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A "quasi maximum principle" for $\mathcal{I}$-surfaces

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Abstract

The result of this paper yields a *maximum principle* for the components of surfaces whose distortion by a certain $\mathrm{GL}_3(\mathbb{R})$ matrix are minimizers of a dominance functional $\mathcal{I}$ of a parametric functional $\mathcal{J}$ with dominant area term within boundary value classes $H^{1,2}_{\varphi}(B,\mathbb{R}^3)$, termed $\mathcal{I}$-surfaces. Finally we derive a *compactness* result for sequences of $\mathcal{I}$-surfaces in $C^0(\bar{B},\mathbb{R}^3)$, which serves as a preparation for the forthcoming article [R. Jakob, Unstable extremal surfaces of the "Shiffman functional" spanning rectifiable boundary curves, Calc. Var., submitted for publication] whose aim is a proof of a sufficient condition for the existence of extremal surfaces of $\mathcal{J}$ which do not furnish global minima of $\mathcal{J}$ within the class $\mathcal{C}^*(\Gamma)$ of $H^{1,2}$-surfaces spanning an arbitrary closed rectifiable boundary curve $\Gamma \subset \mathbb{R}^3$ that merely has to satisfy a chord-arc condition.

1. Introduction and main result

Following Shiffman [12] we consider as in [6] and [8] the functional

$$\mathcal{I}(X) := \int_B F(X_u \wedge X_v) + \frac{k}{2}|DX|^2 \,\mathrm{d}u\,\mathrm{d}v =: \mathcal{F}(X) + k\mathcal{D}(X),$$

on surfaces $X \in H^{1,2}(B,\mathbb{R}^3)$ of the type of the open disc $B := B^2_1(0) \subset \mathbb{R}^2$. The Lagrangian F is assumed to satisfy the following list of requirements (A):

$$F \in C^0\big(\mathbb{R}^3\big) \cap C^2\big(\mathbb{R}^3 \setminus \{0\}\big), \tag{1}$$

$$F(tz) = tF(z) \quad \forall t \geqslant 0,\ \forall z \in \mathbb{R}^3, \tag{2}$$

$$m_1|z| \leqslant F(z) \leqslant m_2|z| \quad \forall z \in \mathbb{R}^3,\ 0 < m_1 \leqslant m_2, \tag{3}$$

$$F \text{ is convex on } \mathbb{R}^3. \tag{4}$$

Moreover we have to impose the following requirement on F:

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(R^*) The restriction of the function $g(z) := F(z) + F(-z)$ to the $\mathbb{S}^2$ shall have three linearly independent critical points, i.e. there have to be at least three linearly independent unit vectors $a_1, a_2, a_3 \in \mathbb{S}^2$ at which $\nabla g(a_j) = r_j a_j^\top$, for some $r_j \in \mathbb{R}$, $j = 1, 2, 3$. Finally we assume that

$$k > \max_{\mathbb{S}^2} F = m_2. \tag{5}$$

Thus $\mathcal{I}$ is a controlled perturbation of the Dirichlet functional $\mathcal{D}$, where F depends only on the normal $X_u \wedge X_v$, but not on the position vector X itself. Now only imposing the requirements (A) it was proved in Lemma 2.2 and Theorem 4.3 of [6] that in every boundary value class $H^{1,2}_\varphi(B, \mathbb{R}^3)$ there exists a unique minimizer of $\mathcal{I}$, termed $\mathcal{I}$-surface, which is additionally of the class $C^0(\bar{B}, \mathbb{R}^3)$ if $\varphi \in C^0(\partial B, \mathbb{R}^3) \cap H^{1/2,2}(\partial B, \mathbb{R}^3)$ by Theorem 5.2 in [6].

Shiffman claimed the results of this paper, Theorems 1.1 and 1.2, in Sections 6 and 7 of [12], but his proof of Theorem 1.1 is incomplete. We emphasize in particular that on p. 552 in [12] Shiffman asserts the incorrect statement that any integrand F meeting (A) satisfies the requirement (R^*) only by the fact that the function g, defined by $g(z) := F(z) + F(-z)$, is even (see the wrong proof in footnote 7 on the mentioned page). In fact one can easily construct counterexamples, see Section 2. On the other hand we will see in Section 2 that any integrand F that satisfies the requirements

$$(\mathrm{A}^*) := \text{ requirements (1)–(3) and } F - \lambda|\cdot| \text{ has to be convex on } \mathbb{R}^3,$$

for some $\lambda > 0$, can be "approximated" by a family of Lagrangians $\{F_\epsilon\}_{\epsilon>0}$ meeting the conditions (A) + (R^*) for sufficiently small ϵ, which will be used in the forthcoming article [8]. Now combining property (R^*) of F with the method of "levelling" real valued functions on $\bar{B}$, used by Shiffman in Section 6 of [12] and by McShane in Theorem 3.1 in [10], the author was able to carry out a rigorous proof of the "quasi maximum principle" for $\mathcal{I}$-surfaces, Theorem 1.1, which will imply a compactness result for sequences of those, Theorem 1.2. Firstly we need

Definition 1.1. Let $f \in C^0(\bar{B})$ and $G \subseteq B$ be an open subset of B. We set

$$\mathrm{m}_G(f) := \max\Bigl\{\max_{\bar{G}} f - \max_{\partial G} f, \min_{\partial G} f - \min_{\bar{G}} f\Bigr\} \tag{6}$$

and call $\mathrm{md}(f) := \sup_{G \subseteq B} \mathrm{m}_G(f)$ the monotonic diefficiency of f, where the supremum is taken over all open subsets $G \subseteq B$.

Now let F be a fixed integrand satisfying (A) + (R^*) and $g(z) := F(z) + F(-z)$. By the requirement (R^*) the function g gives rise to a matrix $A := (a_1, a_2, a_3)^\top \in \mathrm{GL}_3(\mathbb{R})$, having chosen three linearly independent critical points a_1, a_2, a_3 of $g|_{\mathbb{S}^2}$ arbitrarily. The "quasi maximum principle" for $\mathcal{I}$-surfaces reads (see also Theorem 6.1 on p. 554 in [12]):

Theorem 1.1. *Let $\varphi \in C^0(\partial B, \mathbb{R}^3) \cap H^{1/2,2}(\partial B, \mathbb{R}^3)$ be prescribed boundary values. Then the corresponding $\mathcal{I}$-surface $X^* \in H^{1,2}_\varphi(B, \mathbb{R}^3) \cap C^0(\bar{B}, \mathbb{R}^3)$, i.e. the unique minimizer of $\mathcal{I}$ in $H^{1,2}_\varphi(B, \mathbb{R}^3)$, satisfies* $\mathrm{md}((AX^*)_i) = 0$ *for $i = 1, 2, 3$.*

Combining this result with Lemma 1 on p. 719 in [9] one easily obtains the following compactness result:

Theorem 1.2. *Let $\{X^n\}$ be a sequence of $\mathcal{I}$-surfaces with $\mathcal{D}(X^n) \leqslant \mathrm{const}$, $\forall n \in \mathbb{N}$, and with equicontinuous and uniformly bounded boundary values. Then there exists a subsequence $\{X^{n_j}\}$ such that*

$$X^{n_j} \longrightarrow \bar{X} \quad \text{in } C^0\bigl(\bar{B}, \mathbb{R}^3\bigr) \quad \text{and} \quad X^{n_j} \rightharpoonup \bar{X} \quad \text{in } H^{1,2}\bigl(B, \mathbb{R}^3\bigr), \tag{7}$$

for a surface $\bar{X} \in H^{1,2}(B, \mathbb{R}^3) \cap C^0(\bar{B}, \mathbb{R}^3)$ with $\mathrm{md}((A\bar{X})_i) = 0$, $i = 1, 2, 3$.

2. Critical points of even functions on $\mathbb{S}^2$

This section is devoted to a discussion of the requirement (R^*) on the integrand F. Firstly we sketch a construction of a counterexample of Shiffman's assertion that any even C^1-function on the $\mathbb{S}^2$ would possess three linearly independent critical points (see p. 552 in [12]).

