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The inhomogeneous boundary Harnack principle for fully nonlinear and p-Laplace equations

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Abstract. We prove a boundary Harnack principle in Lipschitz domains with small constant for fully nonlinear and p-Laplace-type equations with a right-hand side, as well as for the Laplace equation on nontangentially accessible domains under extra conditions. The approach is completely new and gives a systematic approach for proving similar results for a variety of equations and geometries.

1. Introduction

This work is intended as a sequel to [3] by the first and third authors, where a boundary Harnack principle (BHP) was established for the Laplace equation with right-hand side in Lipschitz domains (with small Lipschitz norm). Here we extend the result to the case of fully nonlinear as well as p-Laplace equations. The novel and very simple approach introduced here also allows us to consider nontangentially accessible (NTA) domains when there is an assumed lower bound on the growth of the solution from the boundary.

In lay terms, the main result in [3] states that (up to a multiplicative constant) a positive harmonic function can dominate a superharmonic function close to a boundary point x^0 of a domain $D \subset \mathbb{R}^n$ ($n \geq 2$), so long as both functions have zero boundary values in a small neighborhood of x^0. Throughout the paper we assume all domains are in $\mathbb{R}^n$ with $n \geq 2$. See below Theorem 1.1 for an exact formulation of the general case in this paper.

The BHP with right-hand side can be used to prove the regularity of free boundaries, for the obstacle problem (see [3]), and the thin obstacle problem (see [12]). Therefore, further study and generalization of the BHP with right-hand side should be emphasized to allow applications to more complicated free boundary problems. This work aims to make progress in this direction.

The reader may find it useful to read the longer introduction and applications mentioned in [3], which we have chosen not to repeat here. Since then there has been some further research on this topic, including [13], as well as [12] which we learned about in the final stages of preparation of this work. The approach taken here is rather different and

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allows treatment of very general configurations (see Theorem 2.2); however, our results do not entirely overlap with the above-mentioned references.

Our main results in this paper are the following theorems:

Theorem 1.1. *Let Ω be a Lipschitz domain with Lipschitz constant L and assume $0 \in \partial\Omega$. Let $u, v \geq 0$ with $u = v = 0$ on $\partial\Omega \cap B_1$ and assume $u(e_n/2) = v(e_n/2) = 1$. Assume the fully nonlinear operator F satisfies the structural conditions (3.1) and (3.2). There exist constants $C, \varepsilon, \eta > 0$ (depending on dimension and the ellipticity constants λ, Λ of F) such that if $L < \eta$ and*

$$-1 \leq F(D^2u, \nabla u), F(D^2v, \nabla v) \leq \varepsilon,$$

then

$$\frac{v}{u} \leq C \quad \text{in } \Omega \cap B_{1/2}.$$

For the p-Laplacian (defined in Section 4) we obtain a similar result for supersolutions:

Theorem 1.2. *Let Ω be a Lipschitz domain with Lipschitz constant L and assume $0 \in \partial\Omega$. Let $u, v \geq 0$ with $u = v = 0$ on $\partial\Omega \cap B_1$ and assume $u(e_n/2) = v(e_n/2) = 1$. If $1 < p < \infty$, then there exist constants $C, \eta > 0$ (depending on dimension and p) such that if $L \leq \eta$ and*

$$-1 \leq \Delta_p v, \Delta_p u \leq 0,$$

then

$$\frac{v}{u} \leq C \quad \text{in } \Omega \cap B_{1/2}.$$

The main ingredients in applying our method are the following:

(A) a boundary Harnack principle for solutions (to the homogeneous equation, with no right-hand side);

(B) an appropriate lower bound on the growth of u, v from the boundary;

(C) a comparison principle for sub- and supersolutions;

(D) solvability of the Dirichlet problem (with continuous data).

Our theorems require a Lipschitz boundary for (A) and (B). However, for operators such as the Laplacian Δ, one has a boundary Harnack principle for NTA domains [9]. Furthermore, for many free boundary problems a lower bound on the growth from the free boundary is often obtained directly (using competitors, barriers, or other techniques). In those cases (B) may be difficult or impossible to verify in general, but will be already available for the specific functions being considered. To handle this situation, one may apply our method on NTA domains and obtain the following conditional theorem, which appears useful in practice:

Theorem 1.3. *Let Ω be an NTA domain with $0 \in \partial\Omega$, and assume that for some $x^0 \in \Omega$ we have $u(x^0) = v(x^0) = 1$ and $u, v \geq 0$ with $u = v = 0$ on $\partial\Omega \cap B_1$. If for some $0 < \beta < 2$*

and some $c > 0$ *one has*

$$u(x), v(x) \geq c(\mathrm{dist}(x, \partial\Omega))^{\beta}$$

then there exists a constant C_0 *(depending on* β*, the NTA constants, and* $\mathrm{dist}(x^0, \partial B_1)$*) such that if*

$$-1 \leq \Delta u, \Delta v \leq 1,$$

then

$$\frac{v}{u} \leq C_0 \quad \textit{in } \Omega \cap B_{1/2}.$$

Unlike in Theorems 1.1 and 1.2, we do not assume that $\Delta u \leq \varepsilon$ small here, nor that the domain is somehow flat: the growth bound is all that is required, though it does carry some indirect implications about the geometry of $\partial\Omega$ and Δu.

2. A metatheorem

Let H_Ω be a family of operators mapping $C(\Omega) \times C(\partial\Omega) \to C(\bar{\Omega})$ for any $\Omega \in Q$ with Q a collection of open sets. The map H should be thought of as a solution operator, mapping boundary data and right-hand sides to solutions of an elliptic PDE. Fix a particular open set $U \in Q$. Let $V \subset C(\bar{U})$ consist of some subset of functions $u \geq 0$ with $H_U[f, u] = u$ on U for some f (generally this may be interpreted as positive functions with f bounded by 1, but only some specific properties below will be relevant; in fact, neither $u \geq 0$ nor $H_U[f, u] = u$ are used explicitly in the proof below). Assume the following properties for H_Ω, V, U, Q:

- (P1) Localization: For every $r > 0$ and $x \in \bar{U}$, there is a set $U_{x,r} \in Q$ such that $U_{x,r} \subset B_{2r}(x)$ and $U \cap B_r(x) = U_{x,r} \cap B_r(x)$.
- (P2) Homogeneity: $H_\Omega[0, 0] = 0$ for every $\Omega \in Q$.
- (P3) Solvability: If $\Omega \in Q$, then $H_\Omega[f, g] = g$ on $\partial\Omega$ for any $g \in C(\partial\Omega)$.
- (P4) Extension: If $\Omega \subset \Omega'$ are in Q, then $H_\Omega[f_\Omega, H_{\Omega'}[f, g]|_{\partial\Omega}] = H_{\Omega'}[f, g]$ on Ω.
- (P5) Comparison:[1] If $f_1 \geq f_2$ and $g_1 \leq g_2$, then $H_\Omega[f_1, g_1] \leq H_\Omega[f_2, g_2]$.
- (P6) Approximation: For any set $\Omega = U_{x,r}$ from (P1) with $r \leq \frac{1}{4}$, $x \in B_{1/2} \cap \partial U_{x,r}$, and $u \in V$, we have $|u - H_\Omega[0, u]| \leq C_1 r^{\zeta}$ for some $\zeta > 1$ on Ω.

[1] It is possible to replace this assumption with a *homogeneous minimum principle*: if $g \geq 0$, then $H_\Omega[0, g] \geq 0$, though without the full comparison principle some of the remarks and typical applications will not follow. In cases where lower-order terms interfere with the comparison principle, it may still be possible to apply the results here by treating the lower-order terms as an inhomogeneity instead. For example, when studying $\Delta u = -\lambda u$ for $\lambda > 0$, our theorem will apply if one first shows u is bounded, and then sets $\Delta u = f = -\lambda u$, with $f \in [-C, 0]$. Here H_Ω should be set to the solution to the Laplace equation, not to the eigenvalue problem.

(P7) Harnack: For any $u \in V$ and $B_{2r}(x) \subset U$,

$$\sup_{B_r(x)} u \leq C_2[\inf_{B_r(x)} u + 1].$$

(P8) Boundary Harnack: For any $a \in \partial U$ and $\Omega = U_{a,r}$ from (P1), let u_1, u_2 satisfy $H_\Omega[0, u_1] = u_1$ and $H_\Omega[0, u_2] = u_2$. Assume, moreover, that $u_1, u_2 \geq 0$ on Ω and $u_1, u_2 = 0$ on $\partial U \cap B_r(a)$. Then

$$\frac{u(x)}{v(x)} \leq \Big(1 + C_3 \frac{|x-y|^\alpha}{r^\alpha}\Big)\frac{u(y)}{v(y)}$$

for any $x, y \in U \cap B_{r/2}(a)$.

In addition, we will use the following concept of a 1-sided NTA (or uniform) domain:

Definition 2.1. A domain $\Omega \subset \mathbb{R}^n$ is a 1-*sided NTA domain* (with constant K) if it satisfies the following two conditions:

(D1) For every $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$, there exists a ball $B_{r/K}(y) \subset \Omega \cap B_r(x)$.

