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Extremal functions for the anisotropic Sobolev inequalities

Fonctions minimales pour des inégalités de Sobolev anisotropiques

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Abstract

The existence of multiple nonnegative solutions to the anisotropic critical problem

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = |u|^{p^*-2}u \quad \text{in } \mathbb{R}^N$$

is proved in suitable anisotropic Sobolev spaces. The solutions correspond to extremal functions of a certain best Sobolev constant. The main tool in our study is an adaptation of the well-known concentration-compactness lemma of P.-L. Lions to anisotropic operators. Furthermore, we show that the set of nontrival solutions $\mathcal{S}$ is included in $L^\infty(\mathbb{R}^N)$ and is located outside of a ball of radius $\tau > 0$ in $L^{p^*}(\mathbb{R}^N)$.

Résumé

Nous montrons l'existence d'une infinité de solutions positives pour le problème anisotropique avec exposant critique. La méthode consiste à regarder la meilleure constante d'une inégalité du type Poincaré–Sobolev et à adapter le fameux principe de concentration-compacité de P. L. Lions. De plus, on montre que l'ensemble des solutions $\mathcal{S}$ est contenu dans $L^\infty(\mathbb{R}^N)$ et est localisé en dehors d'une boule de rayon $\tau > 0$ dans $L^{p^*}(\mathbb{R}^N)$.

Keywords: Quasilinear problems; Concentration-compactness; Anisotropic Sobolev inequalities

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1. Introduction

In this paper, the existence of nontrivial nonnegative solutions to the anisotropic critical problem

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)=|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N \tag{1}$$

is studied, where the exponents p_i and p^* satisfy the following conditions

$$p_i>1, \quad \sum_{i=1}^{N}\frac{1}{p_i}>1,$$

and the critical exponent p^* is defined by

$$p^*:=\frac{N}{\sum_{i=1}^{N}\frac{1}{p_i}-1}.$$

In the best of our knowledge, anisotropic equations with different orders of derivation in different directions, involving critical exponents were never studied before. In the subcritical case, we can refer the reader to the recent paper by I. Fragala et al. [4].

In the special case $p_i=2$, $i\in\{1,2,\dots,N\}$, Problem (1) is reduced to the limiting equation arising in the famous Yamabe problem [13]:

$$-\Delta u=u^{2^*-1}, \quad u>0 \quad \text{in } \mathbb{R}^N. \tag{2}$$

Indeed, let (M,g) be a N-dimensional Riemannian manifold and S_g be the scalar curvature of the metric g. Consider a conformal metric $\tilde{g}$ on M defined by $\tilde{g}:=u^{\frac{4}{N-2}}g$ whose scalar curvature (which is assumed to be constant) is denoted by $S_{\tilde{g}}$, where u is a positive function in $C^\infty(M,\mathbb{R})$. The unknown function u satisfies then

$$-\Delta_g u+\frac{N-2}{4(N-1)}S_g u=\frac{N-2}{4(N-1)}S_{\tilde{g}}u^{2^*-1}, \quad u>0 \quad \text{in } M, \tag{3}$$

where Δ_g denotes the Laplace–Beltrami operator. It is clear that, up to a scaling, the limiting problem of (3) (Eq. (3) without the subcritical term $\frac{N-2}{4(N-1)}S_g u$) is exactly (2). The question of existence of minimizing solutions to (2) was completely solved by Aubin [1] and G. Talenti [9]. Their proofs are based on symmetrization theory. Notice that this theory is not relevant in our context since the radial symmetry of solutions cannot hold true because of the anisotropy of the operator.

In [5], P.-L. Lions introduced the famous concentration-compactness lemma which constitutes a powerful tool for the study of critical nonlinear elliptic equations. The concentration-compactness lemma allows an elegant and simple proof of the existence of solutions to (2) by minimization arguments. In the present work, we will adapt the concentration-compactness lemma to the anisotropic case and show that the infimum

$$\operatorname*{Inf}_{|u|_{L^{p^*}(\mathbb{R}^N)}=1}\left\{\sum_{i=1}^{N}\frac{1}{p_i}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i}^{p_i}\right\}$$

is achieved, of course, the functional space has to be specified.

The motivation of the present work is to give a new result which can provide extremal functions associated to the critical level corresponding to anisotropic problems involving critical exponents. Notice that the genuine extremal functions are obtained by minimization on the Nehari manifold associated to the problem and the critical level is nothing than the energy of these extremal functions.