To this end we consider a linear transformation $A:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which possesses exactly three linearly independent unit eigenvectors a_1, a_2, a_3, such that a_3 lies in a small neighborhood of the great circle G determined by a_1 and a_2. Then we choose some point $b \in G \setminus \{\pm a_1, \pm a_2\}$ near a_3 and construct some smooth tangent vector field V on the $\mathbb{S}^2$ which vanishes outside a small neighborhood U of the shortest arc γ connecting a_3 with b and which induces a global smooth flow $\phi:\mathbb{R}\times\mathbb{S}^2 \longrightarrow \mathbb{S}^2$, by Corollary 10.13 and Theorem 9.5 in [3], taking the point a_3 via γ onto b within a certain time $t^* > 0$ and not effecting the points $\pm a_1$ and $\pm a_2$ in particular. Now we only consider the restriction of the C^∞-diffeomorphism $\phi(t^*, \cdot)$ to some appropriate (closed) hemisphere S, containing U in its interior, and extend it to an uneven diffeomorphism $\tilde{\phi}$ of the $\mathbb{S}^2$ simply by reflection at the origin, i.e.

$$\tilde{\phi}(z) := \begin{cases} \phi(t^*, z) & z \in S, \\ -\phi(t^*, -z) & z \in -S, \end{cases}$$

which is well defined due to $\phi(t, z) \equiv z$ $\forall z \in \partial S$ and $\forall t \in \mathbb{R}$. Then the composition $q \circ \tilde{\phi}^{-1}$ of the quadratic form $q(z) := \langle z, Az\rangle$ with $\tilde{\phi}^{-1}$ is indeed a smooth even function on the $\mathbb{S}^2$ whose critical points are exactly the three linearly dependent unit vectors a_1, a_2 and b, which completes the construction of the asserted counterexample.

On the other hand there holds the following approximation result:

Proposition 2.1. *Let F be an integrand satisfying the requirements* (A*)*, then there exists a family of approximations $\{F_\epsilon\}_{\epsilon>0}$ meeting the requirements* (A) *and additionally* (R*) *if $\epsilon < \bar{\epsilon}$, for some sufficiently small $\bar{\epsilon} > 0$, and with*

$$D^2F_\epsilon \longrightarrow D^2F \quad \text{in } C^0\big(\mathbb{R}^3 \setminus B_\rho(0)\big),\ \forall \rho > 0, \tag{8}$$

$$\nabla F_\epsilon \longrightarrow \nabla F \quad \text{in } C^0\big(\mathbb{R}^3 \setminus \{0\}\big), \tag{9}$$

$$F_\epsilon \longrightarrow F \quad \text{in } C^0\big(\overline{B_R(0)}\big),\ \forall R > 0, \tag{10}$$

for $\epsilon \searrow 0$.

Proof. We set $g(z) := F(z) + F(-z)$ and assume that $g|_{\mathbb{S}^2}$ has only critical points on some great circle which we suppose to be the $\mathbb{S}^1$ without loss of generality, otherwise we were done. Now just arguing in the opposite way as in the above construction of the counterexample we claim the existence of some smooth tangent vector field V on the $\mathbb{S}^2$ which vanishes outside a small neighborhood U of some chosen critical point b of $g|_{\mathbb{S}^2}$ and whose induced flow ϕ, which is globally defined and smooth on $\mathbb{R}\times\mathbb{S}^2$ by Corollary 10.13 and Theorem 9.5 in [3], satisfies $(\phi(t, b))_3 > 0$, $\forall t > 0$, and does not effect the antipodal pairs $\pm a_1$ and $\pm a_2$ of two further linearly independent critical points a_1, a_2 of $g|_{\mathbb{S}^2}$. As above we consider now the restriction of $\phi(t, \cdot)$ to some appropriate (closed) hemisphere S, containing U in its interior, and extend it to an uneven smooth flow $\tilde{\phi}$ on the $\mathbb{S}^2$ by

$$\tilde{\phi}(t, z) := \begin{cases} \phi(t, z) & z \in S, \\ -\phi(t, -z) & z \in -S, \end{cases}$$

which is well defined due to $\phi(t, z) \equiv z$ $\forall z \in \partial S$ and $\forall t \in \mathbb{R}$. We extend this flow homogeneously of first degree onto $\mathbb{R}^3$, i.e. by setting

$$\bar{\phi}(t, z) := |z|\tilde{\phi}\left(t, \frac{z}{|z|}\right) \quad \text{for } z \in \mathbb{R}^3 \setminus \{0\} \tag{11}$$

and $\bar{\phi}(t, 0) \equiv 0$, for any $t \in \mathbb{R}$. As we know that $\tilde{\phi}$ is smooth on $\mathbb{R}\times\mathbb{S}^2$ we infer together with $\bar{\phi}(t, 0) \equiv 0$ that $\bar{\phi} \in C^\infty(\mathbb{R}\times(\mathbb{R}^3 \setminus \{0\}), \mathbb{R}^3) \cap C^0(\mathbb{R}\times\mathbb{R}^3, \mathbb{R}^3)$. Now since $\partial_{ij}\bar{\phi}$ is uniformly continuous on $[-c, c]\times\mathbb{S}^2$, for any $c > 0$, and $\bar{\phi}(0, \cdot) = \mathrm{id}_{\mathbb{R}^3}$ one easily sees that

$$\partial_{ij}\bar{\phi}(t, \cdot)|_{\mathbb{S}^2} \longrightarrow \partial_{ij}\bar{\phi}(0, \cdot)|_{\mathbb{S}^2} = 0 \quad \text{in } C^0\big(\mathbb{S}^2\big),$$

for $t \to 0$ and $i, j \in \{1, 2, 3\}$, where we denote $\partial_i := \frac{\partial}{\partial z_i}$. Now together with the homogeneity of $\partial_{ij}\bar{\phi}(t, \cdot)$ of degree -1 by (11) one infers immediately:

$$\partial_{ij}\bar{\phi}(t, \cdot) \longrightarrow 0 \quad \text{in } C^0\big(\mathbb{R}^3 \setminus B_\rho(0)\big), \tag{12}$$

for $t \to 0$ and any $\rho > 0$. Analogously one obtains

$$D\bar{\phi}(t, \cdot)|_{\mathbb{S}^2} \longrightarrow D\bar{\phi}(0, \cdot)|_{\mathbb{S}^2} = \mathbf{1}_3 \quad \text{in } C^0\big(\mathbb{S}^2\big),$$

and together with the homogeneity of $D\bar{\phi}(t,\cdot)$ of degree 0 by (11):

$$D\bar{\phi}(t,\cdot) \longrightarrow \mathbf{1}_3 \quad \text{in } C^0\big(\mathbb{R}^3 \setminus \{0\}\big), \tag{13}$$

for $t \to 0$. And finally one achieves similarly, using the homogeneity of $\bar{\phi}(t,\cdot)$ of degree 1 and $\bar{\phi}(t,0) \equiv 0$ for any $t \in \mathbb{R}$:

$$\bar{\phi}(t,\cdot) \longrightarrow \mathrm{id}_{\overline{B_R(0)}} \quad \text{in } C^0\big(\overline{B_R(0)}\big), \tag{14}$$

for $t \to 0$ and any $R > 0$. Now we set $F_\epsilon := F(\bar{\phi}(\epsilon,\cdot)^{-1}) \equiv F(\bar{\phi}(-\epsilon,\cdot))$ on $\mathbb{R}^3$, for $\epsilon > 0$. Then we can immediately infer from the homogeneity of degree 1 of $\bar{\phi}(-\epsilon,\cdot)$ and its regularity that F_ϵ inherits the properties (1)–(3) from F. Additionally we see by $(\bar{\phi}(\epsilon,b))_3 > 0$, $\forall \epsilon > 0$, and by the invariance of $a_1, a_2 \in \mathbb{S}^1$ w. r. to $\bar{\phi}$ that a_1, a_2 and $\bar{\phi}(\epsilon,b)$ are three linearly independent critical points of the restriction $g_\epsilon|_{\mathbb{S}^2}$ of $g_\epsilon(z) := F_\epsilon(z) + F_\epsilon(-z) = g(\bar{\phi}(-\epsilon,z))$ on the $\mathbb{S}^2$, for any $\epsilon > 0$, where we used in the last equality that $\bar{\phi}(-\epsilon,\cdot)$ is uneven on $\mathbb{R}^3$. Furthermore we calculate

$$\begin{aligned}
&\partial_i F_\epsilon(z) = \big\langle \nabla F\big(\bar{\phi}(-\epsilon,z)\big), \partial_i \bar{\phi}(-\epsilon,z)\big\rangle \quad \text{and} \\
&\partial_{ij} F_\epsilon(z) = \big\langle \nabla F\big(\bar{\phi}(-\epsilon,z)\big), \partial_{ij} \bar{\phi}(-\epsilon,z)\big\rangle + \big\langle D^2 F\big(\bar{\phi}(-\epsilon,z)\big)\partial_j \bar{\phi}(-\epsilon,z), \partial_i \bar{\phi}(-\epsilon,z)\big\rangle,
\end{aligned}$$

for $z \neq 0$ and $i, j \in \{1,2,3\}$. Hence, on account of (12), (13) and (14) together with the homogeneity of D^2F of degree -1 and of ∇F of degree 0 on $\mathbb{R}^3 \setminus \{0\}$ and of F of degree 1 on $\mathbb{R}^3$ in combination with (11) and with the uniform continuity of D^2F, ∇F and F on $\mathbb{S}^2$ we obtain the asserted convergences (8), (9) and (10). Now by (8) we conclude that

$$\big|\big\langle \xi, \big(D^2F_\epsilon(z) - D^2F(z)\big)\xi\big\rangle\big| \leqslant \big\|D^2F_\epsilon - D^2F\big\|_{C^0(\mathbb{S}^2)} \longrightarrow 0 \tag{15}$$

for $\epsilon \searrow 0$, $\forall z, \xi \in \mathbb{S}^2$. Moreover the required convexity of $F - \lambda|\cdot|$, for some fixed $\lambda > 0$, implies the positive semi-definiteness of $D^2(F(z) - \lambda|z|)$ $\forall z \in \mathbb{R}^3 \setminus \{0\}$ and thus by a short computation:

$$\big\langle \xi, D^2F(z)\xi\big\rangle \geqslant \lambda\big\langle \xi, D^2\big(|z|\big)\xi\big\rangle = \frac{\lambda}{|z|}\bigg(|\xi|^2 - \frac{\langle z,\xi\rangle^2}{|z|^2}\bigg), \tag{16}$$