(D2) For every $x, y \in \Omega$, there is a curve $\gamma\colon [0,1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) = y$, $l(\gamma([0,1])) \leq K|x-y|$, and $\min\{l(\gamma([0,t])), l(\gamma([t,1]))\} \leq K d(\gamma(t), \partial\Omega)$ for all $t \in [0,1]$. Here l denotes length.

Our main theorem can now be phrased as follows:

Theorem 2.2. *Let Q, H_Ω, V, and U satisfy (P1)–(P8), assume that U is a 1-sided NTA domain with constant K, and $0 \in \partial U$. Then there is a constant $c_* = c_*(n, K)$ such that the following holds: let $u_1, u_2 \in V$ with $u_i > 0$ on $U \cap B_1$, $u_i = 0$ on $\partial U \cap B_1$ ($i = 1, 2$), and assume that for some $\beta \in (0, \zeta)$, u_i satisfies the growth condition*

$$u_i(x) \geq C_4 d^\beta(x, \partial U \cap B_1) \quad \forall x \in U. \tag{2.1}$$

In addition, assume that $u_1(x^0) = 1$ for some $x^0 \in B_{c_*}(0)$ with $d(x^0, \partial U) \geq c_*^2$. Then

$$\frac{u_1}{u_2} \leq C_*$$

on $B_{c_*} \cap U$. The constant C_* depends only on n, K, C_1, C_2, C_3, C_4, ζ, β (where C_1, C_2, C_3, and ζ are constants from (P6)–(P8)).

Remark 2.3. While U being a 1-sided NTA domain suffices for our argument, verifying property (P3), and possibly (P1), will often require making stronger assumptions. For the Laplace equation, 2-sided NTA domains (where the complement of U also satisfies (D1)) do have these properties, and in particular (P1) may be found in [9]. If one is working on Lipschitz graph domains $D_{L,r}$ as we do below, some of the details here can be simplified, and the $U_{x,r}$ can simply be chosen to be $U \cap B_r(x)$. It is worth noting, however, that (P8) does hold on 1-sided NTA domains at least for the Laplace equation [1], and this is roughly the most general class of domains on which it might be expected to hold [2].

Remark 2.4. Although we only assume that $u_1(x^0) = 1$ in the above theorem, the lower bound (2.1) automatically implies that $u_2(x^0) \geq c$, while an upper bound for u_2 is unnecessary. An abstract argument shows that for u_1, (2.1) may be replaced with a growth condition on the corresponding homogeneous equation, up to increasing the radii slightly: first, if $u_i = H_U[f_i, u_i]$, let $w = H_{U_{0,1}}[\min\{f_1, 0\}, u_1]$ and use the comparison principle to ensure $u_1 \leq w$. Thus it suffices to prove $w \leq C_* u_2$. Then set $h = H[0, u_1]$: this has $h \leq w$, so a growth estimate for h implies the same for w. Growth estimates for solutions to the homogeneous equation are equivalent to one another, from (P8); therefore in some cases (e.g. $f_2 \geq 0$) this estimate on u_1 may be redundant or easily obtainable. On the other hand, an inspection of the proof shows that if $f_2 = 0$ (and if $f_2 \leq 0$, after applying the comparison principle), then (2.1) for u_2 may be replaced with the condition $u_2(x^0) \geq 1$: the approximating function v_2 is equal to u_2, so (2.2) below is automatic.

Proof of Theorem 2.2. Let $\{r_k\}_{k=0}^{\infty}$ be a decreasing sequence of numbers $r_k \leq \frac{1}{4} r_{k-1}$ to be determined below, with $r_0 = \frac{c_*^2}{2}$, and

$$A_k = \left\{x \in U \cap B_{c_* + \frac{r_{k-1}}{c_*}} : r_k \leq d(x, \partial U) \leq r_{k-1}\right\}, \quad k \geq 1,$$

with $A_0 = \{x \in U \cap B_{2c_*} : d(x, \partial U) \geq r_0\}$; the constant c_* will be chosen below in terms of only n and the NTA constant $K$of U. Let

$$M_k = \sup_{A_k} \frac{u_1}{u_2};$$

as u, v are continuous and positive, we have $M_k < \infty$, for each k. Our main goal is to estimate M_k in terms of M_{k-1}, but we first consider M_0.

Applying (D2), if c_* is sufficiently small in terms of the NTA constant K, any $x, y \in B_{c_*} \cap U$ may be connected by a curve as described there which is contained in $B_{1/2}$. Furthermore, from (D1) for any $x \in \partial U$, and every $r < 1$, there is a ball $B_{r/K}(y) \subset U \cap B_r(x)$; so long as $c_* \leq \frac{1}{2K}$, $B_{rc_*/2}(y)$ has the same property. We now fix c_* so that these properties hold.

To estimate M_0, we first observe that by the lower bound assumption (2.1) we have $u_2 \geq C$ on A_0. On the other hand, we know that $u_1(x^0) = 1$, that $x^0 \in A_0$ and from the NTA property, any other point $x \in A_0$ may be connected to x^0 via a path in $B_{1/2}$ of bounded length. Furthermore, from (D2), if z lies on this path, then $d(z, \partial U) \geq \min\{|x^0 - z|, |x - z|\}/K$. Without loss of generality, assume $|x^0 - z| \leq |x - z|$. From the triangle inequality, we obtain $r_0 \leq d(x^0, \partial U) \leq |x^0 - z| + d(z, \partial U) \leq (K+1) d(z, \partial U)$. Therefore, the path stays a distance at least $c = r_0/(K+1)$ from the boundary ∂U. This path may be covered by finitely many balls of radius $c/2$, and applying the Harnack principle (P7) to each ball consecutively gives that $u_1(x)$ is bounded. Taking the supremum, we see that M_0 is bounded in terms of K and the constant C_2 in (P7).

Now take any point $x \in A_k$, and let $y \in \partial U$ with $|y - x| \leq r_{k-1}$. Use the NTA property to find a ball $B_{r_{k-1}}(z) \subset U \cap B_{r_{k-1}/2c_*}(y)$; then we have that $d(z, \partial U) \geq r_{k-1}$, while

$$|z| \leq |z - y| + |y - x| + |x| \leq \frac{r_{k-1}}{2c_*} + r_{k-1} + c_* + \frac{r_{k-1}}{c_*} \leq c_* + \frac{r_{k-2}}{c_*}$$

if $k \geq 2$, using here that $r_{k-1} \leq \frac{1}{4} r_{k-2}$. If $k = 1$, then using $r_0 = c_*^2/2$ gives

$$|z| \leq \frac{r_0}{2c_*} + r_0 + c_* + \frac{r_0}{c_*} \leq \Big(\frac{1}{4} + \frac{c_*}{2} + 1 + \frac{1}{2}\Big) c_* \leq 2c_*$$

instead. Consider the line segment connecting the points z and y: all points on this line segment must lie inside $B_{c_* + \frac{r_{k-2}}{c_*}}$ and $B_{r_{k-1}/2c_*}(y)$ as well, as both endpoints do and balls are convex. As $d(y, \partial U) = 0$, $d(z, \partial U) \geq r_{k-1}$, and the distance is continuous, we may find some z^1 on the line segment such that $d(z^1, \partial U) = r_{k-1}$. In particular, the two important properties are that $z^1 \in A_{k-1}$, while $x, z^1 \in B_{r_{k-1}/2c_*}(y)$.

Next we fix $U_{y,s}$ with $s \geq \frac{r_{k-1}}{c_*}$ to be chosen below, and use (P1) and (P3) to find v_1, v_2 which satisfy $H_{U_{y,s}}[0, u_i] = v_i$ (recall that this is analogous to solving the homogeneous equation on $U_{y,s}$ with boundary data given by u_i). Note that by comparison (P5), with the function 0, and homogeneity (P2), we have $v_i \geq 0$ on $U_{y,s}$. There are two main estimates we need for u_i and v_i. The first is from (P6): we have that $|v_i - u_i| \leq C_1 s^{\zeta}$ on $U \cap B_s(y)$. We may further combine it with the assumed growth estimate (2.1) to arrive at

$$|u_i - v_i| \leq C_1 s^{\zeta} \leq u_i \frac{C_1 s^{\zeta}}{C_4 r_k^{\beta}} \tag{2.2}$$

on $B_s(y) \cap (A_k \cup A_{k-1})$, which may be rephrased as

$$\Big(1 - C\frac{s^{\zeta}}{r_k^{\beta}}\Big) u_i \leq v_i \leq \Big(1 + C\frac{s^{\zeta}}{r_k^{\beta}}\Big) u_i, \tag{2.3}$$

so after dividing

$$\Big(1 - C\frac{s^{\zeta}}{r_k^{\beta}}\Big) v_i \leq u_i \leq \Big(1 + C\frac{s^{\zeta}}{r_k^{\beta}}\Big) v_i \tag{2.4}$$

on this region, so long as r_k^{β} is much larger than s^{ζ}, which we will ensure below.