The natural functional framework of Problem (1) is the anisotropic Sobolev spaces theory developed by [3,6,11,7,8,10]. Then, let $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ be the completion of the space $\mathcal{D}(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{1,\vec{p}}:=\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i}.$$

It is well known that $(\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N), \|\cdot\|_{1,\vec{p}})$ is a reflexive Banach space which is continuously embedded in $L^{p^*}(\mathbb{R}^N)$.

In what follows, we will assume that

$$p_+ = \max\{p_1, p_2, \ldots, p_N\} < p^*,$$

then p^* is the critical exponent associated to the operator:

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial}{\partial x_i}\right|^{p_i-2}\frac{\partial}{\partial x_i}\right).$$

The space $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ can also be seen as

$$\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N) = \left\{u \in L^{p^*}(\mathbb{R}^N)\colon \left|\frac{\partial u}{\partial x_i}\right| \in L^{p_i}(\mathbb{R}^N)\right\}.$$

In the sequel, we will set $p_- = \min\{p_1, p_2, \ldots, p_N\}$, $p_+ = \max\{p_1, p_2, \ldots, p_N\}$ and $\vec{p} = (p_1, p_2, \ldots, p_n)$. Also, the integral symbol $\int$ will denote $\int_{\mathbb{R}^N}$ and $\|\cdot\|_{p_i}$ will denote the usual Lebesgue norm in $L^{p_i}(\mathbb{R}^N)$. We denote by $\mathfrak{M}(\mathbb{R}^N)$ (resp. $\mathfrak{M}^+(\mathbb{R}^N)$) the space of finite measures (resp. positive finite measures) on $\mathbb{R}^N$, and by $\|\cdot\|$ its usual norm.

2. Existence of extremal functions for a Sobolev type inequality

In this paragraph, we shall prove that a certain best Sobolev constant is achieved.

Theorem 1. *Under the above assumptions on p_i, $i = 1, \ldots, N$, $N \geqslant 2$, there exists at least one function $u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$, $u \geqslant 0$, $u \neq 0$:*

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right) = u^{p^*-1} \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

The proof will need two fundamental lemmas, the first one is a result due to M. Troisi [10]:

Lemma 1. (Troisi [10]) *There is a constant $T_0 > 0$ depending only on $\vec{p}$ and N such that*:

$$T_0\|u\|_{p^*} \leqslant \prod_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i}^{\frac{1}{N}} \quad \text{and} \quad \|u\|_{p^*} \leqslant \frac{1}{NT_0}\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i},$$

for all $u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$.

The second lemma is a rescaling type result ensuring the conservation of suitable norms:

Lemma 2. *Let $\alpha_i = \frac{p^*}{p_i} - 1$, $i = 1, \ldots, N$. For every $y \in \mathbb{R}^N$, $u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$, and $\lambda > 0$, if we write $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N)$, $v(x) \doteq u^{\lambda,y}(x) = \lambda u(\lambda^{\alpha_1}x_1 + y_1, \ldots, \lambda^{\alpha_N}x_N + y_N)$, we get*

$$\|u\|_{p^*} = \|v\|_{p^*},$$
$$\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i} = \left\|\frac{\partial v}{\partial x_i}\right\|_{p_i}, \quad \text{for } i = 1, \ldots, N,$$

thus, $\|u\|_{1,\vec{p}} = \|u^{\lambda,y}\|_{1,\vec{p}}$.

Proof. Noticing that $\sum_{i=1}^{N}\alpha_i = p^*$, a straightforward computation with adequate changes of variables gives the result. □

Lemma 3. *Let*

$$S = \operatorname*{Inf}_{u\in\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N),\ \|u\|_{p^*}=1}\left\{\sum_{i=1}^{N}\frac{1}{p_i}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i}^{p_i}\right\}.$$

Then $S > 0$.