$\forall z \in \mathbb{R}^3 \setminus \{0\}$ and $\forall \xi \in \mathbb{R}^3$. Now we fix some $z \neq 0$, consider the orthogonal decomposition $\mathrm{Span}(z) \oplus \mathrm{Span}(z)^\perp$ of $\mathbb{R}^3$ and note that $\frac{|\langle z,\xi\rangle|}{|z|}$ is just the length of the orthogonal projection $\xi^\parallel := \langle \frac{z}{|z|}, \xi\rangle \frac{z}{|z|}$ of ξ onto $\mathrm{Span}(z)$. Now we conclude by the homogeneity of $\nabla F(z)$ of order 0:

$$D^2F(z)\,z = 0 \cdot \nabla F(z) = 0 \quad \text{for } z \neq 0,$$

showing that $\mathrm{Span}(\xi^\parallel)$ is contained in the kernel of $D^2F(z)$ for any $\xi \in \mathbb{R}^3$ and also of $D^2F_\epsilon(z)$, $\forall \epsilon > 0$, by the same reasoning. Now we introduce $\xi^\perp := \xi - \xi^\parallel$, i.e. the orthogonal projection of ξ onto $\mathrm{Span}(z)^\perp$, and obtain together with the symmetry of $D^2(F_\epsilon - F)(z)$:

$$\big\langle \xi, D^2(F_\epsilon - F)(z)\,\xi\big\rangle = \big\langle \xi^\perp, D^2(F_\epsilon - F)(z)\xi^\perp\big\rangle \tag{17}$$

$\forall z \in \mathbb{R}^3 \setminus \{0\}$ and $\forall \xi \in \mathbb{R}^3$. From (15) we infer in particular the existence of some $\bar{\epsilon} > 0$ such that

$$\big|\big\langle \zeta, D^2(F_\epsilon - F)(z)\zeta\big\rangle\big| \leqslant \frac{\lambda}{2}$$

for any $z \in \mathbb{S}^2$ and $\zeta \in \mathbb{S}^2 \cap \mathrm{Span}(z)^\perp$, if $\epsilon < \bar{\epsilon}$, and thus together with (17):

$$\big|\big\langle \xi, D^2(F_\epsilon - F)(z)\xi\big\rangle\big| = \big|\big\langle \xi^\perp, D^2(F_\epsilon - F)(z)\xi^\perp\big\rangle\big| \leqslant \frac{\lambda}{2}|\xi^\perp|^2$$

for any $z \in \mathbb{S}^2$ and $\xi \in \mathbb{R}^3$, if $\epsilon < \bar{\epsilon}$. Hence, recalling (16) we achieve

$$\begin{aligned}
\big\langle \xi, D^2F_\epsilon(z)\xi\big\rangle &= \big\langle \xi, D^2F(z)\xi\big\rangle + \big\langle \xi, D^2(F_\epsilon - F)(z)\xi\big\rangle \\
&\geqslant \bigg(\lambda - \frac{\lambda}{2}\bigg)|\xi^\perp|^2 = \frac{\lambda}{2}\big(|\xi|^2 - \langle z,\xi\rangle^2\big),
\end{aligned}$$

for any $z \in \mathbb{S}^2$ and $\xi \in \mathbb{R}^3$, if $\epsilon < \bar{\epsilon}$. Thus we obtain together with the homogeneity of $D^2 F_\epsilon$ of degree -1:

$$\big\langle \xi, D^2 F_\epsilon(z)\,\xi \big\rangle \geqslant \frac{\lambda}{2|z|}\left(|\xi|^2 - \frac{\langle z, \xi\rangle^2}{|z|^2}\right) \quad \forall \xi \in \mathbb{R}^3$$

and $\forall z \in \mathbb{R}^3 \setminus \{0\}$, which is equivalent to the positive semi-definiteness of $D^2(F_\epsilon(z) - \frac{\lambda}{2}|z|)$, for any $z \neq 0$, by the second equation in (16). Hence, we obtain the convexity of $F_\epsilon - \frac{\lambda}{2}|\cdot|$ on $\mathbb{R}^3$ from the next lemma and thus especially the asserted convexity of F_ϵ for $\epsilon < \bar{\epsilon}$. $\square$

Lemma 2.1. *Let $q \in C^0(\mathbb{R}^3) \cap C^2(\mathbb{R}^3 \setminus \{0\})$ have a positive semi-definite Hessian pointwise on $\mathbb{R}^3 \setminus \{0\}$, then q is convex on $\mathbb{R}^3$.*

Proof. Let H be an arbitrary open halfspace in $\mathbb{R}^3$ whose boundary contains the origin. A well known argument yields the convexity of q on H on account of the requirement of the lemma, i.e. there holds

$$q\big(tz_1 + (1-t)z_2\big) \leqslant tq(z_1) + (1-t)q(z_2) \qquad \forall t \in [0,1], \tag{18}$$

for any pair $z_1, z_2 \in H$. Now let $z_1^*, z_2^* \in \partial H$ be arbitrarily given. Then we can choose two sequences $\{z_1^i\}, \{z_2^i\} \subset H$ with $z_1^i \to z_1^*$ and $z_2^i \to z_2^*$, for $i \to \infty$, and infer from (18) applied to the pairs z_1^i, z_2^i in combination with the continuity of q on $\mathbb{R}^3$ the convexity relation (18) in the limit also for the pair z_1^*, z_2^*. This proves the convexity of q on $\mathbb{R}^3$. $\square$

3. Preparing propositions

Let F be a fixed integrand meeting (A) and (R*), $g(z) := F(z) + F(-z)$, a_1, a_2, a_3 three linearly independent critical points of $g|_{\mathbb{S}^2}$ and $A := (a_1, a_2, a_3)^\top \in \mathrm{GL}_3(\mathbb{R})$. We choose two vectors b_1, c_1, such that $O_1 := (a_1, b_1, c_1)^\top \in \mathrm{SO}(3)$ and set $F' := F \circ O_1^{-1}$, $g' := g \circ O_1^{-1}$. We prove

Lemma 3.1. *There are real constants k_2 and k_3 such that*

$$F'\big((z_1, z_2, z_3)\big) - F'\big((z_1, 0, 0)\big) \geqslant k_2 z_2 + k_3 z_3 \tag{19}$$

$\forall z_1, z_2, z_3 \in \mathbb{R}$.