On the other hand, we may apply (P8), the homogeneous boundary Harnack principle, to v_i. We apply it specifically with $r = s$, $a = y$, $x = x$, and $y = z^1$, to get

$$v_1(x) \leq v_2(x) \frac{v_1(z^1)}{v_2(z^1)} \Big(1 + C\frac{r_{k-1}^{\alpha}}{s^{\alpha}}\Big). \tag{2.5}$$

Now, $z^1 \in A_{k-1}$, so there we may argue as follows, using (2.3) and (2.4):

$$\frac{v_1(z^1)}{v_2(z^1)} \leq \Big(1 + C\frac{s^{\zeta}}{r_k^{\beta}}\Big)^2 \frac{u_1(z^1)}{u_2(z^1)} \leq \Big(1 + C\frac{s^{\zeta}}{r_k^{\beta}}\Big) M_{k-1}.$$

In the second inequality above we have used that s^{ζ}/r_k^{β} will be small once s and r_k are appropriately chosen. The above inequality along with (2.5) gives

$$\frac{v_1(x)}{v_2(x)} \leq \Big(1 + C\frac{s^{\zeta}}{r_k^{\beta}} + C\frac{r_{k-1}^{\alpha}}{s^{\alpha}}\Big) M_{k-1},$$

and finally using (2.2) again but this time at x,

$$\frac{u_1(x)}{u_2(x)} \le \Big(1 + C\frac{s^\zeta}{r_k^\beta}\Big)^2 \frac{v_1(x)}{v_2(x)} \le \Big(1 + C\frac{s^\zeta}{r_k^\beta} + C\frac{r_{k-1}^\alpha}{s^\alpha}\Big) M_{k-1}.$$

This entire construction can be done at any $x \in A_k$, so taking the supremum gives

$$M_k \le \Big(1 + C\frac{s^\zeta}{r_k^\beta} + C\frac{r_{k-1}^\alpha}{s^\alpha}\Big) M_{k-1}.$$

Now we must choose s and r_k in an appropriate manner; we have already required that $s^\zeta \ll r_k^\beta$, $r_{k-1} \ll s$, and $r_k \le r_{k-1}/4$. To proceed, select a $\gamma > 1$ such that $\beta\gamma < \zeta$, and set $r_k = r_{k-1}^\gamma$. This immediately implies that $r_k \le \frac{r_{k-1}}{4}$. Next, choose a $\sigma < 1$ with $\zeta\sigma > \beta\gamma$, and set $s = r_{k-1}^\sigma = r_k^{\sigma/\gamma}$; this has the other two necessary properties. With these choices, our recurrence relation may be rewritten as

$$M_k \le (1 + C r_k^{\zeta\sigma/\gamma - \beta} + C r_k^{(1-\sigma)\gamma\alpha}) M_{k-1}.$$

As $r_k \le \frac{1}{4} r_{k-1}$, $r_k \le r_0 4^{-k}$, we will have

$$M_k \le (1 + C4^{-ck}) M_{k-1} \le \prod_{i=1}^{\infty} (1 + C4^{-ci}) M_0.$$

This infinite product is finite, giving $M_k \le CM_0$ for all k. As the union of the A_k exhausts $B_{c_*} \cap U$, we have shown that

$$\sup_{B_{c_*} \cap U} \frac{u_1}{u_2} \le CM_0.$$

This completes the proof. ∎

Proof of Theorem 1.3. Set $U = \Omega$, Q the collection of (2-sided, as in Remark 2.3) NTA domains with constant at most K, and $H_{U_{x,r}}[f, g]$ the Perron solution to the Laplace equation on $U_{x,r}$. Set $V = \{u \in C(\bar{U}) : u \ge 0,\ |\Delta u| \le A \text{ on } U\}$, with A to be determined in terms of c_* and the given constants only. Then (P1), (P3), and (P8) follow from [9] as long as K is taken to be a sufficiently large multiple of the NTA constant of Ω, while (P2), (P4), (P5), and (P7) are classical. The approximation property (P6) follows from an elementary barrier argument (as in Lemma 3.3 below). After applying the Harnack inequality repeatedly, we have

$$C \ge u, v \ge c$$

on $U \cap B_2 \cap \{d(x, \partial U) \ge c_*^2\}$. For any x^1 a point in $B_{c_*}(x)$ a distance at least c_*^2 from ∂U, if we define the functions $u_1 = u(\cdot)/u(x^1)$, $u_2 = v(\cdot)/v(x^1)$ on $B_1(x)$, then $|\Delta u_i| \le \max\{\frac{1}{u(x^1)}, \frac{1}{v(x^1)}\} \le c^{-1}$.

Applying Theorem 2.2 to u_1, u_2 on $B_1(x)$ for every $x \in \partial U \cap B_1$ gives

$$\sup_{B_1 \cap U \cap \{z : d(z, \partial U) < c_*\}} \frac{u}{v} \le C,$$

which implies the conclusion. ∎

3. Fully nonlinear equations

Let $S(n)$ be the set of symmetric $n \times n$ matrices, $\Lambda \geq \lambda > 0$ and $M \geq 0$ be constants, and $P^-_{\Lambda,\lambda}$, $P^+_{\Lambda,\lambda}$ the extremal Pucci operators defined by

$$P^-_{\Lambda,\lambda}(R) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i, \quad P^+_{\Lambda,\lambda}(R) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i,$$

where e_i are the eigenvalues of R.

As our method requires that the boundary Harnack principle already holds for solutions to the homogeneous equation, we will require the same structural conditions for fully nonlinear equations as required in [6] where a boundary Harnack principle without right-hand side is shown. We therefore assume that $F : S(n) \times \mathbb{R}^n \to \mathbb{R}$ (the nonlinear operator in our equation $F(D^2u, \nabla u) = f$) satisfies

$$P^-_{\Lambda,\lambda}(R-S) - M|p-q| \leq F(R,p) - F(S,q) \leq P^+_{\Lambda,\lambda}(R-S) + M|p-q| \quad (3.1)$$

for $R, S \in S(n)$ and $p, q \in \mathbb{R}^n$.

We also assume that F is positively homogeneous of degree 1, i.e.

$$F(\gamma R, \gamma p) = \gamma F(R,p) \quad \text{for all } \gamma > 0, \quad R \in S(n), \ p \in \mathbb{R}^n. \quad (3.2)$$

We follow [5,6] when we write $F(D^2u, \nabla u) \leq (\geq) f$ *in the viscosity sense* for a continuous function f. The key property of viscosity solutions is the following comparison-type fact: if $F(D^2u, \nabla u) \geq f$ and $F(D^2v, \nabla v) \leq g$ on a domain Ω, in the viscosity sense, then $P^-_{\Lambda,\lambda}(D^2(v-u)) - M|\nabla v - \nabla u| \leq g - f$ in the viscosity sense on Ω. The proof is straightforward if one of v, u is C^2 from the definitions and (3.1), but the general case may be derived from [5].

We recall the following notation from [3] for Lipschitz domains. We consider Lipschitz domains $D_{L,R}$ where

$$D_{L,R} := \{(x', x_n) \in B_R : x_n > g(x')\},$$

and g is a Lipschitz function with constant at most L, that is, $|g(x') - g(y')| \leq L|x' - y'|$. We will assume $g(0) = 0$, and will write $D_{L,\infty}$ if $R = \infty$.

3.1. Approximation and homogeneous boundary Harnack

We need the following classical boundary Harnack principle, which is [6, Lemma 2.4]:

Lemma 3.1. *Let $F(D^2u, \nabla u) = F(D^2v, \nabla v) = 0$ in $D_{L,1}$ (in the viscosity sense), with $u, v \geq 0$, and $u = v = 0$ on $\partial D_{L,1} \cap B_{3/4}$. If $u(e_n/2) = v(e_n/2) = 1$, then there exists $C(\Lambda, \lambda, M, L, n)$ such that*

$$\frac{u}{v} \leq C \quad \text{in } D_{L,1/2}.$$

In our situation, it will be more convenient to apply the following slight variation of Lemma 3.1.

Lemma 3.2. *Let $F(D^2u,\nabla u)=F(D^2v,\nabla v)=0$ (in the viscosity sense) in $D_{L,1}$, with $u,v\geq 0$, and $u=v=0$ on $\partial D_{L,1}\cap B_{3/4}$. If $u(x^0)=v(x^0)=1$ for some $x^0\in D_{L,1/2}$, then there exists $C(\Lambda,\lambda,M,L,n)$ such that $u/v\leq C$ in $D_{L,1/2}$.*

Proof. We apply Lemma 3.1 to $\tilde{v}$, $\tilde{u}$ in place of u, v, where

$$\tilde{u}:=\frac{u(x)}{u(e_n/2)}\quad\text{and}\quad \tilde{v}:=\frac{v(x)}{v(e_n/2)},$$

and obtain

$$C\geq\frac{\tilde{v}(x)}{\tilde{u}(x)}=\frac{v(x)}{u(x)}\frac{u(e_n/2)}{v(e_n/2)}=\frac{u(e_n/2)}{v(e_n/2)}.$$

Now apply Lemma 3.1 again to $\tilde{u}$, $\tilde{v}$ in the opposite order to get that for any $y\in D_{L,1/2}$,

$$C\geq\frac{\tilde{u}(y)}{\tilde{v}(y)}=\frac{u(y)}{v(y)}\frac{v(e_n/2)}{u(e_n/2)}\geq\frac{u(y)}{v(y)}\frac{1}{C}.$$

This concludes the proof. ■

We will also need the following lemma.

