

A variational approach to frozen planet orbits in helium

Kai Cieliebak, Urs Frauenfelder, and Evgeny Volkov

Abstract. We present variational characterizations of frozen planet orbits for the helium atom in the Lagrangian and the Hamiltonian picture. They are based on a nonlocal Levi-Civita regularization in Barutello, Ortega and Verzini (2021) with different time reparametrizations for the two electrons and lead to nonlocal functionals. Within this variational setup, we deform the helium problem to one where the two electrons interact only by their mean values and use this to deduce the existence of frozen planet orbits.

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1. Introduction

Frozen planet orbits are periodic orbits in the helium atom which play an important role in its semiclassical treatment [10, 12]. In such an orbit both electrons lie on a line on the same side of the nucleus. The inner electron undergoes consecutive collisions with the nucleus while the outer electron (the actual “frozen planet”) stays almost stationary at some distance, as shown below.



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An interesting aspect of frozen planet orbits is that they cannot be obtained using perturbative methods. Indeed, if the interaction between the two electrons is switched off, both electrons just fall into the nucleus. In order to deal with this problem, in [4] the second author replaced the instantaneous interaction between the two electrons by a mean interaction and showed analytically that in this case there exists a unique nondegenerate frozen planet orbit.

Variational setup. In this paper we present a variational approach to frozen planet orbits with instantaneous or mean interaction. One difficulty lies in the collisions of the inner electron with the nucleus, which need to be regularized in order to obtain a good functional analytic setup. A traditional way to regularize two-body collisions is the Levi-Civita regularization [7]. In the case of mean interactions our problem has delay and the application of the traditional Levi-Civita regularization becomes problematic. Fortunately, in a recent interesting paper by Barutello, Ortega and Verzini [2], a new nonlocal regularization was discovered. This regularization is motivated by the traditional Levi-Civita regularization, but in sharp contrast to the latter it is defined on the loop space and therefore fits well with our problem. Even for loops without collisions this transformation is quite intriguing. It is not smooth in the usual sense but scale smooth in the sense of Hofer, Wysocki and Zehnder [6].

We study two functionals $\mathcal{B}_{\text{av}}$ and $\mathcal{B}_{\text{in}}$ arising from regularizing frozen planet configurations for the mean and instantaneous interaction, respectively, as well as their linear interpolation $\mathcal{B}_r = r\mathcal{B}_{\text{in}} + (1-r)\mathcal{B}_{\text{av}}$, $r \in [0, 1]$. In general, it is not clear that critical points of a regularized action functional correspond precisely to the regularized solutions. It might happen that new exotic solutions appear as critical points; see [2] for examples of this phenomenon. Excluding such exotic critical points requires a careful analysis and this analysis occupies the main part of this paper. In particular, we prove (see Theorem 5.1) the following theorem:

Theorem A. *For each $r \in [0, 1]$, critical points of the regularized action functional $\mathcal{B}_r$ correspond precisely to frozen planet orbits for the interpolated interaction.*

Symmetries. There is a special case of frozen planet orbits referred to as *symmetric frozen planet orbits*. These are frozen planet orbits in which the outer electron has vanishing velocity whenever the inner electron collides with the nucleus or is at maximal distance from the nucleus; see [1]. We have the following theorem (see Theorem 6.7):

Theorem B. *The regularized action functional $\mathcal{B}_r$ is invariant under an involution such that the critical points fixed by the involution are precisely the symmetric frozen planet orbits.*

In view of Theorem B one can think of symmetric frozen planet orbits as a nonlocal generalization of brake orbits.

Regularity. In order to study critical points of $\mathcal{B}_r$, we introduce its L^2 -gradient as a map $\nabla \mathcal{B}_r: X \rightarrow Y$ from a suitable Hilbert manifold X to a Hilbert space Y . It satisfies the following theorem (see Theorem A.1):

Theorem C. *For each $r \in [0, 1]$, the L^2 -gradient $\nabla \mathcal{B}_r: X \rightarrow Y$ is a C^1 -Fredholm map of index 0.*

This result is nontrivial because $\nabla \mathcal{B}_r$ involves time reparametrizations depending on points of X as well as singular terms. Inspection of its proof shows that $\nabla \mathcal{B}_r$ is *not* of class C^2 .

Theorem C makes the functionals $\mathcal{B}_r$ amenable to classical variational methods such as index theory or Morse theory. In this paper we content ourselves with computing the mod 2 Euler number $\chi(\nabla \mathcal{B}_r)$, i.e. the count of zeros modulo 2 (after suitable perturbation; see Appendix C). On symmetric loops we find

$$\chi(\nabla \mathcal{B}_{\text{in}}) = \chi(\nabla \mathcal{B}_{\text{av}}) = 1,$$

where the first equality follows from homotopy invariance of the mod 2 Euler number and the second one from a further deformation and explicit computation (see Appendix D). As a consequence, we obtain the following corollary (see Theorem 6.1):

Corollary C. *For every $E < 0$ there exists a symmetric frozen planet orbit of energy E .*

Hamiltonian formulation. The regularized action functional $\mathcal{B}_r$ has an intriguing structure. It consists of two terms. The first term can be interpreted as a kinetic energy, but for a nonlocal metric which depends on the whole loop. The second term can be interpreted as the negative of a nonlocal potential which is defined on the loop space of the configuration space. We explain how in this situation a *nonlocal Legendre transform* can be carried out which produces a nonlocal Hamiltonian (see Section 7):

Theorem D. *After applying the nonlocal Legendre transform to the regularized action functional $\mathcal{B}_r$, the corresponding Hamiltonian delay equation reproduces frozen planet orbits.*

Thus there are two nonlocal approaches to frozen planet orbits, one Lagrangian and one Hamiltonian. This produces food for thought for many interesting research projects. For instance, the Lagrangian action functional has a Morse index at each frozen planet orbit. On the other hand, the Hessian of the Hamiltonian action functional is a Fredholm operator which gives rise to a nonlocal Conley–Zehnder index at each frozen planet orbit. Since in the local case the Morse index corresponds to the Conley–Zehnder index after Legendre transform we may ask the following question:

Question 1. How are the Morse index and the Conley–Zehnder index for frozen planet orbits related?

We remark that for a simple but yet nontrivial delay equation, the regularized free fall, equality has been established recently in [5].

The correspondence between these indices [11] is an important ingredient in the celebrated adiabatic limit argument by Salamon and Weber [8] relating Floer homology with the heat flow on chain level.

Question 2. Is there an analogy of the adiabatic limit argument of Salamon and Weber for frozen planet orbits?

2. The Kepler problem

When the interaction between the electrons is ignored, the system decouples into two one-electron systems, each of which is equivalent to the Kepler problem in celestial mechanics. In this section we recall the regularization procedure of Barutello, Ortega and Verzini [2] for the Kepler problem in the plane. It is based on the Levi-Civita regularization map $\mathcal{L}: \mathbb{C} \rightarrow \mathbb{C}$, $z \mapsto q = z^2$. Since we are interested in the case that the position q of the electron remains on the positive real line, we view the Levi-Civita map as a map to the nonnegative reals $\mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$, $z \mapsto q = z^2$; see (1).

2.1. Levi-Civita transformation

In this subsection we describe the Levi-Civita transformation without worrying about the regularity of the involved maps; precise statements will be given in the following subsections.

We abbreviate by $S^1 = \mathbb{R}/\mathbb{Z}$ the circle. We denote the L^2 -inner product of $z_1, z_2 \in L^2(S^1, \mathbb{R})$ by

$$\langle z_1, z_2 \rangle := \int_0^1 z_1(\tau) z_2(\tau) d\tau$$

and the L^2 -norm of $z \in L^2(S^1, \mathbb{R})$ by

$$\|z\| := \sqrt{\langle z, z \rangle}.$$

In the sequel we will work with Sobolev spaces $H^k = W^{k,2}$, but the only relevant norms and inner products will be the ones from L^2 .

Consider two maps

$$q: S^1 \rightarrow \mathbb{R}_{\geq 0}, \quad z: S^1 \rightarrow \mathbb{R}$$

related by the *Levi-Civita transformation*

$$q(t) = z(\tau)^2 \tag{1}$$

for a time change $t \leftrightarrow \tau$ satisfying $0 \leftrightarrow 0$ and

$$\frac{dt}{q(t)} = \frac{d\tau}{\|z\|^2}. \tag{2}$$

This implies that the mean values of q and $1/q$ are given by

$$\bar{q} := \int_0^1 q(t) dt = \int_0^1 \frac{z(\tau)^4}{\|z\|^2} d\tau = \frac{\|z^2\|^2}{\|z\|^2} \quad (3)$$

and

$$\int_0^1 \frac{dt}{q(t)} = \frac{1}{\|z\|^2}. \quad (4)$$

We will denote derivatives with respect to t by a dot and derivatives with respect to τ by a prime. Then the first and second derivatives of q and z (where they are defined) are related by

$$\dot{q}(t) = 2z(\tau)z'(\tau) \frac{d\tau}{dt} = \frac{2\|z\|^2 z'(\tau)}{z(\tau)} \quad (5)$$

and

$$\ddot{q}(t) = 2\|z\|^2 \frac{z''(\tau)z(\tau) - z'(\tau)^2}{z(\tau)^2} \frac{d\tau}{dt} = \frac{2\|z\|^4}{z(\tau)^4} (z''(\tau)z(\tau) - z'(\tau)^2). \quad (6)$$

Substituting z^2 and z'^2 by (1) and (5) this becomes

$$\ddot{q}(t) = \frac{1}{q(t)} \left(2\|z\|^4 \frac{z''(\tau)}{z(\tau)} - \frac{\dot{q}(t)^2}{2} \right). \quad (7)$$

The L^2 -norm of the derivative of q is given by

$$\|\dot{q}\|^2 = \int_0^1 \dot{q}(t)^2 dt = \int_0^1 \frac{4\|z\|^4 z'(\tau)^2}{z(\tau)^2} \frac{z(\tau)^2}{\|z\|^2} d\tau = 4\|z\|^2 \|z'\|^2. \quad (8)$$

2.2. Inverting the Levi-Civita transformation

In this subsection we prove that, under suitable technical hypotheses, the Levi-Civita transformation defines a 2-to-1 covering.

We begin with a precise definition of the Levi-Civita transformation. Let $z \in C^0(S^1, \mathbb{R})$ be a continuous function with finite zero set

$$Z_z := z^{-1}(0).$$

We associate to z a C^1 -map $t_z: S^1 \rightarrow S^1$ by

$$t_z(\tau) := \frac{1}{\|z\|^2} \int_0^\tau z(\sigma)^2 d\sigma. \quad (9)$$

Note that $t_z(0) = 0$ and

$$t'_z(\tau) = \frac{z(\tau)^2}{\|z\|^2}. \quad (10)$$

Since z has only finitely many zeros, this shows that t_z is strictly increasing and we conclude the following lemma:

Lemma 2.1. *If $z \in C^0(S^1, \mathbb{R})$ has only finitely many zeros, then the map $t_z: S^1 \rightarrow S^1$ defined by (9) is a homeomorphism.*

It follows that $t_z: S^1 \rightarrow S^1$ has a continuous inverse

$$\tau_z := t_z^{-1}: S^1 \rightarrow S^1.$$

Since t_z is of class C^1 , the function τ_z is also of class C^1 on the complement of the finite set $t_z(Z_z)$ with derivative

$$\dot{\tau}_z(t) = \frac{\|z\|^2}{z(\tau_z(t))^2}. \quad (11)$$

We define a continuous map $q: S^1 \rightarrow \mathbb{R}_{\geq 0}$ by

$$q(t) := z(\tau_z(t))^2. \quad (12)$$

Then the two maps z, q are related by the Levi-Civita transformation (1) with $\tau = \tau_z$. Their zero sets

$$Z_z = z^{-1}(0) \quad \text{and} \quad Z_q := q^{-1}(0) = t_z(Z_z)$$

are in bijective correspondence via t_z (or equivalently τ_z). Moreover, by (4) we have

$$\int_0^1 \frac{ds}{q(s)} = \frac{1}{\|z\|^2} < \infty.$$

Conversely, suppose we are given a map $q \in C^0(S^1, \mathbb{R}_{\geq 0})$ with finite zero set Z_q satisfying $\int_0^1 \frac{ds}{q(s)} < \infty$. We associate to q the time reparametrization $\tau_q: S^1 \rightarrow S^1$,

$$\tau_q(t) := \left(\int_0^1 \frac{ds}{q(s)} \right)^{-1} \int_0^t \frac{1}{q(s)} ds. \quad (13)$$

Then $\tau_q(1) = 1$, τ_q is of class C^1 outside the zero set $Z_q = q^{-1}(0)$ with derivative

$$\tau'_q(t) = \left(\int_0^1 \frac{ds}{q(s)} \right)^{-1} \frac{1}{q(t)}, \quad t \in S^1 \setminus Z_q. \quad (14)$$

By [2, Lemma 2.1] the map $\tau_q: S^1 \rightarrow S^1$ is a homeomorphism whose inverse $t_q := \tau_q^{-1}$ is of class C^1 and satisfies $t_q(1) = \tau_q^{-1}(1) = 1$ and

$$t'_q(\tau) = \left(\int_0^1 \frac{ds}{q(s)} \right) q(t_q(\tau)), \quad \tau \in S^1. \quad (15)$$

Suppose that $z: S^1 \rightarrow \mathbb{R}$ is a continuous function satisfying

$$z(\tau)^2 = q(t_q(\tau)). \quad (16)$$

Then z has finite zero set $Z_z = \tau_q(Z_q)$, so we can associate to z the homeomorphism $t_z: S^1 \rightarrow S^1$ defined by (9) and its inverse τ_z . We claim that

$$\tau_q = \tau_z \quad \text{and} \quad t_q = t_z. \quad (17)$$

It is enough to check the second equality. For this we compute

$$\begin{aligned} \int_0^\tau z(\sigma)^2 d\sigma &= \int_0^\tau q(t_q(\sigma)) d\sigma \\ &\stackrel{(*)}{=} \left(\int_0^1 \frac{ds}{q(s)} \right)^{-1} \int_0^{t_q(\tau)} ds \\ &= \left(\int_0^1 \frac{ds}{q(s)} \right)^{-1} t_q(\tau), \end{aligned}$$

where $(*)$ follows from the coordinate change $\sigma = \tau_q(s)$ and (14). Evaluating at $\tau = 1$ gives us

$$\frac{1}{\|z\|^2} = \int_0^1 \frac{ds}{q(s)}. \quad (18)$$

Therefore,

$$t_z(\tau) = \frac{1}{\|z\|^2} \int_0^\tau z(\sigma)^2 d\sigma = t_q(\tau)$$

and (17) is established. Hence q is the Levi-Civita transform of z defined by (12).

Equation (16) does not uniquely determine z for given q because the sign of z can be arbitrarily chosen on each connected component of $S^1 \setminus Z_z$. If Z_z consists of an *even* number of points, then we can determine z up to a global sign by the requirement that z changes its sign at each zero. Therefore, the preceding discussion shows the following lemma:

Lemma 2.2. *The Levi-Civita transformation $z \mapsto q$ given by (12) defines, for each even integer $m \in 2\mathbb{N}$, a surjective 2-to-1 map*

$$\begin{aligned} \mathcal{L}: \{z \in C^0(S^1, \mathbb{R}) \mid z \text{ has precisely } m \text{ zeros and switches sign at each zero}\} \\ \rightarrow \{q \in C^0(S^1, \mathbb{R}_{\geq 0}) \mid z \text{ has precisely } m \text{ zeros and } \int_0^1 \frac{ds}{q(s)} < \infty\}. \end{aligned}$$

The following lemma shows how additional regularity properties translate between z and q . Near each zero t_* of q we define the local sign function

$$s_*(t) := \begin{cases} -1, & t < t_*, \\ +1, & t > t_*. \end{cases}$$

If q is of class C^1 outside its zero set, we denote by

$$E_q(t) := \frac{\dot{q}^2(t)}{2} - \frac{N}{q(t)}, \quad t \in S^1 \setminus Z_q \quad (19)$$

its *Kepler energy* at time t , for some fixed $N > 0$. By (8) it corresponds under the Levi-Civita transformation to

$$E_z(\tau) := \frac{2\|z\|^4 z'(\tau)^2 - N}{z(\tau)^2}, \quad \tau \in S^1 \setminus Z_z. \quad (20)$$

Lemma 2.3. *Let z, q be as in Lemma 2.2 related by the Levi-Civita transform (12), and let k be a nonnegative integer. Then the following hold:*

- (a) $z \in H^1(S^1, \mathbb{R})$ if and only if $q \in H^1(S^1, \mathbb{R}_{\geq 0})$;
- (b) z is of class C^k outside its zeros if and only if q is of class C^k outside its zeros;
- (c) z is of class C^1 on all of S^1 if and only if q is of class C^1 outside Z_q and for each $t_* \in Z_q$ the following limit exists:

$$\lim_{t_* \neq t \rightarrow t_*} s_*(t) \sqrt{q(t)} \dot{q}(t);$$

- (d) z is of class C^1 with transverse zeros if and only if q is of class C^1 outside Z_q and for each $t_* \in Z_q$ the limit in (c) exists and is positive;
- (e) the energy $E_z: S^1 \setminus Z_z \rightarrow \mathbb{R}$ is defined and extends to a continuous function $S^1 \rightarrow \mathbb{R}$ if and only if the energy $E_q: S^1 \setminus Z_q \rightarrow \mathbb{R}$ is defined and extends to a continuous function $S^1 \rightarrow \mathbb{R}$;
- (f) the conditions in (e) imply those in (d).

Proof. Part (a) follows immediately from formula (8). For (b) just note that if z is of class C^k outside Z_z then t_z is of class C^{k+1} outside Z_z , so τ_z and therefore also q is of class C^{k+1} outside $t_z(Z_z) = Z_q$, and the same in the reverse direction.

For (c) and (d) suppose that z, q are of class C^1 outside their zero sets. In the following we will always denote by τ, t times related by the time transformation $t = t_z(\tau) = t_q(\tau)$. Consider a zero $t_* \in Z_q$ with corresponding $\tau_* \in Z_z$. Since z switches sign at τ_* we can write

$$z(\tau) = \varepsilon s_*(t) \sqrt{q(t)}$$

for $\tau \neq \tau_*$ near τ_* , with some sign $\varepsilon \in \{-1, 1\}$. Inserting this into (5) and solving for $z'(\tau)$ yields

$$z'(\tau) = \frac{\varepsilon s_*(t) \sqrt{q(t)} \dot{q}(t)}{2\|z\|^2}, \quad (21)$$

from which (c) and (d) follow.

Part (e) follows immediately from $E_z(\tau) = E_q(t_z(\tau))$. To see that (e) implies (d), note first that the existence and continuity of $E_q: S^1 \setminus Z_q \rightarrow \mathbb{R}$ implies that q is of class C^1 on $S^1 \setminus Z_q$. Suppose now that E_q extends to a continuous function $S^1 \rightarrow \mathbb{R}$, so for each $t_* \in Z_q$ the limit

$$\lim_{t_* \neq t \rightarrow t_*} \left(\frac{\dot{q}(t)^2}{2} - \frac{N}{q(t)} \right)$$

exists. This implies that $\dot{q}(t)^2 \rightarrow \infty$ as $t \rightarrow t_*$, in particular $\dot{q}(t) \neq 0$ for all $t \neq t_*$ close to t_* . Since $q(t) > 0$ for $t \neq t_*$ and $q(t_*) = 0$, this implies that

$$\dot{q}(t) = s_*(t) \sqrt{\dot{q}(t)^2} = s_*(t) \sqrt{2E_q(t) + \frac{2N}{q(t)}}$$

for all $t \neq t_*$ close to t_* . It follows that

$$\lim_{t_* \neq t \rightarrow t_*} s_*(t) \sqrt{q(t)} \dot{q}(t) = \lim_{t_* \neq t \rightarrow t_*} \sqrt{2E(t)q(t) + 2N} = \sqrt{2N} > 0,$$

which is the condition in (d). This proves the lemma. $\blacksquare$

Note that $z \in H^1$ and the extension of E_q to a continuous function $S^1 \rightarrow \mathbb{R}$ implies the existence of the integral

$$\int_0^1 E_q(t) dt = \frac{\|\dot{q}\|^2}{2} - \int_0^1 \frac{dt}{q(t)},$$

and therefore $\int_0^1 \frac{ds}{q(s)} < \infty$. Hence Lemma 2.3 implies the following corollary:

Corollary 2.4. *For each even $m \in 2\mathbb{N}$ the Levi-Civita map $\mathcal{L}$ of Lemma 2.2 restricts to a surjective 2-to-1 map*

$$\mathcal{L}: \mathcal{C}_{\text{ce}}^1(S^1, \mathbb{R}) \rightarrow \mathcal{H}_{\text{ce}}^1(S^1, \mathbb{R}_{\geq 0}),$$

where

- $\mathcal{C}_{\text{ce}}^1(S^1, \mathbb{R})$ denotes the set of $z \in C^1(S^1, \mathbb{R})$ with precisely m zeros such that all zeros are transverse and the energy E_z extends to a continuous function $S^1 \rightarrow \mathbb{R}$, and
- $\mathcal{H}_{\text{ce}}^1(S^1, \mathbb{R}_{\geq 0})$ denotes the set of $q \in H^1(S^1, \mathbb{R}_{\geq 0})$ with precisely m zeros such that q is of class C^1 outside its zeros and the energy E_q extends to a continuous function $S^1 \rightarrow \mathbb{R}$.

2.3. Variational characterization of generalized solutions

An electron moving in the electric field of a fixed nucleus with charge $N > 0$ is described by Newton's equation

$$\ddot{q}(t) = -\frac{N}{q(t)^2}. \quad (22)$$

Alternatively, it describes the Kepler problem of a body of mass 1 moving in the gravitational field of a body of mass N . Its periodic solutions avoiding the origin are the critical points of the Lagrangian action functional

$$\mathcal{S}(q) := \frac{1}{2} \int_0^1 \dot{q}(t)^2 dt + \int_0^1 \frac{N}{q(t)} dt.$$

Since all periodic solutions have collisions there are actually no critical points for this unregularized functional. Let q and z be related by the Levi-Civita transformation (1). Using the relations (2), (5) and (18), we rewrite the Lagrangian action of q in terms of z :

$$\begin{aligned} \mathcal{S}(q) &= \frac{1}{2} \int_0^1 \frac{4\|z\|^4 z'(\tau)^2}{z(\tau)^2} \frac{z(\tau)^2}{\|z\|^2} d\tau + \frac{N}{\|z\|^2} \\ &= 2\|z\|^2 \|z'\|^2 + \frac{N}{\|z\|^2}. \end{aligned}$$

We denote the resulting action functional of z by

$$\mathcal{Q}: H^1(S^1, \mathbb{R}) \setminus \{0\} \rightarrow \mathbb{R}, \quad \mathcal{Q}(z) := 2\|z\|^2\|z'\|^2 + \frac{N}{\|z\|^2}. \quad (23)$$

Following [2] we call $q \in H^1(S^1, \mathbb{R}_{\geq 0})$ a *generalized solution of (22)* if

- (1) the zero set $Z = q^{-1}(0) \subset S^1$ is finite and has an even number of elements;
- (2) on $S^1 \setminus Z$ the map q is smooth and satisfies (22);
- (3) the energy

$$E(t) := \frac{\dot{q}(t)^2}{2} - \frac{N}{q(t)}, \quad t \in S^1 \setminus Z$$

extends to a continuous function $E: S^1 \rightarrow \mathbb{R}$.

Note that the energy E is then constant (by conservation of energy) and negative (for q to be bounded).

Theorem 2.5 (Barutello, Ortega and Verzini [2]). *Under the Levi-Civita transformation (1) with time change (2), critical points $z: S^1 \rightarrow \mathbb{R}$ of the action functional $\mathcal{Q}$ defined in (23) are in 2-to-1 correspondence with generalized solutions $q: S^1 \rightarrow \mathbb{R}_{\geq 0}$ of (22).*

In the remainder of this section we will spell out the proof of this theorem because it uses some ingredients that will also be needed in later sections.

2.4. From critical points to generalized solutions

The differential of $\mathcal{Q}$ at $z \in H^1(S^1, \mathbb{R}) \setminus \{0\}$ in the direction $v \in H^1(S^1, \mathbb{R})$ is given by

$$D\mathcal{Q}(z)v = 4\|z\|^2\langle z', v' \rangle + 4\|z'\|^2\langle z, v \rangle - \frac{2N}{\|z\|^4}\langle z, v \rangle. \quad (24)$$

This shows that a critical point z has a weak second derivative and satisfies the second-order ODE with constant coefficient

$$z''(\tau) = a z(\tau), \quad a = \frac{\|z'\|^2}{\|z\|^2} - \frac{N}{2\|z\|^6}. \quad (25)$$

It follows that z is smooth. Moreover, $z \in H^1(S^1, \mathbb{R})$ implies boundedness of z and thus forces $a < 0$. So z is a shifted sine function. In particular, z has *transverse zeros* in the sense that $z'(\tau) \neq 0$ whenever $z(\tau) = 0$. In particular, its zero set

$$Z := \{\tau \in S^1 \mid z(\tau) = 0\}$$

is finite. We associate to z the smooth map $t_z: S^1 \rightarrow S^1$ defined by (9). By Lemma 2.1, the map t_z is a homeomorphism with continuous inverse $\tau_z = t_z^{-1}: S^1 \rightarrow S^1$. Since t_z is smooth, the function τ_z is also smooth on the complement of the finite set $t_z(Z)$ with derivative given by (11). We define a continuous map $q: S^1 \rightarrow S^1$ by

$$q(t) := z(\tau_z(t))^2.$$

Then the two maps $z, q: S^1 \rightarrow \mathbb{R}$ are smooth except at finitely many points and related by the Levi-Civita transformation (1) with $\tau = \tau_z$. Substituting z'' by (25) in (7) we get the following ODE for q at points $t \in S^1 \setminus t_z(Z)$:

$$\ddot{q}(t) = \frac{1}{q(t)} \left(2\|z\|^4 a - \frac{\dot{q}(t)^2}{2} \right) = \frac{1}{q(t)} \left(c - \frac{\dot{q}(t)^2}{2} \right) \quad (26)$$

with the constant (using (4) and (8))

$$c := 2\|z\|^4 a = 2\|z'\|^2 \|z\|^2 - \frac{N}{\|z\|^2} = \frac{\|\dot{q}\|^2}{2} - \int_0^1 \frac{N}{q(s)} ds.$$

Since at a local maximum t of q we must have $\dot{q}(t) = 0$ and $\ddot{q}(t) < 0$, it follows from (25) that $c < 0$, hence $\ddot{q} < 0$ outside its zeros. Consider now two consecutive zeros $t_- < t_+$ of q and the smooth map

$$\beta := \frac{\ddot{q}}{q}: (t_-, t_+) \rightarrow \mathbb{R}_-.$$

Then (omitting the t) we have $\beta q^2 = q\ddot{q} = c - \dot{q}^2/2$ and taking a time derivative yields

$$\dot{\beta} q^2 + 2\beta q \dot{q} = -\dot{q} \ddot{q} = -\beta q \dot{q},$$

hence

$$\dot{\beta} q = -3\beta \dot{q}. \quad (27)$$

Lemma 2.6. *Equation (27) for functions $q > 0$ and $\beta < 0$ on (t_-, t_+) implies that*

$$\beta = -\frac{\mu}{q^3} \quad (28)$$

on (t_-, t_+) for some constant $\mu > 0$.

Proof. Dividing both sides of (28) by $q\beta$ yields

$$\frac{d}{dt} \log(-\beta) = -3 \frac{d}{dt} \log(q),$$

which by integration implies the lemma. ■

The lemma implies that

$$\ddot{q} = -\mu/q^2 \quad (29)$$

on (t_-, t_+) . Combining this with (26) yields

$$-\frac{\mu}{q} = q\ddot{q} = c - \frac{\dot{q}^2}{2} = \frac{\|\dot{q}\|^2}{2} - \int_0^1 \frac{N}{q(s)} ds - \frac{\dot{q}^2}{2} \quad (30)$$

on (t_-, t_+) . Multiplying this equation by $-q$, the smoothness of q gives

$$\mu = \lim_{t \rightarrow t_{\pm}} -q(t) \left(\frac{\|\dot{q}\|^2}{2} - \int_0^1 \frac{N}{q(s)} ds - \frac{\dot{q}(t)^2}{2} \right).$$

This shows that the constant μ is the same for each interval between consecutive zeros of q , so (30) holds on all of $S^1 \setminus t_z(Z)$. Now integrating (30) over S^1 yields

$$\mu = N,$$

so (29) becomes Newton's equation (22). Inserting $\mu = N$ in (30) shows that the energy

$$E = \frac{\dot{q}^2}{2} - \frac{N}{q} = \frac{\|\dot{q}\|^2}{2} - \int_0^1 \frac{N}{q(s)} ds$$

is constant, so q is a generalized solution of (22).

2.5. From generalized solutions to critical points

Now let $q \in H^1(S^1, \mathbb{R}_{\geq 0})$ be a generalized solution of (22). Integrating the constant energy yields

$$E = \int_0^1 \frac{\dot{q}(t)^2}{2} dt - \int_0^1 \frac{N}{q(t)} dt.$$

Since $q \in H^1$, the first term on the right-hand side is finite and it follows that

$$\int_0^1 \frac{1}{q(t)} dt < \infty.$$

As in Section 2.2, we associate to q the time reparametrization $\tau_q: S^1 \rightarrow S^1$ defined by (13) and its inverse $t_q = \tau_q^{-1}$. Recall that τ_q is smooth outside the zero set $Z_q = q^{-1}(0)$ and t_q is of class C^1 . We define a continuous function $z: S^1 \rightarrow \mathbb{R}$ by the condition

$$z(\tau)^2 = q(t_q(\tau))$$

and the requirement that z changes its sign at each zero. This is possible because q has an even number of zeros, and it determines z up to a global sign. Using the change of variable $\tau = \tau_q(t)$ we find

$$\begin{aligned} \|z\|^2 &= \int_0^1 z(\tau)^2 d\tau = \left(\int_0^1 \frac{1}{q(s)} ds \right)^{-1} \int_0^1 z(\tau_q(t))^2 \frac{1}{q(t)} dt \\ &= \left(\int_0^1 \frac{1}{q(s)} ds \right)^{-1}. \end{aligned}$$

It follows that z and q are related by the Levi-Civita transformation (1) with time change $t \rightarrow \tau = \tau_q$ satisfying (2).