Proof. Since $O_1^{-1} \cdot (1,0,0)^\top = O_1^\top \cdot (1,0,0)^\top = a_1$ and since a_1 is a critical point of $g|_{\mathbb{S}^2}$ we calculate:

$$\nabla g'\big((1,0,0)^\top\big) = \nabla g(a_1) \cdot O_1^{-1} = r_1 a_1^\top \cdot O_1^\top = r_1 (O_1 \cdot a_1)^\top = r_1(1,0,0),$$

for some $r_1 \in \mathbb{R}$. Hence, $(1,0,0)^\top$ is a critical point of $g'|_{\mathbb{S}^2}$, implying in particular the equations:

$$0 = g'_{z_2}\big((1,0,0)\big) = F'_{z_2}\big((1,0,0)\big) - F'_{z_2}\big((-1,0,0)\big),$$
$$0 = g'_{z_3}\big((1,0,0)\big) = F'_{z_3}\big((1,0,0)\big) - F'_{z_3}\big((-1,0,0)\big),$$

where we dropped the "$\top$"-sign. Now using that $\nabla F'$ is homogeneous of degree 0 on $\mathbb{R}^3 \setminus \{0\}$ by (2) we obtain:

$$F'_{z_2} \equiv \text{const} =: k_2, \qquad F'_{z_3} \equiv \text{const} =: k_3$$

on the z_1-axis except $\{0\}$. Furthermore we infer from the convexity of $F' \in C^2(\mathbb{R}^3 \setminus \{0\})$ for $z_1 \neq 0$:

$$\begin{aligned} F'\big((z_1,z_2,z_3)\big) - F'\big((z_1,0,0)\big) &\geqslant \big\langle \nabla F'\big((z_1,0,0)\big), (z_1,z_2,z_3) - (z_1,0,0)\big\rangle \\ &= F'_{z_2}\big((z_1,0,0)\big) z_2 + F'_{z_3}\big((z_1,0,0)\big) z_3 = k_2 z_2 + k_3 z_3, \end{aligned} \tag{20}$$

$\forall z_2, z_3 \in \mathbb{R}$. Now letting $z_1 \longrightarrow 0$ in (20) and using $F' \in C^0(\mathbb{R}^3)$ we achieve the assertion (19) also for $z_1 = 0$. $\square$

If we choose vectors b_2, c_2, and b_3, c_3, such that $O_2 := (b_2, a_2, c_2)^\top$, $O_3 := (b_3, c_3, a_3)^\top \in \mathrm{SO}(3)$ and set $F'^2 := F \circ O_2^{-1}$, $F'^3 := F \circ O_3^{-1}$, then we obtain analogously:

$$F'^2\big((z_1,z_2,z_3)\big) - F'^2\big((0,z_2,0)\big) \geqslant \text{const}\, z_1 + \text{const}\, z_3 \tag{21}$$

and

$$F'^3\big((z_1,z_2,z_3)\big) - F'^3\big((0,0,z_3)\big) \geqslant \mathrm{const}\, z_1 + \mathrm{const}\, z_2 \tag{22}$$

$\forall z_1, z_2, z_3 \in \mathbb{R}$. Next we need

Definition 3.1. Let $\varphi \in C^0(\partial B, \mathbb{R}^3) \cap H^{1/2,2}(\partial B, \mathbb{R}^3)$ be prescribed boundary values. Then we define

$$M(\varphi) := \big\{\{Y^n\} \subset C^0\big(\bar{B}, \mathbb{R}^3\big) \cap H^{1,2}\big(B, \mathbb{R}^3\big) | Y_n|_{\partial B} \longrightarrow \varphi \text{ in } C^0\big(\partial B, \mathbb{R}^3\big)\big\}$$

and

$$m(\varphi) := \inf_{\{Y^n\}\in M(\varphi)} \liminf_{n\to\infty} \mathcal{I}\big(Y^n\big). \tag{23}$$

Clearly one has $m(\varphi) \leqslant \inf_{H^{1,2}_\varphi(B)\cap C^0(\bar{B})} \mathcal{I}$ and

Proposition 3.1. *There exists a minimizing element $\{X^j\}$ for $\mathcal{I}$ in $M(\varphi)$, i.e. $\{X^j\} \in M(\varphi)$ satisfies*

$$\lim_{j\to\infty} \mathcal{I}\big(X^j\big) = m(\varphi).$$

Proof. By the definition of $m(\varphi)$ we can choose a minimizing sequence $\{\{Y^n\}^j\}_{j\in\mathbb{N}}$ of sequences for $\mathcal{I}$ in $M(\varphi)$, i.e. we have $\{\{Y^n\}^j\}_{j\in\mathbb{N}} \subset M(\varphi)$ such that

$$\lim_{j\to\infty} \liminf_{n\to\infty} \mathcal{I}\big(\big\{Y^n\big\}^j\big) = m(\varphi).$$

We set $m_j := \liminf_{n\to\infty} \mathcal{I}(\{Y^n\}^j)$. For each $j \in \mathbb{N}$ we can choose an integer $n(j)$ such that

$$\big|\mathcal{I}\big(\big\{Y^{n(j)}\big\}^j\big) - m_j\big| < \frac{1}{j} \quad \text{and} \quad \big\|\big\{Y^{n(j)}\big\}^j\big|_{\partial B} - \varphi\big\|_{C^0(\partial B)} < \frac{1}{j}.$$

Now we choose $X^j := \{Y^{n(j)}\}^j$ $\forall j \in \mathbb{N}$ and see that $\{X^j\} \in M(\varphi)$ satisfies

$$\big|\mathcal{I}\big(X^j\big) - m(\varphi)\big| \leqslant \big|\mathcal{I}\big(X^j\big) - m_j\big| + \big|m_j - m(\varphi)\big| \longrightarrow 0. \qquad \square$$

Proposition 3.2. *For any $X \in C^0(\bar{B}, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3)$ there is a mollified family $\{X_\epsilon\} \subset C^\infty_c(B_{1+2\delta}(0), \mathbb{R}^3)$, for $\epsilon \in (0, \delta)$ and some $\delta > 0$, that satisfies*:

$$X_\epsilon \longrightarrow X \quad \text{in } C^0\big(\bar{B}, \mathbb{R}^3\big) \cap H^{1,2}\big(B, \mathbb{R}^3\big). \tag{24}$$

Proof. Due to the continuation theorem for Sobolev functions there is a continuation $\hat{X} \in H^{1,2}(B_{1+\delta}(0), \mathbb{R}^3)$ of X, for some $\delta > 0$. An examination of this continuation, explicitly given in [2, p. 256], shows that we also have $\hat{X} \in C^0(\overline{B_{1+\frac{\delta}{2}}(0)}, \mathbb{R}^3)$ on account of $X \in C^0(\bar{B}, \mathbb{R}^3)$. Now we use a family $\{\varphi_\epsilon\}$ of even Dirac kernels, with $\mathrm{supp}(\varphi_\epsilon) = \overline{B_\epsilon(0)}$, to mollify $\hat{X}$:

$$X_\epsilon(\cdot) := \int_{B_{1+\delta}(0)} \varphi_\epsilon(\cdot - w)\hat{X}(w)\,\mathrm{d}w \in C^\infty_c\big(B_{1+2\delta}(0), \mathbb{R}^3\big)$$

for $\epsilon \in (0, \delta)$. Due to $\hat{X} \in H^{1,2}(B_{1+\delta}(0), \mathbb{R}^3)$ we firstly obtain by [2, p. 108]:

$$\|X_\epsilon - X\|_{H^{1,2}(B)} = \|X_\epsilon - \hat{X}|_B\|_{H^{1,2}(B)} \longrightarrow 0 \quad \text{for } \epsilon \searrow 0.$$

Moreover, due to $\mathrm{supp}(\varphi_\epsilon) = \overline{B_\epsilon(0)}$ and $\int_{B_{1+\delta}(0)} \varphi_\epsilon(y - w)\,\mathrm{d}w = 1$, $\forall y \in \bar{B}$, $\forall \epsilon \in (0, \delta)$, we gain:

$$\|X_\epsilon - X\|_{C^0(\bar{B})} = \max_{y\in\bar{B}} \bigg| \int_{B_{1+\delta}(0)} \varphi_\epsilon(y - w)\hat{X}(w)\,\mathrm{d}w - \hat{X}(y)\bigg|$$

$$= \max_{y\in\bar B}\left|\int\limits_{B_{1+\delta}(0)} \varphi_\epsilon(y-w)\big(\hat X(w)-\hat X(y)\big)\,\mathrm{d}w\right|$$
$$\leqslant \max_{y\in\bar B}\int\limits_{B_\epsilon(y)} \varphi_\epsilon(y-w)\big|\hat X(w)-\hat X(y)\big|\,\mathrm{d}w$$
$$\leqslant \max_{y\in\bar B}\max_{w\in\overline{B_\epsilon(y)}}\big|\hat X(w)-\hat X(y)\big| \longrightarrow 0 \quad \text{for } \epsilon\searrow 0,$$

since $\hat X$ is uniformly continuous on $\overline{B_{1+\frac{\delta}{2}}(0)}$, which completes the proof. □

Next we state a proposition due to McShane in [9, p. 719] (see [11, p. 416], for a detailed proof):

Proposition 3.3. *Let $\varphi\in C^0(\partial B)$ be prescribed boundary values and $\{f^n\}$ a sequence in $C^0(\bar B)\cap H^{1,2}(B)$ with the following properties*:

$$f^n|_{\partial B}\longrightarrow\varphi \quad \textit{in } C^0(\partial B), \tag{25}$$
$$\mathrm{md}\big(f^n\big)\longrightarrow 0 \quad \textit{for } n\to\infty, \tag{26}$$
$$\mathcal D\big(f^n\big)\leqslant \mathrm{const}\quad \forall n\in\mathbb N. \tag{27}$$