Lemma 3.3. *Let v satisfy $-1\leq F(D^2v,\nabla v)\leq 1$ in $D_{L,R}$ (in the viscosity sense), where $R\leq 1$. If $v=h+w$ where w solves*

$$\begin{cases} F(D^2w,\nabla w)=0 & \text{in } D_{L,R},\\ w=v & \text{on } \partial D_{L,R},\end{cases}$$

then there exists a constant $C=C(n,\lambda,\Lambda,M)$ such that

$$|h|\leq CR^2.$$

Proof. From earlier remarks we have that $h=v-w$ satisfies

$$P^-_{\Lambda,\lambda}(D^2h)-M|\nabla h|\leq 1,\quad -1\leq P^+_{\Lambda,\lambda}(D^2h)+M|\nabla h| \tag{3.3}$$

in the viscosity sense.

Assume first that $R\leq\frac{\lambda n}{2M}$: we use

$$G(x):=\frac{R^2}{\lambda n}\Big(1-\frac{|x|^2}{R^2}\Big)$$

as an explicit barrier on B_R. We have

$$P^+_{\Lambda,\lambda}(D^2G)+M|\nabla G|=-2+2\frac{MR}{\lambda n}\leq -1.$$

Since $h = 0$ on $\partial D_{L,R}$ and $G \geq 0$ there, using the comparison principle with the right-hand-side inequality in (3.3) we obtain that $h \leq G \leq CR^2$.

On the other hand, if $R \geq \frac{\lambda n}{2M}$, we may use a barrier of the form

$$G(x) = e^{2S} - e^{S(x_1+1)}$$

where $S \geq 0$. Then $G \geq 0$ on $D_{L,1}$, and

$$P^+_{\Lambda,\lambda}(D^2 G) + M|\nabla G| = e^{S(x_1+1)}[-S^2\lambda + SM] \leq S[-S\lambda + M] \leq -1,$$

for S large enough. This gives $h \leq G \leq e^{2S} \leq CR^2$ on $D_{L,R}$ after applying the comparison principle for $C = C(n, \lambda, M)$.

The opposite inequality follows by considering $-h$ instead, using the other viscosity inequality. ∎

3.2. Growth estimates

Lemma 3.4. *There is a number $\varepsilon = \varepsilon(n, \lambda, \Lambda, M) > 0$ such that for every $\eta < \eta_0(n, \lambda, \Lambda, M)$, if $u \geq 0$ on $D_{L,1}$, $u(e_n/2) \geq 1$, $u = 0$ on $\partial D_{L,1} \cap B_1$,*

$$-1 \leq F(D^2 u, \nabla u) \leq \varepsilon$$

in the viscosity sense, and $L \leq \eta$, then

$$u(x) \geq c_*(x_n - \eta)$$

on $D_{L,1/16}$ for a $c_* = c_*(n, \lambda, \Lambda, M)$ (which does not depend on η).

Proof. We will show this using an explicit estimate with a barrier. The barrier argument proceeds in two steps, but they use the same function.

Set $\phi(x) = |x|^{-q}$. Direct computation shows that for q sufficiently large in terms of M, λ, and Λ, we have

$$P^-_{\lambda,\Lambda}(D^2\phi) - M|\nabla\phi| \geq 1$$

for $|x| \leq 1$. Fix q to be such a value, this implies that $F(D^2\phi, \nabla\phi) \geq 1$.

From the Krylov–Safonov Harnack inequality, we have that

$$\inf_{B_\kappa(e_n/2)} u \geq c \sup_{B_\kappa(e_n/2)} u - C\kappa^2 \geq c$$

for a small $\kappa \ll \frac{3}{8}$, c depending only on the ellipticity constants. Consider the barrier function

$$h(x) = c\frac{\phi(x - e_n/2) - (3/8)^{-q}}{\kappa^{-q} - (3/8)^{-q}}$$

defined on the annulus $A = B_{3/8}(e_n/2) \setminus B_\kappa(e_n/2)$. On the outer boundary, we have $h = 0$, while on the inner boundary, $h = c \leq u$. On A we have

$$F(D^2 h, \nabla h) \geq P^-_{\lambda,\Lambda}(D^2 h) - M|\nabla h| \geq \frac{c}{\kappa^{-q} - (3/8)^{-q}} := \varepsilon_1(n, \lambda, \Lambda, M).$$

So long as $\varepsilon < \varepsilon_1$ (and η_0 is small enough so that $A \subset D_{L,1}$), we may apply the comparison principle to h and u to obtain $u \geq h$ on A. In particular, take any point y a distance at most $\frac{1}{32}$ from a point z on the region $D = \{(x', x_n) : |x_n - \frac{1}{2}| \leq \frac{1}{4},\ |x'| < \frac{1}{10}\}$; then

$$|y - e_n/2| \leq |y - z| + |z - e_n/2| \leq \frac{1}{32} + \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{16}\right)^2} \leq \frac{11}{32} < \frac{3}{8}.$$

Hence $u \geq h \geq c_0$ at any such point. To summarize, at any $x \in D$, $u \geq c_0$ on $B_{1/32}(x)$.

Now we apply a similar argument around any $x \in D$ with $x_n = \frac{1}{4}$, except slightly more carefully. Fix η, and note that $D_{L,1/16}$ contains the large region $B_{1/16} \cap \{x_n \geq \eta\}$. Define $r = \frac{1}{4} - \eta$, the annulus $A_x = B_r(x) \setminus B_{\frac{1}{32}}(x)$ contained within this region, and the barrier function

$$h_x(y) = c_0 \frac{\phi(y - x) - (r)^{-q}}{(1/32)^{-q} - (r)^{-q}}$$

defined on A_x. As before, $h_x = 0$ on the outer boundary, $h_x = c_0 \leq u$ on the inner boundary, and

$$F(D^2 h_x, \nabla h_x) \geq \frac{c_0}{(1/32)^{-q} - (r)^{-q}} \geq \frac{c_0}{(1/32)^{-q}} := \varepsilon_2(n, \lambda, \Lambda, M).$$

If $\varepsilon < \varepsilon_2$ (which does not depend on η), we have $u \geq h_x$ on A_x. Using the explicit form of h_x,

$$u(x', t) \geq h_x(x', t) \geq c_1[r - (x_n - t)] = c_1[t - \eta]$$

for $t \in (\eta, \frac{1}{4} - \frac{1}{32})$, where c_1 can be taken independent of η.

So far, we have shown that for any x' with $|x'| \leq \frac{1}{16}$ and any $t \in (\eta, \frac{1}{4} - \frac{1}{32})$,

$$u(x', t) \geq c_1[t - \eta].$$

For $t \leq \eta$, this inequality remains true automatically. In particular, this means that it holds for all $(x', t) \in D_{L,1/16}$, giving the conclusion. ■

Lemma 3.5. *Let $\beta \in (1, 2)$. Then there are constants $\varepsilon, \eta > 0$ such that if $u \geq 0$ on $D_{L,1}$, $u(e_n/2) \geq 1$, $u = 0$ on $\partial D_{L,1} \cap B_1$,*

$$-1 \leq F(D^2 u, \nabla u) \leq \varepsilon$$

in the viscosity sense, and $L \leq \eta$, then

$$u(x) \geq c_1 d^\beta(x, \partial D_{L,1})$$

for $x \in D_{L,1/64}$.

Proof. We begin by ensuring that η, ε are small enough and applying Lemma 3.4 to learn that

$$u(x) \geq c_*[x_n - \eta] \geq \frac{c_*}{2} x_n \tag{3.4}$$

so long as $x_n \geq 2\eta$ and $x \in D_{L,1/16}$. Fix a point $x \in \partial D_{L,1} \cap B_{1/64}$, and define

$$u_1(y) = \frac{u(x + r_1 y)}{u(x + \frac{r_1 e_n}{2})},$$

where $r_1 = 64\eta$. We claim that on B_1, u_1 satisfies all of the assumptions of Lemma 3.4 (using $D'_{L,1} = (D_{L,1} - x)/r_1$). Indeed, most of the assumptions follow immediately: $u_1 \geq 0$ on $D'_{L,1}$ and vanishes on the graphical boundary, has $u_1(e_n/2) = 1$ by construction, and $D'_{L,1}$ has Lipschitz constant bounded by $L \leq \eta$. The main assumption we must check is that it satisfies the relevant differential inequalities. For this, rescaling gives that (for an $\widetilde{F}$ which satisfies the same properties as F)

$$-\frac{r_1^2}{u(x + \frac{r_1 e_n}{2})} \leq \widetilde{F}(D^2 u_1, \nabla u_1) \leq \frac{r_1^2}{u(x + \frac{r_1 e_n}{2})}\varepsilon, \tag{3.5}$$

so we must show that $u(x + \frac{r_1 e_n}{2}) \geq r_1^2$. Note that $x + r_1/2 e_n$ has nth component larger than 2η, so by (3.4),

$$u\Big(x + \frac{r_1 e_n}{2}\Big) \geq \frac{c_*}{2}\Big(x_n + \frac{r_1}{2}\Big) \geq \frac{c_*}{2}\Big(\frac{r_1}{32} + \frac{r_1}{2}\Big) \geq \frac{c_*}{8} r_1.$$

So long as $c_*/8 > 64\eta$ this is larger than r_1^2, so we may proceed so long as η is chosen small enough.