Proof. From Lemma 1, we obtain that if $\|u\|_{p^*} = 1$, then

$$\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i} \geqslant NT_0 > 0. \tag{4}$$

Using standard argument, the infimum

$$\operatorname{Inf}\left\{\sum_{i=1}^{N}\frac{1}{p_i}a_i^{p_i},\ (a_1,\dots,a_n)\in\mathbb{R}^N,\ \sum_{i=1}^{N}a_i \geqslant NT_0,\ a_i\geqslant 0\right\} \doteq S_1$$

is achieved and thus this minimum is positive. By relation (4), one concludes that $S \geqslant S_1 > 0$. □

Corollary 1 of Lemma 3 *(Sobolev type inequality). Let* $p_- = \min(p_1,\dots,p_N)$*,* $p_+ = \max(p_1,\dots,p_N)$ *and* F *be the real valued function defined by*

$$F(\sigma) = \begin{cases}\sigma^{p_+} & \text{if } \sigma \leqslant 1,\\ \sigma^{p_-} & \text{if } \sigma \geqslant 1.\end{cases}$$

Then for every $u \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$*, one has*

$$SF\big(\|u\|_{p^*}\big) \leqslant \sum_{i=1}^{N}\frac{1}{p_i}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i}^{p_i} \doteq P(\nabla u).$$

Proof. Let u be in $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$. If $u = 0$ the inequality is true. If $u \neq 0$, set $w = u/\|u\|_{p^*}$, then from the definition of S one has:

$$\sum_{i=1}^{N}\frac{1}{p_i}\left\|\frac{\partial w}{\partial x_i}\right\|_{p_i}^{p_i} \geqslant S. \tag{5}$$

Since $t^{p_i} \leqslant t^{p_+}$ if $t > 1$ and $t^{p_i} \leqslant t^{p_-}$ otherwise, the result follows from relation (5) and the definition of F. □

Remark 1. Along this paragraph, we only need the inequality:

$$S\|u\|_{p^*}^{p_+} \leqslant P(\nabla u) \quad \text{whenever } \|u\|_{p^*} \leqslant 1.$$

We shall call $(\mathcal{P})$ the minimization problem

$$(\mathcal{P}) \qquad \operatorname*{Inf}_{\|u\|_{p^*}=1}\left\{\sum_{i=1}^{N}\frac{1}{p_i}\left\|\frac{\partial u}{\partial x_i}\right\|_{p_i}^{p_i}\right\} = \operatorname*{Inf}_{\|u\|_{p^*}=1}\big\{P(\nabla u)\big\}.$$

Let $(u_n) \subset \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ be a minimizing sequence for the problem $(\mathcal{P})$. As in [5] and Willem [12], we define the Levy concentration function:

$$Q_n(\lambda) = \sup_{y\in\mathbb{R}^N}\int_{E(y,\lambda^{\alpha_1},\dots,\lambda^{\alpha_N})}|u_n|^{p^*}\,\mathrm{d}x, \quad \lambda > 0.$$

Here $E(y,\lambda^{\alpha_1},\dots,\lambda^{\alpha_N})$ is the ellipse defined by

$$\left\{z = (z_1,\dots,z_N)\in\mathbb{R}^N,\ \sum_{i=1}^{N}\frac{(z_i-y_i)^2}{\lambda^{2\alpha_i}} \leqslant 1\right\}$$

with $y = (y_1, \dots, y_N)$ and $\alpha_i > 0$ as in Lemma 2. Since for every n, $\lim_{\lambda\to 0} Q_n(\lambda) = 0$ and $Q_n(\lambda) \underset{\lambda\to+\infty}{\longrightarrow} 1$. There exists $\lambda_n > 0$ such that $Q_n(\lambda_n) = \frac{1}{2}$. Moreover there exists $y_n \in \mathbb{R}^N$ such that

$$\int\limits_{E(y_n,\lambda_n^{\alpha_1},\dots,\lambda_n^{\alpha_N})} |u_n|^{p^*}\,\mathrm{d}x = \frac{1}{2}.$$

Thus by a change of variables one has for $v_n \doteq u_n^{\lambda_n, y_n}$:

$$\int\limits_{B(0,1)} |v_n|^{p^*}\,\mathrm{d}x = \frac{1}{2} = \sup_{y\in\mathbb{R}^N} \int\limits_{B(y,1)} |v_n|^{p^*}\,\mathrm{d}x.$$

Since

$$\|v_n\|_{p^*} = \|u_n\|_{p^*}, \quad \left\|\frac{\partial v_n}{\partial x_i}\right\|_{p_i} = \left\|\frac{\partial u_n}{\partial x_i}\right\|_{p_i}, \quad P(\nabla u_n) = P(\nabla v_n)$$

we deduce that (v_n) is bounded in $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ and is also a minimizing sequence for $(\mathcal{P})$. We may then assume that:

- $v_n \rightharpoonup v$ in $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$,
- $|\frac{\partial}{\partial x_i}(v_n - v)|^{p_i} \rightharpoonup \mu_i$ in $\mathfrak{M}^+(\mathbb{R}^N)$,
- $|v_n - v|^{p^*} \rightharpoonup \nu$ in $\mathfrak{M}^+(\mathbb{R}^N)$,
- $v_n \to v$ a.e. in $\mathbb{R}^N$.