Then there exists a subsequence $\{f^{n_j}\}$ and a function $f^*\in C^0(\bar B)\cap H^{1,2}(B)$ such that $\mathrm{md}(f^*)=0$ *and*

$$f^{n_j}\longrightarrow f^* \quad \textit{in } C^0(\bar B).$$

In [7, p. 7], the Lipschitz continuity of the integrand F on $\mathbb R^3$, with Lip.-const $= m_2$, is derived from its required properties (A). Together with the Hölder inequality one can easily deduce (see [12, p. 548]):

Proposition 3.4. *For any $X, X'\in H^{1,2}(B,\mathbb R^3)$ and any open subset $\Omega\subseteq B$ there holds*:

$$\big|\mathcal I_\Omega(X)-\mathcal I_\Omega(X')\big|\leqslant(2m_2+k)\Big(\sqrt{\mathcal D_\Omega(X)}+\sqrt{\mathcal D_\Omega(X')}\Big)\sqrt{\mathcal D_\Omega(X-X')}. \tag{28}$$

4. Levelling of $C_c^\infty(\mathbb R^2)$-functions

In this section we discuss the process of "levelling" a function $f\in C_c^\infty(\mathbb R^2)$ on the unit disc $\bar B$ for a given fineness $\delta>0$ (see also [12, p. 553], and [10, p. 558]). To this end let

$$\mathcal Z\colon\ \min_{\bar B} f = l_0 < l_1 < \cdots < l_N < l_{N+1} = \max_{\bar B} f$$

be a partition of the interval $[\min_{\bar B} f, \max_{\bar B} f]$ such that $\triangle\mathcal Z := \max_{i=1,\dots,N+1}\{l_i - l_{i-1}\} < \delta$ and such that $l_1,\dots,l_N$ are regular values of f, which is possible for any choice of δ by Sard's theorem (see [4, p. 205]).

The levelling process starts on the level l_1. Since l_1 is a regular value of $f\in C_c^\infty(\mathbb R^2)$ (especially $l_1\neq 0$) $f^{-1}([l_1,\infty))$ is a compact 2-dimensional C^∞-manifold with boundary by the implicit function theorem (see [5, p. 303]). Hence, $f^{-1}([l_1,\infty))$ is locally connected, in particular, and has therefore only a finite number of connected components. Now we consider the (disjoint) union $U_+^{l_1}$ of those connected components of $f^{-1}([l_1,\infty))$ that are contained in $\bar B$, in particular we have

$$f(w)>l_1 \quad \forall w\in \mathring U_+^{l_1} \quad\text{and}\quad f(w)=l_1 \quad \forall w\in\partial U_+^{l_1}, \tag{29}$$

as l_1 is a regular value of f and as f is continuous, and we set

$$f_+^{l_1}(w) := \begin{cases} l_1 & w\in U_+^{l_1},\\ f(w) & w\in\mathbb R^2\setminus U_+^{l_1}.\end{cases} \tag{$\star$}$$

We go on by considering the compact C^∞-manifold $f^{-1}((-\infty, l_1])$ which again consists of only finitely many connected components, and term $U_-^{l_1}$ the union of those connected components that are contained in $\bar{B}$. By (29) we infer $\mathring{U}_+^{l_1} \cap \mathring{U}_-^{l_1} = \emptyset$ and therefore

$$f_+^{l_1}(w) < l_1 \quad \forall w \in \mathring{U}_-^{l_1} \quad \text{and} \quad f_+^{l_1}(w) = l_1 \quad \forall w \in \partial U_-^{l_1},$$

again since l_1 is a regular value of f, by ($\star$) and as f is continuous, and we set

$$f^{l_1}(w) := \begin{cases} l_1 & w \in U_-^{l_1}, \\ f_+^{l_1}(w) & w \in \mathbb{R}^2 \setminus U_-^{l_1}. \end{cases} \tag{$\star\star$}$$

Next we apply the same process to f^{l_1} on the level l_2 and note that for connected components P^1 of $U_\pm^{l_1}$ and P^2 of $U_+^{l_2}$ we have $P^1 \cap P^2 = \emptyset$ and for connected components P^1 of $U_\pm^{l_1}$ and P^2 of $U_-^{l_2}$ we have either $P^1 \cap P^2 = \emptyset$ or $P^1 \Subset P^2$. After that we apply the process to $(f^{l_1})^{l_2}$ on the level l_3 and so on, until we have performed the last levelling step on the level l_N. Thus after $2 \times N$ steps we arrive at a finite collection of "level sets" $U_\pm^{l_j}$, $j = 1, \ldots, N$, and at a function f^L on $\mathbb{R}^2$, that we term the "levelled" function of f, possessing the following properties:

Lemma 4.1. *Let $f \in C_c^\infty(\mathbb{R}^2)$ and a fineness δ be given arbitrarily. Firstly there holds $U_\pm^{l_j} \subset \bar{B}$ and $\mathring{U}_+^{l_j} \cap \mathring{U}_-^{l_j} = \emptyset$ for $j = 1, \ldots, N$. Secondly for connected components P^j of $U_\pm^{l_j}$ and P^i of $U_+^{l_i}$, with $j < i$, there holds $P^j \cap P^i = \emptyset$ and for connected components P^j of $U_\pm^{l_j}$ and P^i of $U_-^{l_i}$ $(j < i)$ there holds either $P^j \cap P^i = \emptyset$ or $P^j \Subset P^i$. Furthermore $U_\pm^{l_j}$ are compact 2-dimensional C^∞-manifolds with boundary and $\partial U_\pm^{l_j}$ are closed 1-dimensional C^∞-manifolds. In particular, $U_\pm^{l_j}$ consist of only a finite number of connected components and $\partial U_\pm^{l_j}$ are Lebesgue-measurable with $\mathcal{L}^2(\partial U_\pm^{l_j}) = 0$. Moreover f^L satisfies:*

$$f^L \in C^0(\bar{B}) \cap H^{1,2}(B), \quad f^L|_{\partial B} = f|_{\partial B}, \quad \operatorname{md}(f^L|_{\bar{B}}) \leqslant \delta. \tag{30}$$

Proof. The assertions $U_\pm^{l_j} \subset \bar{B}$ and $\mathring{U}_+^{l_j} \cap \mathring{U}_-^{l_j} = \emptyset$ follow immediately from the definition of $U_\pm^{l_j}$ and as the l_j are regular values of f for $j = 1, \ldots, N$. Next one obtains simultaneously $f^L \in C^0(\bar{B})$ and the relations between the connected components P^j of $U_\pm^{l_j}$ and P^i of $U_+^{l_i}$ resp. $U_-^{l_i}$, with $j < i$, by induction during the finite levelling process. As the levels l_j are regular values of $f \in C_c^\infty(\mathbb{R}^2)$ the implicit function theorem yields the assertions about the level sets $U_\pm^{l_j}$ and their boundaries $\partial U_\pm^{l_j}$ at once. Furthermore one has to note that manifolds M are locally connected, thus their connected components are open in M and compact manifolds can only consist of finitely many. Moreover $\mathcal{L}^2(\partial U_\pm^{l_j}) = 0$ follows immediately from the implicit function theorem and Proposition 8 of Section 1.11 in [5, p. 101]. Furthermore by construction of the first levelling step we obtain $f_+^{l_1} \in H^{1,1}(B)$ due to Lemma A 6.9 in [2, p. 254], where we have to use that $\partial U_+^{l_1}$ is a closed C^∞-manifold, thus in particular a Lipschitz boundary. Moreover it is also clear that we have $\nabla f_+^{l_1} \in L^2(B, \mathbb{R}^2)$ as $f_+^{l_1} \equiv f$ on $\mathbb{R}^2 \setminus U_+^{l_1}$ and $\nabla f_+^{l_1} \equiv 0$ on $\mathring{U}_+^{l_1}$ and since $\partial U_+^{l_1}$ especially satisfies $\mathcal{L}^2(\partial U_+^{l_1}) = 0$. Hence, we have $f_+^{l_1} \in H^{1,2}(B)$. Now, using that $\partial U_-^{l_1}$ is a closed C^∞-manifold again, especially with $\mathcal{L}^2(\partial U_-^{l_1}) = 0$ the same reasoning as above yields that $f^{l_1} \in H^{1,2}(B)$ and again using that $\partial U_\pm^{l_2}$ is a C^∞-manifold just the same reasoning as above yields that $(f^{l_1})^{l_2} \in H^{1,2}(B)$. Hence, after $2 \times N$ steps we arrive at $f^L \in H^{1,2}(B)$. Next, if $U_+^{l_1} \cap \partial B = \emptyset$ we have $f_+^{l_1}|_{\partial B} \equiv f|_{\partial B}$, but if $U_+^{l_1} \cap \partial B \neq \emptyset$ we obtain by the construction of $f_+^{l_1}$:

$$f_+^{l_1} \equiv l_1 \equiv f \quad \text{along } \partial U_+^{l_1} \cap \partial B.$$