In the sequel we will drop the arguments t and τ . Combining equations (6) and (22) outside Z_q we obtain

$$-\frac{N}{z^4} = -\frac{N}{q^2} = \ddot{q} = 2 \frac{\|z\|^4}{z^4} (z''z - z'^2),$$

hence

$$z''(\tau)z(\tau) - z'(\tau)^2 = -\frac{N}{2\|z\|^4}, \quad \tau \in S^1 \setminus \tau_q(Z_q). \quad (31)$$

Consider the function

$$\beta := \frac{z''}{z}: S^1 \setminus \tau(Z_q) \rightarrow \mathbb{R}.$$

Inserting this into (31) and using equations (1) and (5) we find

$$-\frac{N}{2\|z\|^4} = \beta z^2 - z'^2 = q\left(\beta - \frac{\dot{q}^2}{4\|z\|^4}\right),$$

and solving for β yields

$$\beta(\tau_q(t)) = \frac{1}{2\|z\|^4} \left(\frac{\dot{q}(t)^2}{2} - \frac{N}{q(t)} \right) = \frac{1}{2\|z\|^4} E(t), \quad t \in S^1 \setminus Z_q.$$

Since q is a generalized solution, the energy E is constant and negative, thus $\beta(\tau) \equiv E/2\|z\|^4 < 0$ and the definition of β implies

$$z''(\tau) = \frac{E}{2\|z\|^4} z(\tau), \quad \tau \in S^1 \setminus \tau_q(Z_q).$$

The solutions of this ODE are shifted sine functions. So the condition that z switches sign at each zero implies that it defines a smooth function $z: S^1 \rightarrow \mathbb{R}$ solving the ODE on all of S^1 . Rewriting the energy via (4) and (8) as

$$E = \frac{\|\dot{q}\|^2}{2} - \int_0^1 \frac{N}{q(t)} dt = 2\|z\|^2 \|z'\|^2 - \frac{N}{\|z\|^2}$$

and inserting this into the ODE, we see that z satisfies ODE (25) and is therefore a critical point of $\mathcal{Q}$.

Together with the previous subsection this concludes the proof of Theorem 2.5.

3. Mean interaction

In this section we consider a “helium atom” in which the two electrons interact by the mean values $\bar{q}_i = \int_0^1 q_i(t) dt$ according to

$$\begin{cases} \ddot{q}_1(t) = -\frac{2}{q_1(t)^2} + \frac{1}{(\bar{q}_1 - \bar{q}_2)^2}, \\ \ddot{q}_2(t) = -\frac{2}{q_2(t)^2} - \frac{1}{(\bar{q}_1 - \bar{q}_2)^2}, \end{cases} \quad (32)$$

where we impose the condition

$$\bar{q}_1 > \bar{q}_2. \quad (33)$$

3.1. Variational characterization of generalized solutions

Solutions of (32) are the critical points of the action functional

$$\mathcal{S}_{\text{av}}(q_1, q_2) := \sum_{i=1}^2 \left(\frac{1}{2} \int_0^1 \dot{q}_i(t)^2 dt + \int_0^1 \frac{2}{q_i(t)} dt \right) - \frac{1}{\bar{q}_1 - \bar{q}_2}.$$

For $i = 1, 2$ let q_i and z_i be related by Levi-Civita transformations

$$q_i(t) = z_i(\tau_i(t))^2 \quad (34)$$

for time changes $\tau_i(t)$ satisfying $\tau_i(0) = 0$ and

$$\frac{dt}{q_i(t)} = \frac{d\tau_i(t)}{\|z_i\|^2}. \quad (35)$$

Note that we perform different time changes for the two electrons. Then all the relations in Section 2.1 hold with $(q, z, \tau) = (q_i, z_i, \tau_i)$. In particular, we can use (3) to rewrite the interaction term in terms of the z_i :

$$\begin{aligned} -\frac{1}{\bar{q}_1 - \bar{q}_2} &= -\frac{1}{\frac{\|z_1^2\|^2}{\|z_1\|^2} - \frac{\|z_2^2\|^2}{\|z_2\|^2}} \\ &= -\frac{\|z_1\|^2 \|z_2\|^2}{\|z_1^2\|^2 \|z_2\|^2 - \|z_2^2\|^2 \|z_1\|^2}. \end{aligned}$$

We denote the resulting *mean interaction* of (z_1, z_2) by

$$\mathcal{A}(z_1, z_2) := -\frac{\|z_1\|^2 \|z_2\|^2}{\|z_1^2\|^2 \|z_2\|^2 - \|z_2^2\|^2 \|z_1\|^2}. \quad (36)$$

This quantity is naturally defined on the space

$$\begin{aligned} \mathcal{H}_{\text{av}}^1 := \{z = (z_1, z_2) \in H^1(S^1, \mathbb{R}^2) \mid &\|z_1\| > 0, \|z_2\| > 0, \\ &\frac{\|z_1^2\|^2}{\|z_1\|^2} > \frac{\|z_2^2\|^2}{\|z_2\|^2}\}. \end{aligned} \quad (37)$$

Note that $\mathcal{H}_{\text{av}}^1$ is an open subset of the Hilbert space $H^1(S^1, \mathbb{R}^2)$ and the last condition in its definition corresponds to condition (33). On $\mathcal{H}_{\text{av}}^1$ we consider the functional

$$\mathcal{B}_{\text{av}}: \mathcal{H}_{\text{av}}^1 \rightarrow \mathbb{R}, \quad \mathcal{B}_{\text{av}}(z_1, z_2) := \mathcal{Q}(z_1) + \mathcal{Q}(z_2) + \mathcal{A}(z_1, z_2), \quad (38)$$

with the functionals

$$\mathcal{Q}(z_i) = 2\|z_i\|^2 \|z_i'\|^2 + \frac{2}{\|z_i\|^2}$$

from (23) with charge $N = 2$.

We call $(q_1, q_2) \in H^1(S^1, \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})$ a *generalized solution of (32)* if for $i = 1, 2$ the following holds:

- (1) $\bar{q}_1 > \bar{q}_2$;
- (2) the zero sets $Z_i = q_i^{-1}(0) \subset S^1$ are finite and each have an even number of elements;
- (3) the restricted maps $q_i: S^1 \setminus Z_i \rightarrow \mathbb{R}_{\geq 0}$ are smooth and satisfy (32);
- (4) the energies

$$E_i(t) := \frac{\dot{q}_i(t)^2}{2} - \frac{2}{q_i(t)}, \quad t \in S^1 \setminus Z_i$$

extend to continuous functions $E_i: S^1 \rightarrow \mathbb{R}$.

Note that the individual energies E_i need not be constant, but their sum is constant and negative.

Theorem 3.1 (Generalized solutions with mean interaction). *Under the Levi-Civita transformations (34) with time changes (35), critical points (z_1, z_2) of the action functional $\mathcal{B}_{\text{av}}$ defined in (38) are in 4-to-1 correspondence with generalized solutions (q_1, q_2) of (32).*

The proof of this theorem will take up the remainder of this section.

3.2. The differential of $\mathcal{B}_{\text{av}}$

The differential of the mean interaction $\mathcal{A}$ at $(z_1, z_2) \in \mathcal{H}_{\text{av}}^1$ in the direction $(v_1, v_2) \in H^1(S^1, \mathbb{R}^2)$ is given by

$$\begin{aligned} D\mathcal{A}[z_1, z_2](v_1, v_2) &= -2 \frac{\|z_2\|^2 \cdot \langle z_1, v_1 \rangle + \|z_1\|^2 \cdot \langle z_2, v_2 \rangle}{\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2} \\ &\quad + 2 \frac{\|z_1\|^2 \cdot \|z_2\|^2 (2\|z_2\|^2 \cdot \langle z_1^3, v_1 \rangle + \|z_1^2\|^2 \cdot \langle z_2, v_2 \rangle)}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} \\ &\quad - 2 \frac{\|z_1\|^2 \cdot \|z_2\|^2 (2\|z_1\|^2 \cdot \langle z_2^3, v_2 \rangle + \|z_2^2\|^2 \cdot \langle z_1, v_1 \rangle)}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} \\ &= -2 \frac{\|z_2\|^4 \cdot \|z_1^2\|^2}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} \langle z_1, v_1 \rangle \\ &\quad + 2 \frac{\|z_1\|^4 \cdot \|z_2^2\|^2}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} \langle z_2, v_2 \rangle \\ &\quad + 4 \frac{\|z_1\|^2 \cdot \|z_2\|^4}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} \langle z_1^3, v_1 \rangle \\ &\quad - 4 \frac{\|z_1\|^4 \cdot \|z_2\|^2}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} \langle z_2^3, v_2 \rangle. \end{aligned} \tag{39}$$

Combined with (24) with $z = z_i$, $v = v_i$ and $N = 2$ for the differentials $D\mathcal{Q}(z_i)v_i$, this yields the differential of $\mathcal{B}_{\text{av}}$:

$$\begin{aligned}
 D\mathcal{B}_{\text{av}}[z_1, z_2](v_1, v_2) = & 4 \sum_{i=1}^2 \left(\|z_i\|^2 \langle z'_i, v'_i \rangle + \|z'_i\|^2 \cdot \langle z_i, v_i \rangle - \frac{\langle z_i, v_i \rangle}{\|z_i\|^4} \right) \\
 & - 2 \frac{\|z_2\|^4 \cdot \|z_1^2\|^2}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} \langle z_1, v_1 \rangle \\
 & + 2 \frac{\|z_1\|^4 \cdot \|z_2^2\|^2}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} \langle z_2, v_2 \rangle \\
 & + 4 \frac{\|z_1\|^2 \cdot \|z_2\|^4}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} \langle z_1^3, v_1 \rangle \\
 & - 4 \frac{\|z_1\|^4 \cdot \|z_2\|^2}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} \langle z_2^3, v_2 \rangle. \tag{40}
 \end{aligned}$$

3.3. Critical points of $\mathcal{B}_{\text{av}}$

Equation (40) leads to the characterization of critical points of $\mathcal{B}_{\text{av}}$:

Proposition 3.2. *A point $(z_1, z_2) \in \mathcal{H}_{\text{av}}^1$ is a critical point of $\mathcal{B}_{\text{av}}$ if and only if (z_1, z_2) is smooth and solves the system of (uncoupled!) ODEs*

$$\begin{cases} z_1'' = a_1 z_1 + b_1 z_1^3, \\ z_2'' = a_2 z_2 + b_2 z_2^3, \end{cases} \tag{41}$$

with the constants

$$\begin{aligned}
 a_1 &= \frac{\|z_1'\|^2}{\|z_1\|^2} - \frac{1}{\|z_1\|^6} - \frac{\|z_2\|^4 \cdot \|z_1^2\|^2}{2\|z_1\|^2 \cdot (\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2}, \\
 b_1 &= + \frac{\|z_2\|^4}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2}, \\
 a_2 &= \frac{\|z_2'\|^2}{\|z_2\|^2} - \frac{1}{\|z_2\|^6} + \frac{\|z_1\|^4 \cdot \|z_2^2\|^2}{2\|z_2\|^2 \cdot (\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2}, \\
 b_2 &= - \frac{\|z_1\|^4}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2}.
 \end{aligned}$$

Proof. From (40) we see that (z_1, z_2) is a critical point of $\mathcal{B}_{\text{av}}$ if and only if z_1 and z_2 have weak second derivatives and satisfy the system of ODEs (41). Bootstrapping these equations we conclude that z_1 and z_2 are smooth and the proposition follows. $\blacksquare$

Corollary 3.3. *Suppose that (z_1, z_2) is a critical point of $\mathcal{B}_{\text{av}}$. Then z_1 and z_2 have transverse zeros. In particular, their zero sets*

$$Z_i := \{\tau \in S^1 \mid z_i(\tau) = 0\}, \quad i = 1, 2$$

are finite.

Proof. Arguing by contradiction, suppose that there exists a point $\tau_0 \in S^1$ such that $z_1(\tau_0) = z'_1(\tau_0) = 0$. Then the function z_1 and the zero function both solve the first equation in (41) with the same initial conditions at τ_0 . By uniqueness of solutions of ODEs we conclude $z_1 \equiv 0$, contradicting the condition $\|z_1\| > 0$ in the definition of $\mathcal{H}_{\text{av}}$. An analogous argument applies to z_2 . ■

3.4. From critical points to generalized solutions

Let $(z_1, z_2) \in \mathcal{H}_{\text{av}}^1$ be a critical point of $\mathcal{B}_{\text{av}}$, so by Proposition 3.2 the maps $z_1, z_2: S^1 \rightarrow \mathbb{R}$ are smooth and satisfy (41). For $i = 1, 2$ we define the smooth maps $t_{z_i}: S^1 \rightarrow S^1$ by

$$t_{z_i}(\tau) := \frac{1}{\|z_i\|^2} \int_0^\tau z_i(\sigma)^2 d\sigma. \quad (42)$$

Since by Corollary 3.3 the map z_i has only finitely many zeros, it follows from Lemma 2.1 that $t_{z_i}: S^1 \rightarrow S^1$ is a homeomorphism with continuous inverse $\tau_{z_i}: S^1 \rightarrow S^1$. We define continuous maps $q_i: S^1 \rightarrow S^1$ by

$$q_i(t) := z_i(\tau_{z_i}(t))^2.$$

Then for $i = 1, 2$ the maps $z_i, q_i: S^1 \rightarrow \mathbb{R}$ are smooth except at finitely many points and related by the Levi-Civita transformation (1) with $\tau = \tau_{z_i}$. As explained in Section 3.1, the last condition in the definition of $\mathcal{H}_{\text{av}}^1$ implies

$$\bar{q}_1 > \bar{q}_2.$$

Let us now focus on $i = 1$. Substituting z_1'' by (41) in (7) with $q = q_1$ and $z = z_1$ we compute at points $t \in S^1 \setminus t_{z_1}(Z_1)$,

$$\begin{aligned} \ddot{q}_1 &= \left(2\|z_1'\|^2 \cdot \|z_1\|^2 - \frac{2}{\|z_1\|^2} - \frac{\|z_1\|^2 \cdot \|z_2\|^4 \cdot \|z_1^2\|^2}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} \right) \frac{1}{q_1} \\ &\quad + \frac{2\|z_1\|^4 \cdot \|z_2\|^4}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} - \frac{\dot{q}_1^2}{2q_1} \\ &= \left(2\|z_1'\|^2 \cdot \|z_1\|^2 - \frac{2}{\|z_1\|^2} - \frac{\|z_1\|^4 \cdot \|z_2\|^4 \cdot \frac{\|z_1^2\|^2}{\|z_1\|^2}}{\|z_1\|^4 \cdot \|z_2\|^4 \cdot \left(\frac{\|z_1^2\|^2}{\|z_1\|^2} - \frac{\|z_2^2\|^2}{\|z_2\|^2} \right)^2} - \frac{\dot{q}_1^2}{2} \right) \frac{1}{q_1} \\ &\quad + \frac{2\|z_1\|^4 \cdot \|z_2\|^4}{\|z_1\|^4 \cdot \|z_2\|^4 \cdot \left(\frac{\|z_1^2\|^2}{\|z_1\|^2} - \frac{\|z_2^2\|^2}{\|z_2\|^2} \right)^2} \\ &= \left(2\|z_1'\|^2 \cdot \|z_1\|^2 - \frac{2}{\|z_1\|^2} - \frac{\frac{\|z_1^2\|^2}{\|z_1\|^2}}{\left(\frac{\|z_1^2\|^2}{\|z_1\|^2} - \frac{\|z_2^2\|^2}{\|z_2\|^2} \right)^2} - \frac{\dot{q}_1^2}{2} \right) \frac{1}{q_1} + \frac{2}{\left(\frac{\|z_1^2\|^2}{\|z_1\|^2} - \frac{\|z_2^2\|^2}{\|z_2\|^2} \right)^2} \\ &= \left(\frac{\|\dot{q}_1\|^2}{2} - \int_0^1 \frac{2}{q_1(s)} ds - \frac{\bar{q}_1}{(\bar{q}_1 - \bar{q}_2)^2} - \frac{\dot{q}_1^2}{2} \right) \frac{1}{q_1} + \frac{2}{(\bar{q}_1 - \bar{q}_2)^2}. \end{aligned}$$

Thus q_1 satisfies the ODE

$$\ddot{q}_1 = \left(c_1 - \frac{\dot{q}_1^2}{2}\right) \frac{1}{q_1} + \frac{2}{(\bar{q}_1 - \bar{q}_2)^2} \quad (43)$$

with the constant

$$c_1 = \frac{\|\dot{q}_1\|^2}{2} - \int_0^1 \frac{2}{q_1(s)} ds - \frac{\bar{q}_1}{(\bar{q}_1 - \bar{q}_2)^2}. \quad (44)$$

At the global maximum $t_{\max}$ of q_1 equation (43) becomes

$$\frac{c_1}{q_1(t_{\max})} + \frac{2}{(\bar{q}_1 - \bar{q}_2)^2} = \ddot{q}_1(t_{\max}) \leq 0,$$

hence

$$c_1 \leq -\frac{2q_1(t_{\max})}{(\bar{q}_1 - \bar{q}_2)^2}. \quad (45)$$

Now let $t_- < t_+$ be adjacent zeros of q_1 and consider the smooth map

$$\beta_1 := \frac{\ddot{q}_1 - \frac{1}{(\bar{q}_1 - \bar{q}_2)^2}}{q_1} : (t_-, t_+) \rightarrow \mathbb{R}.$$

From (43) we obtain

$$\beta_1 q_1^2 = c_1 - \frac{\dot{q}_1^2}{2} + \frac{q_1}{(\bar{q}_1 - \bar{q}_2)^2}.$$

With $q_1 \leq q_1(t_{\max})$ and inequality (45) this implies

$$\beta_1 q_1^2 \leq -\frac{\dot{q}_1^2}{2} - \frac{q_1(t_{\max})}{(\bar{q}_1 - \bar{q}_2)^2} < 0,$$

hence $\beta_1 < 0$ on (t_-, t_+) . Differentiating both sides of the equation for βq_1^2 we get

$$\dot{\beta}_1 q_1^2 + 2\beta_1 q_1 \dot{q}_1 = -\ddot{q}_1 \dot{q}_1 + \frac{\dot{q}_1}{(\bar{q}_1 - \bar{q}_2)^2} = -\beta_1 q_1 \dot{q}_1$$

and therefore

$$\dot{\beta}_1 q_1 = -3\beta_1 \dot{q}_1.$$

By Lemma 2.6 this implies that

$$\beta_1 = -\frac{\mu}{q_1^3}$$

on (t_-, t_+) for some constant $\mu > 0$. By definition of β_1 this yields

$$\ddot{q}_1(t) = -\frac{\mu}{q_1(t)^2} + \frac{1}{(\bar{q}_1 - \bar{q}_2)^2} \quad (46)$$

for $t \in (t_-, t_+)$. Plugging this into (43) we infer

$$\mu = -\left(\frac{\|\dot{q}_1\|^2}{2} - \int_0^1 \frac{2}{q_1(s)} ds - \frac{\bar{q}_1}{(\bar{q}_1 - \bar{q}_2)^2} - \frac{\dot{q}_1(t)^2}{2}\right) q_1(t) - \frac{q_1(t)^2}{(\bar{q}_1 - \bar{q}_2)^2} \quad (47)$$

for $t \in (t_-, t_+)$. In particular, use (5) in the second identity to obtain

$$\mu = \lim_{t \rightarrow t_{\pm}} \frac{\dot{q}_1(t)^2 q_1(t)}{2} = 2 \|z_1\|^4 z_1'(\tau_{z_1}(t_{\pm}))^2. \quad (48)$$

We deduce from this that (46) holds on $S^1 \setminus t_{z_1}(Z_1)$ with a fixed μ independent of the connected component in $S^1 \setminus t_{z_1}(Z_1)$. We divide (47) by q_1 to get

$$\frac{\mu}{q_1(t)} = -\frac{\|\dot{q}_1\|^2}{2} + \int_0^1 \frac{2}{q_1(s)} ds + \frac{\bar{q}_1}{(\bar{q}_1 - \bar{q}_2)^2} + \frac{\dot{q}_1(t)^2}{2} - \frac{q_1(t)}{(\bar{q}_1 - \bar{q}_2)^2}. \quad (49)$$

Integrating this equation yields

$$\mu \int \frac{1}{q_1(t)} dt = 2 \int_0^1 \frac{1}{q_1(s)} ds,$$

and therefore

$$\mu = 2.$$

Thus (46) becomes the first equation in (32). Similarly, one obtains for q_2 the equation

$$\ddot{q}_2 = \left(\frac{\|\dot{q}_2\|^2}{2} - \int_0^1 \frac{2}{q_2} dt + \frac{\bar{q}_2}{(\bar{q}_1 - \bar{q}_2)^2} - \frac{\dot{q}_2^2}{2} \right) \frac{1}{q_2} - \frac{2}{(\bar{q}_1 - \bar{q}_2)^2}.$$

Setting

$$\beta_2 := \frac{\ddot{q}_2 + \frac{1}{(\bar{q}_1 - \bar{q}_2)^2}}{q_2}$$

one gets

$$\dot{\beta}_2 q_2 = -3\beta_2 \dot{q}_2$$

implying that there exists $\mu \in \mathbb{R}$ such that

$$\beta_2 = -\frac{\mu}{q_2^3}.$$

Arguing as above one deduces from this that $\mu = 2$ and thus q_2 satisfies the second equation in (32).

To see the continuity of E_1 , we solve (49) (with $\mu = 2$) for

$$\begin{aligned} E_1(t) &= \frac{\dot{q}_1(t)^2}{2} - \frac{2}{q_1(t)} \\ &= \frac{\|\dot{q}_1\|^2}{2} - \int_0^1 \frac{2}{q_1(s)} ds - \frac{\bar{q}_1}{(\bar{q}_1 - \bar{q}_2)^2} - \frac{q_1(t)}{(\bar{q}_1 - \bar{q}_2)^2} \end{aligned}$$

and note that the right-hand side is continuous as a function of $t \in [0, 1]$. Continuity of E_2 follows similarly, and we have shown that (q_1, q_2) is a generalized solution of (32).

3.5. From generalized solutions to critical points

Now let $(q_1, q_2) \in H^1(S^1, \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})$ be a generalized solution of (32). The definition of a generalized solution implies that $q_1, q_2 \in \mathcal{H}_{\text{ce}}^1(S^1, \mathbb{R}_{\geq 0})$. Corollary 2.4 implies that the set $\mathcal{L}^{-1}(q_1) \times \mathcal{L}^{-1}(q_2)$ consists of four elements. The goal of this section is to show that each $(z_1, z_2) \in \mathcal{L}^{-1}(q_1) \times \mathcal{L}^{-1}(q_2)$ is a critical point of $\mathcal{B}_{\text{av}}$. To see this, observe that smoothness of q_i on the complement of its zero set Z_{q_i} implies smoothness of z_i on the complement of its zero set Z_{z_i} , $i \in \{1, 2\}$. In particular, second derivatives of z_i make sense there and we can make the following statement.

Lemma 3.4. *Any $(z_1, z_2) \in \mathcal{L}^{-1}(q_1) \times \mathcal{L}^{-1}(q_2)$ satisfies the critical point (41) on the complement of the set $Z_{z_1} \cup Z_{z_2}$.*

Assuming this lemma for the moment, recall that z_1 and z_2 are of class C^1 and (41) expresses z_i'' through z_i . Thus bootstrapping (41) implies that z_1 and z_2 are smooth and (41) holds on the whole S^1 . Therefore, it remains to prove Lemma 3.4.

Proof of Lemma 3.4. We will show the desired equation for z_1 . A similar argument will do the job for z_2 . Recall the equation satisfied by q_1 ,

$$\ddot{q}_1 = -\frac{2}{q_1^2} + \frac{1}{(\bar{q}_1 - \bar{q}_2)^2}. \quad (50)$$

Set

$$\beta_1 := \frac{\ddot{q}_1 - \frac{1}{(\bar{q}_1 - \bar{q}_2)^2}}{q_1} \quad (51)$$

on $S^1 \setminus Z_{q_1}$. Then by (50) we have

$$\beta_1 = -\frac{2}{q_1^3},$$

and taking the time derivative we obtain

$$\dot{\beta}_1 = \frac{3 \cdot 2}{q_1^4} \dot{q}_1 = -\frac{3\beta_1 \dot{q}_1}{q_1}.$$

We multiply both sides by q_1^2 to get

$$\dot{\beta}_1 q_1^2 = -3\beta_1 \dot{q}_1 q_1.$$

We bring $-2\beta_1 \dot{q}_1 q_1$ to the other side to continue:

$$\dot{\beta}_1 q_1^2 + 2\beta_1 \dot{q}_1 q_1 = -\beta_1 \dot{q}_1 q_1.$$

We substitute the original definition (51) of β_1 into the right-hand side to get

$$\dot{\beta}_1 q_1^2 + 2\beta_1 \dot{q}_1 q_1 = -\dot{q}_1 \ddot{q}_1 + \frac{\dot{q}_1}{(\bar{q}_1 - \bar{q}_2)^2}.$$

Integrating both sides from 0 to t we get

$$\beta_1 q_1^2 = C - \frac{\dot{q}_1^2}{2} + \frac{q_1}{(\bar{q}_1 - \bar{q}_2)^2}$$

with some constant $C \in \mathbb{R}$. We substitute the original definition (51) of β_1 in the left-hand side to get

$$\ddot{q}_1 q_1 = C - \frac{\dot{q}_1^2}{2} + \frac{2q_1}{(\bar{q}_1 - \bar{q}_2)^2}.$$

Observe that modulo the exact value of C this is exactly (43), which is equivalent to the first equation of (41). Therefore, we are left with computing the constant C . For this we solve the last equation for C and use (50) to obtain

$$C = \frac{\dot{q}_1^2}{2} - \frac{2}{q_1} - \frac{q_1}{(\bar{q}_1 - \bar{q}_2)^2},$$

and integrating from 0 to 1 gives us

$$C = \frac{\|\dot{q}_1\|^2}{2} - \int_0^1 \frac{2}{q_1(s)} ds - \frac{\bar{q}_1}{(\bar{q}_1 - \bar{q}_2)^2},$$

which matches the constant c_1 in (44). This concludes the proof of Lemma 3.4, and therefore of Theorem 3.1. $\blacksquare$

4. Instantaneous interaction

In this section we consider the real helium atom in which the two electrons interact by their Coulomb repulsion according to

$$\begin{cases} \ddot{q}_1(t) = -\frac{2}{q_1(t)^2} + \frac{1}{(q_1(t) - q_2(t))^2}, \\ \ddot{q}_2(t) = -\frac{2}{q_2(t)^2} - \frac{1}{(q_1(t) - q_2(t))^2}, \end{cases} \quad (52)$$

where we impose the condition

$$q_1(t) > q_2(t) \geq 0 \quad \text{for all } t \in S^1. \quad (53)$$

4.1. Variational characterization of generalized solutions

Solutions of (52) are the critical points of the action functional

$$\mathcal{S}_{\text{in}}(q_1, q_2) := \sum_{i=1}^2 \left(\frac{1}{2} \int_0^1 \dot{q}_i(t)^2 dt + \int_0^1 \frac{2}{q_i(t)} dt \right) - \int_0^1 \frac{1}{q_1(t) - q_2(t)} dt.$$

For $i = 1, 2$ let q_i and z_i be related by Levi-Civita transformations

$$q_i(t) = z_i(\tau_{z_i}(t))^2, \quad (54)$$

where $\tau_{z_i}: S^1 \rightarrow S^1$ is the inverse of $t_{z_i}: S^1 \rightarrow S^1$ defined in (9) with $z = z_i$. As in Section 2.1 it follows that

$$\frac{dt}{q_i(t)} = \frac{d\tau_{z_i}(t)}{\|z_i\|^2}, \quad (55)$$

so all the relations in Section 2.1 hold with $(q, z, \tau) = (q_i, z_i, \tau_{z_i})$. In particular, we can rewrite the last integral in terms of the z_i as the *instantaneous interaction*

$$\begin{aligned} \mathcal{I}(z_1, z_2) &:= -\frac{1}{\|z_1\|^2} \int_0^1 \frac{z_1(\tau)^2}{z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau)))} d\tau \\ &= -\int_0^1 \frac{1}{z_1^2(\tau_{z_1}(t)) - z_2^2(\tau_{z_2}(t))} dt \\ &= \frac{1}{\|z_2\|^2} \int_0^1 \frac{z_2(\tau)^2}{z_2^2(\tau) - z_1^2(\tau_{z_1}(t_{z_2}(\tau)))} d\tau, \end{aligned} \quad (56)$$

where in the last two equalities we have changed the integration variable to $\tau = \tau_{z_1}(t)$ (resp. $\tau = \tau_{z_2}(t)$) using (55). Note that the functional $\mathcal{I}$ is *nonlocal* due to the appearance of the time changes, which we have written as τ_{z_i} rather than τ_i to indicate their dependence on z_i .

The instantaneous interaction $\mathcal{I}$ is naturally defined on the space

$$\begin{aligned} \mathcal{H}_{\text{in}}^1 &:= \{z = (z_1, z_2) \in H^1(S^1, \mathbb{R}^2) \mid \|z_1\| > 0, \|z_2\| > 0, \\ &\quad z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))) > 0 \text{ for all } \tau \in S^1\}. \end{aligned} \quad (57)$$

Note that $\mathcal{H}_{\text{in}}^1$ is an open subset of the Hilbert space $H^1(S^1, \mathbb{R}^2)$ and the last condition in its definition corresponds to condition (53). Since integrating condition (53) leads to the averaged condition (33), which is in turn equivalent to the last condition in the definition of $\mathcal{H}_{\text{av}}^1$ in Section 3.1, we have

$$\mathcal{H}_{\text{in}}^1 \subset \mathcal{H}_{\text{av}}^1. \quad (58)$$

On $\mathcal{H}_{\text{in}}^1$ we consider the functional

$$\mathcal{B}_{\text{in}}: \mathcal{H}_{\text{in}}^1 \rightarrow \mathbb{R}, \quad \mathcal{B}_{\text{in}}(z_1, z_2) := \mathcal{Q}(z_1) + \mathcal{Q}(z_2) + \mathcal{I}(z_1, z_2), \quad (59)$$

with the functionals

$$\mathcal{Q}(z_i) = 2\|z_i\|^2 \|z_i'\|^2 + \frac{2}{\|z_i\|^2}$$

from (23) with charge $N = 2$.

We call $(q_1, q_2) \in H^1(S^1, \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})$ a *generalized solution* of (52) if for $i = 1, 2$ the following holds:

- (1) $q_1(t) > q_2(t)$ for all $t \in S^1$;
- (2) the zero sets $Z_i = q_i^{-1}(0) \subset S^1$ are finite and each have an even number of elements;
- (3) the restricted maps $q_i: S^1 \setminus Z_i \rightarrow \mathbb{R}_{\geq 0}$ are smooth and satisfy (52);
- (4) the energies

$$E_i(t) := \frac{\dot{q}_i(t)^2}{2} - \frac{2}{q_i(t)}, \quad t \in S^1 \setminus Z_i$$

extend to continuous functions $E_i: S^1 \rightarrow \mathbb{R}$.

Note that the individual energies E_i need not be constant, but the total energy

$$E = E_1(t) + E_2(t) + \frac{1}{q_1(t) - q_2(t)}$$

is constant and negative.