Applying Lemma 3.4 to u_1 gives that

$$u_1(y) \geq \frac{c_*}{2} y_n$$

for $y_n \geq 2\eta$, which translates to

$$u(x', t) \geq \frac{c_*}{2}(t - x_n)\frac{u(x + \frac{r_1 e_n}{2})}{r_1} \geq \frac{c_*}{16}(t - x_n)$$

for $t \in (x_n + 2\eta r_1, x_n + r_1)$. Note that from our choices, $x_n + r_1 \geq 2\eta$.

We may continue to apply Lemma 3.4 to u_k around x, with $r_k = 64\eta r_{k-1}$, in a similar manner, and we claim that this gives

$$u(x', t) \geq \Big(\frac{c_*}{8}\Big)^k \frac{c_*}{2}(t - x_n)$$

on $t \in (x_n + 2\eta r_k, x_n + r_k)$. Let us verify this claim by induction on k, with the $k = 1$ case already complete. As before we must ensure that u_k satisfies the differential inequalities, which follows from

$$u\Big(x + \frac{r_k e_n}{2}\Big) \geq \Big(\frac{c_*}{8}\Big)^{k-1} \frac{c_*}{2}\frac{r_k}{2} \geq (64\eta)^k r_k \geq r_k^2;$$

we are using here that $r_k/2 \in (2\eta r_{k-1}, r_{k-1})$ by construction and the inductive hypothesis. After applying the lemma and scaling back, we obtain for $t \in (x_n + 2\eta r_k, x_n + r_k)$ that

$$u(x', t) \geq \frac{c_*}{2}(t - x_n)\frac{u(x + \frac{r_k e_n}{2})}{r_k} \geq \Big(\frac{c_*}{8}\Big)^k \frac{c_*}{2}(t - x_n),$$

as claimed.

Now fix $t \in (x_n, 2\eta)$, and find the largest k for which $t \in (x_n + 2\eta r_k, x_n + r_k)$. As these intervals cover $(x_n, 2\eta)$ this is well defined, and as $r_k = (64\eta)^k$, we have

$$k \leq \frac{\log(t - x_n)}{\log 64\eta} \leq k + 1.$$

Using this with our estimate on u,

$$u(x', t) \geq \left(\frac{c_*}{8}\right)^k \frac{c_*}{2}(t - x_n) \geq c(t - x_n)^{-c'/\log\eta}(t - x_n) \geq c(t - x_n)^{1 - c'/\log\eta},$$

where c, c' only depend on c_* and explicit numbers. Select η small enough that $1 - c'/\log\eta < \beta$.

Finally, we observe that if $t \geq 2\eta$, then from our very first estimate

$$u(x', t) \geq \frac{c_*}{2}t \geq \frac{c_*}{4}(t - x_n),$$

as $|x_n| \leq \eta$ from the Lipschitz nature of $D_{L,1}$. For any point $z = (x', t)$ as above, $d(z, \partial D_{L,1}) \leq (t - x_n)$ (as $(x', x_n) \in \partial D_{L,1}$), so this reads

$$u(x', t) \geq c d^\beta((x', t), \partial D_{L,1})$$

for $(x', t) \in D_{L,1/64}$. ∎

Proof of Theorem 1.1. We write $\Omega = D_{L,1}$, as above. Set $U = D_{L,1}$, Q to be the collection of all sets of the form $U \cap B_r(x)$, and $H_{U \cap B_r(x)}[f, g]$ the Perron solution operator, mapping right-hand side f and boundary data g to the unique viscosity solution. Then properties (P1), (P2), and (P4) are immediate, while (P3) and (P5) follow from the viscosity comparison principle (see [6]). Set $V = \{u \in C(\bar{U}) : u \geq 0,\ -A \leq F(D^2u, \nabla u) \leq A \text{ on } U\}$ with constant A to be chosen; then (P6) follows from Lemma 3.3. Finally, (P7) is the Krylov–Safonov Harnack inequality, while (P8) may be found in [6, Lemma 2.4], combined with Lemma 3.2 above.

Fix $\beta \in (1, 2)$ and apply Lemma 3.5 to u and v, selecting η, ε sufficiently small. This gives

$$u(x), v(x) \geq c d^\beta(x, \partial U \cap B_1)$$

on $B_{1/2} \cap U$. This gives that if $x^0 = c_* e_n/4$, $u(x^0), v(x^0) \geq c$. On the other hand, applying the Harnack inequality on a region bounded away from ∂U and containing $e_n/2, x^0$ gives that $u(x^0), v(x^0) \leq C$. The functions $u_1(x) = u(x)/u(x^0)$ and $u_2(x) = v(x)/v(x^0)$ solve $-C \leq F(D^2u_i, \nabla u_i) \leq C$, so in particular in V if A is chosen appropriately. Applying Theorem 2.2 gives

$$\sup_{U \cap B_{c_*/2}} \frac{u}{v} \leq C \quad \sup_{B_{c_*/2} \cap U} \frac{u_1}{u_2} \leq C.$$

The statement as written (on $U \cap B_{1/2}$) then follows after a standard covering argument. ∎

4. p-Laplacian boundary Harnack

We now demonstrate the versatility of this approach by showing the same result for the p-Laplacian ($1 < p < \infty$) defined through $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$. The analogue of Lemma 3.2 (the boundary Harnack principle for the p-Laplace with right-hand side zero) is proven in [11]. We will also need the analogues of Lemmas 3.3 and 3.5 for the p-Laplacian. The analogue of Lemma 3.5 is proven in the same manner for the p-Laplacian. However, proving Lemma 3.3 for the p-Laplacian is more difficult: to see why, note that a difference $u - v$ of two solutions to the (inhomogeneous) p-Laplacian does not satisfy a PDE of the same type; rather, at best it satisfies a kind of linearized equation with coefficients dependent on ∇u and ∇v. Our approach here will be to establish bounds on these gradients and then work with this linearized equation.

4.1. Growth estimates

Lemma 4.1. *Let $\beta \in (1,2)$. Then there are constants $\varepsilon, \eta > 0$ such that if $u \geq 0$ on $D_{L,1}$, $u(e_n/2) \geq 1$, $u = 0$ on $\partial D_{L,1} \cap B_1$,*

$$-1 \leq \Delta_p u \leq \varepsilon,$$

and $L \leq \eta$, then

$$u(x) \geq c_1 d^{\beta}(x, \partial D_{L,1})$$

for $x \in D_{L,1/64}$.

Proof. The proof follows that of Lemma 3.5 with minor modifications, which we explain here. First, in the proof of Lemma 3.4 we used a barrier function $\phi(x) = |x|^{-q}$ which was a subsolution to the equation on $B_1 \setminus \{0\}$. For large values of q (depending on p), this also has $\Delta_p u \geq 1$. Indeed, the p-Laplacian is given by

$$\Delta_p \phi = |\nabla u|^{p-2}\operatorname{Tr}\Big[I + (p-2)\frac{\nabla\phi \otimes \nabla\phi}{|\nabla\phi|^2}\Big]D^2\phi,$$

where Tr is the trace operator. The matrix in square brackets is independent of the form of ϕ for any radial, radially decreasing function, and at the point re_n the ijth entry is given by $\delta_{ij} + \delta_{in}\delta_{jn}(p-2)$. Computing $D^2\phi$ at this point, one may check that this is a diagonal matrix in the e_i basis with $\partial_{ii}\phi = -qr^{-q-2}$ for $i < n$ and $\partial_{nn}\phi = q(q-2)r^{-q-2}$. This gives that at re_n,

$$\operatorname{Tr}\Big[I + (p-2)\frac{\nabla\phi \otimes \nabla\phi}{|\nabla\phi|^2}\Big]D^2\phi \geq c(p,n)q(q-2)r^{-q-2}$$

for all q sufficiently large enough. On the other hand, $|\nabla\phi| = qr^{-q-1}$, so

$$\Delta_p\phi \geq cq(q-2)r^{-q-2}[qr^{-q-1}]^{p-2} \geq cq^{p-1}(q-2)r^{-[q+2+(p-2)(q+1)]}.$$

As $p > 1$, the exponent in square brackets $q + 2 + (p-2)(q+1) > 1$, so

$$\Delta_p\phi \geq cq^{p-1}(q-2)r^{-1} \geq 1$$

so long as q is chosen large enough in terms of c and p.