Since this argument holds true for each step of the levelling process we finally see that $f^L|_{\partial B} \equiv f|_{\partial B}$. If we suppose that there exists an open subset G of B such that $\max_{\bar{G}} f^L - \max_{\partial G} f^L > \delta$, then due to $\triangle \mathcal{Z} < \delta$ there would be some level $l_j \in \mathcal{Z}$ such that $\max_{\partial G} f^L < l_j$ but $\max_{\bar{G}} f^L > l_j$. Hence, together with the continuity of f^L we would have on a connected component G' $(\neq \emptyset)$ of $G \cap (f^L)^{-1}((l_j, \infty)) \Subset G$

$$f^L(w) > l_j \quad \forall w \in G' \quad \text{and} \qquad f^L(w) = l_j \quad \forall w \in \partial G',$$

which implies that $f^L \equiv f$ on G' and $G' \subset U_+^{l_j}$. Therefore we must have $f^L \equiv l_i$ on G' for some $i \geqslant j$ by the construction of f^L and the second part of the assertion of the lemma, which is a contradiction. Similarly one proves that $\min_{\partial G} f^L - \min_{\bar G} f^L \leqslant \delta$ for all open subsets G of B again by the construction of f^L and the second part of the assertion of the lemma, hence $\mathrm{md}(f^L|_{\bar B}) \leqslant \delta$. $\square$

5. Levelling of the components of distorted surfaces $A\pi$

As in Section 3 we consider a fixed integrand F meeting (A) and (R*), some smooth surface $\pi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ and its distortion $\tilde\pi := A\pi$, where $A := (a_1, a_2, a_3)^\top \in \mathrm{GL}_3(\mathbb{R})$ is defined at the beginning of Section 3. Its components satisfy

$$\tilde\pi_i = \langle a_i, \pi\rangle = (O_i\pi)_i = \pi_i'^i \tag{31}$$

for $i = 1, 2, 3$, where we termed $\pi'^i := O_i\pi$. We set $m := \min_{i=1,2,3}\{\min_{\bar B} \tilde\pi_i\}$ and $M := \max_{i=1,2,3}\{\max_{\bar B} \tilde\pi_i\}$ and choose a partition

$$\mathcal{Z} : m = l_0 < l_1 < \cdots < l_N < l_{N+1} = M$$

of the interval $[m, M]$ of fineness $\triangle\mathcal{Z} < \delta$, for an arbitrarily given $\delta > 0$, such that the levels l_j, $j = 1, \dots, N$, are regular values of the three components $\tilde\pi_i$ simultaneously. At first we level the first component, i.e. $\tilde\pi_1 \mapsto (\tilde\pi_1)^L$, abbreviate $(\pi'^1)^L := ((\pi_1'^1)^L, \pi_2'^1, \pi_3'^1)$ and prove (see also (6.6) in [12])

Lemma 5.1. *For an arbitrary $\pi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ there holds*:

$$\mathcal{F}(\pi) \geqslant \mathcal{F}\big(O_1^{-1}\big(\pi'^1\big)^L\big). \tag{32}$$

Proof. We abbreviate $\pi' := \pi'^1 = O_1\pi$. It will suffice to consider only the first step of the levelling process on the level l_1 applied to $\pi_1' = \tilde\pi_1$. Let D be the open kernel of a connected component $\bar D$ of the level set $U_+^{l_1}$ which is a compact C^∞-manifold with boundary by Lemma 4.1. Now we choose an atlas $\mathcal{A} := \{(V_j, \psi_j)_{j=0,\dots,k}\}$ of $\bar D$ such that $\partial D \subset \bigcup_{j=1}^k V_j$ and a subordinate partition of unity $\{\eta_j\}_{j=0,\dots,k}$. Furthermore a careful examination of the implicit function theorem (see [5, p. 303]) shows that we may arrange the charts $\psi_j : B_{r_j}^+(0) \xrightarrow{\cong} V_j \cap \bar D$ such that $\gamma_j := \psi_j|_{[-r_j,r_j]} : [-r_j, r_j] \xrightarrow{\cong} V_j \cap \partial D$ yields a parametrization of $V_j \cap \partial D$ with respect to its arc length, for $j = 1, \dots, k$, implying that $((\gamma_j)_2', -(\gamma_j)_1')$ yields an outward pointing unit normal field ν_j along $V_j \cap \partial D$. Since we have $\pi_1' \equiv l_1$ along ∂D we infer:

$$\frac{\mathrm{d}}{\mathrm{d}s}\pi_1'\big(\gamma_j(s)\big) \equiv 0 \quad \forall s \in [-r_j, r_j], \tag{33}$$

for $j = 1, \dots, k$. Now we consider the vector field $h(z_1, z_2, z_3) := (-z_2, 0, 0)$ on $\mathbb{R}^3$. Firstly we note that $\mathrm{rot}\, h \equiv (0, 0, 1)$, thus setting $N := (N_1, N_2, N_3) := \pi_u' \wedge \pi_v'$ we have $N_3 = \langle \mathrm{rot}\, h(\pi'), \pi_u' \wedge \pi_v'\rangle$ on $\mathbb{R}^2$. Furthermore we set $w := (\langle h(\pi'), \pi_v'\rangle, -\langle h(\pi'), \pi_u'\rangle) \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Using $\pi_{uv}' = \pi_{vu}'$ due to Schwarz one easily calculates:

$$\mathrm{div}\, w = \big\langle \mathrm{rot}\, h(\pi'), \pi_u' \wedge \pi_v'\big\rangle \quad \text{on } \mathbb{R}^2.$$

Now combining this with the divergence theorem for Lipschitz boundaries (see [2, p. 252]) and (33) we arrive at:

$$\begin{aligned}
\int_D N_3\, \mathrm{d}u\, \mathrm{d}v &= \int_D \mathrm{div}\, w\, \mathrm{d}u\, \mathrm{d}v = \int_{\partial D} \langle w, \nu\rangle\, \mathrm{d}s = \sum_{j=1}^k \int_{\partial D \cap V_j} \eta_j \langle w, \nu_j\rangle\, \mathrm{d}s \\
&= \sum_{j=1}^k \int_{-r_j}^{r_j} (\eta_j w_1)\big(\gamma_j(s)\big)(\gamma_j)_2' - (\eta_j w_2)\big(\gamma_j(s)\big)(\gamma_j)_1'\, \mathrm{d}s \\
&= \sum_{j=1}^k \int_{-r_j}^{r_j} \big(\eta_j\big(-\pi_2'(\pi_1')_v\big)\big)\big(\gamma_j(s)\big)(\gamma_j)_2' - \big(\eta_j \pi_2'(\pi_1')_u\big)\big(\gamma_j(s)\big)(\gamma_j)_1'\, \mathrm{d}s
\end{aligned}$$

$$= -\sum_{j=1}^{k} \int_{-r_j}^{r_j} (\eta_j \pi_2')(\gamma_j(s)) \frac{\mathrm{d}}{\mathrm{d}s} \pi_1'(\gamma_j(s))\,\mathrm{d}s = 0. \tag{34}$$

If we use $\tilde{h}(z_1, z_2, z_3) := (z_3, 0, 0)$, with $\operatorname{rot}\tilde{h} = (0, 1, 0)$, we obtain analogously:

$$\int_D N_2\,\mathrm{d}u\,\mathrm{d}v = \sum_{j=1}^{k} \int_{-r_j}^{r_j} (\eta_j \pi_3')(\gamma_j(s)) \frac{\mathrm{d}}{\mathrm{d}s} \pi_1'(\gamma_j(s))\,\mathrm{d}s = 0, \tag{35}$$

on account of (33). Furthermore, as we have $\nabla(\pi_1')_+^{l_1} \equiv 0$ on D we see:

$$N^{l_1} := \begin{pmatrix} N_1^{l_1} \\ N_2^{l_1} \\ N_3^{l_1} \end{pmatrix} := \begin{pmatrix} ((\pi_1')_+^{l_1})_u \\ (\pi_2')_u \\ (\pi_3')_u \end{pmatrix} \wedge \begin{pmatrix} ((\pi_1')_+^{l_1})_v \\ (\pi_2')_v \\ (\pi_3')_v \end{pmatrix}$$
$$= \begin{pmatrix} (\pi_2')_u(\pi_3')_v - (\pi_2')_v(\pi_3')_u \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} N_1 \\ 0 \\ 0 \end{pmatrix}.$$

Thus by Lemma 3.1 we can conclude now:

$$F'(N) - F'(N^{l_1}) = F'(N_1, N_2, N_3) - F'(N_1, 0, 0) \geqslant k_2 N_2 + k_3 N_3.$$