Theorem 4.1 (Generalized solutions with instantaneous interaction). *Under the Levi-Civita transformations (54) with time changes (55), critical points (z_1, z_2) of the action functional $\mathcal{B}_{\text{in}}$ defined in (59) are in 4-to-1 correspondence with generalized solutions (q_1, q_2) of (52).*

The proof of this theorem will take up the remainder of this section.

4.2. The differential of $\mathcal{B}_{\text{in}}$

In this subsection we compute the differential of $\mathcal{B}_{\text{in}}$ at $(z_1, z_2) \in \mathcal{H}_{\text{in}}^1$ in the direction $(v_1, v_2) \in H^1(S^1, \mathbb{R}^2)$. For this we will need for $i = 1, 2$ the derivative of the time change t_{z_i} with respect to z_i in the direction v_i . By (99) it is given by

$$Dt_{z_i}(v_i)(\tau) = \frac{2}{\|z_i\|^2} \int_0^\tau z_i(\sigma) v_i(\sigma) d\sigma - \frac{2\langle z_i, v_i \rangle}{\|z_i\|^4} \int_0^\tau z_i(\sigma)^2 d\sigma. \quad (60)$$

Using this we now compute the differential of the instantaneous interaction $\mathcal{I}$ at $(z_1, z_2) \in \mathcal{H}_{\text{in}}^1$ with respect to z_1 in the direction $v_1 \in H^1(S^1, \mathbb{R})$. Using the first expression for $\mathcal{I}$ in (56) we obtain

$$\begin{aligned} D_1 \mathcal{I}[z_1, z_2](v_1) &= \frac{2\langle z_1, v_1 \rangle}{\|z_1\|^4} \int_0^1 \frac{z_1^2(\tau)}{z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau)))} d\tau \\ &\quad + \frac{2}{\|z_1\|^2} \int_0^1 \frac{z_1(\tau) z_2^2(\tau_{z_2}(t_{z_1}(\tau))) v_1(\tau)}{(z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))))^2} d\tau \\ &\quad - \frac{2}{\|z_1\|^2} \int_0^1 \frac{z_1^2(\tau) z_2(\tau_{z_2}(t_{z_1}(\tau))) z_2'(\tau_{z_2}(t_{z_1}(\tau))) \dot{t}_{z_2}(t_{z_1}(\tau)) Dt_{z_1}(v_1)(\tau)}{(z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))))^2} d\tau. \end{aligned}$$

We rewrite the third term on the right-hand side as

$$\begin{aligned}
& - \frac{2}{\|z_1\|^2} \int_0^1 \frac{z_1^2(\tau) z_2(\tau_{z_2}(t_{z_1}(\tau))) z_2'(\tau_{z_2}(t_{z_1}(\tau))) \dot{t}_{z_2}(t_{z_1}(\tau)) D t_{z_1}(v_1)(\tau)}{(z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))))^2} d\tau \\
& = - \frac{2\|z_2\|^2}{\|z_1\|^2} \int_0^1 \frac{z_1^2(\tau) z_2'(\tau_{z_2}(t_{z_1}(\tau))) D t_{z_1}(v_1)(\tau)}{z_2(\tau_{z_2}(t_{z_1}(\tau))) (z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))))^2} d\tau \\
& = - \frac{4\|z_2\|^2}{\|z_1\|^4} \int_0^1 \frac{z_1^2(\tau) z_2'(\tau_{z_2}(t_{z_1}(\tau)))}{z_2(\tau_{z_2}(t_{z_1}(\tau))) (z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))))^2} \left(\int_0^\tau z_1(\sigma) v_1(\sigma) d\sigma \right) d\tau \\
& \quad + \frac{4\|z_2\|^2 \langle z_1, v_1 \rangle}{\|z_1\|^6} \int_0^1 \frac{z_1^2(\tau) z_2'(\tau_{z_2}(t_{z_1}(\tau))) \int_0^\tau z_1(\sigma)^2 d\sigma}{z_2(\tau_{z_2}(t_{z_1}(\tau))) (z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))))^2} d\tau \\
& = - \frac{4\|z_2\|^2}{\|z_1\|^4} \int_0^1 \left(\int_\sigma^1 \frac{z_1^2(\tau) z_2'(\tau_{z_2}(t_{z_1}(\tau)))}{z_2(\tau_{z_2}(t_{z_1}(\tau))) (z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))))^2} d\tau \right) z_1(\sigma) v_1(\sigma) d\sigma \\
& \quad + \frac{4\|z_2\|^2 \langle z_1, v_1 \rangle}{\|z_1\|^4} \int_0^1 \frac{z_1^2(\tau) z_2'(\tau_{z_2}(t_{z_1}(\tau))) t_{z_1}(\tau)}{z_2(\tau_{z_2}(t_{z_1}(\tau))) (z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))))^2} d\tau.
\end{aligned}$$

Here for the first equality we have used formula (11) for $\dot{t}_{z_i}$, and for the second one we have used (60). For the third equality we have switched the order of integration in the first integral, and we have used (42) to replace $\int_0^\tau z_1(\sigma)^2 d\sigma$ by $\|z_1\|^2 t_{z_1}(\tau)$.

By symmetry, the expression for $D_2 \mathcal{I}[z_1, z_2](v_2)$ is the same with a global minus sign and the roles of (z_1, v_1) and (z_2, v_2) reversed. Putting everything together, we obtain the differential of $\mathcal{B}_{\text{in}}$:

$$\begin{aligned}
& d \mathcal{B}_{\text{in}}[z_1, z_2](v_1, v_2) \\
& = 4 \sum_{i=1}^2 \left(\|z_i'\|^2 \cdot \langle z_i, v_i \rangle + \|z_i\|^2 \langle z_i', v_i' \rangle - \frac{\langle z_i, v_i \rangle}{\|z_i\|^4} \right) \\
& \quad + \frac{2\langle z_1, v_1 \rangle}{\|z_1\|^4} \int_0^1 \frac{z_1^2(\tau)}{z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau)))} d\tau \\
& \quad + \frac{2}{\|z_1\|^2} \int_0^1 \frac{z_1(\tau) z_2^2(\tau_{z_2}(t_{z_1}(\tau))) v_1(\tau)}{(z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))))^2} d\tau \\
& \quad - \frac{4\|z_2\|^2}{\|z_1\|^4} \int_0^1 \left(\int_\sigma^1 \frac{z_1^2(\tau) z_2'(\tau_{z_2}(t_{z_1}(\tau)))}{z_2(\tau_{z_2}(t_{z_1}(\tau))) (z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))))^2} d\tau \right) z_1(\sigma) v_1(\sigma) d\sigma \\
& \quad + \frac{4\|z_2\|^2 \langle z_1, v_1 \rangle}{\|z_1\|^4} \int_0^1 \frac{z_1^2(\tau) z_2'(\tau_{z_2}(t_{z_1}(\tau))) t_{z_1}(\tau)}{z_2(\tau_{z_2}(t_{z_1}(\tau))) (z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))))^2} d\tau \\
& \quad - \frac{2\langle z_2, v_2 \rangle}{\|z_2\|^4} \int_0^1 \frac{z_2^2(\tau)}{z_2^2(\tau) - z_1^2(\tau_{z_1}(t_{z_2}(\tau)))} d\tau \\
& \quad - \frac{2}{\|z_2\|^2} \int_0^1 \frac{z_2(\tau) z_1^2(\tau_{z_1}(t_{z_2}(\tau))) v_2(\tau)}{(z_2^2(\tau) - z_1^2(\tau_{z_1}(t_{z_2}(\tau))))^2} d\tau
\end{aligned}$$

$$\begin{aligned}
& + \frac{4\|z_1\|^2}{\|z_2\|^4} \int_0^1 \left(\int_\sigma^1 \frac{z_2^2(\tau) z_1'(\tau_{z_1}(t_{z_2}(\tau)))}{z_1(\tau_{z_1}(t_{z_2}(\tau)))(z_2^2(\tau) - z_1^2(\tau_{z_1}(t_{z_2}(\tau))))^2} d\tau \right) z_2(\sigma) v_2(\sigma) d\sigma \\
& - \frac{4\|z_1\|^2 \langle z_2, v_2 \rangle}{\|z_2\|^4} \int_0^1 \frac{z_2^2(\tau) z_1'(\tau_{z_1}(t_{z_2}(\tau))) t_{z_2}(\tau)}{z_1(\tau_{z_1}(t_{z_2}(\tau)))(z_2^2(\tau) - z_1^2(\tau_{z_1}(t_{z_2}(\tau))))^2} d\tau. \tag{61}
\end{aligned}$$

4.3. Critical points of $\mathcal{B}_{\text{in}}$

Equation (61) leads to the characterization of critical points of $\mathcal{B}_{\text{in}}$:

Proposition 4.2. *A point $(z_1, z_2) \in \mathcal{H}_{\text{in}}^1$ is a critical point of $\mathcal{B}_{\text{in}}$ if and only if (z_1, z_2) is smooth and solves the following system of coupled nonlocal integral–differential equations:*

$$\begin{aligned}
z_1''(\tau) &= \frac{\|z_1'\|^2 z_1(\tau)}{\|z_1\|^2} - \frac{z_1(\tau)}{\|z_1\|^6} \\
&+ \frac{z_1(\tau)}{2\|z_1\|^6} \int_0^1 \frac{z_1^2(\sigma)}{z_2^2(\sigma) - z_1^2(\tau_{z_2}(t_{z_1}(\sigma)))} d\sigma \\
&+ \frac{z_1(\tau) z_2^2(\tau_{z_2}(t_{z_1}(\tau)))}{2\|z_1\|^4 (z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))))^2} \\
&- \frac{\|z_2\|^2 z_1(\tau)}{\|z_1\|^6} \int_\tau^1 \frac{z_1^2(\sigma) z_2'(\tau_{z_2}(t_{z_1}(\sigma)))}{z_2(\tau_{z_2}(t_{z_1}(\sigma)))(z_1^2(\sigma) - z_2^2(\tau_{z_2}(t_{z_1}(\sigma))))^2} d\sigma \\
&+ \frac{\|z_2\|^2 z_1(\tau)}{\|z_1\|^6} \int_0^1 \frac{z_1^2(\sigma) z_2'(\tau_{z_2}(t_{z_1}(\sigma))) t_{z_1}(\sigma)}{z_2(\tau_{z_2}(t_{z_1}(\sigma)))(z_1^2(\sigma) - z_2^2(\tau_{z_2}(t_{z_1}(\sigma))))^2} d\sigma, \\
z_2''(\tau) &= \frac{\|z_2'\|^2 z_2(\tau)}{\|z_2\|^2} - \frac{z_2(\tau)}{\|z_2\|^6} \\
&- \frac{z_2(\tau)}{2\|z_2\|^6} \int_0^1 \frac{z_2^2(\sigma)}{z_2^2(\sigma) - z_1^2(\tau_{z_1}(t_{z_2}(\sigma)))} d\sigma \\
&- \frac{z_2(\tau) z_1^2(\tau_{z_1}(t_{z_2}(\tau)))}{2\|z_2\|^4 (z_2^2(\tau) - z_1^2(\tau_{z_1}(t_{z_2}(\tau))))^2} \\
&+ \frac{\|z_1\|^2 z_2(\tau)}{\|z_2\|^6} \int_\tau^1 \frac{z_2^2(\sigma) z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))(z_2^2(\sigma) - z_1^2(\tau_{z_1}(t_{z_2}(\sigma))))^2} d\sigma \\
&- \frac{\|z_1\|^2 z_2(\tau)}{\|z_2\|^6} \int_0^1 \frac{z_2^2(\sigma) z_1'(\tau_{z_1}(t_{z_2}(\sigma))) t_{z_2}(\sigma)}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))(z_2^2(\sigma) - z_1^2(\tau_{z_1}(t_{z_2}(\sigma))))^2} d\sigma. \tag{62}
\end{aligned}$$

Proof. From (61) we see that (z_1, z_2) is a critical point of $\mathcal{B}_{\text{in}}$ if and only if z_1 and z_2 have weak second derivatives and satisfy the system of equations (62). Bootstrapping these equations we conclude that z_1 and z_2 are smooth and the proposition follows. ■

Corollary 4.3. *Suppose that (z_1, z_2) is a critical point of $\mathcal{B}_{\text{in}}$. Then z_1 has no zeros and z_2 has transverse zeros. In particular, their zero sets*

$$Z_i := \{\tau \in S^1 \mid z_i(\tau) = 0\}, \quad i = 1, 2$$

are finite.

Proof. Let $(z_1, z_2) \in \mathcal{H}_{\text{in}}^1$ be a critical point of $\mathcal{B}_{\text{in}}$. It follows directly from the definition of $\mathcal{H}_{\text{in}}^1$ that $z_1(t) > 0$ for all $t \in S^1$. Next note that (z_1, z_2) solves the system of coupled nonlinear integral–differential equations (62) which has the form

$$\begin{cases} z_1''(\tau) &= f_1(\tau)z_1(\tau), \\ z_2''(\tau) &= f_2(\tau)z_2(\tau), \end{cases} \quad (63)$$

where $f_i: S^1 \rightarrow \mathbb{R}$, $i = 1, 2$ are smooth functions depending on (z_1, z_2) . Disregarding the dependence of the f_i on (z_1, z_2) , we can view (z_1, z_2) as a solution to the system of *uncoupled linear ODEs* (63).

Arguing by contradiction, suppose now that there exists a point $\tau_0 \in S^1$ such that $z_2(\tau_0) = z_2'(\tau_0) = 0$. Then the function z_2 and the zero function both solve the second equation in (63) with the same initial conditions at τ_0 . By uniqueness of solutions of ODEs we conclude $z_2 \equiv 0$, contradicting the condition $\|z_2\| > 0$ in the definition of $\mathcal{H}_{\text{in}}$. ■

4.4. From critical points to generalized solutions

Let $(z_1, z_2) \in \mathcal{H}_{\text{in}}^1$ be a critical point of $\mathcal{B}_{\text{in}}$, so by Proposition 4.2 the maps $z_1, z_2: S^1 \rightarrow \mathbb{R}$ are smooth and satisfy (62). As in the previous section, for $i = 1, 2$ we define the smooth maps $t_{z_i}: S^1 \rightarrow S^1$ by (42). Since by Corollary 4.3 the map z_i has only finitely many zeros, it follows from Lemma 2.1 that $t_{z_i}: S^1 \rightarrow S^1$ is a homeomorphism with continuous inverse $\tau_{z_i}: S^1 \rightarrow S^1$. We define continuous maps $q_i: S^1 \rightarrow S^1$ by

$$q_i(t) := z_i(\tau_{z_i}(t))^2. \quad (64)$$

Then for $i = 1, 2$ the maps $z_i, q_i: S^1 \rightarrow \mathbb{R}$ are smooth except at finitely many points and related by the Levi-Civita transformation (1) with $\tau = \tau_{z_i}$. The last condition in the definition of $\mathcal{H}_{\text{in}}^1$ implies

$$q_1(t) > q_2(t) \quad \text{for all } t \in S^1.$$

Let us now focus on $i = 1$. Although by Corollary 4.3 the function z_1 has no zeros, in the following argument we will allow z_1 to have a finite set Z_1 of zeros; this will ensure that the same argument carries over to z_2 (which may have zeros). At points $t \in S^1 \setminus t_{z_1}(Z_1)$ we compute the following:

$$\ddot{q}_1(t)q_1(t) = 2\|z_1\|^4 \frac{z_1''(\tau_{z_1}(t))}{z_1(\tau_{z_1}(t))} - \frac{\dot{q}_1(t)^2}{2}$$

$$\begin{aligned}
&= 2\|z_1\|^2 \cdot \|z_1'\|^2 - \frac{2}{\|z_1\|^2} - \frac{\dot{q}_1^2(t)}{2} \\
&\quad + \frac{1}{\|z_1\|^2} \int_0^1 \frac{z_1(\tau_{z_1}(s))^2}{z_1^2(\tau_{z_1}(s)) - z_2^2(\tau_{z_2}(s))} \dot{\tau}_{z_1}(s) ds \\
&\quad + \frac{z_2^2(\tau_{z_2}(t))}{(z_1^2(\tau_{z_1}(t)) - z_2^2(\tau_{z_2}(t)))^2} \\
&\quad - \frac{2\|z_2\|^2}{\|z_1\|^2} \int_t^1 \frac{z_1^2(\tau_{z_1}(s))z_2'(\tau_{z_2}(s))}{z_2(\tau_{z_2}(s))(z_1^2(\tau_{z_1}(s)) - z_2^2(\tau_{z_2}(s)))^2} \dot{\tau}_{z_1}(s) ds \\
&\quad + \frac{2\|z_2\|^2}{\|z_1\|^2} \int_0^1 \frac{z_1^2(\tau_{z_1}(s))z_2'(\tau_{z_2}(s))s}{z_2(\tau_{z_2}(s))(z_1^2(\tau_{z_1}(s)) - z_2^2(\tau_{z_2}(s)))^2} \dot{\tau}_{z_1}(s) ds \\
&= \frac{\|\dot{q}_1\|^2}{2} - \int_0^1 \frac{2}{q_1(s)} ds - \frac{\dot{q}_1^2(t)}{2} + \int_0^1 \frac{1}{q_1(s) - q_2(s)} ds \\
&\quad + \frac{q_2(t)}{(q_1(t) - q_2(t))^2} - 2\|z_2\|^2 \int_t^1 \frac{\frac{d}{ds}z_2(\tau_{z_2}(s))}{\dot{\tau}_{z_2}(s)z_2(\tau_{z_2}(s))(q_1(s) - q_2(s))^2} ds \\
&\quad + 2\|z_2\|^2 \int_0^1 \frac{s \frac{d}{ds}z_2(\tau_{z_2}(s))}{\dot{\tau}_{z_2}(s)z_2(\tau_{z_2}(s))(q_1(s) - q_2(s))^2} ds \\
&= \frac{\|\dot{q}_1\|^2}{2} - \int_0^1 \frac{2}{q_1(s)} ds - \frac{\dot{q}_1^2(t)}{2} + \int_0^1 \frac{1}{q_1(s) - q_2(s)} ds \\
&\quad + \frac{q_2(t)}{(q_1(t) - q_2(t))^2} - \int_t^1 \frac{2z_2(\tau_{z_2}(s)) \frac{d}{ds}z_2(\tau_{z_2}(s))}{(q_1(s) - q_2(s))^2} ds \\
&\quad + \int_0^1 \frac{2sz_2(\tau_{z_2}(s)) \frac{d}{ds}z_2(\tau_{z_2}(s))}{(q_1(s) - q_2(s))^2} ds \\
&= \frac{\|\dot{q}_1\|^2}{2} - \int_0^1 \frac{2}{q_1(s)} ds - \frac{\dot{q}_1^2(t)}{2} + \int_0^1 \frac{1}{q_1(s) - q_2(s)} ds \\
&\quad + \frac{q_2(t)}{(q_1(t) - q_2(t))^2} - \int_t^1 \frac{\frac{d}{ds}z_2^2(\tau_{z_2}(s))}{(q_1(s) - q_2(s))^2} ds \\
&\quad + \int_0^1 \frac{s \frac{d}{ds}z_2^2(\tau_{z_2}(s))}{(q_1(s) - q_2(s))^2} ds \\
&= \frac{\|\dot{q}_1\|^2}{2} - \int_0^1 \frac{2}{q_1(s)} ds - \frac{\dot{q}_1^2(t)}{2} + \int_0^1 \frac{1}{q_1(s) - q_2(s)} ds \\
&\quad + \frac{q_2(t)}{(q_1(t) - q_2(t))^2} - \int_t^1 \frac{\dot{q}_2(s)}{(q_1(s) - q_2(s))^2} ds \\
&\quad + \int_0^1 \frac{s\dot{q}_2(s)}{(q_1(s) - q_2(s))^2} ds. \tag{65}
\end{aligned}$$

Here, the first equality comes from (7) with $q = q_1$ and $z = z_1$. In the second one we substitute z_1'' by (62) and change variables $\sigma = \tau_{z_1}(s)$ in the integrals. In the third one we

use the following replacements from (64), (4), (8) and (11):

$$\begin{aligned} z_i(\tau_{z_i}(s))^2 &= q_i(s), \quad \frac{1}{\|z_1\|^2} = \int_0^1 \frac{1}{q_1(s)} ds, \\ 4\|z_1\|^2 \cdot \|z_1'\|^2 &= \|\dot{q}_1\|^2, \quad \dot{\tau}_{z_1}(s) = \frac{\|z_1\|^2}{z_1(\tau_{z_1}(s))^2} \end{aligned}$$

and the chain rule for $\frac{d}{ds} z_2(\tau_{z_2}(s))$. In the fourth one we use (11) to replace $\dot{\tau}_{z_2}(s)$, in the fifth one the chain rule for $\frac{d}{ds} z_2^2(\tau_{z_2}(s))$, and in the sixth one (64) to insert $q_2(s)$. Thus q_1 satisfies the integral–differential equation

$$\ddot{q}_1 = \left(c_1 - \frac{\dot{q}_1^2}{2} + \frac{q_2}{(q_1 - q_2)^2} - \int_t^1 \frac{\dot{q}_2(s)}{(q_1(s) - q_2(s))^2} ds \right) \frac{1}{q_1} \quad (66)$$

with the constant

$$c_1 = \frac{\|\dot{q}_1\|^2}{2} - \int_0^1 \frac{2}{q_1(s)} ds + \int_0^1 \frac{1}{q_1(s) - q_2(s)} ds + \int_0^1 \frac{s \dot{q}_2(s)}{(q_1(s) - q_2(s))^2} ds.$$

Now let $t_- < t_+$ be adjacent zeros of q_1 and consider the smooth map

$$\beta_1 := \frac{\ddot{q}_1 - \frac{1}{(q_1 - q_2)^2}}{q_1} : (t_-, t_+) \rightarrow \mathbb{R}.$$

From (66) we obtain

$$\begin{aligned} \beta_1 q_1^2 &= \ddot{q}_1 q_1 - \frac{q_1}{(q_1 - q_2)^2} \\ &= c_1 - \frac{\dot{q}_1^2}{2} + \frac{1}{q_2 - q_1} - \int_t^1 \frac{\dot{q}_2(s)}{(q_1(s) - q_2(s))^2} ds. \end{aligned}$$

Taking the time derivative of this expression for $\beta_1 q_1^2$ we obtain

$$\begin{aligned} \dot{\beta}_1 q_1^2 + 2\beta_1 q_1 \dot{q}_1 &= -\ddot{q}_1 \dot{q}_1 - \frac{\dot{q}_2 - \dot{q}_1}{(q_2 - q_1)^2} + \frac{\dot{q}_2}{(q_1 - q_2)^2} \\ &= -\ddot{q}_1 \dot{q}_1 + \frac{\dot{q}_1}{(q_1 - q_2)^2} \\ &= -\beta_1 q_1 \dot{q}_1. \end{aligned}$$

Therefore, dividing both sides by q_1 , we get

$$\dot{\beta}_1 q_1 = -3\beta_1 \dot{q}_1.$$

This is exactly (27). We apply Lemma 2.6 to get

$$\beta_1 = -\frac{\mu}{q_1^3}$$

on (t_-, t_+) for some constant $\mu > 0$. Here positivity of μ follows again because by definition $\beta_1 < 0$ at the maximum of q_1 on (t_-, t_+) . We substitute the definition of β_1 in the last displayed equation and solve for $\ddot{q}_1$ to get

$$\ddot{q}_1(t) = -\frac{\mu}{q_1(t)^2} + \frac{1}{(q_1(t) - q_2(t))^2}, \quad t \in (t_-, t_+). \quad (67)$$

We solve the last equation for μ/q_1 and substitute $\ddot{q}_1 q_1$ by (65) to get

$$\begin{aligned} \frac{\mu}{q_1(t)} &= -\ddot{q}_1(t)q_1(t) + \frac{q_1(t)}{(q_1(t) - q_2(t))^2} \\ &= -\frac{\|\dot{q}_1\|^2}{2} + \int_0^1 \frac{2ds}{q_1(s)} + \frac{\dot{q}_1(t)^2}{2} - \int_0^1 \frac{ds}{q_1(s) - q_2(s)} + \frac{1}{q_1(t) - q_2(t)} \\ &\quad + \int_t^1 \frac{\dot{q}_2(s) ds}{(q_1(s) - q_2(s))^2} - \int_0^1 \frac{s \dot{q}_2(s) ds}{(q_1(s) - q_2(s))^2}. \end{aligned} \quad (68)$$

We multiply both sides of the last equation by $q_1(t)$, take the limit $t \rightarrow t_{\pm}$ and recall that $q_1(t_{\pm}) = 0$ to get

$$\mu = \lim_{t \rightarrow t_{\pm}} \frac{\dot{q}_1(t)^2 q_1(t)}{2} = 2\|z_1\|^4 z_1'(\tau_{z_1}(t_{\pm}))^2. \quad (69)$$

Therefore, μ is a global constant independent of the interval (t_-, t_+) . (This argument is actually only needed for q_2 in place of q_1 ; for q_1 itself, since it has no zeros, we can replace (t_-, t_+) by all of S^1 from the outset).

Now note that, by Fubini's theorem, for every integrable function $f: [0, 1] \rightarrow \mathbb{R}$ we have

$$\int_0^1 dt \int_t^1 f(s) ds = \int_0^1 ds f(s) \int_0^s dt = \int_0^1 s f(s) ds.$$

Integrating both sides of (68) from 0 to 1 and applying this identity to

$$f(s) := \frac{\dot{q}_2}{(q_1(s) - q_2(s))^2},$$

we obtain

$$\mu = 2.$$

Thus (67) becomes the first equation in (52). Similarly, we deduce that q_2 satisfies the second equation in (52) outside its zero set.

To see the continuity of the energy E_1 , we solve (68) (with $\mu = 2$) for

$$\begin{aligned} E_1(t) &= \frac{\dot{q}_1(t)^2}{2} - \frac{2}{q_1(t)} \\ &= \frac{\|\dot{q}_1\|^2}{2} - \int_0^1 \frac{2ds}{q_1(s)} + \int_0^1 \frac{ds}{q_1(s) - q_2(s)} - \frac{1}{q_1(t) - q_2(t)} \\ &\quad - \int_t^1 \frac{\dot{q}_2(s) ds}{(q_1(s) - q_2(s))^2} + \int_0^1 \frac{s \dot{q}_2(s) ds}{(q_1(s) - q_2(s))^2} \end{aligned}$$

and note that the right-hand side is continuous as a function of $t \in [0, 1]$. Continuity of E_2 follows similarly, and we have shown that (q_1, q_2) is a generalized solution of (52).

4.5. From generalized solutions to critical points

Now let $(q_1, q_2) \in H^1(S^1, \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})$ be a generalized solution of (52). The definition of a generalized solution implies that $q_1, q_2 \in \mathcal{H}_{\text{ce}}^1(S^1, \mathbb{R}_{\geq 0})$. Corollary 2.4 implies that the set $\mathcal{L}^{-1}(q_1) \times \mathcal{L}^{-1}(q_2)$ consists of four elements. The goal of this section is to show that each $(z_1, z_2) \in \mathcal{L}^{-1}(q_1) \times \mathcal{L}^{-1}(q_2)$ is a critical point of $\mathcal{B}_{\text{in}}$. To see this, recall that q_1 has no zeros, so smoothness of q_1 implies smoothness of z_1 . Smoothness of q_2 on the complement of its zero set Z_{q_2} implies smoothness of z_2 on the complement of its zero set Z_{z_2} . In particular, second derivatives of z_2 make sense there and we can make the following statement.

Lemma 4.4. *Any $(z_1, z_2) \in \mathcal{L}^{-1}(q_1) \times \mathcal{L}^{-1}(q_2)$ satisfies the critical point equation (62) on the complement of the set Z_{z_2} .*

Assuming this lemma for the moment, recall from Corollary 2.4 that z_2 is of class C^1 and that (62) expresses z_2'' through z_1, z_1', z_2, z_2' . Thus bootstrapping (62) implies that z_2 is smooth and (62) holds on the whole of S^1 . Therefore, it remains to prove Lemma 4.4.

Proof of Lemma 4.4. We will show the desired equation for z_1 pretending that it has a possibly nonempty zero set Z_{q_1} . A similar argument will do the job for z_2 . Recall the equation satisfied by q_1 ,

$$\ddot{q}_1 = -\frac{2}{q_1^2} + \frac{1}{(q_1 - q_2)^2}. \quad (70)$$

Set

$$\beta_1 := \frac{\ddot{q}_1 - \frac{1}{(q_1 - q_2)^2}}{q_1} \quad (71)$$

on $S^1 \setminus Z_{q_1}$. Then by (70) we have

$$\beta_1 = -\frac{2}{q_1^3},$$

and taking a time derivative we obtain

$$\dot{\beta}_1 = \frac{3 \cdot 2}{q_1^4} \dot{q}_1 = -\frac{3\beta_1 \dot{q}_1}{q_1}.$$

We multiply both sides by q_1^2 to get

$$\dot{\beta}_1 q_1^2 = -3\beta_1 \dot{q}_1 q_1.$$

We bring $-2\beta_1 \dot{q}_1 q_1$ to the other side to continue:

$$\dot{\beta}_1 q_1^2 + 2\beta_1 \dot{q}_1 q_1 = -\beta_1 \dot{q}_1 q_1.$$

We substitute the original definition (71) of β_1 into the right-hand side to get

$$\dot{\beta}_1 q_1^2 + 2\beta_1 \dot{q}_1 q_1 = -\dot{q}_1 \ddot{q}_1 + \frac{\dot{q}_2}{(q_1 - q_2)^2} - \frac{\dot{q}_2 - \dot{q}_1}{(q_2 - q_1)^2}.$$

Integrating both sides from t to 1 we get

$$\beta_1 q_1^2 = -\frac{\dot{q}_1^2}{2} - \int_t^1 \frac{\dot{q}_2(s)}{(q_1(s) - q_2(s))^2} ds + \frac{q_2 - q_1}{(q_2 - q_1)^2} + C \quad (72)$$

for some constant $C \in \mathbb{R}$. We bring $-\frac{q_1}{(q_2 - q_1)^2}$ to the left-hand side and rewrite (71) in the form

$$\beta_1 q_1^2 + \frac{q_1}{(q_2 - q_1)^2} = \ddot{q}_1 q_1.$$

Therefore (72) transforms to

$$\ddot{q}_1 q_1 = -\frac{\dot{q}_1^2}{2} - \int_t^1 \frac{\dot{q}_2(s)}{(q_1(s) - q_2(s))^2} ds + \frac{q_2}{(q_2 - q_1)^2} + C. \quad (73)$$

Observe that modulo the exact value of C this is exactly (66), which is equivalent to the first equation of (62). An analogous discussion applied to q_2 will lead to the second equation of (62). Therefore, we are left with computing the constant C .