The only other modification needed is in (3.5) in the proof of Lemma 3.5, where we rescale $u_k(y) = u(x + r_k y)/u(x + r_k e_n/2)$ and compute the PDE. Here our equation is different, but we still have

$$\Delta_p u_k(x) = \frac{r_k^p}{u^{p-1}(x + r_k e_n/2)}(\Delta_p u)(x + r_k y),$$

which implies $-1 \le \Delta_p u_k \le \varepsilon$ so long as $u(x + r_k e_n/2) \ge r_k^{\frac{p}{p-1}}$. This may be ensured by replacing the condition $c_*/8 > 64\eta$ with $c_*/8 > (64\eta)^{\frac{1}{p-1}}$ on η. ■

We have an elementary Harnack principle at the boundary:

Lemma 4.2. *Assume $L \le \frac{1}{100}$. Let $u \ge 0$ on $D_{L,1}$ and satisfy $-1 \le \Delta_p u \le 1$. Assume, moreover, that $u = 0$ on $\partial D_{L,1} \cap B_1$, the graph part of the boundary, and $u(e_n/2) = 1$. Then there is a constant $C = C(n, p)$ such that*

$$\sup_{D_{L,1/2}} u \le C.$$

Proof. The proof is standard, and we follow the outline in [4, Theorem 11.5], highlighting the differences for including a right-hand side. The first tool is the interior Harnack inequality (see for instance [8, Theorem 7.10])

$$\sup_{B_{r/2}} u \le C_1\big(\inf_{B_{r/2}} u + C_2 r^{\gamma_1}\big)$$

for some positive constants C_1, C_2, γ_1 as long as $B_r \subset D_{L,1}$. From utilizing the interior Harnack inequality on a chain of balls, one obtains for a large enough $\gamma_2 > 0$ that

$$u(x) \le s^{-\gamma_2} \quad \text{whenever } d(x, \partial D_{L,1}) \ge s \text{ and } x \in B_{3/4}. \tag{4.1}$$

The second main tool follows from extending u to be zero on $B_1 \setminus D_{L,1}$ and noting that u is then a subsolution on all of B_1 to $\Delta_p w(x) = f(x)1_{D_{L,1}}(x)$. Applying the oscillation decay estimate for subsolutions ([8, Chapter 7]) leads to

$$\sup_{B_{r/2}} u \le \mu \sup_{B_r} u + C_3 r^{\gamma_3} \tag{4.2}$$

for constants $0 < \mu < 1$ and $C_3, \gamma_3 > 0$. We now let $M = \sup_{B_{1/2}} u$: we will show that if M is chosen large enough, then $\sup_{B_{3/4}} u = +\infty$, contradicting that u is continuous. We first choose M large enough so that $M - C_3 r^{\gamma_3} \ge M(1 + \mu)/2$. This in turn would imply that if $\sup_{B_{r/2}} u \ge M$, then by (4.2) we have

$$\sup_{B_r} u \ge \frac{1 + \mu}{2\mu} \sup_{B_{r/2}} u. \tag{4.3}$$

We set $T := (1 + \mu)/(2\mu) > 1$. Now let $M = u(x^0)$, and $y^0 \in \partial D_{L,1}$ such that $x^i = y^i$ for $1 \le i \le n - 1$. From (4.1) we have $M = u(x^0) \le d(x^0, \partial D_{L,1})^{-\gamma_2}$, so that $d(x^0, \partial D_{L,1}) \le$

M^{-1/γ_2}. Utilizing the Lipschitz constant L we have $d_0 := |x^0 - y^0| \le C_1 d(x^0, \partial D_{L,1}) \le C_1 M^{-1/\gamma_2}$. Applying (4.3) we have $\sup_{B_{2d_0}} u = u(x^1) \ge TM$. Next we define y^1 in a similar way as y^0 was defined, and apply the same reasoning as before, so that by (4.1) we have $d_1 := |x^1 - y^1| \le C_1(TM)^{-1/\gamma_2}$. Applying (4.3) we obtain $u(x^2) \ge T^2 M$. Repeating the process inductively, we get a sequence of points x^k satisfying

(i) $u(x^k) \ge T^k M$,

(ii) $|x^k - y^k| \le C(T^k M)^{-1/\gamma_2}$,

(iii) $|x^k - x^{k-1}| \le 4C(T^k M)^{-1/\gamma_2}$.

Choosing M large enough, we have by (iii) that $\sum_{k=1}^{\infty} |x^k - x^{k-1}| \le 1/16$, so that each $x^k \in B_{3/4}$. From (i) it then follows that $\sup_{B_{3/4}} u = +\infty$ which contradicts that u is continuous on compact subsets of B_1. ■

We use the following Liouville-type result.

Lemma 4.3. *If $u, v \ge 0$ and u, v satisfy $\Delta_p u, \Delta_p v = 0$ on $D_{L,\infty}$, with $u, v = 0$ on $\partial D_{L,\infty}$, then $u = cv$ for some constant c.*

Proof. We assume that u, v are both not identically zero. We use the classical boundary Harnack principle from [10] to show that u/v is uniformly Hölder continuous up to the boundary on any compact subset of $\mathbb{R}^n$. We normalize u so that $\lim_{x\to 0} u(x)/v(x) = 1$, and let $x^0 \in D_{L,\infty}$. The rescaled functions

$$u_R(x) := \frac{u(Rx)}{u(Re_n)}, \quad v_R := \frac{v(Rx)}{v(Re_n)}$$

are also p-harmonic. Furthermore, we have that $\lim_{x\to 0} u_R(x)/v_R(x) = 1$, and that u_R/v_R is Hölder continuous on $D_{L,2}$ with norms independent of R. Then by continuity of the quotient, for any $\varepsilon > 0$ there exists R large enough, so that

$$\left| \frac{u_R(x^0/R)}{v_R(x^0/R)} - \frac{u_R(0)}{v_R(0)} \right| \le C\left(\frac{|x^0|}{R}\right)^{\beta} < \varepsilon.$$

We note that

$$\frac{u_R(x^0/R)}{v_R(x^0/R)} = \frac{u(x^0)}{v(x^0)} \frac{v(Re_n)}{u(Re_n)},$$

and therefore $\lim_{R\to\infty} v(Re_n)/u(Re_n) = u(x^0)/v(x^0)$, and this limit is independent of the chosen x^0, so $u(x)/v(x)$ is constant. Combining with the normalization $u(0)/v(0) = 1$, we conclude $u = v$. ■

As a consequence, on $D_{L,\infty} = \mathbb{R}^n_+$ we have $u = c(x_n)_+$ in the configuration above. We will exploit this fact below and later.

Lemma 4.4. *Let $\beta \in (0,1)$. Then there is a constant $\eta > 0$ such that if $u \geq 0$ on $D_{L,1}$, $u(e_n/2) = 1$, $u = 0$ on $\partial D_{L,1} \cap B_1$,*

$$-1 \leq \Delta_p u \leq 1,$$

and $L \leq \eta$, then

$$u(x) \leq C_1 d^\beta(x, \partial D_{L,1/2} \cap B_1)$$

for $x \in D_{L,1/64}$.

Proof. We start by considering the solution w_L to

$$\begin{cases} \Delta_p w_L = -1 & \text{in } \mathcal{C}_{L,2}, \\ w_L = \phi & \text{on } \partial\mathcal{C}_{L,2}, \end{cases}$$

where $\mathcal{C}_{L,2} := \{(x', x_n) \in B_2 \mid x_n \geq -L|x'|\}$. Here ϕ is a continuous function with $\phi(x) = 0$ for $x_n = -L|x'|$ and $\phi(x) = x_n - \overline{L}$ for $x_n \geq -L|x'|$ where $\overline{L} = \sqrt{4L^2/(L^2+1)}$. As in the proof of Lemma 4.2, by extending w_L to be zero on $B_2 \setminus \mathcal{C}_{L,2}$, we have that w_L is a subsolution, and we can therefore apply the 1-sided oscillation decay estimate to obtain that w_L is Hölder continuous of order α up to the boundary.

Now fix $\beta \in (0,1)$. We wish to first prove that $w_L(te_n) \leq Mt^\beta$ for small enough L. Suppose this is not true; then there exists $L_k \to 0$ and $t_k \to 0$, such that $w_{L_k}(t_k e_n/2) > 2^{-\beta} w_{L_k}(t_k e_n)$. By rescaling with $w_k = w_{L_k}(t_k x)/w_k(t_k e_n/2)$, we have that $w_k(e_n/2) = 1$ and $|\Delta_p w_k| \leq C t_k^{p-\beta_1}$ where β_1 is the exponent from Lemma 4.1. Choosing L small enough we have that $p > \beta_1$. Using $C_2 w_{2,2}$ as a universal barrier at the boundary, we have that $w_k \to w$ with $w \geq 0$, $\Delta_p w = 0$ in $\{x_n > 0\}$, $w = 0$ on $\{x_n = 0\}$, $w(e_n) = 1$, and $w(e_n/2) \geq 2^{-\beta}$. From Lemma 4.3 we have that $w = 2x_n^+$, but this contradicts the fact that $w(e_n) \leq 2^\beta$. Thus, there is an $\eta_1 > 0$ such that if $L \leq \eta_1$, then our claim for w_L is true.