Integration of this inequality over D yields

$$\int_D F'(N)\,\mathrm{d}u\,\mathrm{d}v - \int_D F'(N^{l_1})\,\mathrm{d}u\,\mathrm{d}v \geqslant k_2 \int_D N_2\,\mathrm{d}u\,\mathrm{d}v + k_3 \int_D N_3\,\mathrm{d}u\,\mathrm{d}v = 0,$$

where we used (34) and (35). Hence, by $(\pi_1')_+^{l_1} \equiv \pi_1'$ on $B \setminus U_+^{l_1}$ we obtain

$$\int_B F'(N)\,\mathrm{d}u\,\mathrm{d}v \geqslant \int_B F'(N^{l_1})\,\mathrm{d}u\,\mathrm{d}v.$$

Thus due to $O_1 \in \mathrm{SO}(3)$ we finally achieve after $2 \times N$ levelling steps:

$$\mathcal{F}(\pi) = \int_B F\big(O_1^{-1}(O_1\pi_u \wedge O_1\pi_v)\big)\,\mathrm{d}u\,\mathrm{d}v = \int_B F'(N)\,\mathrm{d}u\,\mathrm{d}v$$
$$\geqslant \int_B F\big(O_1^{-1}\big((\pi')_u^L \wedge (\pi')_v^L\big)\big)\,\mathrm{d}u\,\mathrm{d}v = \mathcal{F}\big(O_1^{-1}(\pi')^L\big). \qquad \square$$

Furthermore we shall also level the second and third component of $\tilde{\pi}$, i.e. $\tilde{\pi}_i \mapsto (\tilde{\pi}_i)^L$ for $i = 2, 3$. Abbreviating $(\pi'^2)^L := (\pi_1'^2, (\pi_2'^2)^L, \pi_3'^2)$ and $(\pi'^3)^L := (\pi_1'^3, \pi_2'^3, (\pi_3'^3)^L)$ we gain by (21) and (22) analogously for $i = 2, 3$:

$$\mathcal{F}(\pi) \geqslant \mathcal{F}\big(O_i^{-1}\big(\pi'^i\big)^L\big), \tag{36}$$

where one has to use the vector fields $h^2 := (0, -z_3, 0)$, $\tilde{h}^2 := (0, z_1, 0)$ for $i = 2$ and $h^3 := (0, 0, z_2)$, $\tilde{h}^3 := (0, 0, -z_1)$ for $i = 3$ to obtain the counterparts of the central equations (34) and (35). Next we prove (see also (6.7) in [12])

Lemma 5.2. *For an arbitrary $\pi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ there holds*

$$\mathcal{D}(\pi) - \mathcal{D}\big(O_i^{-1}\big(\pi'^i\big)^L\big) = \mathcal{D}\big(\pi_i'^i - \big(\pi_i'^i\big)^L\big), \tag{37}$$

for $i = 1, 2, 3$.

Proof. For $i = 1$ we abbreviate again $\pi' := \pi'^1$. We consider the union $\mathcal{L} := \bigcup_{j=1}^N \mathring{U}_\pm^{l_j}$ of all level sets that arise during the levelling process applied to $\tilde{\pi}_1 = \pi_1'$. Now combining the facts that π_2' and π_3' remain unchanged on B and that π_1' remains unchanged on $B \setminus \mathcal{L}$, while we level π_1', and that $\nabla \pi_1' \equiv 0$ on $\mathcal{L}$ we infer:

$$\begin{aligned}\mathcal{D}(\pi') - \mathcal{D}\big((\pi')^L\big) &= \mathcal{D}_\mathcal{L}(\pi_1') - \mathcal{D}_\mathcal{L}\big((\pi_1')^L\big) = \mathcal{D}_\mathcal{L}(\pi_1') \\ &= \mathcal{D}_\mathcal{L}\big(\pi_1' - (\pi_1')^L\big) = \mathcal{D}\big(\pi_1' - (\pi_1')^L\big).\end{aligned}$$

Together with the invariance of the Euclidean scalar product with respect to the action of SO(3) we finally achieve the assertion (37) for $i = 1$. For $i = 2, 3$ the proof works analogously. □

A combination of (32), (36) and (37) yields

$$\mathcal{D}\big(\pi_i'^i - \big(\pi_i'^i\big)^L\big) \leqslant \frac{1}{k}\big(\mathcal{I}(\pi) - \mathcal{I}\big(O_i^{-1}\big(\pi'^i\big)^L\big)\big), \tag{38}$$

for $i = 1, 2, 3$. Furthermore we define $\tilde{\pi}^L := ((\tilde{\pi}_1)^L, (\tilde{\pi}_2)^L, (\tilde{\pi}_3)^L)$ and $\pi^L := A^{-1}\tilde{\pi}^L$ $(= A^{-1}(A\pi)^L)$ and state (see also Lemma 6.3 in [12])

Lemma 5.3. *The surface π^L has the following properties*:

(i) $\pi^L \in C^0(\bar{B}, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3)$,

(ii) $\pi^L|_{\partial B} = \pi|_{\partial B}$,

(iii) $\mathrm{md}((A\pi^L)_i|_{\bar{B}}) \leqslant \delta$ *for* $i = 1, 2, 3$.

(iv) *Using the matrix norm* $\|B\| := \sup_{x\in\mathbb{S}^2} |Bx|$ *on* $\mathrm{Mat}_{3,3}(\mathbb{R})$ *we have*:

$$\mathcal{D}\big(\pi^L - \pi\big) \leqslant \frac{\|A^{-1}\|^2}{k}\left(\sum_{i=1}^3 \mathcal{I}(\pi) - \mathcal{I}\big(O_i^{-1}\big(\pi'^i\big)^L\big)\right). \tag{39}$$

Proof. The points (i), (ii) and (iii) follow immediately from Lemma 4.1 and the definition of π^L. Moreover we calculate by (31) and (38):

$$\begin{aligned}\mathcal{D}\big(\pi^L - \pi\big) &= \mathcal{D}\big(A^{-1}\big(\tilde{\pi}^L - \tilde{\pi}\big)\big) \leqslant \big\|A^{-1}\big\|^2\left(\sum_{i=1}^3 \mathcal{D}\big((\tilde{\pi}_i)^L - \tilde{\pi}_i\big)\right) \\ &= \big\|A^{-1}\big\|^2\left(\sum_{i=1}^3 \mathcal{D}\big(\big(\pi_i'^i\big)^L - \pi_i'^i\big)\right) \leqslant \frac{\|A^{-1}\|^2}{k}\left(\sum_{i=1}^3 \mathcal{I}(\pi) - \mathcal{I}\big(O_i^{-1}\big(\pi'^i\big)^L\big)\right).\end{aligned}$$ □

6. Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Now let $\varphi \in C^0(\partial B, \mathbb{R}^3) \cap H^{1/2,2}(\partial B, \mathbb{R}^3)$ be prescribed boundary values. By Proposition 3.1 there exists a minimizing element $\{X^n\}$ for $\mathcal{I}$ in $M(\varphi)$, i.e. $\{X^n\} \in M(\varphi)$ satisfies

$$\lim_{n\to\infty} \mathcal{I}\big(X^n\big) = m(\varphi). \tag{40}$$

By Proposition 3.2 there exists a mollified sequence $\{\pi^n\} := \{X^n_{\epsilon_n}\} \subset C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ such that

$$\big\|\pi^n - X^n\big\|_{C^0(\bar{B})} + \big\|\pi^n - X^n\big\|_{H^{1,2}(B)} < \frac{1}{n} \quad \forall n \in \mathbb{N}. \tag{41}$$