To compute C we use (70) to get rid of $\ddot{q}$ on the left-hand side of (73) and obtain

$$-\frac{2}{q_1} + \frac{q_1}{(q_1 - q_2)^2} = -\frac{\dot{q}_1^2}{2} - \int_t^1 \frac{\dot{q}_2(s)}{(q_1(s) - q_2(s))^2} ds + \frac{q_2}{(q_2 - q_1)^2} + C.$$

We solve the last equation for C ,

$$C = -\frac{2}{q_1} + \frac{1}{q_1 - q_2} + \frac{\dot{q}_1^2}{2} + \int_t^1 \frac{\dot{q}_2}{(q_1 - q_2)^2} ds,$$

and integrate both sides from 0 to 1. Noting that

$$\begin{aligned} \int_0^1 dt \int_t^1 \frac{\dot{q}_2(s)}{(q_1(s) - q_2(s))^2} ds &= \int_0^1 ds \int_0^s dt \frac{\dot{q}_2(s)}{(q_1(s) - q_2(s))^2} \\ &= \int_0^1 \frac{s \dot{q}_2(s) ds}{(q_1(s) - q_2(s))^2}, \end{aligned}$$

this gives us

$$C = -\int_0^1 \frac{2 ds}{q_1(s)} + \int_0^1 \frac{ds}{q_1(s) - q_2(s)} + \frac{\|\dot{q}_1\|^2}{2} + \int_0^1 \frac{s \dot{q}_2 ds}{(q_1 - q_2)^2},$$

which matches the constant in (65). This concludes the proof of Lemma 4.4, and therefore of Theorem 4.1. ■

5. Interpolation

We now interpolate linearly between the instantaneous and mean interactions. That is, for $r \in [0, 1]$ we consider the system of coupled ODEs

$$\begin{cases} \ddot{q}_1(t) = -\frac{2}{q_1(t)^2} + \frac{r}{(q_1(t) - q_2(t))^2} + \frac{1-r}{(\bar{q}_1 - \bar{q}_2)^2}, \\ \ddot{q}_2(t) = -\frac{2}{q_2(t)^2} - \frac{r}{(q_1(t) - q_2(t))^2} - \frac{1-r}{(\bar{q}_1 - \bar{q}_2)^2}. \end{cases} \quad (74)$$

For $r = 0$ this agrees with system (32) for mean interaction, and for $r = 1$ with system (52) for instantaneous interaction. Solutions of (74) are critical points of the functional $r\mathcal{S}_{\text{in}} + (1-r)\mathcal{S}_{\text{av}}$, which under Levi-Civita transformation corresponds to the functional

$$\mathcal{B}_r := r\mathcal{B}_{\text{in}} + (1-r)\mathcal{B}_{\text{av}}: \mathcal{H}_{\text{av}}^1 \stackrel{(58)}{\supset} \mathcal{H}_{\text{in}}^1 \rightarrow \mathbb{R} \quad (75)$$

which computes as

$$\begin{aligned} \mathcal{B}_r(z_1, z_2) &= r\mathcal{B}_{\text{in}}(z_1, z_2) + (1-r)\mathcal{B}_{\text{av}}(z_1, z_2) \\ &= 2 \sum_{i=1}^2 \left(\|z_i\|^2 \cdot \|z_i'\|^2 + \frac{1}{\|z_i\|^2} \right) - \frac{r\|z_1\|^2 \cdot \|z_2\|^2}{\|z_1\|^2 \cdot \|z_2\|^2 - \|z_2\|^2 \cdot \|z_1\|^2} \\ &\quad - \frac{1-r}{\|z_1\|^2} \int_0^1 \frac{z_1(\tau)^2}{z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau)))} d\tau. \end{aligned}$$

We define a *generalized solution* (q_1, q_2) of (74) as in Section 4.1, only instead of (52) it now solves (74). Then we have the following generalization of Theorems 3.1 and 4.1.

Theorem 5.1 (Generalized solutions for interpolated interaction). *Under the Levi-Civita transformations (54) with time changes (55), critical points (z_1, z_2) of the action functional $\mathcal{B}_r$, $r \in [0, 1]$ are in 4-to-1 correspondence with generalized solutions (q_1, q_2) of (74).*

Proof. The proof is very similar to the proofs of Theorems 3.1 and 4.1. Critical points of $\mathcal{B}_r$ are solutions of the problem

$$\begin{aligned} z_1''(\tau) &= \frac{\|z_1'\|^2 z_1(\tau)}{\|z_1\|^2} - \frac{z_1(\tau)}{\|z_1\|^6} + \frac{(1-r)z_1(\tau)}{2\|z_1\|^6} \int_0^1 \frac{z_1(\sigma)^2}{z_1^2(\sigma) - z_2^2(\tau_{z_2}(t_{z_1}(\sigma)))} d\sigma \\ &\quad + \frac{(1-r)z_1(\tau)z_2^2(\tau_{z_2}(t_{z_1}(\tau)))}{2\|z_1\|^4(z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))))^2} \\ &\quad - \frac{(1-r)\|z_2\|^2 z_1(\tau)}{\|z_1\|^6} \int_\tau^1 \frac{z_1^2(\sigma)z_2'(\tau_{z_2}(t_{z_1}(\sigma)))}{z_2(\tau_{z_2}(t_{z_1}(\sigma)))(z_1^2(\sigma) - z_2^2(\tau_{z_2}(t_{z_1}(\sigma))))^2} d\sigma \\ &\quad + \frac{(1-r)\|z_2\|^2 z_1(\tau)}{\|z_1\|^6} \int_0^1 \frac{z_1^2(\sigma)z_2'(\tau_{z_2}(t_{z_1}(\sigma)))t_{z_1}(\sigma)}{z_2(\tau_{z_2}(t_{z_1}(\sigma)))(z_1^2(\sigma) - z_2^2(\tau_{z_2}(t_{z_1}(\sigma))))^2} d\sigma \end{aligned}$$

$$\begin{aligned}
& - \frac{r \|z_2\|^4 \|z_1^2\|^2 z_1(\tau)}{2 \|z_1\|^2 (\|z_1^2\|^2 \|z_2\|^2 - \|z_2^2\|^2 \|z_1\|^2)^2} \\
& + \frac{r \|z_2\|^4 z_1^3(\tau)}{(\|z_1^2\|^2 \|z_2\|^2 - \|z_2^2\|^2 \|z_1\|^2)^2}, \tag{76} \\
z_2''(\tau) = & \frac{\|z_2'\|^2 z_2(\tau)}{\|z_2\|^2} - \frac{z_2(\tau)}{\|z_2\|^6} - \frac{(1-r)z_2(\tau)}{2\|z_2\|^6} \int_0^1 \frac{z_2(\sigma)^2}{z_2^2(\sigma) - z_1^2(\tau_{z_1}(t_{z_2}(\sigma)))} d\sigma \\
& - \frac{(1-r)z_2(\tau)z_1^2(\tau_{z_1}(t_{z_2}(\tau)))}{2\|z_2\|^4 (z_2^2(\tau) - z_1^2(\tau_{z_1}(t_{z_2}(\tau))))^2} \\
& + \frac{(1-r)\|z_1\|^2 z_2(\tau)}{\|z_2\|^6} \int_\tau^1 \frac{z_2^2(\sigma)z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))(z_2^2(\sigma) - z_1^2(\tau_{z_1}(t_{z_2}(\sigma))))^2} d\sigma \\
& - \frac{(1-r)\|z_1\|^2 z_2(\tau)}{\|z_2\|^6} \int_0^1 \frac{z_2^2(\sigma)z_1'(\tau_{z_1}(t_{z_2}(\sigma)))t_{z_2}(\sigma)}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))(z_2^2(\sigma) - z_1^2(\tau_{z_1}(t_{z_2}(\sigma))))^2} d\sigma \\
& + \frac{r \|z_1\|^4 \|z_2^2\|^2 z_2(\tau)}{2 \|z_2\|^2 (\|z_1^2\|^2 \|z_2\|^2 - \|z_2^2\|^2 \|z_1\|^2)^2} \\
& - \frac{r \|z_1\|^4 z_2^3(\tau)}{(\|z_1^2\|^2 \|z_2\|^2 - \|z_2^2\|^2 \|z_1\|^2)^2} \tag{77}
\end{aligned}$$

which interpolates between problems (41) and (62).

Suppose now that (z_1, z_2) is a critical point of $\mathcal{B}_r$ and define

$$q_i(t) := z_i^2(\tau_{z_i}(t)), \quad i \in \{1, 2\}.$$

From (76) we obtain for q_1 the equation

$$\begin{aligned}
2\ddot{q}_1(t)q_1(t) = & \|\dot{q}_1\|^2 - \int_0^1 \frac{4}{q_1(s)} ds - \dot{q}_1^2(t) + \int_0^1 \frac{2(1-r)}{q_1(s) - q_2(s)} ds \\
& + \frac{2(1-r)q_2(t)}{(q_1(t) - q_2(t))^2} - \int_t^1 \frac{2(1-r)\dot{q}_2(s)}{(q_1(s) - q_2(s))^2} ds \\
& + \int_0^1 \frac{2(1-r)s\dot{q}_2(s)}{(q_1(s) - q_2(s))^2} ds - \frac{2r\bar{q}_1}{(\bar{q}_1 - \bar{q}_2)^2} + \frac{4rq_1(t)}{(\bar{q}_1 - \bar{q}_2)^2}, \tag{78}
\end{aligned}$$

which interpolates between (43) and (65). As before, outside collisions we define

$$\beta_1 := \frac{\ddot{q}_1 - \frac{r}{(q_1 - q_2)^2} - \frac{1-r}{(\bar{q}_1 - \bar{q}_2)^2}}{q_1}.$$

Again β_1 solves the ODE

$$\dot{\beta}_1 q_1 = -3\beta_1 \dot{q}_1$$

and it follows that

$$\beta_1 = -\frac{\mu}{q_1^3},$$

where μ is locally constant on the complement of collisions on the circle. Using the continuity of z'_1 , it follows again that μ is constant and from (78) we conclude that $\mu = 2$. Therefore, outside of collisions q_1 solves the first equation in (74). Similarly, it follows that outside collisions q_2 solves the second equation in (74). The converse direction is proved similarly to the previous cases. ■

6. Existence of symmetric frozen planet orbits

A *simple frozen planet orbit of period $T > 0$* is a map $q = (q_1, q_2) \in H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^2)$ with the following properties:

- (1) $q_1(t) > q_2(t) \geq 0$ for all $t \in \mathbb{R}/T\mathbb{Z}$;
- (2) q_2 has a unique zero at $t = 0$;
- (3) $(q_1, q_2): (0, T) \rightarrow \mathbb{R}^2$ is smooth and satisfies (52);
- (4) the energies

$$E_i(t) := \frac{\dot{q}_i(t)^2}{2} - \frac{2}{q_i(t)}, \quad t \in (0, T)$$

extend to continuous functions $E_i: \mathbb{R}/T\mathbb{Z} \rightarrow \mathbb{R}$.

Here simplicity corresponds to the second condition, and every frozen planet orbit is a multiple cover of a simple one. Recall that the individual energies E_i need not be constant, but the total energy

$$E = E_1(t) + E_2(t) + \frac{1}{q_1(t) - q_2(t)}$$

is constant and negative. A simple frozen planet orbit of period $T > 0$ is called *symmetric* if it satisfies in addition

$$q(t) = q(T - t) \quad \text{for all } t \in \mathbb{R}/T\mathbb{Z}.$$

In this section we prove the following result which corresponds to Theorem C in the introduction.

Theorem 6.1. *For every $E < 0$ there exists a symmetric simple frozen planet orbit of energy E .*

Rescaling. Let $q = (q_1, q_2): \mathbb{R} \rightarrow (\mathbb{R}_{\geq 0})^2$ be a generalized solution of (52) of period T and energy E . Direct computation shows that for each $c > 0$,

$$q_c(t) := c^2 q(c^{-3}t)$$

is again a generalized solution of (52) of period $c^3 T$ and energy $c^{-2} E$. Therefore, given a generalized solution q of (52) of period 1 and negative energy, rescaling yields similar solutions with any prescribed period, or alternatively with any prescribed negative energy. As a consequence, we will from now on restrict our discussion to generalized solutions of period 1.

6.1. Symmetries

In this subsection we describe the symmetries of the variational problems in Sections 2, 3 and 4.

The Kepler problem. The functional $\mathcal{Q}$ in (23) is obviously invariant under the following transformations:

- time shift $T_s z(\tau) := z(s + \tau)$, $s \in S^1$;
- time reversal $Rz(\tau) := z(-\tau)$;
- sign reversal $Sz(\tau) := -z(\tau)$;

where $z \in H^1(S^1, \mathbb{R}) \setminus \{0\}$.

Lemma 6.2. *The homeomorphism $t_z: S^1 \rightarrow S^1$ defined in (9) and its inverse τ_z transform under time shift and reversal as*

$$t_{T_s z}(\tau) = t_z(s + \tau) - t_z(s), \quad \tau_{T_s z}(t) = \tau_z(t + t_z(s)) - s, \quad (79)$$

$$t_{Rz}(\tau) = -t_z(-\tau), \quad \tau_{Rz}(t) = -\tau_z(-t). \quad (80)$$

Proof. For the first equation of (79) we compute

$$\begin{aligned} t_{T_s z}(\tau) &= \frac{1}{\|T_s z\|^2} \int_0^\tau z(s + \sigma)^2 d\sigma \\ &= \frac{1}{\|z\|^2} \left(\int_0^{s+\tau} z(\sigma)^2 d\sigma - \int_0^s z(\sigma)^2 d\sigma \right) \\ &= t_z(s + \tau) - t_z(s). \end{aligned}$$

For the second equation of (79) note that $t_{T_s z}$ can be written as a composition of three homeomorphisms of the circle:

$$t_{T_s z} = T_{-t_z(s)} \circ t_z \circ T_s.$$

The inverse $\tau_{T_s z}$ of $t_{T_s z}$ is the composition of the three inverse homeomorphisms in the opposite order:

$$\tau_{T_s z} = T_{-s} \circ \tau_z \circ T_{t_z(s)}.$$

The first equation of (80) follows from

$$t_{Rz}(\tau) = \frac{\int_0^\tau (Rz)^2(\sigma) d\sigma}{\|Rz\|^2} = \frac{\int_0^\tau z^2(-\sigma) d\sigma}{\|z\|^2} = -\frac{\int_0^{-\tau} z^2(\sigma) d\sigma}{\|z\|^2} = -t_z(-\tau),$$

and the second equation of (80) follows from the first one by splitting t_{Rz} into three homeomorphisms in analogy with the above. ■

We discuss what these symmetries correspond to under the Levi-Civita transformation $q(t) = z(\tau_z(t))^2$. Let $q_s(t)$ denote the Levi-Civita transformation of $T_s z$. Then

$$q_s(t) = (T_s z(\tau_{T_s z}(t)))^2 = (T_s z(\tau_z(t_z(s) + t) - s))^2 = z(\tau_z(t + t_z(s)))^2 = q(t_z(s) + t).$$

We see that the time shift T_s applied to z corresponds to the time shift $T_{t_z(s)}$ applied to the Levi-Civita transformation q of z . Analogous but simpler arguments show that time reversal corresponds to time reversal $Rq(t) = q(-t)$, and sign reversal corresponds to the identity $Sq(t) = q(t)$. Thus, the first two symmetries of z correspond to actual symmetries of the Kepler problem, while the sign change $Sz = -z$ just expresses the fact that the Levi-Civita transformation defines a 2-to-1 correspondence.

Mean and instantaneous interactions. The problems with interaction have the following symmetries.

Lemma 6.3. *The functionals $\mathcal{B}_{\text{av}}$ and $\mathcal{B}_{\text{in}}$ in (38) and (59) are invariant under the following transformations of $z = (z_1, z_2)$:*

- joint time shift $T_s z(\tau) := z(s + \tau)$, $s \in S^1$;
- joint time reversal $Rz(\tau) := z(-\tau)$;
- separate sign reversals $S_1(z_1, z_2) := (-z_1, z_2)$ and $S_2(z_1, z_2) := (z_1, -z_2)$.

Proof. Invariance of $\mathcal{B}_{\text{av}}$ and $\mathcal{B}_{\text{in}}$ under joint time shift T_s and separate sign changes S_i is obvious, and so is invariance of $\mathcal{B}_{\text{av}}$ under joint time reversal R (in fact, $\mathcal{B}_{\text{av}}$ is even invariant under separate time reversals). For invariance of $\mathcal{B}_{\text{in}}$ under R we write the instantaneous interaction from (56) in the form

$$\mathcal{I}(z_1, z_2) = \int_0^1 \frac{dt}{z_2^2(\tau_{z_2}(t)) - z_1^2(\tau_{z_1}(t))}.$$

Using Lemma 6.2 we compute

$$(Rz_i)(\tau_{Rz_i}(t)) = z_i(-\tau_{Rz_i}(t)) = z_i(\tau_{z_i}(-t))$$

for $i = 1, 2$, and therefore

$$\begin{aligned} \mathcal{I}(R(z_1, z_2)) &= \int_0^1 \frac{dt}{z_2^2(\tau_{z_2}(-t)) - z_1^2(\tau_{z_1}(-t))} \\ &= \int_0^1 \frac{dt}{z_2^2(\tau_{z_2}(t)) - z_1^2(\tau_{z_1}(t))} = \mathcal{I}(z_1, z_2). \end{aligned} \quad \blacksquare$$

6.2. Twisted loops

Theorems 2.5, 3.1 and 4.1 establish a correspondence between critical points of the functionals $\mathcal{Q}$, $\mathcal{B}_{\text{av}}$ and $\mathcal{B}_{\text{in}}$ and generalized solutions with an even number of zeros. In this subsection we explain how to deal with generalized solutions with an *odd* number of zeros.

The Kepler problem. Consider $q \in H^1(S^1, \mathbb{R}_{\geq 0})$ satisfying all conditions on a generalized solution of (22) except that it has an odd number of zeros. By slight abuse of notation we will still refer to such q as a “generalized solution”. Recall that $S^1 = \mathbb{R}/\mathbb{Z}$. Then we can view q as a map $\tilde{q} \in H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{R}_{\geq 0})$ with an even number of zeros and associate to

it as in Section 2.5 a homeomorphism $\tau_q: \mathbb{R}/2\mathbb{Z} \rightarrow \mathbb{R}/2\mathbb{Z}$ and a map $z \in H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{R})$ such that

$$z(\tau_q(t))^2 = \tilde{q}(t), \quad t \in \mathbb{R}/2\mathbb{Z}.$$

By construction these maps satisfy

$$\tilde{q}(t+1) = \tilde{q}(t), \quad \tau_q(t+1) = \tau_q(t), \quad z(t+1) = -z(t), \quad t \in \mathbb{R}/2\mathbb{Z}.$$

This leads us to introduce for each $k \in \mathbb{N}_0$ the Hilbert space of *twisted loops*,

$$H_{\text{twist}}^k(S^1, \mathbb{R}) := \{z \in H^k(\mathbb{R}/2\mathbb{Z}, \mathbb{R}) \mid z(\tau+1) = -z(\tau) \text{ for all } \tau\}$$

with the inner product

$$\langle z, v \rangle := \int_0^1 z(\tau)v(\tau) d\tau = \frac{1}{2} \int_0^2 z(\tau)v(\tau) d\tau.$$

Note that $H_{\text{twist}}^k(S^1, \mathbb{R})$ is the fixed point set of the linear involution

$$\sigma = S \circ T_1: H^k(\mathbb{R}/2\mathbb{Z}, \mathbb{R}) \rightarrow H^k(\mathbb{R}/2\mathbb{Z}, \mathbb{R}), \quad \sigma z(\tau) = -z(\tau+1),$$

where T_1 and S are the time shift by 1 and the sign reversal from Section 6.1. Thus $H_{\text{twist}}^k(S^1, \mathbb{R})$ is a closed linear subspace of $H^k(\mathbb{R}/2\mathbb{Z}, \mathbb{R})$, with inner product the one induced from $H^k(\mathbb{R}/2\mathbb{Z}, \mathbb{R})$ divided by 2. We define

$$\mathcal{Q}: H_{\text{twist}}^1(S^1, \mathbb{R}) \setminus \{0\} \rightarrow \mathbb{R}, \quad \mathcal{Q}(z) := 2\|z\|^2\|z'\|^2 + \frac{N}{\|z\|^2}$$

by the same formula as in (23). As in Section 2.4 it follows that $z \in H_{\text{twist}}^1(S^1, \mathbb{R}) \setminus \{0\}$ is a critical point of $\mathcal{Q}$ if and only if

$$\langle z', v' \rangle + \langle az, v \rangle = 0 \quad \text{for all } v \in H_{\text{twist}}^1(S^1, \mathbb{R}), \quad (81)$$

with the constant

$$a = \frac{\|z'\|^2}{\|z\|^2} - \frac{N}{2\|z\|^6}.$$

To proceed we need the following lemma.

Lemma 6.4. *Suppose that $f, g \in H_{\text{twist}}^0(S^1, \mathbb{R})$ satisfy*

$$\langle f, v' \rangle + \langle g, v \rangle = 0 \quad \text{for all } v \in H_{\text{twist}}^1(S^1, \mathbb{R}).$$

Then $f \in H_{\text{twist}}^1(S^1, \mathbb{R})$ with weak derivative $f' = g$.

Proof. By definition of $H_{\text{twist}}^0(S^1, \mathbb{R})$ we have $f, g \in H^0(\mathbb{R}/2\mathbb{Z}, \mathbb{R})$ with $f(\tau+1) = -f(\tau)$ and $g(\tau+1) = -g(\tau)$, so the hypothesis of the lemma reads

$$\int_0^2 f(\tau)v'(\tau) d\tau + \int_0^2 g(\tau)v(\tau) d\tau = 0 \quad (82)$$

for all $v \in H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{R})$ with $v(\tau + 1) = -v(\tau)$. By definition of the weak derivative, we need to show that (82) holds for all $v \in H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{R})$ (not necessarily satisfying $v(\tau + 1) = -v(\tau)$). Multiplying by bump functions and using linearity, it suffices to show this for $v \in H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{R})$ with support in an interval $I \subset \mathbb{R}/2\mathbb{Z}$ of length less than 1. Given such v we define $\tilde{v} \in H^1(\mathbb{R}/2\mathbb{Z}, \mathbb{R})$ by

$$\tilde{v}(\tau) := \begin{cases} v(\tau), & \tau \in I, \\ -v(\tau - 1), & \tau - 1 \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Using $f(\tau + 1) = -f(\tau)$ and $\tilde{v}'(\tau + 1) = -\tilde{v}'(\tau)$ we compute

$$\begin{aligned} \int_0^2 f(\tau) \tilde{v}'(\tau) d\tau &= \int_I f(\tau) \tilde{v}'(\tau) d\tau + \int_I f(\tau + 1) \tilde{v}'(\tau + 1) d\tau \\ &= \int_I f(\tau) \tilde{v}'(\tau) d\tau + \int_I f(\tau) \tilde{v}'(\tau) d\tau \\ &= 2 \int_0^2 f(\tau) v'(\tau) d\tau, \end{aligned}$$

and similarly

$$\int_0^2 g(\tau) \tilde{v}(\tau) d\tau = 2 \int_0^2 g(\tau) v(\tau) d\tau.$$

Since (82) holds with $\tilde{v}$ in place of v , we conclude

$$0 = \int_0^2 f(\tau) \tilde{v}'(\tau) d\tau + \int_0^2 g(\tau) \tilde{v}(\tau) d\tau = 2 \left(\int_0^2 f(\tau) v'(\tau) d\tau + \int_0^2 g(\tau) v(\tau) d\tau \right)$$

and the lemma is proved. $\blacksquare$

Let us now return to the critical point $z \in H_{\text{twist}}^1(S^1, \mathbb{R}) \setminus \{0\}$ of $\mathcal{Q}$ satisfying (81) above. Applying Lemma 6.4 with $f = z'$ and $g(z) = az$, we conclude that $z \in H_{\text{twist}}^2(S^1, \mathbb{R})$ and its second weak derivative satisfies

$$z'' = az.$$

This is the same ODE (25) as for critical points in the untwisted case. Now the arguments in Section 2 carry over without further changes to show the following twisted version of Theorem 2.5.

Theorem 6.5. *Under the Levi-Civita transformation (1) with time change (2), critical points z of the action functional $\mathcal{Q}: H_{\text{twist}}^1(S^1, \mathbb{R}) \setminus \{0\} \rightarrow \mathbb{R}$ on twisted loops are in 2-to-1 correspondence with generalized solutions $q: S^1 \rightarrow \mathbb{R}_{\geq 0}$ of (22) having an odd number of zeros.*

Mean and instantaneous interactions. Consider now the helium atom. Recall that in a frozen planet configuration the inner electron q_2 should undergo repeated collisions with the nucleus while the outer electron q_1 experiences no collisions. If in period 1 the inner electron undergoes an odd number of collisions (for example a single one), the Levi-Civita transformed maps z_1, z_2 will be 2-periodic and satisfy

$$z_1(\tau + 1) = z_1(\tau), \quad z_2(\tau + 1) = -z_2(\tau).$$

This leads us to introduce for each $k \in \mathbb{N}_0$ the Hilbert space of *twisted loops*

$$H_{\text{twist}}^k(S^1, \mathbb{R}^2) := \{z = (z_1, z_2) \in H^k(\mathbb{R}/2\mathbb{Z}, \mathbb{R}) \mid z_1(\tau + 1) = z_1(\tau), \\ z_2(\tau + 1) = -z_2(\tau) \text{ for all } \tau\}$$

with the inner product

$$\langle z, v \rangle := \sum_{i=1}^2 \int_0^1 z(\tau) v(\tau) d\tau = \frac{1}{2} \sum_{i=1}^2 \int_0^2 z(\tau) v(\tau) d\tau.$$

Note that $H_{\text{twist}}^k(S^1, \mathbb{R}^2)$ is the fixed point set of the linear involution

$$\sigma = S_2 \circ T_1: H^k(\mathbb{R}/2\mathbb{Z}, \mathbb{R}^2) \rightarrow H^k(\mathbb{R}/2\mathbb{Z}, \mathbb{R}^2), \quad \sigma z(\tau) = (z_1(\tau + 1), -z_2(\tau + 1)),$$

where T_1 and S_2 are the joint time shift by 1 and sign reversal in the second component from Section 6.1. Thus $H_{\text{twist}}^k(S^1, \mathbb{R}^2)$ is a closed linear subspace of $H^k(\mathbb{R}/2\mathbb{Z}, \mathbb{R}^2)$, with inner product the one induced from $H^k(\mathbb{R}/2\mathbb{Z}, \mathbb{R}^2)$ divided by 2. We define open subsets

$$\tilde{\mathcal{H}}_{\text{av}}^1 := \{z = (z_1, z_2) \in H_{\text{twist}}^1(S^1, \mathbb{R}^2) \mid \|z_1\| > 0, \|z_2\| > 0, \frac{\|z_1^2\|^2}{\|z_1\|^2} > \frac{\|z_2^2\|^2}{\|z_2\|^2}\}$$

and

$$\tilde{\mathcal{H}}_{\text{in}}^1 := \{z = (z_1, z_2) \in H_{\text{twist}}^1(S^1, \mathbb{R}^2) \mid \|z_1\| > 0, \|z_2\| > 0, \\ z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))) > 0 \text{ for all } \tau \in S^1\} \quad (83)$$

as in Sections 3 and 4, and we define the mean and instantaneous interaction functionals and their interpolation

$$\mathcal{B}_{\text{av}}: \tilde{\mathcal{H}}_{\text{av}}^1 \rightarrow \mathbb{R}, \quad \mathcal{B}_{\text{in}}, \mathcal{B}_r: \tilde{\mathcal{H}}_{\text{in}}^1 \rightarrow \mathbb{R}$$

by the same formulas as in (38), (59) and (75). Note that in formulas (40) and (61) for the derivatives of $\mathcal{B}_{\text{av}}$ and $\mathcal{B}_{\text{in}}$, the leading-order terms $\langle z'_i, v'_i \rangle$ are the same as in the Kepler problem. Arguing as above using Lemma 6.4, we thus conclude that critical points of $\mathcal{B}_{\text{av}}$ and $\mathcal{B}_r$ on twisted loops still satisfy the same equations (41) and (77) as in the untwisted case. Therefore, in analogy with Theorem 6.5, we obtain the following twisted version of Theorems 3.1 and 5.1.

Theorem 6.6. *Under the Levi-Civita transformations (34) with time changes (35), critical points (z_1, z_2) of the action functional $\mathcal{B}_r: \tilde{\mathcal{H}}_{\text{av}}^1 \rightarrow \mathbb{R}$ on twisted loops are in 4-to-1 correspondence with generalized solutions (q_1, q_2) of (32) with q_1 having an even and q_2 an odd number of zeros. Similarly, for each $r \in [0, 1]$ critical points (z_1, z_2) of the action functional $\mathcal{B}_r: \tilde{\mathcal{H}}_{\text{in}}^1 \rightarrow \mathbb{R}$ on twisted loops are in 4-to-1 correspondence with generalized solutions (q_1, q_2) of (74) with q_1 having an even and q_2 an odd number of zeros.*

6.3. Symmetric loops

In order to study symmetric frozen planet orbits, we introduce for each $k \in \mathbb{N}_0$ the Hilbert space of *symmetric loops*

$$H_{\text{sym}}^k(S^1, \mathbb{R}^2) := \{z = (z_1, z_2) \in H_{\text{twist}}^k(S^1, \mathbb{R}^2) \mid z(\tau) = z(1 - \tau) \text{ for all } \tau\}$$

with the inner product induced from $H_{\text{twist}}^k(S^1, \mathbb{R}^2)$. Thus $H_{\text{sym}}^k(S^1, \mathbb{R}^2)$ is the fixed point set of the linear involution

$$\rho = R \circ T_1: H_{\text{twist}}^k(S^1, \mathbb{R}^2) \rightarrow H_{\text{twist}}^k(S^1, \mathbb{R}^2), \quad \rho z(\tau) = z(1 - \tau),$$

where T_1 and R are the joint time shift by 1 and joint time reversal from Section 6.1.

Elements $(z_1, z_2) \in H_{\text{sym}}^k(S^1, \mathbb{R}^2)$ are 2-periodic loops satisfying for all $\tau \in \mathbb{R}/2\mathbb{Z}$ the conditions

$$z_1(\tau) = z_1(\tau + 1) = z_1(1 - \tau), \quad z_2(\tau) = -z_2(\tau + 1) = z_2(1 - \tau).$$

Taking derivatives they imply

$$z_1'(\tau) = z_1'(\tau + 1) = -z_1'(1 - \tau), \quad z_2'(\tau) = -z_2'(\tau + 1) = -z_2'(1 - \tau).$$

In particular, at $\tau = 0$ and $\tau = 1/2$ they imply

$$z_1'(0) = z_2(0) = 0, \quad z_1'(1/2) = z_2'(1/2) = 0.$$

Thus z_1 is 1-periodic with critical points at $\tau = 0$ and $\tau = 1/2$, while z_2 is 2-periodic with zeros at $\tau = 0$ and $\tau = 1$ and with critical points at $\tau = 1/2$ and $\tau = 3/2$.