We now choose $\eta = \eta_1/2$. By employing Lemma 4.2, we have that $u \leq C' w_{\eta_1}(x+z)$ on $D_{L,1/2}$ at every point of $\partial D_{L,1/2}$. The conclusion then follows from the Hölder growth of w_{η_1}. ∎

4.2. Derivative lower bounds

Theorem 4.5. *There exist constants $C, \varepsilon, \eta, r > 0$ depending only on n and p such that if $L < \eta$,*

$$\begin{cases} -1 \leq \Delta_p u \leq \varepsilon & \text{on } D_{L,2}, \\ u = 0 & \text{on } \partial D_{L,2} \cap B_2, \\ u(e_n) = 1, & \\ u \geq 0 & \text{in } D_{L,2}, \end{cases}$$

then

$$\frac{1}{C}\frac{u(x)}{d(x,\partial D_{L,2})} \leq |\nabla u(x)| \leq C\frac{u(x)}{d(x,\partial D_{L,2})} \quad \text{whenever } x \in B_r.$$

Proof. Suppose by way of contradiction that the theorem is not true. Then there exist u_k, $D^k_{L_k,2}$, ε_k satisfying the assumptions with $\varepsilon_k \to 0$ and $x^k \in B_{r_k}$ and satisfying either

$$|\nabla u_k(x^k)| \leq \frac{1}{C}\frac{u_k(x^k)}{d(x^k, \partial D^k_{L_k,2})} \quad \text{or} \quad |\nabla u_k(x^k)| \geq C\frac{u_k(x^k)}{d(x^k, \partial D^k_{L_k,2})}. \tag{4.4}$$

Apply Lemma 4.1 to u_k to obtain (for some $c > 0$)

$$c d^{\beta_1}(z, \partial D^k_{L_k,2}) \leq u_k(z)$$

on $B_{1/32}$ for a $\beta_1 > 1$ to be chosen. Set $y^k = (x'_k, g_k(x'_k)) \in \partial D^k_{L_k,2}$, the projection of x^k onto the graphical part of the boundary of $D^k_{L_k,2}$ and $s_k = |x^k - y^k| \leq r_k$. We rescale with

$$\tilde{u}_k(x) := \frac{u_k(y^k + s_k x)}{u_k(y^k + s_k e_n)}.$$

Note that $y^k + s_k e_n = x^k$.

Let us verify the differential inequalities satisfied by $\tilde{u}_k$: if $A_k = \frac{(2s_k)^p}{u_k^{p-1}(y^k + s_k e_n)}$, then

$$-A_k \leq \Delta_p \tilde{u}_k \leq A_k \varepsilon_k$$

on $\widetilde{D}^k_{L_k,1/s_k} = (D^k_{L_k,2} - y^k)/s_k \cap B_{1/s_k}$. We claim that $A_k \to 0$. Indeed,

$$A_k \leq \frac{(2s_k)^p}{c s_k^{\beta_1(p-1)}} \leq C r_k^{p-\beta_1(p-1)},$$

which converges to 0 so long as $\beta_1 < p/(p-1)$.

Next, fix any large R. We have $\tilde{u}_k \geq 0$ and $\tilde{u}_k(e_n) = 1$ by construction. Applying Lemma 4.4 a finite number of times to $\tilde{u}_k$ on progressively larger balls which exhaust $\widetilde{D}^k_{L_k,R}$, we obtain

$$\tilde{u}_k(z) \leq C(R) d^{\beta_2}(z, \partial \widetilde{D}^k_{L_k,2}) \tag{4.5}$$

for a fixed $\beta_2 < 1$. Meanwhile, on any $U = B_R \cap \{x_n > \delta\}$, which lies entirely inside $\widetilde{D}^k_{L_k,1/s_k}$ for k large, from standard interior $C^{1,\alpha}$ estimates we have

$$\|\tilde{u}_k\|_{C^{1,\alpha}(U)} \leq C.$$

We may extract a subsequence along which $\tilde{u}_k$ converges in $C^{1,\alpha}(U)$ for every set U to a limiting function $u \geq 0$ on $\mathbb{R}^n_+$. From (4.5), we have that u is continuous up to $\{x_n = 0\}$ and vanishes along that set. The PDE passes to the limit to give $\Delta_p u = 0$. We also have $u(e_n) = 1$, and

$$\frac{\nabla u_k(x^k)}{u_k(x^k)} = \nabla \tilde{u}_k(e_n) \to \nabla u(e_n),$$

meaning that $|\nabla u(e_n)| \notin [\frac{1}{C}, C]$. Applying Lemma 4.3, however, gives that $u(x) = x_n$ (using $u(e_n) = 1$ here), so this is impossible. ∎

4.3. The approximation lemma

Lemma 4.6. *Let u be an H_0^1 function on $D_{L,1}$, $L \leq \frac{1}{10}$, which satisfies*

$$\int_{D_{L,1}} A\nabla u \cdot \nabla \phi \leq \int_{D_{L,1}} \phi$$

for all nonnegative $\phi \in C_0^1(D_{L,1})$, where A is a measurable matrix-valued function with

$$\lambda d^{\varepsilon}(x, \partial D_{L,1} \cap B_1) I \leq A \leq \lambda^{-1} d^{-\varepsilon}(x, \partial D_{L,1} \cap B_1) I.$$

Then if $\varepsilon < \varepsilon_0$ small enough, we have

$$\sup_{D_{L,1}} u \leq C(\lambda).$$

Proof. Let $u_k = (u - l_k)_+$, where $\{l_k\}$ is a strictly increasing sequence of real numbers. A straightforward approximation argument shows that u_k may be used as test functions, i.e.

$$\int \lambda d^{\varepsilon} |\nabla u_k|^2 \leq \int A\nabla u \cdot \nabla u_k \leq \int u_k, \tag{4.6}$$

where $d(x) = d(x, \partial D_{L,1} \cap B_1)$. Applying the Hölder inequality,

$$\int |\nabla u_k|^{2\alpha} = \int |\nabla u_k|^{2\alpha} \frac{d^{\varepsilon\alpha}}{d^{\varepsilon\alpha}} \leq \left(\int d^{\varepsilon} |\nabla u_k|^2\right)^{\alpha} \left(\int d^{\varepsilon\alpha/(\alpha-1)}(x)\right)^{(1-\alpha)}.$$

For any $\alpha < 1$, we may choose ε small enough that the rightmost factor is bounded. Now from the Sobolev embedding,

$$\|u_k\|_{L^{\frac{2\alpha n}{n-2\alpha}}} \leq C\|\nabla u_k\|_{L^{2\alpha}} \leq C\left(\int d^{\varepsilon} |\nabla u_k|^2\right)^{\frac{1}{2}}.$$

Choose $\alpha < 1$ so the exponent $q := \frac{2\alpha n}{n-2\alpha} > 2$. Applying (4.6) to the right-hand side and raising to the qth power,

$$\int u_k^q \leq C\left(\int d^{\varepsilon} |\nabla u_k|^2\right)^{q/2} \leq C\left(\int u_k\right)^{q/2}.$$

In particular, applying Hölder's inequality to the right-hand side and dividing gives

$$\int u_k^q \leq C. \tag{4.7}$$

Alternatively, we can obtain the recursion formula

$$\int u_{k+1} \leq \int_{u_k > l_{k+1} - l_k} u_k \leq \frac{1}{(l_k - l_{k-1})^{q-1}} \int u_k^q \leq \frac{C}{(l_k - l_{k-1})^{q-1}} \left(\int u_k\right)^{q/2}. \tag{4.8}$$

Next, select $l_k = 2^k$: for any $K > 2$, choosing m so that $2^m \leq K \leq 2^{m+1}$ and then combining (4.7) and (4.8) gives

$$\int (u-K)_+ \leq \int (u-2^m)_+ = \int u_m \leq \frac{C}{2^{(m-1)(q-1)}}\left(\int u_{m-1}\right)^{q/2} \leq \frac{C}{K^{q-1}}. \quad (4.9)$$

Now we make a different selection of l_k: $l_k = K + 1 - 2^{-k}$ with $K > 2$ large. From (4.8),

$$\int u_{k+1} \leq C 2^{k(q-1)}\left(\int u_k\right)^{q/2}.$$

If $\int u_0 \leq \delta$ for some δ depending on C and q here, the sequence $\{\int u_k\}_{k=1}^{\infty}$ converges to 0, which would give that $u \leq K+1$. Using (4.9), though,

$$\int u_0 = \int (u-K)_+ \leq \frac{C}{K^{q-1}} \leq \delta$$

if K is chosen large enough in terms of C and q. Thus for a large enough K, $u \leq K+1$, which implies the conclusion. ∎

Lemma 4.7. *There exist constants η, ε, r small such that if $L \leq \eta$, $u \geq 0$ on $D_{L,1}$, $u = 0$ on $\partial D_{L,1} \cap B_1$, $u(e_1/2) = 1$, and $-A_0 \leq \Delta_p u \leq A_0 \varepsilon$ on $D_{L,1}$ for some $A_0 \leq 1$, the following holds: if w satisfies*

$$\begin{cases} \Delta_p w = 0 & \text{on } D_{L,r}, \\ w = u & \text{on } \partial D_{L,r}, \end{cases}$$

then $|w-u| \leq CA_0$.

Proof. Set $d(x) = d(x, \partial D_{L,1} \cap B_1)$ below. Let f solve the following PDE:

$$\begin{cases} \Delta_p f = -1 & \text{on } D_{L,1}, \\ f = u & \text{on } \partial D_{L,1}. \end{cases}$$

From the maximum principle, $f \geq u$ and $f \geq w$. In particular, $C \geq f(e_1/2) \geq 1$, with the upper bound from the Harnack inequality, so applying Lemma 4.4 to $f/f(e_1/2)$ gives

$$u(x), w(x) \leq f(x) \leq C d^{\beta_1}(x)$$

on $D_{L,1/64}$ for $\beta_1 < 1$ fixed.