We define:

$$\mu = \sum_{i=1}^N \frac{1}{p_i}\mu_i,$$

$$\mu_\infty = \lim_{R\to+\infty} \overline{\lim_n} \sum_{i=1}^N \frac{1}{p_i} \int\limits_{|x|>R} \left|\frac{\partial v_n}{\partial x_i}\right|^{p_i} \mathrm{d}x, \tag{6}$$

$$\nu_\infty = \lim_{R\to+\infty} \overline{\lim_n} \int\limits_{|x|>R} |v_n|^{p^*}\,\mathrm{d}x. \tag{7}$$

We start with some general lemmas. First by the Brezis–Lieb's Lemma [2], direct computations give the following

Lemma 4.

$$|v_n|^{p^*} \rightharpoonup |v|^{p^*} + \nu \quad \textit{in } \mathfrak{M}^+(\mathbb{R}^N).$$

The lemma which follows gives some reverse Hölder type inequalities connecting the measures ν, μ and μ_i, $1 \leqslant i \leqslant N$.

Lemma 5. *Under the above statement, one has for all $\varphi \in C_c^\infty(\mathbb{R}^N)$*

$$\left(\int |\varphi|^{p^*}\,\mathrm{d}\nu\right)^{\frac{1}{p^*}} \leqslant \frac{1}{T_0} \prod_{i=1}^N \left(\int |\varphi|^{p_i}\,\mathrm{d}\mu_i\right)^{\frac{1}{Np_i}},$$

$$\left(\int |\varphi|^{p^*}\,\mathrm{d}\nu\right)^{\frac{1}{p^*}} \leqslant p_+^{\frac{1}{N}+\frac{1}{p^*}} \|\mu\|^{\frac{1}{N}+\frac{1}{p^*}-\frac{1}{p_+}} \cdot \frac{1}{T_0} \left(\int |\varphi|^{p_+}\,\mathrm{d}\mu\right)^{\frac{1}{p_+}}.$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ and set $w_n = v_n - v$. Since $\int |\varphi_{x_i}|^{p_i} |w_n|^{p_i}\,\mathrm{d}x \underset{n\to+\infty}{\longrightarrow} 0$, we then have:

$$\lim_n \int \left|\frac{\partial}{\partial x_i}(\varphi w_n)\right|^{p_i} \mathrm{d}x = \lim_n \int |\varphi|^{p_i} \left|\frac{\partial w_n}{\partial x_i}\right|^{p_i} \mathrm{d}x = \int |\varphi|^{p_i}\,\mathrm{d}\mu_i. \tag{8}$$

Thus from Lemma 1, it follows that

$$\left(\int |\varphi|^{p^*}\,\mathrm{d}\nu\right)^{\frac{1}{p^*}} = \lim_n \left(\int |\varphi w_n|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}} \leqslant \frac{1}{T_0}\prod_{i=1}^N \left(\int |\varphi|^{p_i}\,\mathrm{d}\mu_i\right)^{\frac{1}{Np_i}}. \tag{9}$$

On the other hand, since

$$\int |\varphi|^{p_i}\,\mathrm{d}\mu_i \leqslant p_+ \int |\varphi|^{p_i}\,\mathrm{d}\mu \leqslant p_+ \|\mu\|^{1-\frac{p_i}{p_+}} \left(\int |\varphi|^{p_+}\,\mathrm{d}\mu\right)^{\frac{p_i}{p_+}} \tag{10}$$

applying the estimates (9) and (10) and knowing that $\sum_{i=1}^N \frac{1}{p_i} = 1 + \frac{N}{p^*}$, we deduce

$$\left(\int |\varphi|^{p^*}\,\mathrm{d}\nu\right)^{\frac{1}{p^*}} \leqslant p_+^{\frac{1}{N}+\frac{1}{p^*}} \|\mu\|^{\frac{1}{N}+\frac{1}{p^*}-\frac{1}{p_+}} \cdot \frac{1}{T_0}\left(\int |\varphi|^{p_+}\,\mathrm{d}\mu\right)^{\frac{1}{p_+}}.$$

This ends the proof. □

We then have $\|v\|_{p^*} \leqslant 1$. So if $\|v\|_{p^*} = 1$ then v is an extremal function since $P(\nabla v) \leqslant \liminf_n P(\nabla v_n) = S$ and $S \leqslant P(\nabla v)$. Thus, we want to show that fact, by proving that if it is not true then we have a concentration of ν at a single point and therefore $v = 0$.

