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Corrigendum

Corrigendum to "An isoperimetric inequality for a nonlinear eigenvalue problem" [Ann. I. H. Poincaré – AN 29 (1) (2012) 21–34]

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Recently it came to our attention that our proof proposed in [1] about the minimizing set of the eigenvalue

$$\lambda^{p,q}(\Omega) = \inf\left\{ \frac{\|\nabla v\|_{L^p(\Omega)}}{\|v\|_{L^q(\Omega)}}, v \neq 0, v \in W_0^{1,p}(\Omega), \int_\Omega |v|^{q-2} v \, dx = 0 \right\} \tag{1}$$

among the bounded open sets $\Omega \subset \mathbb{R}^N$ of given volume is not correct. Indeed the argument that we used to write the Euler–Lagrange equation in Theorem 6 cannot be applied. More precisely in the last paragraph of the proof we need to test the minimality with the function $u + t_n(\varphi + c_{t_n})$ which is not in $W_0^{1,p}$ but merely constant on the boundary.

The statement of Theorem 1 in [1] is not true for every p, q satisfying

$$1 < p < \infty \quad \text{and} \quad \begin{cases} 2 \le q < p^*, & \text{if } 1 < p < N, \\ 2 \le q < \infty, & \text{if } p \ge N \end{cases} \tag{2}$$

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as remarked also by Nazarov in [3].[1] For a correct statement we should replace $\lambda^{p,q}(\Omega)$ in [1] by

$$\lambda^{p,q}_{per}(\Omega) = \inf\left\{ \frac{\|\nabla v\|_{L^p(\Omega)}}{\|v\|_{L^q(\Omega)}}, v \neq 0, v \in W^{1,p}_{per}(\Omega), \int_\Omega |v|^{q-2} v\,dx = 0 \right\} \tag{3}$$

where $W^{1,p}_{per}(\Omega)$ stands for the set of $W^{1,p}(\Omega)$ functions with constant boundary value. As in [2] for the case $p = q = 2$, $\lambda^{p,q}_{per}(\Omega)$ is minimized by the union of two equal balls for every p, q satisfying (2).

The proof goes exactly as in [1], up to these modifications:

(1) In Theorem 5 one needs to use the Schwarz symmetrization on $\Omega_+ = \{x \in \Omega\colon u(x) > c\}$ and $\Omega_- = \{x \in \Omega\colon u(x) < c\}$, where c is the value of the eigenfunction u at $\partial\Omega$.
(2) Theorem 7 is no longer true in general for this eigenvalue, nevertheless we can prove Theorem 8 in a similar way. There are two differences in the proof:
   (a) we have to deal with the boundary condition which is not standard when performing the domain derivative;
   (b) even when the optimal domain has a multiple first eigenvalue, we can still deduce that the derivative of any eigenfunction has to vanish. For this we use the existence of directional derivatives along with the minimality of the eigenvalue.
(3) The ordinary differential equation satisfied by the radial component w of u_1, that is,

$$-\left[(p-1)z''(t) + \frac{N-1}{t} z'(t)\right]\left|z'(t)\right|^{p-2} = \left[\lambda^{p,q}_{per}(\Omega)\right]^p \|u\|^{p-q}_{L^q(\Omega)} \left|z(t)\right|^{q-2} z(t),$$

implies that if $\frac{\partial u_1}{\partial \nu_1} = 0$ then $u_1 = 0$ on the boundary. Therefore the proof of Theorem 9 is valid.

We end this erratum with a remark about $\lambda^{p,q}(\Omega)$. For q small enough, it turns out that the eigenfunction of $\lambda^{p,q}_{per}(\Omega)$ seems to be symmetric and satisfies $u = 0$ on the boundary (at this time, we are only able to prove it for $q = p$).[2] In this case, our result about $\lambda^{p,q}(\Omega)$ still holds, since

$$\lambda^{p,q}(\Omega) \geq \lambda^{p,q}_{per}(\Omega) \geq \lambda^{p,q}_{per}(B \cup B) = \lambda^{p,q}(B \cup B)$$

where the last equality is due to the fact that if u is an eigenfunction of $\lambda^{p,q}_{per}(\Omega)$ satisfying $u = 0$ on $\partial\Omega$, then u is an eigenfunction of $\lambda^{p,q}(\Omega)$.

Conflict of interest statement

There are no known conflicts of interest associated to this publication.

[1] Nazarov considers the following test functions on domains Ω given by union of balls B_+, B_- of radii R_+, R_-. Let v be the restriction to one ball of the eigenfunction realizing $\lambda^{p,q}(B_+ \cup B_+)$. For $x \mapsto c_+ v(\frac{R_{1/2}x}{R_+})\chi_{B_+} - c_- v(\frac{R_{1/2}x}{R_-})\chi_{B_-}$ (where $c_+, c_- \geq 0$, $c_+^{q-1} R_+^N = c_-^{q-1} R_-^N$ and $2R_{1/2}^N = \frac{|\Omega|}{\omega_N}$) one has

$$\lambda^{p,q}(B_+ \cup B_-) \leq \frac{\|\nabla v\|_{L^p(B_{1/2})}}{\|v\|_{L^q(B_{1/2})}} R_{1/2}^{1-\frac{N}{p}+\frac{N}{q}} \frac{[c_+^p R_+^{N-p} + c_-^p R_-^{N-p}]^{\frac{1}{p}}}{[c_+^q R_+^N + c_-^q R_-^N]^{\frac{1}{q}}}.$$

Then he finds an explicit value $\hat{q}$ such that, for $q > \hat{q}$, a simple first order analysis shows that the last function of (R_+, R_-) does not attain its minimum value in $(R_{1/2}, R_{1/2})$. This proves that the minimizing set is not given by a pair of two equal balls.

[2] Indeed, let $u = u_1(|x - x_1|)\chi_{B_1} + u_2(|x - x_2|)\chi_{B_2}$ be an eigenfunction of $\lambda = \lambda^{p,q}_{per}(B_1 \cup B_2)$ and w be the solution to

$$-\left[(p-1)w''(t) + \frac{N-1}{t} w'(t)\right]\left|w'(t)\right|^{p-2} = \left|w(t)\right|^{q-2} w(t), \qquad w(0) = 1, \qquad w'(0) = 0.$$

Let $\alpha_i = u(x_i)$ and $w_{\alpha_i}(t) = \alpha_i w(c(\alpha_i)t)$, where $c(\alpha) = \frac{\lambda \|u\|_{L^q}^{1-q/p}}{|\alpha|^{1-\frac{q}{p}}}$, for $\alpha \neq 0$. Then it is not difficult to prove that $u_i = w_{\alpha_i}(|x - x_i|)$. Since u is constant on the boundary, in the case $q = p$ one has $\alpha_1 w(\lambda R_1) = \alpha_2 w(\lambda R_2)$. Assuming w.l.o.g. that $u = c \geq 0$ on the boundary, we get $w(\lambda R_1) = 0 = w(\lambda R_2)$ since $\alpha_1 > 0 > \alpha_2$.

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