The images $q_i(t) = z_i(\tau_{z_i}(t))^2$ under the Levi-Civita transformation are 1-periodic and *symmetric*, i.e.

$$q_i(t) = q_i(1 - t), \quad t \in S^1.$$

In particular, if in addition (q_1, q_2) satisfies ODE (52), then (q_1, q_2) is a symmetric frozen planet orbit.

Again, we define open subsets

$$\hat{\mathcal{H}}_{\text{av}}^k := \{z = (z_1, z_2) \in H_{\text{sym}}^k(S^1, \mathbb{R}^2) \mid \|z_1\| > 0, \|z_2\| > 0, \frac{\|z_1\|^2}{\|z_1\|^2} > \frac{\|z_2\|^2}{\|z_2\|^2}\}$$

and

$$\begin{aligned}\hat{\mathcal{H}}_{\text{in}}^k &:= \{z = (z_1, z_2) \in H_{\text{sym}}^k(S^1, \mathbb{R}^2) \mid \|z_1\| > 0, \|z_2\| > 0, \\ &\quad z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))) > 0 \text{ for all } \tau \in S^1\},\end{aligned}\quad (84)$$

and we define the mean and instantaneous interaction functionals and their interpolation

$$\mathcal{B}_{\text{av}}: \hat{\mathcal{H}}_{\text{av}}^1 \rightarrow \mathbb{R}, \quad \mathcal{B}_{\text{in}}: \hat{\mathcal{H}}_{\text{in}}^1 \rightarrow \mathbb{R}, \quad \mathcal{B}_r: \hat{\mathcal{H}}_{\text{av}}^1 \stackrel{(58)}{\supset} \hat{\mathcal{H}}_{\text{in}}^1 \rightarrow \mathbb{R}$$

by the same formulas as in (38), (59) and (75). Arguing as in the previous subsection, using a variant of Lemma 6.4, we conclude that critical points of $\mathcal{B}_{\text{av}}$ and $\mathcal{B}_r$ on symmetric loops still satisfy the same equations (41) and (77) and we obtain the following symmetric version of Theorems 3.1 and 5.1.

Theorem 6.7. *Under the Levi-Civita transformations (34) with time changes (35), critical points (z_1, z_2) of the action functional $\mathcal{B}_{\text{av}}: \hat{\mathcal{H}}_{\text{av}}^1 \rightarrow \mathbb{R}$ on symmetric loops are in 4-to-1 correspondence with symmetric generalized solutions (q_1, q_2) of (32) with q_1 having an even and q_2 an odd number of zeros. Similarly, for each $r \in [0, 1]$ critical points (z_1, z_2) of the action functional $\mathcal{B}_r: \hat{\mathcal{H}}_{\text{in}}^1 \rightarrow \mathbb{R}$ on symmetric loops are in 4-to-1 correspondence with symmetric generalized solutions (q_1, q_2) of (74) with q_1 having an even and q_2 an odd number of zeros (i.e. with symmetric frozen planet orbits).*

6.4. Proof of the existence Theorem 6.1

Now we are ready to prove Theorem 6.1. By the rescaling discussion following the theorem, it suffices to consider the case of period 1 without prescribing the energy.

Using the notation of the previous subsection we define

$$X := \{z = (z_1, z_2) \in \hat{\mathcal{H}}_{\text{in}}^2 \mid z_i(\tau) > 0 \text{ for all } \tau \in (0, 1) \text{ and } i = 1, 2\}. \quad (85)$$

This is an open subset of the Hilbert space $H_{\text{sym}}^2(S^1, \mathbb{R}^2)$ and thus a Hilbert manifold. Note that the twisting conditions $z_1(\tau + 1) = z_1(\tau)$ and $z_2(\tau + 1) = -z_2(\tau)$ imply that $z_1(\tau) > 0$ for all $\tau \in \mathbb{R}$ and $z_2(\tau) < 0$ for $\tau \in (1, 2)$. This ensures that z_2 is simple of minimal period 2 and it removes the symmetries $z_i \mapsto \pm z_i$. We consider the Hilbert space

$$Y := H_{\text{sym}}^0(S^1, \mathbb{R}^2)$$

and the L^2 -gradient (see beginning of Appendix A) of the interpolation functional $\mathcal{B}_r = (1 - r)\mathcal{B}_{\text{in}} + r\mathcal{B}_{\text{av}}$,

$$\nabla \mathcal{B}_r = (1 - r)\nabla \mathcal{B}_{\text{in}} + r\nabla \mathcal{B}_{\text{av}}: X \rightarrow Y, \quad r \in [0, 1].$$

According to Theorem A.1, for each $r \in [0, 1]$ this is a C^1 -Fredholm map of index zero. Thus

$$F: [0, 1] \times X \rightarrow Y, \quad (r, z) \mapsto \nabla \mathcal{B}_r(z)$$

is a C^1 -Fredholm map of index 1.

According to Theorem 6.7, for each $r \in [0, 1]$ zeros $z \in X$ of $\nabla \mathcal{B}_r$ correspond under the Levi-Civita transformation to symmetric generalized solutions $q = (q_1, q_2): S^1 \rightarrow \mathbb{R}^2$ of (74). The condition $z_2(\tau) > 0$ for all $\tau \in (0, 1)$ in the definition of X implies that q_2 has a unique zero at $t = 0$. By the main result in [1] there exists a constant κ such that

$$\max_{t \in S^1} \left\{ q_1(t), \frac{1}{q_1(t) - q_2(t)} \right\} \leq \kappa$$

for all such solutions q of (74) and all $r \in [0, 1]$. Thus on $(r, z) \in F^{-1}(0)$ the function $z_1^2(t)$ is uniformly bounded from above and the difference $z_1^2(\tau_{z_1}(t)) - z_2^2(\tau_{z_2}(t))$ is uniformly bounded away from zero. In view of ODE (77), this implies that the zero set $F^{-1}(0) \subset [0, 1] \times X$ is compact. Hence $F: [0, 1] \times X \rightarrow Y$ is a homotopy as in Theorem C.1 between $f_0 = \nabla \mathcal{B}_{\text{in}}$ and $f_1 = \nabla \mathcal{B}_{\text{av}}$, and it follows that f_0, f_1 have well-defined mod 2 Euler numbers satisfying

$$\chi(\nabla \mathcal{B}_{\text{in}}) = \chi(\nabla \mathcal{B}_{\text{av}}).$$

By Theorem D.1, the mod 2 Euler number of $\nabla \mathcal{B}_{\text{av}}$ satisfies

$$\chi(\nabla \mathcal{B}_{\text{av}}) = 1.$$

Together with the previous displayed equation this shows that $\chi(\nabla \mathcal{B}_{\text{in}}) = 1$, so $\nabla \mathcal{B}_{\text{in}}$ possesses a zero whose Levi-Civita transform is the desired symmetric frozen planet orbit. This concludes the proof of Theorem 6.1.

7. Hamiltonian formulation

In this section we present the Hamiltonian formulations of the problems described in the previous sections. They will all be derived from a general result proved in the first subsection.

7.1. Legendre transform

In this subsection we describe an abstract Legendre transform which will be applied to the helium atom in the following subsections. For concreteness we restrict to the case that the configuration space is $\mathbb{R}^n$, but everything could be easily extended to more general configuration manifolds.

For $k \in \mathbb{N}_0$ we abbreviate $H^k := H^k(S^1, \mathbb{R}^n)$. We denote the derivative of $q \in H^1$ by $\dot{q}$. Suppose we are given an open subset $\mathcal{U}^1 \subset H^1$ and a Lagrange function

$$\mathcal{L}: \mathcal{U}^1 \times H^0 \rightarrow \mathbb{R}, \quad (q, v) \mapsto \mathcal{L}(q, v).$$

We say that $\mathcal{L}$ possesses a *continuous L^2 -gradient* if $\mathcal{L}$ is of class C^1 and there exists a continuous map

$$\nabla \mathcal{L} = (\nabla_1 \mathcal{L}, \nabla_2 \mathcal{L}): \mathcal{U}^1 \times H^0 \rightarrow H^0 \times H^0$$

uniquely defined by the conditions

$$\langle \nabla_i \mathcal{L}(q, v), w \rangle = D_i \mathcal{L}(q, v)w \quad \text{for all } w \in H^1,$$

where $\langle \cdot, \cdot \rangle$ is the L^2 -inner product and $D_i \mathcal{L}$ denotes the derivative with respect to the i th variable. We associate to such $\mathcal{L}$ its *Lagrangian action*

$$\mathcal{S}_{\mathcal{L}}: \mathcal{U}^1 \rightarrow \mathbb{R}, \quad q \mapsto \mathcal{L}(q, \dot{q}).$$

This is a C^1 -function whose Fréchet derivative at $q \in \mathcal{U}^1$ in the direction $w \in H^1$ is

$$D\mathcal{S}_{\mathcal{L}}(q)w = D_1 \mathcal{L}(q, \dot{q})w + D_2 \mathcal{L}(q, \dot{q})\dot{w} = \langle \nabla_1 \mathcal{L}(q, \dot{q}), w \rangle + \langle \nabla_2 \mathcal{L}(q, \dot{q}), \dot{w} \rangle.$$

It follows that $q \in \mathcal{U}^1$ is a critical point of $\mathcal{S}$ if and only if $\nabla_2 \mathcal{L}(q, \dot{q}) \in H^1$ and the following *Euler–Lagrange equation* holds:

$$\frac{d}{dt} \nabla_2 \mathcal{L}(q, \dot{q}) = \nabla_1 \mathcal{L}(q, \dot{q}). \quad (86)$$

Let us impose the following condition on $\mathcal{L}$:

(L) *There exists a differentiable map*

$$F: \mathcal{U}^1 \times H^0 \rightarrow H^0, \quad (q, p) \mapsto F(q, p)$$

such that for each $q \in \mathcal{U}^1$ the map $H^0 \rightarrow H^0$, $v \mapsto \nabla_2 \mathcal{L}(q, v)$ is a homeomorphism with inverse $p \mapsto F(q, p)$.

In particular, the map F then satisfies

$$\nabla_2 \mathcal{L}(q, F(q, p)) = p. \quad (87)$$

Then we associate to $\mathcal{L}$ its *fibrewise Legendre transform*

$$\mathcal{H}: \mathcal{U}^1 \times H^0 \rightarrow \mathbb{R}, \quad \mathcal{H}(q, p) := \langle p, F(q, p) \rangle - \mathcal{L}(q, F(q, p)).$$

Using (87) we compute, for $w \in H^1$,

$$\begin{aligned} D_1 \mathcal{H}(q, p)w &= \langle p, D_1 F(q, p)w \rangle - D_1 \mathcal{L}(q, F(q, p))w - D_2 \mathcal{L}(q, F(q, p))D_1 F(q, p)w \\ &= \langle p - \nabla_2 \mathcal{L}(q, F(q, p)), D_1 F(q, p)w \rangle - \langle \nabla_1 \mathcal{L}(q, F(q, p)), w \rangle \\ &= -\langle \nabla_1 \mathcal{L}(q, F(q, p)), w \rangle, \\ D_2 \mathcal{H}(q, p)w &= \langle F(q, p), w \rangle + \langle p, D_2 F(q, p)w \rangle - D_2 \mathcal{L}(q, F(q, p))D_2 F(q, p)w \\ &= \langle F(q, p), w \rangle + \langle p - \nabla_2 \mathcal{L}(q, F(q, p)), D_2 F(q, p)w \rangle \\ &= \langle F(q, p), w \rangle. \end{aligned}$$

This shows that $\mathcal{H}$ has a continuous L^2 -gradient which is related to that of $\mathcal{L}$ by

$$\begin{aligned}\nabla_1 \mathcal{H}(q, p) &= -\nabla_1 \mathcal{L}(q, F(q, p)), \\ \nabla_2 \mathcal{H}(q, p) &= F(q, p).\end{aligned}\tag{88}$$

On the other hand, to any Hamilton function $\mathcal{H}: \mathcal{U}^1 \times H^0 \rightarrow \mathbb{R}$ with continuous L^2 -gradient we can associate its *Hamiltonian action*

$$\mathcal{A}_{\mathcal{H}}: \mathcal{U}^1 \times H^0 \rightarrow \mathbb{R}, \quad \mathcal{A}_{\mathcal{H}}(q, p) := \langle p, \dot{q} \rangle - \mathcal{H}(q, p).$$

Its derivatives in the direction $w \in H^1$ are given by

$$\begin{aligned}D_1 \mathcal{A}_{\mathcal{H}}(q, p)w &= \langle p, \dot{w} \rangle - \langle \nabla_1 \mathcal{H}(q, p), w \rangle, \\ D_2 \mathcal{A}_{\mathcal{H}}(q, p)w &= \langle \dot{q}, w \rangle - \langle \nabla_2 \mathcal{H}(q, p), w \rangle.\end{aligned}$$

It follows that $(q, p) \in \mathcal{U}^1 \times H^0$ is a critical point of $\mathcal{A}_{\mathcal{H}}$ if and only if $p \in H^1$ and the following *Hamilton equations* hold:

$$\begin{aligned}\dot{p} &= -\nabla_1 \mathcal{H}(q, p), \\ \dot{q} &= \nabla_2 \mathcal{H}(q, p).\end{aligned}\tag{89}$$

Proposition 7.1. *Let $\mathcal{L}: \mathcal{U}^1 \times H^0 \rightarrow \mathbb{R}$ be a Lagrange function with continuous gradient satisfying condition (L) and $\mathcal{H}: \mathcal{U}^1 \times H^0 \rightarrow \mathbb{R}$ its fibrewise Legendre transform. Then the assignments $(q, p) \mapsto q$ and $q \mapsto (q, p = \nabla_2 \mathcal{L}(q, \dot{q}))$ define a 1-to-1 correspondence between critical points (q, p) of $\mathcal{A}_{\mathcal{H}}$ and critical points q of $\mathcal{S}_{\mathcal{L}}$.*

Proof. Let $q \in \mathcal{U}^1$ be a critical point of $\mathcal{S}_{\mathcal{L}}$, so $p := \nabla_2 \mathcal{L}(q, \dot{q}) \in H^1$ and q solves (86). Then condition (L) and the second equation in (88) give

$$\dot{q} = F(q, p) = \nabla_2 \mathcal{H}(q, p),$$

and (86) and the first equation in (88) give

$$\dot{p} = \frac{d}{dt} \nabla_2 \mathcal{L}(q, \dot{q}) = \nabla_1 \mathcal{L}(q, \dot{q}) = -\nabla_1 \mathcal{L}(q, F(q, p)) = \nabla_1 \mathcal{H}(q, p).$$

So (q, p) solves (89) and is therefore a critical point of $\mathcal{A}_{\mathcal{H}}$.

Conversely, let $(q, p) \in \mathcal{U}^1 \times H^0$ be a critical point of $\mathcal{A}_{\mathcal{H}}$, so $p \in H^1$ and (q, p) solves (89). Then the second equation in (89) and the second equation in (88) give

$$\dot{q} = \nabla_2 \mathcal{H}(q, p) = F(q, p),$$

which by condition (L) implies $\nabla_2 \mathcal{L}(q, \dot{q}) = p \in H^1$. Now the first equation in (89) and the first equation in (88) give

$$\frac{d}{dt} \nabla_2 \mathcal{L}(q, \dot{q}) = \dot{p} = \nabla_1 \mathcal{H}(q, p) = -\nabla_1 \mathcal{L}(q, F(q, p)) = \nabla_1 \mathcal{L}(q, \dot{q}).$$

So q solves (86) and is therefore a critical point of $\mathcal{S}_{\mathcal{L}}$. ■

Remark 7.2. Inspection of the preceding proof shows that formulas for the derivatives of $\mathcal{H}$ do not involve derivatives of F . This suggests that Proposition 7.1 should still hold if in condition (L) we only assume continuity of F rather than differentiability.

Example 7.3. Classically, the Lagrangian $\mathcal{L}: \mathcal{U}^1 \times H^0 \rightarrow \mathbb{R}$ has the form

$$\mathcal{L}(q, v) = \int_0^1 L(q(t), v(t)) dt$$

for a smooth Lagrangian density $L: U \times \mathbb{R}^n \rightarrow \mathbb{R}$, where $U \subset \mathbb{R}^n$ is an open subset and $\mathcal{U}^1 = H^1(S^1, U)$. Then

$$\mathcal{H}(q, p) = \int_0^1 H(q(t), p(t)) dt$$

with $H: U \times \mathbb{R}^n \rightarrow \mathbb{R}$ the fibrewise Legendre transform of L , and the Euler–Lagrange and Hamilton equations take the traditional form for $i = 1, \dots, n$:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

and

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}.$$

This example covers the instantaneous interaction Lagrangian $\mathcal{L}_{\text{in}}$ for helium in the original coordinates $q = (q_1, q_2) \in \mathbb{R}_+^2$ away from collisions. The more general setting in Proposition 7.1 will be needed in the following subsections to deal with the Lagrangians $\mathcal{B}_{\text{av}}$ and $\mathcal{B}_{\text{in}}$ in the new coordinate $z = (z_1, z_2)$, which do not have the form in Example 7.3.

7.2. The Kepler problem

For the Kepler problem, the function $\mathcal{Q}$ defined in (23) is the Lagrangian action $\mathcal{S}_{\mathcal{L}}$ associated to the Lagrange function

$$\mathcal{L}: \mathcal{U}^1 \times H^0(S^1, \mathbb{R}) \rightarrow \mathbb{R}, \quad \mathcal{L}(z, w) = 2\|z\|^2\|w\|^2 + \frac{N}{\|z\|^2} \quad (90)$$

with $\mathcal{U}^1 = H^1(S^1, \mathbb{R}) \setminus \{0\}$. The computation of the differential in Section 2.4 shows that $\mathcal{L}$ has a continuous L^2 -gradient. The associated momentum η is given by

$$\eta = \nabla_2 \mathcal{L}(z, w) = 4\|z\|^2 w,$$

which can be solved for w as

$$w = \frac{\eta}{4\|z\|^2} = F(z, \eta).$$

Note that the map F is smooth in (z, η) . It follows that $\|\eta\|^2 = 16\|z\|^4\|w\|^2$ and the associated Hamilton function becomes

$$\mathcal{H}(z, \eta) = \langle \eta, F(z, \eta) \rangle - \mathcal{L}(z, F(z, \eta)) = \frac{\|\eta\|^2}{8\|z\|^2} - \frac{N}{\|z\|^2}, \quad (91)$$

with Hamiltonian action

$$\mathcal{A}_{\mathcal{H}}(z, \eta) = \langle \eta, z' \rangle - \frac{\|\eta\|^2}{8\|z\|^2} + \frac{N}{\|z\|^2}. \quad (92)$$

By Proposition 7.1, critical points of $\mathcal{A}_{\mathcal{H}}$ are in 1-to-1 correspondence to critical points of $\mathcal{Q}$.

7.3. Mean interaction

For the helium atom with mean interaction, the function $\mathcal{B}_{\text{av}}$ defined in (38) is the Lagrangian action $\mathcal{S}_{\mathcal{L}_{\text{av}}}$ associated to the Lagrange function

$$\mathcal{L}_{\text{av}}: \mathcal{H}_{\text{av}}^1 \times H^0(S^1, \mathbb{R}^2) \rightarrow \mathbb{R}, \quad \mathcal{L}_{\text{av}}(z, w) = \mathcal{L}(z_1, w_1) + \mathcal{L}(z_2, w_2) + \mathcal{A}(z_1, z_2),$$

where $\mathcal{H}_{\text{av}}^1$ and $\mathcal{A}$ are defined in (37) and (36), and $\mathcal{L}$ is the Kepler Lagrangian from (90) with charge $N = 2$. The computation of the differential in Section 3.2 shows that $\mathcal{L}_{\text{av}}$ has a continuous L^2 -gradient. Since the interaction term $\mathcal{A}$ does not depend on the w_i , the associated momenta η_i are given as in the Kepler case by

$$\begin{aligned} \eta_i &= 4\|z_i\|^2 w_i, \\ w_i &= \frac{\eta_i}{4\|z_i\|^2} = F(z_i, \eta_i) \end{aligned}$$

and the associated Hamilton function becomes

$$\begin{aligned} \mathcal{H}_{\text{av}}(z, \eta) &= \mathcal{H}(z_1, \eta_1) + \mathcal{H}(z_2, \eta_2) - \mathcal{A}(z_1, z_2) \\ &= \sum_{i=1}^2 \left(\frac{\|\eta_i\|^2}{8\|z_i\|^2} - \frac{2}{\|z_i\|^2} \right) + \frac{\|z_1\|^2\|z_2\|^2}{\|z_1^2\|^2\|z_2\|^2 - \|z_2^2\|^2\|z_1\|^2}, \end{aligned}$$

with Hamiltonian action

$$\begin{aligned} \mathcal{A}_{\mathcal{H}_{\text{av}}}(z, \eta) &= \langle \eta_1, z'_1 \rangle + \langle \eta_2, z'_2 \rangle - \mathcal{H}_{\text{av}}(z, \eta) \\ &= \sum_{i=1}^2 \left(\langle \eta_i, z'_i \rangle - \frac{\|\eta_i\|^2}{8\|z_i\|^2} + \frac{2}{\|z_i\|^2} \right) - \frac{\|z_1\|^2\|z_2\|^2}{\|z_1^2\|^2\|z_2\|^2 - \|z_2^2\|^2\|z_1\|^2}. \end{aligned}$$

By Proposition 7.1, critical points of $\mathcal{A}_{\mathcal{H}_{\text{av}}}$ are in 1-to-1 correspondence to critical points of $\mathcal{B}_{\text{av}}$.

7.4. Instantaneous interaction

For the helium atom with instantaneous interaction, the function $\mathcal{B}_{\text{in}}$ defined in (59) is the Lagrangian action $\mathcal{S}_{\mathcal{L}_{\text{in}}}$ associated to the Lagrange function

$$\mathcal{L}_{\text{in}}: \mathcal{H}_{\text{in}}^1 \times H^0(S^1, \mathbb{R}^2) \rightarrow \mathbb{R}, \quad \mathcal{L}_{\text{in}}(z, w) = \mathcal{L}(z_1, w_1) + \mathcal{L}(z_2, w_2) + \mathcal{I}(z_1, z_2),$$

where $\mathcal{H}_{\text{in}}^1$ and $\mathcal{I}$ are defined in (57) and (56), and $\mathcal{L}$ is the Kepler Lagrangian from (90) with charge $N = 2$. The computation of the differential in Section 4.2 shows that $\mathcal{L}_{\text{in}}$ has a continuous L^2 -gradient. Since the interaction term $\mathcal{I}$ does not depend on the w_i , the associated momenta η_i are given as in the Kepler case by

$$\eta_i = 4\|z_i\|^2 w_i, \quad w_i = \frac{\eta_i}{4\|z_i\|^2} = F(z_i, \eta_i)$$

and the associated Hamilton function becomes

$$\begin{aligned} \mathcal{H}_{\text{in}}(z, \eta) &= \mathcal{H}(z_1, \eta_1) + \mathcal{H}(z_2, \eta_2) - \mathcal{I}(z_1, z_2) \\ &= \sum_{i=1}^2 \left(\frac{\|\eta_i\|^2}{8\|z_i\|^2} - \frac{2}{\|z_i\|^2} \right) + \int_0^1 \frac{1}{z_1^2(\tau_{z_1}(t)) - z_2^2(\tau_{z_2}(t))} dt, \end{aligned}$$

with Hamiltonian action

$$\begin{aligned} \mathcal{A}_{\mathcal{H}_{\text{in}}}(z, \eta) &= \langle \eta_1, z'_1 \rangle + \langle \eta_2, z'_2 \rangle - \mathcal{H}_{\text{in}}(z, \eta) \\ &= \sum_{i=1}^2 \left(\langle \eta_i, z'_i \rangle - \frac{\|\eta_i\|^2}{8\|z_i\|^2} + \frac{2}{\|z_i\|^2} \right) - \int_0^1 \frac{1}{z_1^2(\tau_{z_1}(t)) - z_2^2(\tau_{z_2}(t))} dt. \end{aligned}$$

By Proposition 7.1, critical points of $\mathcal{A}_{\mathcal{H}_{\text{in}}}$ are in 1-to-1 correspondence to critical points of $\mathcal{B}_{\text{in}}$.

A. Differentiability and Fredholm property

Let $W \subset H^1(S^1, \mathbb{R}^2)$ be an open subset and

$$\mathcal{B}: W \rightarrow \mathbb{R}$$

be a Fréchet differentiable map. We say that $\mathcal{B}$ possesses an L^2 -gradient if for each $z \in W$ the derivative $D\mathcal{B}(z): H^1(S^1, \mathbb{R}^2) \rightarrow \mathbb{R}$ extends to a continuous linear functional $L^2(S^1, \mathbb{R}^2) \rightarrow \mathbb{R}$. In this case the L^2 -gradient

$$\nabla \mathcal{B}: W \rightarrow L^2(S^1, \mathbb{R}^2)$$

is uniquely defined by the condition

$$\langle \nabla \mathcal{B}(z), v \rangle = D\mathcal{B}(z)(v) \quad \text{for all } v \in L^2(S^1, \mathbb{R}^2),$$

where $\langle \cdot, \cdot \rangle$ is the L^2 -inner product and $D\mathcal{B}$ denotes the Fréchet derivative.

The natural domain of definition for such a functional $\mathcal{B}$ is an open subset of H^1 . However, for the applications in this article we are only interested in functionals that possess L^2 -gradients of class C^1 . Since for the functionals we consider this is only the case when their domain of definition is restricted to H^2 , we restrict the following discussion to subsets of H^2 .

Recall the setup from Section 4. The set

$$\mathcal{H}_{\text{in}} = \{z = (z_1, z_2) \in H^2(S^1, \mathbb{R}^2) \mid \|z_1\| > 0, \|z_2\| > 0, \\ z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau))) > 0 \text{ for all } \tau \in S^1\}$$

is an open subset of the Hilbert space $H^2(S^1, \mathbb{R}^2) = W^{2,2}(S^1, \mathbb{R}^2)$ and equipped with the H^2 -topology. We consider the instantaneous interaction functional

$$\mathcal{B}_{\text{in}}: \mathcal{H}_{\text{in}} \rightarrow \mathbb{R}, \quad z = (z_1, z_2) \mapsto Q(z_1, z_2) + I(z_1, z_2)$$

with the free (noninteracting) term

$$Q(z_1, z_2) = 2 \sum_{i=1}^2 \left(\|z_i\|^2 \cdot \|z'_i\|^2 + \frac{1}{\|z_i\|^2} \right)$$

and the instantaneous interaction term

$$I(z_1, z_2) = -\frac{1}{\|z_1\|^2} \int_0^1 \frac{z_1(\tau)^2}{z_1^2(\tau) - z_2^2(\tau_{z_2}(t_{z_1}(\tau)))} d\tau \\ = \frac{1}{\|z_2\|^2} \int_0^1 \frac{z_2(\tau)^2}{z_2^2(\tau) - z_1^2(\tau_{z_1}(t_{z_2}(\tau)))} d\tau.$$

For each $z \in \mathcal{H}_{\text{in}}$ the derivative $D\mathcal{B}_{\text{in}}(z): H^2(S^1, \mathbb{R}^2) \rightarrow \mathbb{R}$ extends to a continuous linear functional $L^2(S^1, \mathbb{R}^2) \rightarrow \mathbb{R}$ (see the formulas for $DQ(z)$ in the proof of Theorem A.1 and for $DI(z)$ in (105) below), so we can define its L^2 -gradient as above. An analogous discussion applies to the mean interaction functional

$$\mathcal{B}_{\text{av}}: \mathcal{H}_{\text{av}} \rightarrow \mathbb{R}, \quad z = (z_1, z_2) \mapsto Q(z_1, z_2) + \mathcal{A}(z_1, z_2)$$

from Section 4 with the mean interaction term

$$\mathcal{A}(z_1, z_2) = -\frac{\|z_1\|^2 \|z_2\|^2}{\|z_1^2\|^2 \|z_2\|^2 - \|z_2^2\|^2 \|z_1\|^2},$$

defined on the open subset

$$\mathcal{H}_{\text{av}} = \{z = (z_1, z_2) \in H^2(S^1, \mathbb{R}^2) \mid \|z_1\| > 0, \|z_2\| > 0, \frac{\|z_1^2\|^2}{\|z_1\|^2} > \frac{\|z_2^2\|^2}{\|z_2\|^2}\}$$

of the Hilbert space $H^2(S^1, \mathbb{R}^2)$. Intersecting (58) with $H^2(S^1, \mathbb{R}^2)$ gives us

$$\mathcal{H}_{\text{in}} \subset \mathcal{H}_{\text{av}}.$$

The goal of this appendix is the proof of the following theorem:

Theorem A.1. *On neighbourhoods of their respective zero sets, the L^2 -gradients $\nabla \mathcal{B}_{\text{in}}: \mathcal{H}_{\text{in}} \rightarrow L^2(S^1, \mathbb{R}^2)$ and $\nabla \mathcal{B}_{\text{av}}: \mathcal{H}_{\text{av}} \rightarrow L^2(S^1, \mathbb{R}^2)$, as well as their interpolation $\nabla \mathcal{B}_r = (1-r)\nabla \mathcal{B}_{\text{in}} + r\nabla \mathcal{B}_{\text{av}}: \mathcal{H}_{\text{av}} \supset \mathcal{H}_{\text{in}} \rightarrow L^2(S^1, \mathbb{R}^2)$ are C^1 -Fredholm maps of index zero. The same holds for their restrictions to the spaces of symmetric orbits $\hat{\mathcal{H}}_{\text{av}}^2 \rightarrow L_{\text{sym}}^2(S^1, \mathbb{R}^2)$ (resp. $\hat{\mathcal{H}}_{\text{in}}^2 \rightarrow L_{\text{sym}}^2(S^1, \mathbb{R}^2)$) defined in (84).*

Proof. The derivative of the free term applied to $v = (v_1, v_2) \in L^2(S^1, \mathbb{R}^2)$ is

$$\begin{aligned} DQ(z)v &= 4 \sum_{i=1}^2 \left(\|z_i\|^2 \langle z'_i, v'_i \rangle + \|z'_i\|^2 \langle z_i, v_i \rangle - \frac{1}{\|z_i\|^4} \langle z_i, v_i \rangle \right) \\ &= 4 \sum_{i=1}^2 \left(-\|z_i\|^2 z''_i + \|z'_i\|^2 z_i - \frac{1}{\|z_i\|^4} z_i, v_i \right), \end{aligned}$$

hence its L^2 -gradient has components

$$\nabla_i Q(z) = 4 \left(-\|z_i\|^2 z''_i + \|z'_i\|^2 z_i - \frac{1}{\|z_i\|^4} z_i \right), \quad i = 1, 2. \quad (93)$$

This obviously defines a C^1 -map $\nabla Q: \mathcal{H}_{\text{av}} \rightarrow L^2(S^1, \mathbb{R}^2)$. To see that the derivative of ∇Q at $w = (w_1, w_2)$ is Fredholm of index zero we write the leading term of ∇Q near w as $z \mapsto -4(\|w_1\|^2 z''_1, \|w_2\|^2 z''_2)$. The latter map is a restriction of the obvious linear index zero Fredholm map $H^2(S^1, \mathbb{R}^2) \rightarrow L^2(S^1, \mathbb{R}^2)$ (the kernel is spanned and the image is complemented by constants). The remaining lower-order terms give a compact perturbation, so ∇Q is a nonlinear C^1 -Fredholm map $\mathcal{H}_{\text{av}} \rightarrow L^2(S^1, \mathbb{R}^2)$ of index zero.

