\ell = G\delta_\ell x_1^{s+1} \oplus \mathbf{C}(x_1x_2x_3)^2x_1^d\delta_{\ell+1}x_1^{s+1} \oplus \mathbf{C}(x_1x_2x_3)^4x_1\delta_{\ell+1}x_1^{s+1}$$

where $G \subset E$ is the subspace generated by the monomials:

$(x_1x_2x_3)^\varepsilon x_1$	$2 \le \varepsilon \le 4$
$(x_1x_2x_3)^\varepsilon x_1^i$	$\varepsilon=0,\ i=d,$ or $\varepsilon=1,\ i=d-1,d,$ or $\varepsilon=2,\ 2 \le i \le d$
$(x_1x_2x_3)^\varepsilon x_1x_\theta^i$	$\varepsilon=1,\ i=d-1$ or $\varepsilon=2,\ 1 \le i \le d-1$ $(\theta=2,3)$
$(x_1x_2x_3)^\varepsilon x_2^ix_3^j$	$\varepsilon=0,\ i=j=d-2$ or $\varepsilon=1,\ d-3 \le i,j \le d-2$
$x_1^ix_\theta^j$	$i=d,\ 1 \le j \le d-1$ or $d-2 \le i,j \le d-1$ $(\theta=2,3)$.

The determination of the sequence $(H'_{\ell,m})$ associated with $g'_{x_1}\delta_\ell x_1^{s+1}$ is similar (for more details, see [22]). So we obtain that the quotient space $\mathcal{Z}'_\ell/\mathcal{Z}_\ell$ may be identified to:

$$G'\delta_\ell x_1^{s+1} \oplus \mathbf{C}(x_1x_2x_3)^2x_1^{d-1}\delta_{\ell+1}x_1^{s+1}$$

where $G' \subset E$ is the $\mathbf{C}$-vector space generated by the $d(d-2)$ monomials:

$$\begin{array}{ll} (x_1x_2x_3)^\varepsilon x_1^i & \varepsilon=0,\ i=d-1,\ \text{or}\ \varepsilon=1,\ i=d-2 \\ (x_1x_2x_3)x_2^ix_3^j & 1\le i,j\le d-2 \text{ except } d-3\le i,j\le d-2 \\ x_1^ix_\theta^j & i=d-1,\ 1\le j\le d-3,\ \text{or}\ i=d-3,\ d-1\le j\le d-2 \end{array}$$

for every $\ell \in \mathbf{N}^*$, expect if d is even and $\ell = 2$. In this case, the four monomials $x_1^{d-1}x_\theta^{d/2+1}$, $x_\theta^{d/2+1}x_2x_3(x_1x_2x_3)$, $\theta = 2,3$, do not belong to G', and G' have the following two vectors in addition $x_\theta^{d/2+1}g'_{x_1} = dx_1^{d-1}x_\theta^{d/2+1} + 2x_\theta^{d/2+1}x_2x_3(x_1x_2x_3)$, $\theta = 2,3$.

In order to study the action of s on nonzero spaces $\mathcal{Z}'^*_{\ell,q}/\mathcal{Z}^*_{\ell,q}$, we use the relation:

$$(\alpha_{F_i,1}(s+2)+\rho^*(u)-k)u\delta_k x_1^{s+1} = \frac{6-d}{2}ux_i^d\delta_{k+1}x_1^{s+1}$$

where u is a monomial and $F_i \in \mathcal{F}$ such that $\rho^*(u) = \rho^*_{F_i}(u)$, and we compute the image by c_ℓ after rewriting by division. For every $u\delta_\ell x_1^{s+1}$, $u \in G'$, the computation gives zero - in $\mathrm{gr}^*_{\rho^*(u)-\ell}\mathcal{Z}'_\ell/\mathcal{Z}_\ell$ - with one exception if $u = x_1^{d-1}$:

$$\begin{aligned} &\Big(\frac{1}{d}(s+2)+\frac{3}{2}-\frac{2}{d}-\ell\Big)x_1^{d-1}\delta_\ell x_1^{s+1} \\ &= \frac{d-6}{2d}(x_1^{d-1}\delta_\ell x_1^{s+1} + 2(x_1x_2x_3)^2x_1^{d-1}\delta_{\ell+1}x_1^{s+1}) \end{aligned}$$

and $((1/d)(s+2)+3/2-2/d-\ell)^2\delta_\ell x_1^{s+1} = 0$. Consequently, $\tilde{b}_\ell(s)$ is the l.c.m. of $((1/d)(s+2)+3/2-2/d-\ell)^2$ and of $(\alpha_{F,1}(s+2)+\rho^*(u)-\ell)$ with $F \in \mathcal{F}^*(u)$, $u \neq x_1^{d-1}$ in the given basis of G'. Then in the general case, we have:

$$\begin{aligned} \tilde{b}_\ell(s) = \text{l.c.m.}\Bigg\{ & s+d(2-\ell)-1, \Big(s+d\Big(\frac{3}{2}-\ell\Big)\Big)^2 \prod_{i=1}^{d-3}\Big(s+d\Big(\frac{3}{2}-\ell\Big)+i\Big), \\ & \prod_{i=0}^{2d-8}\Big(s+\frac{d(3-2\ell)+2i}{d-4}\Big)\Bigg\} \end{aligned}$$

where the last polynomial is the one of the monomials u with $\mathcal{F}^*(u) = \{F_1\}$.

References

[1] Barlet, D. and Kashiwara, M., Le réseau L^2 d'un système holonome régulier, *Invent. Math.*, **86** (1986), 35-62.

[2] Briançon, J., Granger, M., Maisonobe, Ph. and Miniconi, M., Algorithme de calcul du polynôme de Bernstein: cas non dégénéré, *Ann. Inst. Fourier*, **39** (1989), 553-610.

[3] Briançon, J. and Skoda, H., Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de $\mathbf{C}^n$, *C. R. Acad. Sci. Paris*, **278** (1974), 949-951.

[4] Ginsburg V., Characteristic varieties and vanishing cycles, *Invent. Math.*, **84** (1986), 327-402.

[5] Grothendieck, A., On the de Rham cohomology of algebraic varieties, *Pub. Math. I.H.E.S.*, **29** (1966), 95-105.

[6] Kashiwara, M., On the holonomic systems of differential equations II, *Invent. Math.*, **49** (1978), 121-135.

[7] Kashiwara, M., Vanishing cycle sheaves and holonomic systems of differential equations, *Lect. Notes in Math.*, **1016** (1983), 134-142.

[8] ———, Quasi-unipotent constructible sheaves, *J. Fac. Sci. Univ. Tokyo, Sect. I A*, **28** (1981), 757-773.

[9] Kouchnirenko, A. G., Polyèdres de Newton et nombres de Milnor, *Invent. Math.*, **32** (1976), 1-31.

[10] Lê, D.T., Le théorème de la monodromie singulier, *C. R. Acad. Sci., Paris, Ser. A*, **288** (1979), 985-988.

[11] Malgrange, B., Le polynôme de Bernstein d'une singularité isolée, *Lect. Notes in Math.*, **459** (1975), 98-119.

[12] ———, Polynôme de Bernstein-Sato et cohomologie évanescente, *Astérisque*, **101-102** (1983), 243-328.

[13] Maynadier, H., Polynômes de Bernstein-Sato associés à une intersection complète quasi-homogène à singularité isolée, *Bull. Soc. Math. France*, **125** (1997), 547-571.

[14] Mebkhout, Z., Local cohomology of analytic spaces, *Publ. RIMS, Kyoto Univ.*, **12** (1977), 247-256.

[15] Sabbah, C., Proximité évanescente. I. La structure polaire d'un $\mathcal{D}$-module, *Compositio Math.*, **62** (1987), 283-328.

[16] ———, Proximité évanescente. II. Équations fonctionnelles pour plusieurs fonctions analytiques, *Compositio Math.*, **64** (1987), 213-241.

[17] Sawaya, J., *Divisions selon les puissances fractionnaires d'un idéal*, Thèse UNSA, 2001.

[18] Torrelli, T., *Equations fonctionnelles pour une fonction sur un espace singulier*, Thèse UNSA, 1998.

[19] ———, Équations fonctionnelles pour une fonction sur une intersection complète quasi homogène à singularité isolée, *C. R. Acad. Sci. Paris*, **330** (2000), 577-580.

[20] ———, Un calcul de polynôme Bernstein associé à un faisceau de coniques non dégénéré, *C. R. Acad. Sci. Paris*, **331** (2000), 47-50.

[21] ———, Polynômes de Bernstein associés à une fonction sur une intersection complète à singularité isolée, *Ann. Inst. Fourier*, **52** (2002), 221-244

[22] ———, Polynômes de Bernstein d'une fonction lisse restreinte à une hypersurface à singularité isolée, *prépublication* 22/2002 *de l'Institut Elie Cartan de Nancy.*

[23] ———, Sur les germes de fonctions méromorphes définis par un système différentiel d'ordre 1, *prépublication* 45/2002 *de l'Institut Elie Cartan de Nancy.*