Main Lemma.

$$\|v\|_{p^*} = 1.$$

The remainder of this section is devoted to the proof of the main Lemma

Lemma 6. *If $v \neq 0$ then*

$$\lim_n \|v_n - v\|_{p^*}^{p^*} = 1 - \|v\|_{p^*}^{p^*} < 1.$$

Proof. From Brezis–Lieb's Lemma we have:

$$\lim_n\left(\|v_n\|_{p^*}^{p^*} - \|v_n - v\|_{p^*}^{p^*}\right) = \|v\|_{p^*}^{p^*},$$

Since $\|v_n\|_{p^*} = 1$, we derive the result. □

Lemma 7.

$$S\|v\|^{\frac{p_+}{p^*}} \leqslant \|\mu\|.$$

Proof. For large n, according to Lemma 6, we have:

$$\int |v_n - v|^{p^*}\,\mathrm{d}x \leqslant 1.$$

Thus for all $\varphi \in C_c^\infty(\mathbb{R}^N)$, $|\varphi|_\infty \leqslant 1$, it holds:

$$S\left(\int |\varphi|^{p^*} |v_n - v|^{p^*}\right)^{\frac{p_+}{p^*}} \leqslant \sum_{i=1}^N \frac{1}{p_i}\int |\varphi|^{p_i} \left|\frac{\partial(v_n - v)}{\partial x_i}\right|^{p_i} \mathrm{d}x + \mathrm{o}_n(1).$$

Letting $n \to +\infty$, one gets:

$$S\left(\int |\varphi|^{p^*}\,\mathrm{d}\nu\right)^{\frac{p_+}{p^*}} \leqslant \sum_{i=1}^{N} \frac{1}{p_i} \int |\varphi|^{p_i}\,\mathrm{d}\mu_i \leqslant \|\mu\|. \tag{11}$$

Using the density of $C_c^\infty(\mathbb{R}^N)$ in $C_c(\mathbb{R}^N)$, we get then

$$S\left(\sup_{\varphi\in C_c(\mathbb{R}^N),\ |\varphi|_\infty=1} \int |\varphi|^{p^*}\,\mathrm{d}\nu\right)^{\frac{p_+}{p^*}} \leqslant \|\mu\|,$$

that is the desired result. □

Lemma 8. *Let ψ_R be in $C^1(\mathbb{R})$, $0 \leqslant \psi_R \leqslant 1$, $\psi_R = 1$ if $|x| > R+1$, $\psi_R(x) = 0$ if $|x| < R$. Then for any $\gamma_i > 0$, $i = 0, \dots, N$, the two equalities*

$$\nu_\infty = \lim_{R\to+\infty} \overline{\lim_n} \int |v_n|^{p^*} \psi_R^{\gamma_0}\,\mathrm{d}x,$$

$$\mu_\infty = \lim_{R\to+\infty} \overline{\lim_n} \sum_{i=1}^{N} \frac{1}{p_i} \int \left|\frac{\partial v_n}{\partial x_i}\right|^{p_i} \psi_R^{\gamma_i}\,\mathrm{d}x$$

hold true, where ν_∞ and μ_∞ are defined by (6), (7).

Proof. As in Willem [12], one has:

$$\int_{|x|>R+1} |v_n|^{p^*}\,\mathrm{d}x \leqslant \int |v_n|^{p^*} \psi_R^{\gamma_0}\,\mathrm{d}x \leqslant \int_{|x|>R} |v_n|^{p^*}\,\mathrm{d}x,$$

$$\int_{|x|>R+1} \left|\frac{\partial v_n}{\partial x_i}\right|^{p_i}\,\mathrm{d}x \leqslant \int \left|\frac{\partial v_n}{\partial x_i}\right| \psi_R^{\gamma_i} \leqslant \int_{|x|>R} \left|\frac{\partial v_n}{\partial x_i}\right|^{p_i}\,\mathrm{d}x.$$