For the mean interaction, the components of the L^2 -gradient are read off from (39) to be

$$\begin{aligned} \nabla_1 \mathcal{A}[z_1, z_2] &= -2 \frac{\|z_2\|^4 \cdot \|z_1^2\|^2}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} z_1 \\ &\quad + 4 \frac{\|z_1\|^2 \cdot \|z_2\|^4}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} z_1^3, \\ \nabla_2 \mathcal{A}[z_1, z_2] &= +2 \frac{\|z_1\|^4 \cdot \|z_2^2\|^2}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} z_2 \\ &\quad - 4 \frac{\|z_1\|^4 \cdot \|z_2\|^2}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} z_2^3. \end{aligned} \quad (94)$$

Since these are C^1 -maps whose derivatives at each point are compact linear operators, this proves the assertion for $\mathcal{B}_{\text{av}}$ (which also follows from Theorem D.1).

Existence of the restriction of ∇Q to $\hat{\mathcal{H}}_{\text{av}}^2 \rightarrow L_{\text{sym}}^2(S^1, \mathbb{R}^2)$ as a Fredholm map of index zero is straightforward. So Theorem A.1 follows from Proposition A.2 below. ■

Proposition A.2. *On a neighbourhood of the zero set of $\mathcal{B}$, the gradient $\nabla I: \mathcal{H}_{\text{in}} \rightarrow L^2(S^1, \mathbb{R}^2)$ of the interaction term defines a C^1 -map whose derivative at each point is a compact linear operator $H^2(S^1, \mathbb{R}^2) \rightarrow L^2(S^1, \mathbb{R}^2)$.*

The proof of this proposition will occupy the rest of this appendix. It uses some technical lemmas from Appendix B.

A.1. Reparametrizations of the circle

Here we collect some facts about reparametrizations of the circle that are used throughout this article. In this subsection we consider a map $z \in C^1(S^1, \mathbb{R})$ with finite zero set

$$Z := \{\tau \in S^1 \mid z(\tau) = 0\}.$$

As in Section 2.2, we associate to z a C^2 -map $t_z: S^1 \rightarrow S^1$ by

$$t_z(\tau) := \frac{1}{\|z\|^2} \int_0^\tau z(\sigma)^2 d\sigma \quad (95)$$

with derivative

$$t'_z(\tau) = \frac{z(\tau)^2}{\|z\|^2}. \quad (96)$$

By Lemma 2.1, the map t_z is a homeomorphism with continuous inverse

$$\tau_z := t_z^{-1}: S^1 \rightarrow S^1. \quad (97)$$

Since t_z is of class C^2 , the function τ_z is also of class C^2 on the complement of the finite set $t_z(Z)$ with derivative

$$\dot{\tau}_z(t) = \frac{\|z\|^2}{z(\tau_z(t))^2}. \quad (98)$$

In Section 4 we need the Fréchet derivatives of t_z and τ_z with respect to z . The derivative of t_z with respect to z in the direction $v \in L^2(S^1, \mathbb{R})$ is given by

$$Dt_z(v)(\tau) = \frac{2}{\|z\|^2} \int_0^\tau z(\sigma)v(\sigma) d\sigma - \frac{2\langle z, v \rangle}{\|z\|^4} \int_0^\tau z(\sigma)^2 d\sigma. \quad (99)$$

For future use observe that Dt_z defines a bounded operator $L^2(S^1, \mathbb{R}) \rightarrow H^1(S^1, \mathbb{R})$ depending continuously on z . Indeed, the first summand is a composition of multiplication with a continuous function and integrating from 0 to τ . The multiplication is a continuous operator to L^2 , and integration is a continuous operator to H^1 . Boundedness of the second summand can be seen analogously. Continuous dependence on z is clear.

Using (99) and (96), we derive a formula for the derivative of τ_z :

$$\begin{aligned} 0 &= D(t_z \circ \tau_z)(v)(t) \\ &= Dt_z(v)(\tau_z(t)) + t'_z(\tau_z(t))D\tau_z(v)(t) \\ &= \frac{2}{\|z\|^2} \int_0^{\tau_z(t)} z(\sigma)v(\sigma) d\sigma - \frac{2\langle z, v \rangle}{\|z\|^4} \int_0^{\tau_z(t)} z(\sigma)^2 d\sigma + \frac{z(\tau_z(t))^2}{\|z\|^2} D\tau_z(v)(t), \end{aligned}$$

thus

$$D\tau_z(v)(t) = \frac{2\langle z, v \rangle}{\|z\|^2 z(\tau_z(t))^2} \int_0^{\tau_z(t)} z(\sigma)^2 d\sigma - \frac{2}{z(\tau_z(t))^2} \int_0^{\tau_z(t)} z(\sigma)v(\sigma) d\sigma. \quad (100)$$

The instantaneous interaction term in Section 4 involves the expression

$$\Delta(\tau) := z_2^2(\tau) - z_1^2(\tau_{z_1}(t_{z_2}(\tau))), \quad (101)$$

whose derivative with respect to z_1 in the direction $v \in L^2(S^1, \mathbb{R})$ is given by

$$\begin{aligned} (D_1 \Delta)(v)(\tau) &= -2z_1(\tau_{z_1}(t_{z_2}(\tau)))v(\tau_{z_1}(t_{z_2}(\tau))) \\ &\quad - 2z_1(\tau_{z_1}(t_{z_2}(\tau)))z_1'(\tau_{z_1}(t_{z_2}(\tau)))D\tau_{z_1}(v)(t_{z_2}(\tau)). \end{aligned} \quad (102)$$

We also sometimes need the transformation of Δ under time change

$$\Delta(\tau_{z_2}(t_{z_1}(\tau))) = z_2^2(\tau_{z_2}(t_{z_1}(\tau))) - z_1^2(\tau) < 0 \quad (103)$$

and its derivative

$$\begin{aligned} D_1(\Delta(\tau_{z_2}(t_{z_1}(\cdot))))(v)(\tau) &= 2z_2(\tau_{z_2}(t_{z_1}(\tau)))z_2'(\tau_{z_2}(t_{z_1}(\tau)))\dot{t}_{z_2}(t_{z_1}(\tau))D\tau_{z_1}(v)(\tau) - 2z_1(\tau)v(\tau) \\ &= 2\|z_2\|^2 \frac{z_2'(\tau_{z_2}(t_{z_1}(\tau)))}{z_2(\tau_{z_2}(t_{z_1}(\tau)))} D\tau_{z_1}(v)(\tau) - 2z_1(\tau)v(\tau). \end{aligned} \quad (104)$$

A.2. L^2 -gradient of the instantaneous interaction term

We write the interaction term as

$$I(z_1, z_2) = \frac{1}{\|z_2\|^2} \mathcal{I}(z_1, z_2),$$

with

$$\mathcal{I}(z_1, z_2) := \int_0^1 \frac{z_2(\tau)^2}{z_2^2(\tau) - z_1^2(\tau_{z_1}(t_{z_2}(\tau)))} d\tau = \int_0^1 \frac{z_2(\tau)^2}{\Delta(\tau)} d\tau.$$

Since $\|z_2\| > 0$, it is enough to prove Proposition A.2 with $\mathcal{I}$ in place of I . In the remainder of this appendix we will prove compactness and continuous dependence for the z_1 -derivative of the z_1 -component of the L^2 -gradient $\nabla \mathcal{I}$; the treatments of the z_2 -component and the z_2 -derivatives of both components are analogous and will be omitted.

Recall that $\Delta(\tau)$ never vanishes; we will use this without further mention in the computations below.

Derivative of $\mathcal{I}$. Let us compute the derivative of $\mathcal{I}$ with respect to z_1 in the direction $v \in L^2(S^1, \mathbb{R})$:

$$D_1 \mathcal{I}(z_1, z_2)(v) = - \int_0^1 \frac{z_2^2(\tau)}{\Delta(\tau)^2} D_1 \Delta(v)(\tau) d\tau$$

$$\begin{aligned}
&= 2 \int_0^1 \frac{z_2^2(\tau)}{\Delta(\tau)^2} z_1(\tau_{z_1}(t_{z_2}(\tau))) v(\tau_{z_1}(t_{z_2}(\tau))) d\tau \\
&\quad + 2 \int_0^1 \frac{z_2^2(\tau)}{\Delta(\tau)^2} z_1(\tau_{z_1}(t_{z_2}(\tau))) z_1'(\tau_{z_1}(t_{z_2}(\tau))) D\tau_{z_1}(v)(t_{z_2}(\tau)) d\tau \\
&= 2 \int_0^1 \frac{z_2^2(\tau)}{\Delta(\tau)^2} z_1(\tau_{z_1}(t_{z_2}(\tau))) v(\tau_{z_1}(t_{z_2}(\tau))) d\tau \\
&\quad + \frac{4\langle z_1, v \rangle}{\|z_1\|^2} \int_0^1 \frac{z_2^2(\tau)}{\Delta(\tau)^2} \frac{z_1'(\tau_{z_1}(t_{z_2}(\tau)))}{z_1(\tau_{z_1}(t_{z_2}(\tau)))} \int_0^{\tau_{z_1}(t_{z_2}(\tau))} z_1(\sigma)^2 d\sigma d\tau \\
&\quad - 4 \int_0^1 \frac{z_2^2(\tau)}{\Delta(\tau)^2} \frac{z_1'(\tau_{z_1}(t_{z_2}(\tau)))}{z_1(\tau_{z_1}(t_{z_2}(\tau)))} \int_0^{\tau_{z_1}(t_{z_2}(\tau))} z_1(\sigma) v(\sigma) d\sigma d\tau \\
&=: \sum_{i=1}^3 D_1^i \mathcal{I}(z_1, z_2)(v). \tag{105}
\end{aligned}$$

Here, in the second equality we have used (102), in the third equality we have substituted $D\tau_{z_1}(v)$ using (100) with $z = z_1$, and we denote the resulting three summands by $D_1^i \mathcal{I}(z_1, z_2)(v)$, $i = 1, 2, 3$.

Our next goal is to rewrite $D_1 \mathcal{I}(z_1, z_2)(v)$ as the L^2 -inner product of v with the first component of the L^2 -gradient of $\mathcal{I}$,

$$D_1 \mathcal{I}(z_1, z_2)v = \langle \nabla_1 \mathcal{I}(z_1, z_2), v \rangle \quad \text{for all } v \in L^2(S^1, \mathbb{R}).$$

Coordinate change in the integrals. In order to write the first term in (105) as an L^2 -inner product with v , we perform the following coordinate change that will also be used later. For $\sigma \in S^1$ we set

$$\xi := \tau_{z_1}(t_{z_2}(\sigma)) \in S^1,$$

so that

$$\sigma = \tau_{z_2}(t_{z_1}(\xi))$$

and from equations (96) and (98) we get

$$d\sigma = \dot{\tau}_{z_2}(t_{z_1}(\xi)) t_{z_1}'(\xi) d\xi = \frac{\|z_2\|^2}{\|z_1\|^2} \frac{z_1^2(\xi)}{z_2^2(\sigma)} d\xi. \tag{106}$$

So, after renaming the integration variable from τ to σ , the first term in (105) becomes

$$\begin{aligned}
D_1^1 \mathcal{I}(z_1, z_2)(v) &= 2 \int_0^1 \frac{z_2^2(\tau)}{\Delta(\tau)^2} z_1(\tau_{z_1}(t_{z_2}(\sigma))) v(\tau_{z_1}(t_{z_2}(\sigma))) d\sigma \\
&= 2 \frac{\|z_2\|^2}{\|z_1\|^2} \int_0^1 \frac{z_1^3(\xi) v(\xi)}{\Delta(\tau_{z_2}(t_{z_1}(\xi)))^2} d\xi.
\end{aligned}$$

The second term in (105) already has the form of an L^2 -inner product with v . Using (95) to insert $t_z(\tau)$ and renaming the integration variable τ to σ it becomes

$$D_1^2 \mathcal{I}(z_1, z_2)(v) = 4\langle z_1, v \rangle \int_0^1 \frac{z_2^2(\sigma)}{\Delta(\sigma)^2} \frac{z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))} t_{z_1}(\tau_{z_1}(t_{z_2}(\sigma))) d\sigma.$$

Switching the order of integration. To write the third term as an inner product, we need to switch the order of integration in the double integral. The general setup for this is the following. Let $f, F: S^1 \rightarrow \mathbb{R}$ be continuous functions. Let $\tau_1: S^1 \rightarrow S^1$ be a C^1 -homeomorphism with $\tau(0) = 0$ and finitely many critical points. Then

$$\begin{aligned} \int_0^1 d\tau F(\tau) \int_0^{\tau_1(\tau)} f(\sigma) d\sigma &= \int_0^1 d\sigma f(\sigma) \int_{\tau_1^{-1}(\sigma)}^1 F(\tau) d\tau \\ &= \int_0^1 d\tau f(\tau) \int_{\tau_1^{-1}(\tau)}^1 F(\sigma) d\sigma. \end{aligned}$$

We apply this formula with

$$\tau_1(\tau) := \tau_{z_1}(t_{z_2}(\tau)), \quad f(\sigma) := z_1(\sigma)v(\sigma)$$

to the third term in (105). To deal with the integration limits observe that the inverse of τ_1 is given by $\tau_1^{-1}(\tau) = \tau_{z_2}(t_{z_1}(\tau))$. Thus we find

$$\begin{aligned} D_1^3 \mathcal{I}(z_1, z_2)(v) &= -4 \int_0^1 d\tau \frac{z_2^2(\tau)}{\Delta(\tau)^2} \frac{z_1'(\tau_{z_1}(t_{z_2}(\tau)))}{z_1(\tau_{z_1}(t_{z_2}(\tau)))} \int_0^{\tau_{z_1}(t_{z_2}(\tau))} z_1(\sigma)v(\sigma) d\sigma \\ &= -4 \int_0^1 z_1(\tau)v(\tau) \int_{\tau_{z_2}(t_{z_1}(\tau))}^1 \frac{z_2^2(\sigma)}{\Delta(\sigma)^2} \frac{z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))} d\sigma. \end{aligned}$$

Altogether this gives us the formula for the first component of the L^2 -gradient

$$\begin{aligned} \nabla_1 \mathcal{I}(z_1, z_2) &= 2 \frac{\|z_2\|^2}{\|z_1\|^2} \frac{z_1^3(\tau)}{\Delta(\tau_{z_2}(t_{z_1}(\tau)))^2} \\ &\quad + 4z_1(\tau) \int_0^1 \frac{z_2^2(\sigma)}{\Delta(\sigma)^2} \frac{z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))} t_{z_1}(\tau_{z_1}(t_{z_2}(\sigma))) d\sigma \\ &\quad - 4z_1(\tau) \int_{\tau_{z_2}(t_{z_1}(\tau))}^1 \frac{z_2^2(\sigma)}{\Delta(\sigma)^2} \frac{z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))} d\sigma \\ &=: \sum_{i=1}^3 \mathcal{V}_1^i(z_1, z_2)(\tau). \end{aligned} \tag{107}$$

Note that, since Δ and z_1 never vanish, $\nabla_1 \mathcal{I}(z_1, z_2)$ exists as an L^2 -function and depends continuously on $(z_1, z_2) \in \mathcal{H}_{\text{in}}$.

A.3. Hessian part 1

In this subsection we consider the first part of the gradient in (107),

$$\mathcal{V}_1^1(z_1, z_2)(\tau) = 2 \frac{\|z_2\|^2}{\|z_1\|^2} \frac{z_1^3(\tau)}{\Delta(\tau_{z_2}(t_{z_1}(\tau)))^2}.$$

Differentiating it with respect to z_1 in the direction $v \in H^2(S^1, \mathbb{R})$ we obtain

$$\begin{aligned}
\frac{\|z_1\|^2}{2\|z_2\|^2} D_1 \mathcal{V}_1^{(z_1, z_2)}(v)(\tau) &= -\frac{2}{\|z_1\|^2} \frac{z_1^3(\tau)}{\Delta(\tau_{z_2}(t_{z_1}(\tau)))^2} \langle z_1, v \rangle \\
&\quad + 3 \frac{z_1^2(\tau)v(\tau)}{\Delta(\tau_{z_2}(t_{z_1}(\tau)))^2} \\
&\quad - 2 \frac{z_1^3(\tau)}{\Delta(\tau_{z_2}(t_{z_1}(\tau)))^3} D_1(\Delta(\tau_{z_2}(t_{z_1}(\cdot))))(v)(\tau) \\
&= -\frac{2}{\|z_1\|^2} \frac{z_1^3(\tau)}{\Delta(\tau_{z_2}(t_{z_1}(\tau)))^2} \langle z_1, v \rangle \\
&\quad + 3 \frac{z_1^2(\tau)v(\tau)}{\Delta(\tau_{z_2}(t_{z_1}(\tau)))^2} \\
&\quad - 4 \frac{z_1^3(\tau)\|z_2\|^2 z_2'(\tau_{z_2}(t_{z_1}(\tau)))}{\Delta(\tau_{z_2}(t_{z_1}(\tau)))^3 z_2(\tau_{z_2}(t_{z_1}(\tau)))} D t_{z_1}(v)(\tau) \\
&\quad + 4 \frac{z_1^3(\tau)z_1(\tau)v(\tau)}{\Delta(\tau_{z_2}(t_{z_1}(\tau)))^3}, \tag{108}
\end{aligned}$$

where we have used (104) to replace $D_1(\Delta(\tau_{z_2}(t_{z_1}(\cdot))))(v)(\tau)$. We need to show that each of the four summands on the right-hand side as a function of v defines a compact linear operator $H^2(S^1, \mathbb{R}) \rightarrow L^2(S^1, \mathbb{R})$ that depends continuously on $(z_1, z_2) \in \mathcal{H}_{\text{in}}$ with respect to the operator norm.

The first summand is a one-dimensional operator (hence compact) whose image is spanned by a function that lies in $H^1(S^1, \mathbb{R})$ and depends continuously on (z_1, z_2) by Lemmas B.8 and B.7.

The second and fourth summands are multiplication operators with functions that lie in $H^1(S^1, \mathbb{R})$ and depend continuously on (z_1, z_2) by Lemmas B.8 and B.7. They are compact because they are compositions

$$H^2(S^1, \mathbb{R}) \hookrightarrow H^1(S^1, \mathbb{R}) \rightarrow H^1(S^1, \mathbb{R}) \hookrightarrow L^2(S^1, \mathbb{R}),$$

where the middle map is multiplication with a fixed H^1 -function and the two inclusions are compact.

For the third summand first note that by formula (99) with $z = z_1$ the map $v \mapsto D t_{z_1}(v)$ defines a bounded linear operator $H^2(S^1, \mathbb{R}) \rightarrow H^1(S^1, \mathbb{R})$ depending continuously on z_1 . The third term is the composition of this operator and several multiplication operators. The functions by which we multiply lie in $H^1(S^1, \mathbb{R})$ except for $\frac{1}{z_2(\tau_{z_2}(t_{z_1}(\tau)))}$, which lies in $L^2(S^1, \mathbb{R})$. They depend continuously on (z_1, z_2) by Lemmas B.8, B.7 and B.9. To show compactness of this operator we write it as the composition of

continuous linear maps

$$H^2(S^1, \mathbb{R}) \rightarrow H^1(S^1, \mathbb{R}) \hookrightarrow C^0(S^1, \mathbb{R}) \rightarrow L^2(S^1, \mathbb{R}),$$

where the first map sends $v \mapsto Dt_{z_1}(v)$, the third map is multiplication by $\frac{1}{z_2(\tau_{z_2}(t_{z_1}(\tau)))}$, and the canonical inclusion in the middle is compact by the Rellich embedding theorem. This concludes the discussion of $\mathcal{V}_1^1$.

A.4. Hessian part 2

The second part of the gradient in (107) has the form

$$\mathcal{V}_1^2(z_1, z_2)(\tau) = 4z_1(\tau) \int_0^1 g(z_1, z_2)(\sigma) t_{z_2}(\sigma) d\sigma,$$

with

$$g(z_1, z_2)(\sigma) := \frac{z_2^2(\sigma)}{\Delta(\sigma)^2} \frac{z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))}.$$

Differentiating it with respect to z_1 in the direction $v \in H^2(S^1, \mathbb{R})$ we obtain

$$\begin{aligned} \frac{1}{4} D_1 \mathcal{V}_1^2(z_1, z_2)(v)(\tau) &= v(\tau) \int_0^1 g(z_1, z_2)(\sigma) t_{z_2}(\sigma) d\sigma \\ &\quad + z_1(\tau) \int_0^1 D_1 g(z_1, z_2)(v)(\sigma) t_{z_2}(\sigma) d\sigma \\ &\quad + z_1(\tau) \int_0^1 g(z_1, z_2)(\sigma) Dt_{z_2}(v)(\sigma) d\sigma. \end{aligned}$$

Note that $g(z_1, z_2)(\sigma)$ agrees with the integrand of the third part of the gradient in (107). It is shown in the next subsection that $g(z_1, z_2) \in C^0(S^1, \mathbb{R})$ depends continuously on (z_1, z_2) , and $v \mapsto D_1 g(z_1, z_2)(v)$ defines a bounded linear operator $H^2(S^1, \mathbb{R}) \rightarrow L^2(S^1, \mathbb{R})$ that depends continuously on (z_1, z_2) . Equations (95) and (99) show that $t_{z_2} \in C^0(S^1, \mathbb{R})$ depends continuously on z_2 , and $v \mapsto Dt_{z_2}(v)$ defines a bounded linear operator $H^2(S^1, \mathbb{R}) \rightarrow L^2(S^1, \mathbb{R})$ that depends continuously on z_2 . This shows that the right-hand side as a function of v defines a bounded linear operator $H^2(S^1, \mathbb{R}) \rightarrow L^2(S^1, \mathbb{R})$ that depends continuously on $(z_1, z_2) \in \mathcal{H}_{\text{in}}$. It is compact because the first term is a scalar multiplication operator composed with the compact inclusion $H^2(S^1, \mathbb{R}) \hookrightarrow L^2(S^1, \mathbb{R})$, and the other two terms have one-dimensional images.

A.5. Hessian part 3

In this subsection we consider the third part of the gradient in (107),

$$\mathcal{V}_1^3(z_1, z_2)(\tau) = -4z_1(\tau) \int_{\tau_{z_2}(t_{z_1}(\tau))}^1 \frac{z_2^2(\sigma)}{\Delta(\sigma)^2} \frac{z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))} d\sigma.$$

Differentiating it with respect to z_1 in the direction $v \in H^2(S^1, \mathbb{R})$ we obtain

$$\begin{aligned}
-\frac{1}{4}D_1\mathcal{V}_1^3(z_1, z_2)(v)(\tau) &= v(\tau) \int_{\tau_{z_2}(t_{z_1}(\tau))}^1 \frac{z_2^2(\sigma)}{\Delta(\sigma)^2} \frac{z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))} d\sigma \\
&\quad - z_1(\tau) \frac{z_2^2(\tau_{z_2}(t_{z_1}(\tau)))}{\Delta(\tau_{z_2}(t_{z_1}(\tau)))^2} \frac{z_1'(\tau)}{z_1(\tau)} \dot{\tau}_{z_2}(t_{z_1}(\tau)) Dt_{z_1}(v)(\tau) \\
&\quad + z_1(\tau) \int_{\tau_{z_2}(t_{z_1}(\tau))}^1 z_2^2(\sigma) D_1\left(\frac{1}{\Delta(\sigma)^2}\right)(v) \frac{z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))} d\sigma \\
&\quad + z_1(\tau) \int_{\tau_{z_2}(t_{z_1}(\tau))}^1 \frac{z_2^2(\sigma)}{\Delta(\sigma)^2} D_1\left(\frac{z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))}\right)(v) d\sigma \\
&=: \sum_{i=1}^4 T_i v(\tau).
\end{aligned}$$

Again, we need to show that each T_i defines a compact linear operator $H^2(S^1, \mathbb{R}) \rightarrow L^2(S^1, \mathbb{R})$ that depends continuously on $(z_1, z_2) \in \mathcal{H}_{\text{in}}$. In the integrals we will use the change of variables $\xi := \tau_{z_1}(t_{z_2}(\sigma))$ from Section A.2. The integrand transforms according to formula (106). To change the limits of integration note that $\sigma \in [\tau_{z_2}(t_{z_1}(\tau)), 1]$ corresponds to $\xi \in [\tau, 1]$.

Now we discuss the four terms one by one. We will omit the arguments $(S^1, \mathbb{R})$ and simply write H^2 instead of $H^2(S^1, \mathbb{R})$ etc.

The first term. A change of integration variable turns the first term into

$$\begin{aligned}
T_1 v(\tau) &= v(\tau) \int_{\tau_{z_2}(t_{z_1}(\tau))}^1 \frac{z_2^2(\sigma)}{\Delta(\sigma)^2} \frac{z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))} d\sigma \\
&= v(\tau) \frac{\|z_2\|^2}{\|z_1\|^2} \int_{\tau}^1 \frac{z_1'(\xi) z_1(\xi)}{\Delta(\tau_{z_2}(t_{z_1}(\xi)))^2} d\xi.
\end{aligned}$$

By Lemmas B.8 and B.7 the integrand is continuous and depends continuously on (z_1, z_2) as a map $\mathcal{H}_{\text{in}} \rightarrow C^0$. Therefore, T_1 defines a bounded linear operator $H^2 \rightarrow H^1$ depending continuously on (z_1, z_2) , and composition with the inclusion $H^1 \hookrightarrow L^2$ makes it compact.

The second term. Using (95) to replace $\dot{\tau}_{z_2}(t_{z_1}(\tau))$ turns the second term into

$$\begin{aligned}
T_2 v(\tau) &= -z_1(\tau) \frac{z_2^2(\tau_{z_2}(t_{z_1}(\tau)))}{\Delta(\tau_{z_2}(t_{z_1}(\tau)))^2} \frac{z_1'(\tau)}{z_1(\tau)} \frac{\|z_2\|^2}{z_2^2(\tau_{z_2}(t_{z_1}(\tau)))} Dt_{z_1}(v)(\tau) \\
&= -\frac{z_1'(\tau) \|z_2\|^2}{\Delta(\tau_{z_2}(t_{z_1}(\tau)))^2} Dt_{z_1}(v)(\tau).
\end{aligned}$$

Formula (99) with $z = z_1$ shows that $v \mapsto Dt_{z_1}(v)$ defines a bounded operator $H^2 \rightarrow H^1$ depending continuously on (z_1, z_2) . Together with Lemmas B.8 and B.7 this implies that T_2 defines a bounded operator $H^2 \rightarrow H^1$ depending continuously on (z_1, z_2) , and composition with the inclusion $H^1 \hookrightarrow L^2$ makes it compact.

The third term. We rewrite the third term as

$$\begin{aligned}
T_3 v(\tau) &= z_1(\tau) \int_{\tau_{z_2}(t_{z_1}(\tau))}^1 z_2^2(\sigma) \frac{-2(D_1 \Delta)(v)(\sigma)}{\Delta(\sigma)^3} \frac{z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))} d\sigma \\
&= -2z_1(\tau) \int_{\tau_{z_2}(t_{z_1}(\tau))}^1 \frac{z_2^2(\sigma)}{\Delta(\sigma)^3} \frac{z_1'(\tau_{z_1}(t_{z_2}(\sigma)))}{z_1(\tau_{z_1}(t_{z_2}(\sigma)))} [-2z_1(\tau_{z_1}(t_{z_2}(\tau)))v(\tau_{z_1}(t_{z_2}(\tau))) \\
&\quad - 2z_1(\tau_{z_1}(t_{z_2}(\tau)))z_1'(\tau_{z_1}(t_{z_2}(\tau)))D\tau_{z_1}(v)(t_{z_2}(\sigma))] d\sigma \\
&= 4 \frac{\|z_2\|^2}{\|z_1\|^2} z_1(\tau) \int_{\tau}^1 \frac{z_1(\xi)z_1'(\xi)}{\Delta(\tau_{z_2}(t_{z_1}(\xi)))^3} [z_1(\xi)v(\xi) + z_1(\xi)z_1'(\xi)Xv(\xi)] d\xi.
\end{aligned}$$

Here in the second equality we have used (102) to replace $(D_1 \Delta)(v)(\sigma)$, in the third equality we change the integration variable, and we have abbreviated (replacing $D\tau_{z_1}(v)$ via (100))

$$Xv(\xi) := D\tau_{z_1}(v)(t_{z_1}(\xi)) = \frac{2}{z_1^2(\xi)} \left(\frac{\|z_1\|^2(\xi)}{\|z_1\|^2} \langle z_1, v \rangle - \langle z_1, v \rangle(\xi) \right). \quad (109)$$

The map $v \mapsto Xv$ defines a bounded operator $H^2 \rightarrow H^1$ depending continuously on (z_1, z_2) . Together with Lemmas B.8 and B.7 this implies that T_3 defines a bounded operator $H^2 \rightarrow H^1$ depending continuously on (z_1, z_2) , and composition with the inclusion $H^1 \hookrightarrow L^2$ makes it compact.