Apply Lemmas 4.1 and 4.5 to u for $\beta_2 > 1$ fixed, choosing η and ε so the assumptions are satisfied regardless of A_0. Set r to the smaller of the r in Theorem 4.5 and $1/64$; then we have

$$c d^{\beta_2}(x) \leq u(x)$$

and

$$|\nabla u(x)| \approx \frac{u(x)}{d(x)},$$

so

$$cd^{\beta_2-1}(x) \le |\nabla u(x)| \le Cd^{\beta_1-1}(x)$$

for $x \in D_{L,r}$. Now, take any $x \in D_{L,r}$ and $B_{d(x)/2}(x)$: on this ball, we may apply either the boundary or interior form of the $C^{1,\alpha}$ estimate for p-harmonic functions [7] to give that

$$|\nabla w(x)| \le C\frac{\max\{d^{\frac{p}{p-1}}(x), \sup_{B_{d(x)/2}} w\}}{d(x)} \le Cd^{\beta_1-1}(x).$$

Next, set

$$a(x) = \int_0^1 |\nabla u(x)t + \nabla w(x)(1-t)|^{p-2}\, dt.$$

The quantities ∇u, ∇w are locally bounded on the set $D_{L,r}$, so when $p \ge 2$ this is well defined on this region. When $p < 2$, note that $\nabla u \ne 0$, and so the integrand is an integrable function regardless of the value of ∇w, meaning a is still well defined. In a similar vein, we estimate a from above and below. If $p \ge 2$, then

$$a(x) \le C[|\nabla u(x)|^{p-2} + |\nabla w(x)|^{p-2}] \le Cd^{(\beta_1-1)(p-2)} \le Cd^{-\alpha}$$

so long as β_1 is chosen large enough relative to α, which will be determined below. When $p < 2$, the same computation instead gives

$$a(x) \ge [|\nabla u(x)| + |\nabla w(x)|]^{p-2} \ge cd^{(\beta_1-1)(p-2)} \ge cd^{\alpha}.$$

On the other hand, we have

$$|\nabla u(x)t + \nabla w(x)(1-t)| \ge t|\nabla u| - (1-t)|\nabla w| \ge \frac{1}{4}|\nabla u|$$

for $t \ge \frac{3}{4}$ if $|\nabla w| \le |\nabla u|$. If instead $|\nabla w| \ge |\nabla u|$, we get

$$|\nabla u(x)t + \nabla w(x)(1-t)| \ge (1-t)|\nabla w| - t|\nabla u| \ge \frac{1}{4}|\nabla w| \ge \frac{1}{4}|\nabla u|$$

for $t < \frac{1}{4}$. In either case this holds on an interval of length $\frac{1}{4}$, so if $p > 2$,

$$a(x) \ge c|\nabla u|^{p-2} \ge cd^{(\beta_2-1)(p-2)}(x) \ge cd^{\alpha}(x)$$

if β_2 is small enough. Finally, for $p < 2$ one may check that

$$\int_0^1 |\nabla u(x)t + \nabla w(x)(1-t)|^{p-2}\, dt \le C|\nabla u|^{p-2} \le Cd^{(\beta_2-1)(p-2)} \le Cd^{-\alpha}$$

by directly computing the integral. To summarize, we have shown that

$$cd^{\alpha} \le a(x) \le Cd^{-\alpha}.$$

Now consider the matrix

$$a_{ij}(x) = \int_0^1 |\nabla u(x)t + \nabla w(x)(1-t)|^{p-2} m_{ij}^t \, dt,$$

where

$$m_{ij}^t = \delta_{ij} + (p-2)\frac{(u_i t + w_i(1-t))(u_j t + w_j(1-t))}{|\nabla u(x)t + \nabla w(x)(1-t)|^2}.$$

For any fixed t and $\xi \in \mathbb{R}^n$, the sum $m_{ij}^t \xi_i \xi_j$ is

$$m_{ij}^t \xi_i \xi_j = |\xi|^2 + (p-2)\frac{|\langle \nabla(ut + w(1-t)), \xi\rangle|^2}{|\nabla u(x)t + \nabla w(x)(1-t)|^2},$$

so we have $(p-1)|\xi|^2 \leq m_{ij}^t \xi_i \xi_j \leq |\xi|^2$ for $1 < p \leq 2$, and $|\xi|^2 \leq m_{ij}^t \xi_i \xi_j \leq (p-1)|\xi|^2$ for $p > 2$. Now let $\lambda = p-1$ for $1 < p \leq 2$ and $\lambda^{-1} = p-1$ for $2 < p < \infty$. Using this,

$$a_{ij}(x)\xi_i \xi_j = \int_0^1 |\nabla u(x)t + \nabla w(x)(1-t)|^{p-2} m_{ij}^t \xi_i \xi_j \, dt \geq a(x)\lambda|\xi|^2,$$

and similarly $a_{ij}(x) \leq \lambda^{-1}|\xi|^2 a(x)$.

The point of this a_{ij} is that, if $F(z) = |z|^{p-2}z$,

$$F_i(\nabla u) - F_i(\nabla w) = \int_0^1 \partial_t F(\nabla u(x)t + \nabla w(x)(1-t)) \, dt = a_{ij}(x)(u_j - w_j).$$

Setting $h = u - w$, we have shown that

$$\Delta_p u - \Delta_p w = \operatorname{div}[F(\nabla u) - F(\nabla w)] = \partial_i(a_{ij}h_j)$$

on $D_{L,r}$ (in the distributional sense). In particular,

$$-A_0 \leq \partial_i(a_{ij}h_j) \leq A_0$$

from the equations on u and w. Apply Lemma 4.6 to $\pm\frac{h(r\cdot)}{A_0}$, using our bounds on a_{ij} and choosing α small enough, to get that

$$|h| \leq CA_0$$

on $D_{L,1}$. This completes the argument. ∎

We may reformulate this approximation lemma in a more helpful way:

Lemma 4.8. *For every $\alpha > 0$, there exist constants η, ε, r_0 small such that if $L \leq \eta$, $u \geq 0$ on $D_{L,1}$, $u = 0$ on $\partial D_{L,1} \cap B_1$, $u(e_1/2) = 1$, and $-1 \leq \Delta_p u \leq \varepsilon$ on $D_{L,1}$, the following holds: if w satisfies*

$$\begin{cases} \Delta_p w = 0 & \text{on } D_{L,r}, \\ w = u & \text{on } \partial D_{L,r}, \end{cases}$$

with $r \leq r_0$, then $|w - u| \leq Cr^{2-\alpha}$.

Proof. First, apply Lemma 4.1 to u to obtain that $cd^\beta(x) \le u(x)$ for a β to be determined shortly on D_{L,r_0}. Set

$$u_1(y) = \frac{u(sy)}{u(se_n/2)},$$

where $s = \frac{r}{r_1}$, where we set r_1 to be the r in Lemma 4.7's conclusion and ask that $r_0 \le r_1^2$. Set $w_1(y) = \frac{w(sy)}{u(se_n/2)}$. Let us check the equation satisfied by u_1 on $D'_{L,1}$ (the rescaled domain):

$$\Delta_p u_1(y) = \frac{s^p}{u^{p-1}(se_n/2)} \Delta_p u(y/s) := A_0 \Delta_p u(y/s).$$

We wish to arrange to have $A_0 \le 1$. This may be done, as

$$u^{p-1}(se_n/2) \ge cs^{(p-1)\beta} \ge s^{p-\frac{1}{2}},$$

where we choose β sufficiently close to 1, and then r_0 small enough so as to have $s \le r_0/r_1$ absorb the constant. Apply Lemma 4.7 to deduce that

$$|u_1 - w_1| \le CA_0,$$

which scales back to

$$|u - w| \le CA_0 u(se_n/2) \le C \frac{s^p}{u^{p-2}(se_n/2)}.$$

As before, we may estimate

$$u^{p-2}(se_n/2) \ge cs^{(p-2)\beta} \ge cs^{p-2+\alpha},$$

by choosing β close to 1, so that

$$|u - w| \le Cs^{2-\alpha} \le Cr^{2-\alpha}.$$ ∎

Proof of Theorem 1.2. We apply Theorem 2.2 with H the solution mapping for the p-Laplacian, U our Lipschitz graph domain $D_{L,1}$, $U_{x,r} = U \cap B_r(x)$, and V the set of all u with $u > 0$ on U, $u = 0$ on ∂U, $u(e_n/2) = 1$, and $-1 \le \Delta_p u \le \varepsilon$ for ε small. Then all of the properties (P1)–(P5) and (P7)–(P8) follow in a standard way. For property (P6), we apply Lemma 4.8 to $u \in V$ to see that at least it is valid when centered at $x = 0$ and $r < r_0$. For $r \ge r_0$, the property is automatic from the bound in Lemma 4.2 instead. For other $x \in \partial U \cap B_{1/2}$, it then follows from a simple translation argument.

Lemma 4.1 ensures that the growth assumptions on u, v hold, so we may apply Theorem 2.2 to $u, v \in V$. The rest follows as in the proof of Theorem 1.3 or 1.1. ∎

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