We conclude with the definition of ν_∞ and μ_∞. □

Lemma 9. *Let $w_n = v_n - v$. Then, for any $\gamma_i > 0$, $i = 0, \dots, N$, we get*

$$\nu_\infty = \lim_{R\to\infty} \overline{\lim_n} \int |w_n|^{p^*} \psi_R^{\gamma_0}\,\mathrm{d}x,$$

and

$$\mu_\infty = \lim_{R\to\infty} \overline{\lim_n} \int \left|\frac{\partial w_n}{\partial x_i}\right|^{p_i} \psi_R^{\gamma_i}\,\mathrm{d}x.$$

Proof. Since

$$\lim_{R\to+\infty} \int |v|^{p^*} \psi_R^{\gamma_0} = \lim_{R\to+\infty} \int \left|\frac{\partial v}{\partial x_i}\right|^{p_i} \psi_R^{\gamma_i}\,\mathrm{d}x = 0.$$

Thus

$$\lim_{R\to\infty} \overline{\lim_n} \int |w_n|^{p^*} \psi_R^{\gamma_0}\,\mathrm{d}x = \lim_{R\to\infty} \overline{\lim_n} \int |v_n|^{p^*} \psi_R^{\gamma_0}\,\mathrm{d}x = \nu_\infty$$

and

$$\lim_{R\to\infty} \overline{\lim_n} \sum_{i=1}^{N} \frac{1}{p_i} \int \left|\frac{\partial w_n}{\partial x_i}\right|^{p_i} \psi_R^{\gamma_i}\,\mathrm{d}x = \lim_{R\to\infty} \overline{\lim_n} \sum_{i=1}^{N} \frac{1}{p_i} \int \left|\frac{\partial v_n}{\partial x_i}\right|^{p_i} \psi_R^{\gamma_i}\,\mathrm{d}x. \qquad \square$$

Lemma 10.

$$S\nu_\infty^{\frac{p_+}{p^*}} \leqslant \mu_\infty.$$

Proof. From Lemma 6, we know that for n large enough, we have

$$\int \psi_R^{p^*}|w_n|^{p^*} \leqslant \int |w_n|^{p^*}\,\mathrm{d}x \leqslant 1.$$

Thus by Sobolev inequality (Corollary 4 of Lemma 3), it follows

$$S\left(\int |\psi_R w_n|^{p^*}\,\mathrm{d}x\right)^{\frac{p_+}{p^*}} \leqslant \sum_{i=1}^{N}\frac{1}{p_i}\int\left|\frac{\partial}{\partial x_i}(\psi_R w_n)\right|^{p_i},$$

$$S\left(\lim_{R\to+\infty}\overline{\lim_n}\int |\psi_R w_n|^{p^*}\,\mathrm{d}x\right)^{\frac{p_+}{p^*}} \leqslant \lim_{R\to+\infty}\overline{\lim_n}\sum_{i=1}^{N}\frac{1}{p_i}\int\left|\frac{\partial}{\partial x_i}(\psi_R w_n)\right|^{p_i}. \tag{12}$$

Since

$$\lim_n \sum_{i=1}^{N}\frac{1}{p_i}\int\left|\frac{\partial\psi_R}{\partial x_i}\right|^{p_i}|w_n|^{p_i} = 0,$$

then

$$\lim_{R\to+\infty}\overline{\lim_n}\sum_{i=1}^{N}\frac{1}{p_i}\int\left|\frac{\partial}{\partial x_i}(\psi_R w_n)\right|^{p_i} = \lim_{R\to+\infty}\overline{\lim_n}\sum_{i=1}^{N}\frac{1}{p_i}\int\left|\frac{\partial w_n}{\partial x_i}\right|^{p_i}\psi_R^{p_i} = \mu_\infty.$$

Relation (12) and Lemma 9 give:

$$S\nu_\infty^{\frac{p_+}{p^*}} \leqslant \mu_\infty. \qquad \square$$

Following again the arguments used in [12] we claim that:

Lemma 11.

$$1 = \lim_n \|v_n\|_{p^*}^{p^*} = \|v\|_{p^*}^{p^*} + \|\nu\| + \nu_\infty.$$

Proof. From Lemma 4, we have:

$$|v_n|^{p^*} \rightharpoonup |v|^{p^*} + \nu.$$

Thus

$$\lim_{R\to+\infty}\lim_n \int \left(1-\psi_R^{p^*}\right)|v_n|^{p^*}\,\mathrm{d}x = \int |v|^{p^*}\,\mathrm{d}x + \int \mathrm{d}\nu.$$