The fourth term. We rewrite the fourth term as

$$\begin{aligned}
T_3 v(\tau) &= z_1(\tau) \int_{\tau_{z_2}(t_{z_1}(\tau))}^1 \frac{z_2^2(\sigma)}{\Delta(\sigma)^2} D_1((\log z_1)'(\tau_{z_1}(t_{z_2}(\sigma))))(v) d\sigma \\
&= z_1(\tau) \int_{\tau_{z_2}(t_{z_1}(\tau))}^1 \frac{z_2^2(\sigma)}{\Delta(\sigma)^2} \left[\left(\frac{v'(\xi)}{z_1(\xi)} - \frac{z_1'(\xi)}{z_1^2(\xi)} v(\xi) \right) \Big|_{\xi=\tau_{z_1}(t_{z_2}(\sigma))} \right. \\
&\quad \left. + (\log z_1)''(\tau_{z_1}(t_{z_2}(\sigma))) D\tau_{z_1}(v)(t_{z_2}(\sigma)) \right] d\sigma \\
&= \frac{\|z_2\|^2}{\|z_1\|^2} z_1(\tau) \int_{\tau}^1 \frac{z_1^2(\xi)}{\Delta(\tau_{z_2}(t_{z_1}(\xi)))^2} \left[\left(\frac{v'(\xi)}{z_1(\xi)} - \frac{z_1'(\xi)}{z_1^2(\xi)} v(\xi) \right) \right. \\
&\quad \left. + (\log z_1)''(\xi) Xv(\xi) \right] d\xi,
\end{aligned}$$

with $Xv(\xi)$ from (109). By Lemma B.7 the function

$$\xi \mapsto (\log z_1)''(\xi) = \frac{z_1''(\xi)z_1(\xi) - z_1'(\xi)^2}{z_1(\xi)^2}$$

lies in L^2 and depends continuously on $z_1 \in H^2$. All other terms in the integrand are continuous functions of ξ that depend continuously on (z_1, z_2) by Lemmas B.8 and B.7 together with (109). So the integrand belongs to $L^2(S^1, \mathbb{R})$, hence the integral belongs to $H^1(S^1, \mathbb{R})$, thus T_4 defines a bounded operator $H^2 \rightarrow H^1$ depending continuously on (z_1, z_2) , and composition with the inclusion $H^1 \hookrightarrow L^2$ makes it compact.

This finishes the discussion of $\mathcal{V}_1^3$, and thus of the z_1 -derivative of the z_1 -component of the L^2 -gradient $\nabla \mathcal{I}$. The treatments of the z_2 -component and the z_2 -derivatives of both components are analogous and will be omitted. This concludes the proof of Proposition A.2.

B. Some lemmas on continuous dependence

In this appendix we prove some technical lemmas on continuous dependence that were used in Appendix A. We will freely use the notation from Appendix A.

B.1. The basic lemma on continuous dependence

We say that a function $f \in C^1(S^1, \mathbb{R})$ has *transverse zeros* if for all $t \in S^1$ with $f(t) = 0$ we have $f'(t) \neq 0$. We define the following open subset of $H^2(S^1, \mathbb{R})$:

$$\mathcal{H}_0^2 := \{z \in H^2(S^1, \mathbb{R}) \mid z \text{ has transverse zeros}\}.$$

Note that by Proposition 4.2 and Corollary 4.3, for a critical point (z_1, z_2) of the functional $\mathcal{B}_{\text{in}}$ from Section 4 both components z_1, z_2 belong to $\mathcal{H}_0^2$. Similarly, we define

$$\mathcal{C}_0^1 := \{f \in C^1(S^1, \mathbb{R}) \mid f \text{ has transverse zeros}\}.$$

We introduce the maps

$$\mathcal{F}: \mathcal{H}_0^2 \rightarrow \mathcal{C}_0^1, \quad \mathcal{F}(z) := z^3 \circ \tau_z,$$

with τ_z defined in (97), and

$$\mathcal{G}: \mathcal{C}_0^1 \rightarrow L^2(S^1, \mathbb{R}), \quad \mathcal{G}(f) := \frac{1}{f^{1/3}}$$

(defined outside the zero set of f). Our goal in this section is to prove the following lemma:

Lemma B.1. *The map*

$$\mathcal{G} \circ \mathcal{F}: \mathcal{H}_0^2 \rightarrow L^2(S^1, \mathbb{R}), \quad z \mapsto \frac{1}{z \circ \tau_z}$$

is continuous.

The statement naturally splits in two – continuity of $\mathcal{F}$ and continuity of $\mathcal{G}$.

Lemma B.2. *The map $\mathcal{F}$ is continuous.*

Lemma B.3. *The map $\mathcal{G}$ is continuous.*

In the proofs we will use the following standard fact whose easy proof we omit.

Lemma B.4. *Denote by $\text{Homeo}(S^1)$ the space of homeomorphisms of S^1 equipped with the C^0 -topology. Then the inversion $h \mapsto h^{-1}$ defines a continuous map $\text{Homeo}(S^1) \rightarrow \text{Homeo}(S^1)$.*

Proof of Lemma B.2. Formula (95) for the homeomorphism t_z shows that it depends continuously on z . Therefore, by Lemma B.4, its inverse τ_z depends continuously on z as well. This shows that $\mathcal{F}$ lands in $C^0(S^1, \mathbb{R})$ and is continuous as a map to $C^0(S^1, \mathbb{R})$. Next we set $f := \mathcal{F}(z)$ and write out its derivative with respect to t :

$$f'(t) = 3z^2(\tau_z(t))z'(\tau_z(t))\dot{t}_z(t) = 3z^2(\tau_z(t))z'(\tau_z(t))\frac{\|z\|^2}{z(\tau_z(t))^2} = \|z\|^2 z'(\tau_z(t)).$$

Since $z' \in C^0(S^1, \mathbb{R})$ and τ_z depend continuously on z , we see that f' lands in $C^0(S^1, \mathbb{R})$ and depends continuously on z . Altogether this shows that $\mathcal{F}$ is a continuous map to $C^1(S^1, \mathbb{R})$. Transversality of zeros for z and the above formula for $f'(t)$ imply transversality of zeros for f . This completes the proof of Lemma B.2. ■

B.2. Proof of Lemma B.3

Desingularization procedure. The key point in the proof of Lemma B.3 is the question of how to deal with integrals of the type

$$J = \int_{S^1} \frac{dt}{f^{2/3}(t)}$$

for a function $f \in \mathcal{C}_0^1$. The idea is to apply a coordinate change that turns the integrand into a continuous one. For this consider a C^1 -homeomorphism $\rho: S^1 \mapsto S^1$ that restricts to a C^1 -diffeomorphism $S^1 \setminus f^{-1}(0) \mapsto S^1 \setminus f^{-1}(0)$ and satisfies

$$\rho(\tau) = t_i + (\tau - t_i)^3 \quad \text{near each zero } t_i \text{ of } f.$$

We perform the coordinate change $t = \rho(\tau)$ in the integral J . Then near a zero t_i of f , using $t - t_i = (\tau - t_i)^3$, the integrand in J becomes

$$\begin{aligned} \frac{dt}{f^{2/3}(t)} &= \frac{\rho'(\tau) d\tau}{f^{2/3}(t)} = \frac{3(\tau - t_i)^2 d\tau}{f^{2/3}(t)} = 3 \left(\frac{(t - t_i)^2}{f^2(t)} \right)^{1/3} d\tau \\ &= 3 \left(\frac{f(t)}{t - t_i} \right)^{-2/3} d\tau. \end{aligned} \quad (110)$$

Since f is a C^1 -function, the quotient $f(t)/(t - t_i)$ extends continuously over $t = t_i$ by the derivative $f'(t_i)$, which is nonzero because $f \in \mathcal{C}_0^1$. Therefore, the function $t \mapsto 3 \left(\frac{f(t)}{t - t_i} \right)^{-2/3}$ extends continuously over $t = t_i$ by $3f'(t_i)^{-2/3}$. Composing this with the continuous function ρ , we conclude the following lemma:

Lemma B.5. *For f , ρ as above the coefficient in front of $d\tau$ in the pullback $\rho^*(\frac{dt}{f^{2/3}(t)})$ extends uniquely to a continuous function*

$$g: S^1 \rightarrow \mathbb{R}, \quad g(\tau) = \begin{cases} \frac{\rho'(\tau)}{f^{2/3}(\rho(\tau))}, & f(\tau) \neq 0, \\ 3f'(t_i)^{-2/3}, & \tau = t_i \text{ zero of } f. \end{cases} \quad (111)$$

We need to globalize this procedure, assigning to each $f \in \mathcal{C}_0^1$ a map $\rho = \rho_f$ with the properties above in a continuous fashion. For this, we introduce some notation. We write $\mathcal{C}_0^1$ as the disjoint union

$$\mathcal{C}_0^1 = \coprod_{m \in \mathbb{N}_0} \mathcal{U}_m,$$

where $\mathcal{U}_m$ is the set of $f \in \mathcal{C}_0^1$ with precisely m zeros. Let

$$X_m := \{(t_1, \dots, t_m) \in (S^1)^m \mid t_1 < t_2 < \dots < t_m < t_1\} / \mathbb{Z}_m$$

be the configuration space of m cyclically ordered points on S^1 modulo cyclic permutations (with the quotient topology). Assigning to a function its cyclically ordered zero set defines a canonical continuous map

$$Z: \mathcal{U}_m \rightarrow X_m.$$

Let $\mathcal{G}_m$ be the set of C^1 -homeomorphisms $\rho: S^1 \rightarrow S^1$ with the following properties:

- ρ has precisely m critical points (i.e. zeros of ρ') $t_1, \dots, t_m$.
- For each $l = 1, \dots, m$ let

$$\delta_i := \frac{1}{4} \min\{t_i - t_{i-1}, t_{i+1} - t_i\} \quad (112)$$

be the distance of t_i to the nearest zero, where we stipulate $t_0 := t_m \in S^1$. Then we require that

$$\rho(\tau) = (\tau - t_i)^3 + t_i$$

for all $\tau \in (t_i - \delta_i, t_i + \delta_i)$.

We equip $\mathcal{G}_m$ with the C^1 -topology. Assigning to a function its cyclically ordered set of critical points defines a canonical continuous map

$$\pi: \mathcal{G}_m \rightarrow X_m.$$

Since π is a fibration with contractible fibres, there exists a continuous section

$$s: X_m \rightarrow \mathcal{G}_m.$$

So we obtain a continuous map

$$s \circ Z: \mathcal{U}_m \rightarrow \mathcal{G}_m, \quad f \mapsto \rho_f := s \circ Z(f).$$

For different $m \in \mathbb{N}_0$ these maps together give a continuous map

$$\mathcal{C}_0^1 = \coprod_{m \in \mathbb{N}_0} \mathcal{U}_m \rightarrow \mathcal{G} := \coprod_{m \in \mathbb{N}_0} \mathcal{G}_m, \quad f \mapsto \rho_f \quad (113)$$

with the property that the critical points of ρ_f are precisely the zeros of f . It follows that f, ρ_f satisfy the hypotheses of Lemma B.5; we denote the resulting continuous function by $g_f: S^1 \rightarrow \mathbb{R}$.

Lemma B.6. *The map $\mathcal{C}_0^1 \rightarrow C^0(S^1, \mathbb{R})$, $f \mapsto g_f$ is continuous.*

Proof. We will use the following criterion for uniform convergence of a sequence of functions $g_n: S^1 \rightarrow \mathbb{R}$ to a function $g: S^1 \rightarrow \mathbb{R}$ (which holds more generally for functions on any compact metric space):

$$g_n \rightarrow g \text{ uniformly} \Leftrightarrow g_n(\tau_n) \rightarrow g(\tau) \text{ for every sequence } \tau_n \rightarrow \tau. \quad (114)$$

Consider now a converging sequence $f_n \rightarrow f$ in $\mathcal{C}_0^1$ and denote $g_n := g_{f_n}$, $g := g_f$, $\rho_n := \rho_{f_n}$, $\rho := \rho_f$. Let $\tau_n \rightarrow \tau$ be a converging sequence in S^1 . Then by the criterion we need to show that $g_n(\tau_n) \rightarrow g(\tau)$. We distinguish two cases.

Case 1: $f(\tau) \neq 0$. Then uniform convergence $f_n \rightarrow f$ implies $f_n(\tau_n) \rightarrow f(\tau)$, so $f_n(\tau_n) \neq 0$ for all sufficiently large n . Hence by Lemma B.5 we have

$$g_n(\tau_n) = \frac{\rho'_n(\tau_n)}{f_n^{2/3}(\rho_n(\tau_n))} \quad \text{and} \quad g(\tau) = \frac{\rho'(\tau)}{f^{2/3}(\rho(\tau))}.$$

Note that $f(\rho(\tau)) \neq 0$ and $f_n(\rho_n(\tau_n)) \neq 0$ for large n . Continuity of the map $f \mapsto \rho_f$ implies that $\rho_n \rightarrow \rho$ in $C^1(S^1, S^1)$, hence $\rho_n \rightarrow \rho$ and $\rho'_n \rightarrow \rho'$ uniformly. Applying the above criterion repeatedly it follows that $\rho'_n(\tau_n) \rightarrow \rho'(\tau)$, $\rho_n(\tau_n) \rightarrow \rho(\tau)$, $f_n(\rho_n(\tau_n)) \rightarrow f(\rho(\tau)) \neq 0$, and therefore $g_n(\tau_n) \rightarrow g(\tau)$.

Case 2: $f(\tau) = 0$. In this case let $t_1 < \dots < t_m < t_1$ be the zeros of f . Then for large n the function f_n also has m zeros $t_{1,n} < \dots < t_{m,n} < t_{1,n}$ such that $t_{i,n} \rightarrow t_i$ as $n \rightarrow \infty$ for each i . Hence the positive numbers δ_i and $\delta_{i,n}$ defined via (112) (the latter using the $t_{i,n}$) also satisfy $\delta_{i,n} \rightarrow \delta_i$ as $n \rightarrow \infty$. We have $\tau = t_i$ for some i . Since $\tau_n \rightarrow \tau = t_i$ and $\rho_n \rightarrow \rho$, it follows that for large n both τ_n and $t_n := \rho_n(\tau_n)$ lie in the interval $(t_{i,n} - \delta_{i,n}, t_{i,n} + \delta_{i,n})$.

Let us assume first that $\tau_n \neq t_{i,n}$, hence also $t_n \neq t_{i,n}$ for all sufficiently large n . Then $f_n(\tau_n) \neq 0$ for all sufficiently large n , and by Lemma B.5 and (110) we have

$$g_n(\tau_n) = \frac{\rho'_n(\tau_n)}{f_n^{2/3}(\tau_n)} = 3 \left(\frac{f_n(t)}{t_n - t_{i,n}} \right)^{-2/3} \quad \text{and} \quad g(\tau) = 3f'(t_i)^{-2/3}.$$

By the mean value theorem we have $f_n(t_n)/(t_n - t_{i,n}) = f'(\xi_n)$ for some ξ_n between $t_{i,n}$ and t_n . Then $t_{i,n} \rightarrow t_i$ and $t_n \rightarrow t_i$ implies $\xi_n \rightarrow t_i$, so uniform convergence $f'_n \rightarrow f'$ yields $f'_n(\xi_n) \rightarrow f'(t_i) \neq 0$ and thus $g_n(\tau_n) \rightarrow g(\tau)$.

If $\tau_n = t_{i,n}$ for some arbitrarily large n , then for these n by Lemma B.5 we have $g_n(\tau_n) = 3f'_n(t_{i,n})^{-2/3}$, which also converges to $g(\tau) = 3f'(t_i)^{-2/3}$ as $n \rightarrow \infty$. This concludes the proof of Lemma B.6. ■

Proof of Lemma B.3. Consider a sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{C}_0^1$ converging to $f \in \mathcal{C}_0^1$ in $C^1(S^1, \mathbb{R})$. We need to show $f_n^{-1/3} \rightarrow f^{-1/3}$ in $L^2(S^1, \mathbb{R})$. We write the squared L^2 -distance as

$$\|f^{-1/3} - f_n^{-1/3}\|_{L^2}^2 = A + A_n - 2B_n \quad (115)$$

with

$$A := \int_0^1 \frac{dt}{f^{2/3}(t)}, \quad A_n = \int_0^1 \frac{dt}{f_n^{2/3}(t)}, \quad B_n = \int_0^1 \frac{dt}{f_n^{1/3}(t)f^{1/3}(t)}.$$

By (110) we have

$$A = \int_0^1 g_f(\tau) d\tau, \quad A_n = \int_0^1 g_{f_n}(\tau) d\tau,$$

where $g_f, g_{f_n}: S^1 \rightarrow \mathbb{R}$ are the continuous functions assigned to f, f_n in Lemma B.5. From the convergence $f_n \rightarrow f$ in $\mathcal{C}_0^1$ and Lemma B.6 we obtain $g_{f_n} \rightarrow g_f$ in $C^0(S^1, \mathbb{R})$ and thus $A_n \rightarrow A$. So in view of (115) we are done if we can show $B_n \rightarrow A$.

To prove this we introduce some notation. For $h \in \mathcal{C}_0^1$ and a measurable subset $I \subset S^1$ we denote

$$A(h, I) := \int_I \frac{dt}{h^{2/3}(t)} dt.$$

By (110) we have

$$A(h, I) = \int_{\rho_h^{-1}(I)} g_h(\tau) d\tau \leq |\rho_h^{-1}(I)| \|g_h\|_{C^0(S^1, \mathbb{R})},$$

where $h \mapsto \rho_h$ and $h \mapsto g_h$ are the continuous maps from (113) and Lemma B.6, respectively, and $|I|$ denotes the Lebesgue measure of I . It follows from the definition of ρ_h that $|\rho_h^{-1}(I)|$ depends continuously on $h \in \mathcal{C}_0^1$ and can be made arbitrarily small by making $|I|$ small.

Now let $\varepsilon > 0$ be given and consider the compact subset

$$Z := \{f_n\}_{n \in \mathbb{N}} \cup \{f\} \subset \mathcal{C}_0^1.$$

Since the zero set of f_n converges to that of f , there exists for each $\gamma > 0$ a union $I \subset S^1$ of open intervals containing the zero set of f with the following properties:

- (i) there exist $\delta > 0$ and $N \in \mathbb{N}$ such that $|f_n| \geq \delta$ on $S^1 \setminus I$ for all $n \geq N$;
- (ii) $|I| \leq \gamma$.

By the preceding discussion we can choose $\gamma > 0$ so small that

$$\max_{h \in Z} A(h, I) \leq \max_{h \in Z} |\rho_h^{-1}(I)| \max_{h \in Z} \|g_h\|_{C^0(S^1, \mathbb{R})} < \varepsilon/3. \quad (116)$$

We set

$$B_n(I) := \int_I \frac{dt}{f_n^{1/3}(t) f^{1/3}(t)}.$$

Using the Cauchy–Schwarz inequality and (116) we estimate

$$|B_n(I)|^2 \leq \int_I \frac{dt}{f_n^{2/3}(t)} \int_{I_k} \frac{dt}{f^{2/3}(t)} = A(f_n, I)A(f, I) \leq \varepsilon^2/9 \quad (117)$$

for all $n \geq N$. Now we split $B_n - A$ into summands

$$B_n - A = B_n(I) - A(f, I) + \int_{S^1 \setminus I} \left(\frac{1}{f_n^{1/3}(t) f^{1/3}(t)} - \frac{1}{f^{2/3}(t)} \right) dt$$

and estimate it by the triangle inequality:

$$|B_n - A| \leq |B_n(I)| + |A(f, I)| + \int_{S^1 \setminus I} \left| \frac{1}{f_n^{1/3}(t) f^{1/3}(t)} - \frac{1}{f^{2/3}(t)} \right| dt.$$

By property (i) above we have $|f_n| \geq \delta$ and $|f| \geq \delta > 0$ on the compact set $S^1 \setminus I$, so the integrand in the last integral converges uniformly to zero on $S^1 \setminus I$. Therefore, there exists an integer $N_1 \geq N$ such that the last integral is smaller than $\varepsilon/3$ for all $n \geq N_1$. Together with equations (116) and (117) this implies

$$|B_n - A| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

for all $n \geq N_1$. This proves $B_n \rightarrow A$, which concludes the proof of Lemma B.3 and thus of Lemma B.1. ■

B.3. Further lemmas on continuous dependence

We will frequently use the following standard result for which we denote

$$H_{\neq 0}^1(S^1, \mathbb{R}) := \{f \in H^1(S^1, \mathbb{R}) \mid f(t) \neq 0 \text{ for all } t\}.$$

Lemma B.7. *There are continuous maps*

$$H^1(S^1, \mathbb{R}) \times H^1(S^1, \mathbb{R}) \rightarrow H^1(S^1, \mathbb{R}), \quad (f, g) \mapsto fg$$

and

$$H_{\neq 0}^1(S^1, \mathbb{R}) \rightarrow H^1(S^1, \mathbb{R}), \quad f \mapsto 1/f.$$

Proof. The first assertion is just the well-known fact that $H^1(S^1, \mathbb{R})$ is a Banach algebra. For the second assertion we abbreviate $H_{\neq 0}^1 := H_{\neq 0}^1(S^1, \mathbb{R})$ etc. The map $H_{\neq 0}^1 \rightarrow L^2$, $f \mapsto 1/f$ is the composition of continuous maps

$$H_{\neq 0}^1 \hookrightarrow C_{\neq 0}^0 \rightarrow C_{\neq 0}^0 \hookrightarrow L^2,$$

where the middle map sends $f \mapsto 1/f$ and the other two maps are the canonical inclusions. The map $H_{\neq 0}^1 \rightarrow L^2$, $f \mapsto (1/f)' = -f'/f^2$ is the composition of continuous maps

$$H_{\neq 0}^1 \rightarrow C_{\neq 0}^0 \oplus L^2 \rightarrow L^2,$$

where the first map sends $f \mapsto (-1/f^2, f')$ and the second map is multiplication. Together this proves the lemma. ■

Lemma B.8. *The following maps are continuous:*

$$\begin{aligned} \mathcal{H}_0^2 &\rightarrow H^1(S^1, \mathbb{R}), & z &\mapsto z^2 \circ \tau_z, \\ (H^2(S^1, \mathbb{R}) \times \mathcal{H}_0^2) \cap \mathcal{H}_{\text{in}} &\rightarrow H^1(S^1, \mathbb{R}), & (z_1, z_2) &\mapsto (\tau \mapsto z_2^2(\tau_{z_2}(t_{z_1}(\tau)))), \\ (H^2(S^1, \mathbb{R}) \times \mathcal{H}_0^2) \cap \mathcal{H}_{\text{in}} &\rightarrow H^1(S^1, \mathbb{R}), & (z_1, z_2) &\mapsto (\tau \mapsto \Delta(\tau_{z_2}(t_{z_1}(\tau)))), \end{aligned}$$

where Δ is defined in (101).

Proof. For the first map, continuous dependence of τ_z on z (which follows from Lemma B.4) implies that $z^2 \circ \tau_z$ depends continuously on z as an element of $L^2(S^1, \mathbb{R})$. For its time derivative

$$\frac{d}{dt} z^2(\tau_z(t)) = 2z(\tau_z(t)) \dot{\tau}_z(t) = 2z(\tau_z(t)) \frac{\|z\|^2}{z(\tau_z(t))^2} = 2 \frac{\|z\|^2}{z(\tau_z(t))},$$

continuous dependence follows from Lemma B.1.

For the second map, observe that z_1 never vanishes. Therefore, t_{z_1} is a C^1 -diffeomorphism of S^1 depending continuously on z_1 . Together with the statement about the first map this concludes the argument. Continuity of the third map now follows directly from (103). ■

Lemma B.9. *The map*

$$(H^2(S^1, \mathbb{R}) \times \mathcal{H}_0^2) \cap \mathcal{H}_{\text{in}} \rightarrow L^2(S^1, \mathbb{R})$$

defined by

$$(z_1, z_2) \mapsto \left(\tau \mapsto \frac{1}{z_2(\tau_{z_2}(t_{z_1}(\tau)))} \right)$$

is continuous.

Proof. Recall from the previous proof that t_{z_1} is a C^1 -diffeomorphism of S^1 depending continuously on z_1 . This together with Lemma B.1 completes the proof. ■

C. The mod 2 Euler number

Throughout this appendix X denotes a Hilbert manifold (an open subset of a Hilbert space will suffice for our purposes), Y an open neighbourhood of 0 in a Hilbert space and

$k \in \mathbb{N}_0$. We say that a C^1 -map $f: X \rightarrow Y$ is *transverse to 0* if 0 is a regular value of f , i.e. $Df(x): T_x X \rightarrow T_0 Y$ is surjective for all $x \in f^{-1}(0)$. Our goal is to prove the following theorem:

Theorem C.1. *To each C^1 -Fredholm map $f: X \rightarrow Y$ of index 0 with compact zero set $f^{-1}(0)$ we can associate its mod 2 Euler number $\chi(f) \in \mathbb{Z}/2\mathbb{Z}$ which is uniquely characterized by the following axioms:*

- (Transversality) *If f is transverse to 0, then $\chi(f) = |f^{-1}(0)| \bmod 2$.*
- (Excision) *For any open neighbourhoods $\tilde{X} \subset X$ of $f^{-1}(0)$ and $\tilde{Y} \subset Y$ of 0 such that $f(\tilde{X}) \subset \tilde{Y}$ we have $\chi(f) = \chi(f|_{\tilde{X}}: \tilde{X} \rightarrow \tilde{Y})$.*
- (Cobordism) *If W is a Hilbert manifold with boundary and $F: W \rightarrow Y$ a C^1 -Fredholm map of index 1 with compact zero set $F^{-1}(0)$, then $\chi(F|_{\partial W}: \partial W \rightarrow Y) = 0$.*
- (Homotopy) *Let $f_0, f_1: X \rightarrow Y$ be C^1 -Fredholm maps of index 0 with compact zero sets. If there exists a C^1 -Fredholm map $F: [0, 1] \times X \rightarrow Y$ of index 1 with compact zero set $F^{-1}(0)$ such that $F|_{\{i\} \times X} = f_i$ for $i = 0, 1$, then $\chi(f_0) = \chi(f_1)$.*

Note that the (homotopy) axiom is just a special case of the (cobordism) axiom.

The invariant $\chi(f)$ in Theorem C.1 can be viewed as a special case of either the Caccioppoli–Smale degree defined in [9], or of the Euler class of a G -moduli problem defined in [3] (with trivial group G). The main improvement of Theorem C.1 over those results is the fact that in the (cobordism) we require only regularity C^1 instead of C^2 . This is not entirely obvious because the Sard–Smale theorem [9] for a Fredholm map of index 1 requires regularity C^2 . While this improvement may seem boring from the viewpoint of general theory, it is crucial for the application in this paper because the functional $\nabla \mathcal{B}_{\text{in}}$ is of class C^1 but not C^2 .

The main idea of the proof of Theorem C.1 is the following. Given a C^1 -Fredholm map $f: X \rightarrow Y$ of index 0 with compact zero set $f^{-1}(0)$ we use the Sard–Smale theorem [9] to C^1 -approximate f by a C^1 -Fredholm map $g: X \rightarrow Y$ of index 0 transverse to 0. We define $\chi(f) := \chi(g) := |g^{-1}(0)|$. To see that this is well defined we join g_0 to g_1 by a convex linear combination $G: [0, 1] \times X \rightarrow Y$. The map G is a C^1 -Fredholm map of index 1. We want to approximate G by a C^1 -Fredholm map $\tilde{G}$ transverse to 0 and such that $\tilde{G}(i, x) = G(i, x)$ for $i = 0, 1$. The direct application of the Sard–Smale theorem [9] is not sufficient, since Fredholm maps of index 1 produce a loss of regularity by 1. On the other hand, Lemma C.2 below applies and we conclude by the standard argument that $M := \tilde{G}^{-1}(0)$ is a compact C^1 -manifold with boundary $\partial M = g_0^{-1}(0) \amalg g_1^{-1}(0)$. Therefore, $\chi(g_0) = |g_0^{-1}(0)| = |g_1^{-1}(0)| = \chi(g_1)$.

The proof of Lemma C.2 below uses a three-step approximation, where the key application of the Sard–Smale theorem [9] occurs on step (3) to a C^∞ -map defined on an open subset of the domain, so the loss of regularity does not happen. To get to this nice situation of step (3) we use the fact that the restriction of F to ∂W (notation of Lemma C.2) has index 0 and can therefore be approximated by the Sard–Smale theorem [9] without the loss of regularity.

Lemma C.2. *Let W be a Hilbert manifold with boundary and $F: W \rightarrow Y$ a C^k -Fredholm map of index $k \in \mathbb{N}$ with compact zero set $F^{-1}(0)$ such that $F|_{\partial W}$ is transverse to 0. Then F can be C^k -approximated by a C^k -Fredholm map $\tilde{F}: W \rightarrow Y$ of index k with compact zero set $\tilde{F}^{-1}(0)$ such that $F|_{\partial W} = \tilde{F}|_{\partial W}$ and $\tilde{F}: W \rightarrow Y$ is transverse to 0. In particular, $\tilde{F}^{-1}(0)$ is a compact C^k -manifold of dimension k with boundary $\partial \tilde{F}^{-1}(0) = (F|_{\partial W})^{-1}(0)$.*

Proof. (1) Let $g: \partial W \rightarrow Y$ be a C^∞ -map which is sufficiently C^k -close to $f := F|_{\partial W}$ so that the map $(1-t)f + tg$ is transverse to zero for each $t \in [0, 1]$ and the map

$$[0, 1] \times \partial W \rightarrow Y, \quad (t, x) \mapsto (1-t)f(x) + tg(x)$$

is Fredholm of index k with compact zero set. Let $\mathcal{N} \cong [0, 2) \times \partial W \subset W$ be a collar neighbourhood of $\partial W \cong \{0\} \times \partial W$. Pick a smooth cutoff function $\varphi: \mathbb{R} \rightarrow [0, 1]$ with compact support in $(0, 2)$ which equals 1 in a neighbourhood of 1. Define $G: W \rightarrow Y$ by

$$G(t, x) := (1 - \varphi(t))F(t, x) + \varphi(t)g(x)$$

for $(t, x) \in \mathcal{N}$ and $G := F$ on $W \setminus \mathcal{N}$. Then $G: W \rightarrow Y$ is a C^k -Fredholm map of index k such that $G|_{\partial W} = F|_{\partial W}$ and G is C^∞ in a neighbourhood of $\{1\} \times \partial W$. By choosing $\mathcal{N}$ sufficiently small we can ensure that G has compact zero set and $G|_{\mathcal{N}}$ is transverse to 0.

(2) Set $\tilde{\mathcal{N}} := [0, 1) \times \partial W \subset \mathcal{N}$. We C^k -approximate G by a C^k -Fredholm map $H: W \rightarrow Y$ of index k with compact zero set such that $H = G$ on $\tilde{\mathcal{N}}$ and H is C^∞ on $W \setminus \tilde{\mathcal{N}}$.