Firstly we infer from (41) and $\{X^n\} \in M(\varphi)$:

$$\big\|\pi^n|_{\partial B} - \varphi\big\|_{C^0(\partial B)} \leqslant \big\|\pi^n|_{\partial B} - X^n|_{\partial B}\big\|_{C^0(\partial B)} + \big\|X^n|_{\partial B} - \varphi\big\|_{C^0(\partial B)} \longrightarrow 0, \tag{42}$$

for $n \to \infty$, which shows that $\{\pi^n\} \in M(\varphi)$. Secondly a combination of (41) with Proposition 3.4 and (40) yields

$$\big|\mathcal{I}\big(\pi^n\big) - m(\varphi)\big| \leqslant \big|\mathcal{I}\big(\pi^n\big) - \mathcal{I}\big(X^n\big)\big| + \big|\mathcal{I}\big(X^n\big) - m(\varphi)\big| \longrightarrow 0, \tag{43}$$

where we also used that $\mathcal{D}(X^n) \leqslant \frac{1}{k}\mathcal{I}(X^n) \leqslant \text{const}\ \forall n \in \mathbb{N}$ due to (40). Hence, $\{\pi^n\}$ is a minimizing element for $\mathcal{I}$ in $M(\varphi)$ again. Now we level the components of $\tilde{\pi}^n := A\pi^n$, i.e. $\tilde{\pi}^n \mapsto (\tilde{\pi}^n)^L$, with decreasing fineness $\delta_n \searrow 0$. Firstly by (30) and (42) we have

$$O_i^{-1}\big(\big(\pi^n\big)'^i\big)^L\big|_{\partial B} = \pi^n|_{\partial B} \longrightarrow \varphi \quad \text{in } C^0\big(\partial B, \mathbb{R}^3\big), \tag{44}$$

and therefore $\{O_i^{-1}((\pi^n)'^i)^L\} \in M(\varphi)$, for $i = 1, 2, 3$. Furthermore by (32), (36) and (37) we obtain

$$\mathcal{I}\big(O_i^{-1}\big(\big(\pi^n\big)'^i\big)^L\big) \leqslant \mathcal{I}\big(\pi^n\big) \quad \forall n \in \mathbb{N},$$

for $i = 1, 2, 3$. Combining this with (23) and (43) we conclude:

$$m(\varphi) \leqslant \liminf_{n\to\infty} \mathcal{I}\big(O_i^{-1}\big(\big(\pi^n\big)'^i\big)^L\big) \leqslant \lim_{n\to\infty} \mathcal{I}\big(\pi^n\big) = m(\varphi),$$

implying that $\{O_i^{-1}((\pi^n)'^i)^L\}$ is a minimizing element for $\mathcal{I}$ in $M(\varphi)$, for $i = 1, 2, 3$. If we insert this and (43) into (39), applied to π^n, we obtain:

$$0 \leqslant \mathcal{D}\big(\big(\pi^n\big)^L - \pi^n\big) \leqslant \frac{\|A^{-1}\|^2}{k}\left(\sum_{i=1}^{3} \mathcal{I}\big(\pi^n\big) - \mathcal{I}\big(O_i^{-1}\big(\big(\pi^n\big)'^i\big)^L\big)\right) \longrightarrow 0, \tag{45}$$

for $n \to \infty$. Combining this with (28) and noting that $\{\mathcal{D}(\pi^n)\}$ and $\{\mathcal{D}((\pi^n)^L)\}$ are bounded due to (43) and (45) we arrive at:

$$\big|\mathcal{I}\big(\big(\pi^n\big)^L\big) - m(\varphi)\big| \leqslant \big|\mathcal{I}\big(\big(\pi^n\big)^L\big) - \mathcal{I}\big(\pi^n\big)\big| + \big|\mathcal{I}\big(\pi^n\big) - m(\varphi)\big| \longrightarrow 0, \tag{46}$$

for $n \to \infty$. Moreover by Lemma 5.3(ii) and (42) we know that

$$\big(\pi^n\big)^L|_{\partial B} = \pi^n|_{\partial B} \longrightarrow \varphi \quad \text{in } C^0\big(\partial B, \mathbb{R}^3\big).$$

Hence, together with Lemma 5.3(i) and (46) we see that $\{(\pi^n)^L\}$ is a minimizing element for $\mathcal{I}$ in $M(\varphi)$. Now recalling Lemma 5.3(iii) we gather the following facts about the sequence $\{A(\pi^n)^L\}$:

$$A\big(\pi^n\big)^L\big|_{\partial B} \longrightarrow A\varphi \quad \text{in } C^0\big(\partial B, \mathbb{R}^3\big),$$
$$\mathrm{md}\big(\big(A\big(\pi^n\big)^L\big)_i\big|_{\bar{B}}\big) \leqslant \delta_n \searrow 0 \quad \text{for } i = 1, 2, 3,$$
$$\mathcal{D}\big(A\big(\pi^n\big)^L\big) \leqslant \|A\|^2 \mathcal{D}\big(\big(\pi^n\big)^L\big) \leqslant \text{const} \quad \forall n \in \mathbb{N}.$$

Hence, we can apply Proposition 3.3 and obtain a subsequence $\{A(\pi^{n_j})^L\}$ and a surface $\pi^* \in C^0(\bar{B}, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3)$ such that

$$A\big(\pi^{n_j}\big)^L\big|_{\bar{B}} \longrightarrow \pi^* \quad \text{in } C^0\big(\bar{B}, \mathbb{R}^3\big),$$

$\mathrm{md}(\pi_i^*) = 0$, for $i = 1, 2, 3$, and $\pi^*|_{\partial B} \equiv A\varphi$. Thus, if we rename $\{A(\pi^{n_j})^L\}$ into $\{A(\pi^n)^L\}$ we conclude:

$$\big(\pi^n\big)^L\big|_{\bar{B}} \longrightarrow A^{-1}\pi^* \quad \text{in } C^0\big(\bar{B}, \mathbb{R}^3\big), \tag{47}$$

with $A^{-1}\pi^*|_{\partial B} \equiv \varphi$. As we already know $\mathcal{D}((\pi^n)^L) \leqslant \text{const}$ this entails in particular $\|(\pi^n)^L\|_{H^{1,2}(B)} \leqslant \text{const}, \forall n \in \mathbb{N}$, implying the existence of a further subsequence $\{(\pi^{n_j})^L\}$ with

$$\big(\pi^{n_j}\big)^L\big|_B \rightharpoonup A^{-1}\pi^* \quad \text{in } H^{1,2}\big(B, \mathbb{R}^3\big).$$

We set $X^* := A^{-1}\pi^*$. Now using the weak lower semicontinuity of $\mathcal{I}$ due to [1], Theorem II.4, (see [7, p. 12]) we conclude together with (46) and (23):

$$j(\varphi) := \inf_{H_\varphi^{1,2}(B)\cap C^0(\bar{B})} \mathcal{I} \leqslant \mathcal{I}(X^*) \leqslant \liminf_{j\to\infty} \mathcal{I}\big(\big(\pi^{n_j}\big)^L\big) = m(\varphi) \leqslant j(\varphi). \tag{48}$$

Moreover in [7, p. 34], it is proved that the (unique) minimizer Y of $\mathcal{I}$ within the class $H_\varphi^{1,2}(B, \mathbb{R}^3)$ lies already in $C^0(\bar{B}, \mathbb{R}^3)$, if $\varphi \in C^0(\partial B, \mathbb{R}^3) \cap H^{1/2,2}(\partial B, \mathbb{R}^3)$, which implies

$$\mathcal{I}(Y) = \inf_{H_\varphi^{1,2}(B)} \mathcal{I} \leqslant \inf_{H_\varphi^{1,2}(B)\cap C^0(\bar{B})} \mathcal{I} \leqslant \mathcal{I}(Y).$$

Combining this with (48) we finally obtain:

$$\mathcal{I}(X^*) = j(\varphi) = \inf_{H_\varphi^{1,2}(B)} \mathcal{I},$$

with $\mathrm{md}((AX^*)_i) = \mathrm{md}(\pi_i^*) = 0$, for $i = 1, 2, 3$. $\square$

Proof of Theorem 1.2. Firstly by hypothesis we have the equicontinuity and uniform boundedness of the distorted boundary values $\{AX^n|_{\partial B}\}$, thus we gain a convergent subsequence $\{AX^{n_j}|_{\partial B}\}$ in $C^0(\partial B, \mathbb{R}^3)$ by Arzelà–Ascoli's theorem, which we rename again $\{AX^n|_{\partial B}\}$. Now we infer by Theorem 1.1 that $\{AX^n\} \subset C^0(\bar{B}, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3)$ satisfies $\mathrm{md}((AX^n)_i) = 0$ for $i = 1, 2, 3$. Hence, together with $\mathcal{D}(AX^n) \leqslant \|A\|^2 \mathcal{D}(X^n) \leqslant \mathrm{const}$ we see that Proposition 3.3 implies the existence of a further subsequence $\{AX^{n_j}\}$ and some surface $Y \in C^0(\bar{B}, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3)$ such that

$$AX^{n_j} \longrightarrow Y \quad \text{in } C^0\bigl(\bar{B}, \mathbb{R}^3\bigr)$$

and $\mathrm{md}(Y_i) = 0$ for $i = 1, 2, 3$. Thus the subsequence $\{X^{n_j}\}$ converges uniformly to $\bar{X} := A^{-1}Y \in C^0(\bar{B}, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3)$ and $\mathrm{md}((A\bar{X})_i) = 0$ for $i = 1, 2, 3$. Together with the required boundedness of $\{\mathcal{D}(X^n)\}$ we obtain $\|X^{n_j}\|_{H^{1,2}(B)} \leqslant \mathrm{const}, \forall j \in \mathbb{N}$, and therefore the asserted weak $H^{1,2}$-convergence in (7) for a further subsequence. $\square$

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