Rewriting $\|v_n\|_{p^*}^{p^*}$ as

$$\|v_n\|_{p^*}^{p^*} = \int \left(1-\psi_R^{p^*}\right)|v_n|^{p^*} + \int \psi_R^{p^*}|v_n|^{p^*},$$

we obtain

$$\lim_n \|v_n\|_{p^*}^{p^*} = \lim_{R\to+\infty}\lim_n \int\left(1-\psi_R^{p^*}\right)|v_n|^{p^*} + \lim_{R\to+\infty}\overline{\lim_n}\int \psi_R^{p^*}|v_n|^{p^*} = \|v\|_{p^*}^{p^*} + \|\nu\| + \nu_\infty. \qquad \square$$

Next, we shall prove the following corollary:

Corollary 1 *(Of Lemma 5). There exists an at most countable index set J of distinct points $\{x_j\}_{j\in J} \subset \mathbb{R}^N$ and nonnegative weights a_j and b_j, $j \in J$, such that*:

(1) $\nu = \sum_{j\in J} a_j\delta_{x_j}$.

(2) $\mu \geqslant \sum_{j\in J} b_j\delta_{x_j}$.

(3) $Sa_j^{\frac{p_+}{p^*}} \leqslant b_j,\ \forall j \in J$.

Proof. The proof follows essentially the concentration compactness principle of P.L. Lions [5] because we have the reverse Hölder type inequalities of Lemma 5.

Indeed, the second statement of this lemma implies that for all Borelian sets $E \subset \mathbb{R}^N$, one has:

$$\nu(E) \leqslant c_\mu \mu(E)^{\frac{p^*}{p_+}}. \tag{13}$$

Since the set $D = \{x \in \mathbb{R}^N\colon\ \mu(\{x\}) > 0\}$ is at most countable because $\mu \in \mathfrak{M}(\mathbb{R}^N)$, therefore $D = \{x_j,\ j \in J\}$ and $b_j \doteq \mu(\{x_j\})$ satisfies $\mu \geqslant \sum_{j\in J} b_j\delta_{x_j}$.

Relation (13) implies that ν is absolutely continuous with respect to μ, i.e., $\nu \ll \mu$ and

$$\frac{\nu(B(x,r))}{\mu(B(x,r))} \leqslant c_\mu \mu\big(B(x,r)\big)^{\frac{p^*}{p_+}-1},$$

provided that $\mu(B(x,r)) \neq 0$ (remember that $p^* > p_+$). Thus, we have:

$$\nu(E) = \int_E \lim_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))}\,\mathrm{d}\mu(x),$$

and

$$D_\mu\nu(x) = \lim_{r\to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))} = 0, \quad \mu \text{ a.e. on } \mathbb{R}^N \setminus D.$$

Setting $a_j = D_\mu\nu(x_j)b_j$, relation (13) implies that ν has only atoms that are given by $\{x_j\}$, that we have already get. □

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$, $\varphi(x_j) = 1$, $\|\varphi\|_\infty = 1$. Then, using statement (1) of this corollary and relation (11), we have

$$Sa_j^{\frac{p_+}{p^*}} \leqslant S\biggl(\int |\varphi|^{p^*}\,\mathrm{d}\nu\biggr)^{\frac{p_+}{p^*}} \leqslant \sum_{i=1}^N \frac{1}{p_i}\int |\varphi|^{p_i}\,\mathrm{d}\mu_i. \tag{14}$$

We shall consider $\phi \in C_c^\infty(\mathbb{R}^N)$, $0 \leqslant \phi \leqslant 1$, support$(\phi) \subset B(0,1)$, $\phi(0) = 1$. We fix $j \in J$ and set $x_j = (x_{j,1},\dots,x_{j,N})$, $q_i = p_ip^*/(p^* - p_i)$, $i = 1,\dots,N$. Then $\alpha_i \doteq \frac{1}{q_i}$ satisfy $\sum_{k=1}^N \alpha_k - \alpha_i q_i = 0$. For $\varepsilon > 0$, we define, for every $z \in \mathbb{R}^N$, $z = (z_1,\dots,z_N)$:

$$\phi_\varepsilon(z) = \phi\biggl(\frac{z_1 - x_{j,1}}{\varepsilon^{\alpha_1}},\dots,\frac{z_N - x_{j,N}}{\varepsilon^{\alpha_N}}\biggr). \tag{15}$$