(3) Using the Sard–Smale theorem [9], we C^k -approximate the restriction $H|_{W \setminus \tilde{\mathcal{N}}}$ by a C^∞ -Fredholm map $\tilde{F}: W \setminus \tilde{\mathcal{N}} \rightarrow Y$ with compact zero set which agrees with H near $\{1\} \times \partial W$ and is transverse to zero. This map extends via H over $\tilde{\mathcal{N}}$ to the desired map $\tilde{F}: W \rightarrow Y$. ■

Proof of Theorem C.1. Uniqueness: Let $f: X \rightarrow Y$ be a C^1 -Fredholm map of index 0 with compact zero set $f^{-1}(0)$. By the Sard–Smale theorem [9], we can C^1 -approximate f by a C^1 -Fredholm map $g: X \rightarrow Y$ of index 0 which is transverse to 0. By choosing g sufficiently C^1 -close to f we can ensure that

$$F: [0, 1] \times X \rightarrow Y, \quad (t, x) \mapsto (1-t)f(x) + tg(x)$$

is a C^1 -Fredholm map of index 1 with compact zero set. So by the (homotopy) and (transversality) axioms $\chi(f)$ is uniquely determined by

$$\chi(f) = \chi(g) = |g^{-1}(0)| \mod 2.$$

Existence: On maps f as in the theorem which are transverse to 0 we define $\chi(f) := |f^{-1}(0)| \mod 2$ by the (transversality) axiom. We claim that then the (cobordism) axiom

holds under the additional assumption that $f := F|_{\partial W}$ is transverse to 0. To see this, we apply Lemma C.2 to find a C^1 -Fredholm map $\tilde{F}: W \rightarrow Y$ of index 1 whose zero set $M := \tilde{F}^{-1}(0)$ is a compact C^1 -manifold of dimension 1 with boundary $\partial M = f^{-1}(0)$. Since M has an even number of boundary points, we conclude $|f^{-1}(0)| = 0 \bmod 2$ and the claim is proved. As a special case, the (homotopy) axiom also holds under the additional assumption that f_0 and f_1 are transverse to 0.

Now let $f: X \rightarrow Y$ be any C^1 -Fredholm map of index 0 with compact zero set $f^{-1}(0)$. We choose g, F as in the proof of uniqueness and define $\chi(f) := \chi(g) = |g^{-1}(0)|$. To see that this is well defined, let $g_i, F_i, i = 0, 1$ be two such choices. Then the maps F_0, F_1 can be joined by a cutoff construction to a C^1 -Fredholm map $G: [0, 1] \times X \rightarrow Y$ of index 1 with compact zero set such that $G|_{\{i\} \times X} = g_i$ for $i = 0, 1$, and the special case of the (homotopy) axiom implies $\chi(f_0) = \chi(f_1)$. So $\chi(f)$ is well defined. By construction, it satisfies the (transversality) and (excision) axioms. The (cobordism) axiom follows from the special case above, and the (homotopy) axiom is a special case of this. ■

D. The mod 2 Euler number of the mean interaction functional

In this section we prove the following result which is used in the proof of the existence Theorem 6.1, but which may also be of independent interest.

Theorem D.1. *The L^2 -gradient of the mean interaction functional on symmetric loops,*

$$\nabla \mathcal{B}_{\text{av}}: \hat{\mathcal{H}}_{\text{in}}^2 \rightarrow H_{\text{sym}}^0(S^1, \mathbb{R}^2),$$

is a C^1 -Fredholm map whose mod 2 Euler number equals 1.

It follows from the main result in [4] that $\nabla \mathcal{B}_{\text{av}}$ has a unique zero (z_1, z_2) . To conclude that its mod 2 Euler number equals 1, we would need to prove invertibility of the Hessian $D\nabla \mathcal{B}_{\text{av}}(z_1, z_2)$, or equivalently (since the Fredholm index is zero) triviality of its kernel. This is still complicated because the Hessian has many terms and the equations for an element (v_1, v_2) in its kernel are coupled. Therefore, we will instead further deform the mean interaction functional to one for which the equations decouple, and then compute the mod 2 Euler number of the latter.

D.1. Decoupling the mean interaction

In this subsection we describe the deformation of the mean interaction functional on symmetric loops to one for which the equations decouple. We will first phrase it in terms of the original (physical) coordinates $q(t)$; the Levi-Civita transformation to the new coordinates $z(\tau)$ will be considered in the next subsection.

In the following discussion we adapt some arguments from [1] to our situation. In order to be consistent with the notation in that paper, we replace the period 1 used in this paper before by period 2 as in [1], and consider the restriction of a symmetric loop to half

a period. Thus we consider

$$q_1 \in C^\infty([0, 1], (0, \infty)), \quad q_2 \in C^0([0, 1], [0, \infty)) \cap C^\infty([0, 1), (0, \infty))$$

satisfying

$$\dot{q}_1(0) = \dot{q}_1(1) = \dot{q}_2(0) = q_2(1) = 0 \quad \text{and} \quad q_1(t) > q_2(t) \geq 0 \quad \text{for all } t \in [0, 1]. \quad (118)$$

We assume that (q_1, q_2) solves a system of second-order ODEs coupled through their means,

$$\begin{cases} \ddot{q}_1(t) = -\frac{2}{q_1(t)^2} + f_1(\bar{q}_1, \bar{q}_2), \\ \ddot{q}_2(t) = -\frac{2}{q_2(t)^2} - f_2(\bar{q}_1, \bar{q}_2), \end{cases} \quad (119)$$

where $f_i(\bar{q}_1, \bar{q}_2)$ are continuous functions defined for $\bar{q}_1 > \bar{q}_2 \geq 0$ satisfying

$$f_1(\bar{q}_1, \bar{q}_2) > 0 \quad \text{and} \quad f_2(\bar{q}_1, \bar{q}_2) \geq 0 \quad \text{for all } \bar{q}_1 > \bar{q}_2 \geq 0. \quad (120)$$

Lemma D.2. *Under the above assumptions the following hold.*

- (a) *The map q_1 is constant, $q_1(t) \equiv \bar{q}_1$, where $\bar{q}_1 > 0$ solves the equation*

$$\bar{q}_1^2 f_1(\bar{q}_1, \bar{q}_2) = 2. \quad (121)$$

- (b) *The map q_2 is strictly concave and strictly decreasing with maximum $q_2^{\max} = q_2(0)$ satisfying the estimates*

$$\frac{1}{2} \leq \frac{q_2^{\max}}{2} \leq \bar{q}_2 \leq q_2^{\max} \leq 2 + \frac{f_2(\bar{q}_1, \bar{q}_2)}{2}. \quad (122)$$

Proof. We abbreviate $f_1 = f_1(\bar{q}_1, \bar{q}_2)$.

- (a) Note first that, since q_1 solves the first equation in (119), it is actually smooth and 1-periodic. It attains its maximum at some time $t_{\max} \in [0, 1]$ satisfying $\dot{q}(t_{\max}) = 0$ and

$$\ddot{q}_1(t_{\max}) = -\frac{2}{q_1(t_{\max})^2} + f_1 \leq 0.$$

It follows that for all $t \in [0, 1]$ we have

$$\ddot{q}_1(t) = -\frac{2}{q_1(t)^2} + f_1 \leq -\frac{2}{q_1(t_{\max})^2} + f_1 \leq 0.$$

By periodicity this implies $\ddot{q}_1 \equiv 0$, so $\dot{q}_1$ is constant. Again by periodicity this implies $\dot{q}_1 \equiv 0$, so $q_1 \equiv \bar{q}_1$ is constant and the first equation in (119) becomes (121).

- (b) Since $f_2 \geq 0$, the second equation in (119) implies $\ddot{q}_2(t) < 0$ for all $t \in [0, 1]$, so q_2 is strictly concave. Together with $\dot{q}_2(0) = 0$ this implies $\dot{q}_2(t) < 0$ for all $t \in (0, 1)$, so q_2 is strictly decreasing with maximum $q_2^{\max} = q_2(0) > 0$.

For the first inequality in (122), we use the second equation in (119) and $f_2 \geq 0$ to estimate for all $t \in [0, 1]$:

$$\dot{q}_2(t) = \int_0^t \ddot{q}_2(s) ds \leq - \int_0^t \frac{2}{q_2(s)^2} ds \leq - \int_0^t \frac{2}{(q_2^{\max})^2} ds = - \frac{2}{(q_2^{\max})^2} t.$$

Together with $q_2(1) = 0$ this implies

$$-q_2^{\max} = q_2(1) - q_2(0) = \int_0^1 \dot{q}_2(t) dt \leq - \frac{2}{(q_2^{\max})^2} \int_0^1 t dt = - \frac{1}{(q_2^{\max})^2},$$

hence $(q_2^{\max})^3 \geq 1$ and thus $q_2^{\max} \geq 1$.

The second inequality in (122) follows from concavity of q_2 : for all $t \in [0, 1]$ we have

$$q_2(t) \geq (1-t)q_2(0) + tq_2(1) = (1-t)q_2^{\max},$$

and it follows that

$$\bar{q}_2 = \int_0^1 q_2(t) dt \geq q_2^{\max} \int_0^1 (1-t) dt = \frac{q_2^{\max}}{2}.$$

The third inequality in (122) is clear, so it remains to prove the fourth one. Since q_2 is strictly decreasing and $q_2(0) \geq 1$, there exists a unique $t_0 \in [0, 1]$ with $q_2(t_0) = 1$. Then for all $t \in [0, t_0]$ we have $q_2(t) \geq 1$, and therefore

$$\begin{aligned} \ddot{q}_2(t) &= -\frac{2}{q_2(t)^2} - f_2 \geq -2 - f_2, \\ \dot{q}_2(t) &= \int_0^t \ddot{q}_2(s) ds \geq -(2 + f_2)t \end{aligned}$$

for all $t \in [0, t_0]$. This implies

$$\begin{aligned} 1 - q_2^{\max} &= q_2(t_0) - q_2(0) = \int_0^{t_0} \dot{q}_2(t) dt \geq -(2 + f_2) \int_0^{t_0} t dt \\ &= -\frac{t_0^2}{2} (2 + f_2), \end{aligned}$$

and with $t_0 \leq 1$ we obtain

$$q_2^{\max} \leq 1 + \frac{t_0^2}{2} (2 + f_2) \leq 1 + \frac{1}{2} (2 + f_2) = 2 + \frac{f_2}{2}. \quad \blacksquare$$

Compactness. We wish to consider families of problems (119) as above parametrized by pairs of functions (f_1, f_2) satisfying (120). For compactness of the corresponding space of solutions (q_1, q_2) we need

- a uniform lower bound $\bar{q}_1 - q_2^{\max} \geq \delta > 0$, and
- a uniform upper bound $\bar{q}_1 \leq c < \infty$,

where “uniform” means independent of the parameter. In view of Lemma D.2, this is ensured by the following *sufficient condition*: for all solutions $(\bar{q}_1, \bar{q}_2)$ of

$$\bar{q}_1^2 f_1(\bar{q}_1, \bar{q}_2) = 2, \quad \frac{1}{2} \leq \bar{q}_2 \leq 2 + \frac{f_2(\bar{q}_1, \bar{q}_2)}{2}, \quad \bar{q}_1 > \bar{q}_2 \quad (123)$$

we have uniform lower and upper bounds

$$\bar{q}_1 - 2\bar{q}_2 \geq \delta > 0 \quad \text{and} \quad \bar{q}_1 \leq c < \infty. \quad (124)$$

Indeed, by Lemma D.2 the averages $(\bar{q}_1, \bar{q}_2)$ of a solution (q_1, q_2) of problem (119) satisfy conditions (123), and in view of $q_2^{\max} \leq 2\bar{q}_2$ the first inequality in (124) implies the uniform lower bound $\bar{q}_1 - q_2^{\max} \geq \delta > 0$. The following lemma describes a situation where this sufficient condition is satisfied.

Lemma D.3. *Suppose that*

$$f_1(\bar{q}_1, \bar{q}_2) = \frac{1}{(\bar{q}_1 - \bar{q}_2)^2} \quad \text{and} \quad 0 \leq f_2(\bar{q}_1, \bar{q}_2) \leq \frac{1}{(\bar{q}_1 - \bar{q}_2)^2}.$$

Then each solution $(\bar{q}_1, \bar{q}_2)$ of (123) satisfies

$$\bar{q}_1 = (2 + \sqrt{2})\bar{q}_2,$$

as well as the lower and upper bounds

$$\bar{q}_1 - 2\bar{q}_2 \geq \frac{1}{\sqrt{2}} \quad \text{and} \quad \bar{q}_1 \leq (2 + \sqrt{2})\left(2 + \frac{2}{(1 + \sqrt{2})^2}\right).$$

Proof. In this case the first equation in (123) becomes the homogeneous quadratic equation

$$\bar{q}_1^2 = \frac{2}{f_1(\bar{q}_1, \bar{q}_2)} = 2(\bar{q}_1 - \bar{q}_2)^2,$$

which has the solutions $\bar{q}_1 = (2 \pm \sqrt{2})\bar{q}_2$. The condition $\bar{q}_1 > \bar{q}_2$ enforces $\bar{q}_1 = (2 + \sqrt{2})\bar{q}_2$. Together with $\bar{q}_2 \geq 1/2$ this implies the lower bound

$$\bar{q}_1 - 2\bar{q}_2 = \sqrt{2}\bar{q}_2 \geq \frac{\sqrt{2}}{2},$$

as well as

$$\bar{q}_1 - \bar{q}_2 = (1 + \sqrt{2})\bar{q}_2 \geq \frac{1 + \sqrt{2}}{2}.$$

With the condition on f_2 this yields an upper bound on $\bar{q}_2$,

$$\bar{q}_2 \leq 2 + \frac{f_2(\bar{q}_1, \bar{q}_2)}{2} \leq 2 + \frac{1}{2(\bar{q}_1 - \bar{q}_2)^2} \leq 2 + \frac{2}{(1 + \sqrt{2})^2},$$

and thus on $\bar{q}_1$,

$$\bar{q}_1 = (2 + \sqrt{2})\bar{q}_2 \leq (2 + \sqrt{2})\left(2 + \frac{2}{(1 + \sqrt{2})^2}\right). \quad \blacksquare$$

Fixing $f_1(\bar{q}_1, \bar{q}_2) = \frac{1}{(\bar{q}_1 - \bar{q}_2)^2}$, Lemma D.3 allows us to linearly interpolate between $f_2(\bar{q}_1, \bar{q}_2) = \frac{1}{(\bar{q}_1 - \bar{q}_2)^2}$ and $f_2 = 0$. We are thus led to consider the *decoupled mean interaction problem*

$$\begin{cases} \ddot{q}_1(t) = -\frac{2}{q_1(t)^2} + \frac{1}{(\bar{q}_1 - \bar{q}_2)^2}, \\ \ddot{q}_2(t) = -\frac{2}{q_2(t)^2}. \end{cases} \quad (125)$$

Note that the second equation is a pure Kepler problem which is not coupled to the first one. It has a unique solution $q_2: [0, 1] \rightarrow [0, \infty)$ with $\dot{q}_2(0) = \dot{q}_2(1) = 0$ and we denote by

$$a := \bar{q}_2$$

its average. Note that by Lemma D.2 it satisfies

$$1/2 \leq a \leq 2.$$

Inserting $\bar{q}_2 = a$ into the first equation, Lemma D.3 shows that it has a unique solution q_1 , which is constant and given by

$$q_1(t) \equiv \bar{q}_1 = (2 + \sqrt{2})a.$$

This concludes our discussion of compactness. In the next subsection we will consider its Levi-Civita transformation and use it to prove Theorem D.1.

D.2. A Fredholm homotopy

Let X and Y be as in Section 6.4. For $r \in [0, 1]$ we consider the map

$$F^r = (F_1, F_2^r): X \rightarrow Y$$

given by

$$\begin{aligned} F_1(z_1, z_2) &:= -z_1'' + a_1(z_1, z_2)z_1 + b_1(z_1, z_2)z_1^3, \\ F_2^r(z_1, z_2) &:= -z_2'' + a_2^r(z_1, z_2)z_2 + b_2^r(z_1, z_2)z_2^3 \end{aligned}$$

with the functions

$$\begin{aligned} a_1(z_1, z_2) &= \frac{\|z_1'\|^2}{\|z_1\|^2} - \frac{1}{\|z_1\|^6} - \frac{\|z_2\|^4 \cdot \|z_1^2\|^2}{2\|z_1\|^2 \cdot (\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2}, \\ b_1(z_1, z_2) &= + \frac{\|z_2\|^4}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2}, \\ a_2^r(z_1, z_2) &:= \frac{\|z_2'\|^2}{\|z_2\|^2} - \frac{1}{\|z_2\|^6} + r \frac{\|z_1\|^4 \cdot \|z_2^2\|^2}{2\|z_2\|^2 \cdot (\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2}, \\ b_2^r(z_1, z_2) &:= -r \frac{\|z_1\|^4}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2}. \end{aligned}$$

For $r = 1$ comparison with equations (93) and (94) shows that

$$\nabla_1 \mathcal{B}_{\text{av}}(z_1, z_2) = 4\|z_1\|^2 F_1(z_1, z_2), \quad \nabla_2 \mathcal{B}_{\text{av}}(z_1, z_2) = 4\|z_2\|^2 F_2^1(z_1, z_2).$$

Thus, up to the irrelevant positive factors $4\|z_i\|^2$, F^1 agrees with $\nabla \mathcal{B}_{\text{av}}$. In particular, the zeros of F^1 satisfy the coupled ODEs (41).

For $r = 0$ the first component remains unchanged and comparison with (93) shows that

$$\nabla \mathcal{Q}(z_2) = 4\|z_2\|^2 F_2^0(z_1, z_2). \quad (126)$$

So the second component of F^0 is decoupled from the first one and corresponds to a pure Kepler problem.

The discussion in Section 3 shows that under the Levi-Civita transformations $q_i(t) = z_i(\tau z_i(t))^2$, zeros of F^r correspond to generalized solutions of the coupled ODEs

$$\begin{cases} \ddot{q}_1(t) = -\frac{2}{q_1(t)^2} + \frac{1}{(\bar{q}_1 - \bar{q}_2)^2}, \\ \ddot{q}_2(t) = -\frac{2}{q_2(t)^2} - \frac{r}{(\bar{q}_1 - \bar{q}_2)^2}. \end{cases} \quad (127)$$

By definition of the space X , the q_i are symmetric and therefore, after replacing their period 1 by 2, they satisfy conditions (118) in the previous subsection. By the discussion in that subsection, the q_i satisfy a lower bound $q_1(t) \geq q_2(t) \geq \delta > 0$ and an upper bound $q_1(t) \leq c < \infty$, uniform in $r \in [0, 1]$. This implies that the zero set of the C^1 -Fredholm homotopy

$$F: [0, 1] \times X \rightarrow Y, \quad (r, z_1, z_2) \mapsto F^r(z_1, z_2)$$

is compact, so by the (homotopy) axiom in Theorem C.1, the mod 2 Euler numbers satisfy

$$\chi(\nabla \mathcal{B}_{\text{av}}) = \chi(F^1) = \chi(F^0).$$

To prove Theorem D.1, it thus remains to compute $\chi(F^0)$. By the discussion in the previous subsection, F^0 has a unique zero (z_1, z_2) whose components correspond under the Levi-Civita transformation $q_i(t) = z_i(\tau z_i(t))^2$ to the unique solution q_2 of the pure Kepler problem and the constant solution $q_1(t) \equiv \bar{q}_2 = (2 + \sqrt{2})a$, where $a = \bar{q}_2$. In particular, the first component is constant and given by

$$z_1(t) \equiv \bar{z}_1 = \sqrt{(2 + \sqrt{2})a}. \quad (128)$$

It thus remains to prove that the derivative $DF^0(z_1, z_2)$ at its unique zero (z_1, z_2) has trivial kernel. Suppose $(v_1, v_2) \in \ker DF^0(z_1, z_2)$. Since the second component $F_2^0(z_1, z_2) = F_2^0(z_2)$ is independent of z_1 , this implies $DF_2^0(z_2)v_2 = 0$. In the next subsection we will show the following proposition:

Proposition D.4. *The derivative $DF_2^0(z_2)$ at the Kepler orbit z_2 has trivial kernel.*

It follows that $v_2 = 0$ and v_1 satisfies $D_1 F_1(z_1, z_2)v_1 = 0$. In Section D.4 we will show the following proposition:

Proposition D.5. *For the Kepler orbit z_2 , the derivative of the map $z_1 \mapsto F_1(z_1, z_2)$ at its unique zero has trivial kernel.*

This implies $v_1 = 0$ and thus concludes the proof of Theorem D.1.

D.3. Hessian of the Kepler problem

In this subsection we prove Proposition D.4. Consider the map

$$F: Z \rightarrow H_{\text{sym}}^0(S^1, \mathbb{R}), \quad F(z) = -z'' + a(z)z, \quad a(z) = \frac{\|z'\|^2}{\|z\|^2} - \frac{1}{\|z\|^6} \quad (129)$$

defined on the space

$$Z := \{z \in H_{\text{sym}}^2(S^1, \mathbb{R}) \mid z(\tau) > 0 \text{ for all } \tau \in (0, 1)\}.$$

Thus F corresponds to the map F_2^0 of the previous subsection describing simple symmetric solutions of the Kepler problem, where we have renamed z_2 to z and a_2^0 to a . The unique zero of F is given by

$$z(\tau) = \zeta \sin(\pi\tau),$$

where $\zeta > 0$ is uniquely determined by the equation $F(z) = 0$, or equivalently

$$a(z) = -\pi^2.$$

We need to show that the derivative of F at its zero z has trivial kernel. In the direction $v \in H_{\text{sym}}^2(S^1, \mathbb{R})$ it is given by

$$DF(z)v = -v'' + a(z)v + (Da(z)v)z$$

with

$$\begin{aligned} Da(z)v &= \frac{2\langle z', v' \rangle}{\|z\|^2} - \frac{2\|z'\|^2 \langle z, v \rangle}{\|z\|^4} + \frac{6\langle z, v \rangle}{\|z\|^8} \\ &= \frac{2}{\|z\|^2} \left(-z'' - \frac{\|z'\|^2}{\|z\|^2} z + \frac{3}{\|z\|^6} z, v \right) \\ &= \frac{2}{\|z\|^2} \left(-2a(z) + \frac{2}{\|z\|^6} \right) \langle z, v \rangle, \end{aligned}$$

where for the last equality we have used $F(z) = 0$. Using $a(z) = -\pi^2$, it follows that an element v in the kernel of $DF(z)$ satisfies

$$-v'' - \pi^2 v + b\langle z, v \rangle z = 0 \quad (130)$$

with the constant

$$b = \frac{4}{\|z\|^2} \left(\pi^2 + \frac{1}{\|z\|^6} \right) > 0.$$

It follows that v is smooth. Multiplying (130) by v and integrating from 0 to 1 yields

$$\langle -v'', v \rangle - \pi^2 \|v\|^2 + b \langle z, v \rangle^2 = 0. \quad (131)$$

Since v extends to an odd 2-periodic function, it has a Fourier expansion

$$v(\tau) = \sum_{k=1}^{\infty} c_k \sin(\pi k \tau), \quad c_k \in \mathbb{R}.$$

We deduce $v'(\tau) = \sum_{k=1}^{\infty} \pi k c_k \cos(\pi k \tau)$ and thus the Poincaré inequality

$$\langle -v'', v \rangle = \|v'\|^2 = \frac{1}{2} \sum_{k=1}^{\infty} \pi^2 k^2 c_k^2 \geq \frac{1}{2} \sum_{k=1}^{\infty} \pi^2 c_k^2 = \pi^2 \|v\|^2.$$

Hence (131) can only hold if $b \langle z, v \rangle^2 \leq 0$, i.e. $\langle z, v \rangle = 0$, and equality holds in the Poincaré inequality, i.e. $c_k = 0$ for all $k \geq 2$. Thus $v(\tau) = c_1 \sin(\pi \tau)$, and $\langle v, z \rangle = 0$ implies $v = 0$. This concludes the proof of Proposition D.4.

D.4. Hessian of the Kepler problem with constant force

In this subsection we prove Proposition D.5. Denote by z_2 the unique symmetric Kepler orbit from the previous subsection and by $q_2(t) = z_2(\tau(t))^2$ its Levi-Civita transform. We denote its average using (3) by

$$a := \bar{q}_2 = \frac{\|z_2^2\|}{\|z_2\|^2} > 0.$$

Using this, we rewrite the functions a_1 and b_1 from Section D.2 as

$$\begin{aligned} a_1(z_1, z_2) &= \frac{\|z_1'\|^2}{\|z_1\|^2} - \frac{1}{\|z_1\|^6} - \frac{\|z_2\|^4 \cdot \|z_1^2\|^2}{2\|z_1\|^2 \cdot (\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} \\ &= \frac{\|z_1'\|^2}{\|z_1\|^2} - \frac{1}{\|z_1\|^6} - \frac{\|z_1^2\|^2}{2\|z_1\|^2 \cdot (\|z_1^2\|^2 - a \cdot \|z_1\|^2)^2}, \\ b_1(z_1, z_2) &= \frac{\|z_2\|^4}{(\|z_1^2\|^2 \cdot \|z_2\|^2 - \|z_2^2\|^2 \cdot \|z_1\|^2)^2} \\ &= \frac{1}{(\|z_1^2\|^2 - a \cdot \|z_1\|^2)^2}. \end{aligned}$$

Renaming z_1 to z , we thus consider the map

$$F: W \rightarrow H_{\text{sym}}^0(S^1, \mathbb{R}), \quad F(z) = -z'' + a_1(z)z + b_1(z)z^3 \quad (132)$$

with

$$a_1(z) := \frac{\|z'\|^2}{\|z\|^2} - \frac{1}{\|z\|^6} - \frac{\|z^2\|^2}{2\|z\|^2 \cdot (\|z^2\|^2 - a \cdot \|z\|^2)^2},$$

$$b_1(z) := \frac{1}{(\|z^2\|^2 - a \cdot \|z\|^2)^2} > 0,$$

defined on the space

$$W := \{z \in H^2(S^1, \mathbb{R}) \mid z(\tau) > 0 \text{ for all } \tau \in S^1, \|z^2\|^2 > a\|z\|^2\}.$$

So F corresponds to the map $z_1 \mapsto F_1(z_1, z_2)$ in Proposition D.5. By the discussion in Section D.1, the unique zero of F is the constant function

$$z(\tau) \equiv \bar{z},$$

where $\bar{z} > 0$ is uniquely determined by the equation $F(z) = 0$, or equivalently

$$a_1(\bar{z}) + b_1(\bar{z})\bar{z}^2 = 0. \quad (133)$$

We need to show that the derivative of F at its zero $\bar{z}$ has trivial kernel. In the direction $v \in H_{\text{sym}}^2(S^1, \mathbb{R})$ it is given by

$$DF(\bar{z})v = -v'' + (a_1 + 3b_1\bar{z}^2)v + c_1$$

with the constants $a_1 = a_1(\bar{z})$, $b_2 = b_1(\bar{z})$ and

$$c_1 := \langle \nabla a_1(\bar{z}), v \rangle \bar{z} + \langle \nabla b_1(\bar{z}), v \rangle \bar{z}^3.$$

Using (133) the equation $DF(\bar{z})v = 0$ thus becomes

$$v'' = 2b_1\bar{z}^2v + c_1. \quad (134)$$

As in the proof of Lemma D.2 it follows that v is constant: it attains its maximum at some time $\tau_{\max} \in S^1$ satisfying

$$v''(\tau_{\max}) = 2b_1\bar{z}^2v(\tau_{\max}) + c_1 \leq 0.$$

From $b_1 > 0$ it follows that for all $\tau \in S^1$ we have

$$v''(\tau) = 2b_1\bar{z}^2v(\tau) + c_1 \leq 2b_1\bar{z}^2v(\tau_{\max}) + c_1 \leq 0,$$

which by periodicity implies that $v(\tau) \equiv \bar{v}$ is constant. Plugging this into (134) yields

$$\lambda \bar{v} = 0 \quad (135)$$

with the constant

$$\lambda := 2b_1(\bar{z})\bar{z}^2 + \nabla a_1(\bar{z})\bar{z} + \nabla b_1(\bar{z})\bar{z}^3.$$

To compute λ , we plug $z = \bar{z}$ into a_1 and b_1 to get

$$a_1(\bar{z}) = -\frac{1}{\bar{z}^6} - \frac{1}{2\bar{z}^2(\bar{z}^2 - a)^2},$$

$$b_1(\bar{z}) = \frac{1}{\bar{z}^4(\bar{z}^2 - a)^2}$$

and compute their derivatives (as functions $\mathbb{R} \rightarrow \mathbb{R}$)

$$\begin{aligned}\nabla a_1(\bar{z}) &= \frac{6}{\bar{z}^7} + \frac{2\bar{z}(\bar{z}^2 - a)^2 + \bar{z}^2 \cdot 2(\bar{z}^2 - a) \cdot 2\bar{z}}{2\bar{z}^4(\bar{z}^2 - a)^4} \\ &= \frac{6}{\bar{z}^7} + \frac{3\bar{z}^2 - a}{\bar{z}^3(\bar{z}^2 - a)^3}, \\ \nabla b_1(\bar{z}) &= -\frac{4\bar{z}^3(\bar{z}^2 - a)^2 + \bar{z}^4 \cdot 2(\bar{z}^2 - a) \cdot 2\bar{z}}{\bar{z}^8(\bar{z}^2 - a)^4} \\ &= \frac{4a - 8\bar{z}^2}{\bar{z}^5(\bar{z}^2 - a)^3}.\end{aligned}$$

From (133) we obtain

$$0 = a_1(\bar{z}) + b_1(\bar{z})\bar{z}^2 = -\frac{1}{\bar{z}^6} - \frac{1}{2\bar{z}^2(\bar{z}^2 - a)^2} + \frac{1}{\bar{z}^2(\bar{z}^2 - a)^2},$$

and therefore

$$\frac{1}{\bar{z}^6} = \frac{1}{2\bar{z}^2(\bar{z}^2 - a)^2}.$$

Using this and the preceding formulas we compute

$$\begin{aligned}\lambda &= 2b_1(\bar{z})\bar{z}^2 + \nabla a_1(\bar{z})\bar{z} + \nabla b_1(\bar{z})\bar{z}^3 \\ &= \frac{2}{\bar{z}^2(\bar{z}^2 - a)^2} + \frac{6}{\bar{z}^6} + \frac{3\bar{z}^2 - a + 4a - 8\bar{z}^2}{\bar{z}^3(\bar{z}^2 - a)^3} \\ &= \frac{2 + 3}{\bar{z}^2(\bar{z}^2 - a)^2} + \frac{3a - 5\bar{z}^2}{\bar{z}^3(\bar{z}^2 - a)^3} \\ &= \frac{5\bar{z}^2 - 5a + 3a - 5\bar{z}^2}{\bar{z}^3(\bar{z}^2 - a)^3} \\ &= \frac{-2a}{\bar{z}^3(\bar{z}^2 - a)^3} < 0.\end{aligned}$$

Hence (135) implies $\bar{v} = 0$. This concludes the proof of Proposition D.5, and therefore of Theorem D.1.

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Kai Cieliebak

Universität Augsburg, Universitätsstraße 14, 86159 Augsburg, Germany;
kai.cieliebak@math.uni-augsburg.de

Urs Frauenfelder

Universität Augsburg, Universitätsstraße 14, 86159 Augsburg, Germany;
urs.frauenfelder@math.uni-augsburg.de

Evgeny Volkov

Universität Augsburg, Universitätsstraße 14, 86159 Augsburg, Germany;
evgeny.volkov@math.uni-augsburg.de