Thus we have:

$$\int \left|\frac{\partial\phi_\varepsilon}{\partial x_i}\right|^{q_i} = \int \left|\frac{\partial\phi}{\partial x_i}\right|^{q_i}(z)\,\mathrm{d}z \tag{16}$$

and then

$$\int \left|\frac{\partial\phi_\varepsilon}{\partial x_i}\right|^{p_i} |v|^{p_i} \leqslant \biggl(\int \left|\frac{\partial\phi}{\partial x_i}\right|^{q_i}\mathrm{d}z\biggr)^{1-\frac{p_i}{p^*}} \biggl(\int_{B(x_j,\max_i \varepsilon^{\frac{1}{q_i}})} |v|^{p^*}\,\mathrm{d}z\biggr)^{\frac{p_i}{p^*}} \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{17}$$

Lemma 12. *Let $x_j \in D$ and ϕ_ε be the function defined above associated to x_j. Then*:

$$Sa_j^{\frac{p_+}{p^*}} \leqslant \overline{\lim_{\varepsilon\to 0}}\, \overline{\lim_{n}} \sum_{i=1}^N \frac{1}{p_i}\int \phi_\varepsilon^{p_i}\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i}\mathrm{d}x.$$

Proof. Since $0 \leqslant \phi_\varepsilon \leqslant 1$ then $\int \phi_\varepsilon^{p_*} |v_n|^{p^*}\,\mathrm{d}x \leqslant 1$. From Corollary 4 of Lemma 3, it follows

$$S\left(\int \phi_\varepsilon^{p_*} |v_n|^{p^*}\,\mathrm{d}x\right)^{\frac{p_+}{p^*}} \leqslant \sum_{i=1}^N \frac{1}{p_i} \int \left|\frac{\partial}{\partial x_i}(\phi_\varepsilon v_n)\right|^{p_i}. \tag{18}$$

From relation (17), we have

$$\lim_{\varepsilon\to 0} \int \left|\frac{\partial \phi_\varepsilon}{\partial x_i}\right|^{p_i} |v|^{p_i}\,\mathrm{d}x = 0. \tag{19}$$

Since

$$\lim_{n\to+\infty} \int \left|\frac{\partial \phi_\varepsilon}{\partial x_i}\right|^{p_i} |v_n - v|^{p_i}\,\mathrm{d}x = 0, \tag{20}$$

then one has:

$$\overline{\lim_{\varepsilon\to 0}}\, \overline{\lim_{n}} \sum_{i=1}^N \frac{1}{p_i} \int \left|\frac{\partial}{\partial x_i}(\phi_\varepsilon v_n)\right|^{p_i}\mathrm{d}x = \overline{\lim_{\varepsilon\to 0}}\, \overline{\lim_{n}} \sum_{i=1}^N \frac{1}{p_i} \int \left|\frac{\partial v_n}{\partial x_i}\right|^{p_i} \phi_\varepsilon^{p_i}\,\mathrm{d}x. \tag{21}$$

From relations (18) and (21), knowing that $|v_n|^{p^*} \rightharpoonup |v|^{p^*} + \nu$ (see Lemma 4), we obtain

$$S a_j^{\frac{p_+}{p^*}} \leqslant \overline{\lim_{\varepsilon\to 0}}\, \overline{\lim_{n}} \sum_{i=1}^N \frac{1}{p_i} \int \phi_\varepsilon^{p_i} \left|\frac{\partial v_n}{\partial x_i}\right|^{p_i}\mathrm{d}x. \qquad \square$$

Lemma 13. *Assume that*

$$\sum_{i=1}^N \frac{1}{p_i}\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i} \rightharpoonup \tilde\mu \quad \text{in } \mathfrak{M}^+(\mathbb{R}^N).$$

Then

(1) *For all* $j \in J$, $S a_j^{\frac{p_+}{p^*}} \leqslant \lim_{\varepsilon\to 0} \tilde\mu(\textit{support } \phi_\varepsilon)$ (*one has support* $\phi_\varepsilon \subset B(x_j, \max_i \varepsilon^{\frac{1}{q_i}})$).
(2) $\|\tilde\mu\| \geqslant S\|\nu\|^{\frac{p_+}{p^*}} + P(\nabla v)$.
(3) $S = \lim_{n\to+\infty} P(\nabla v_n) \doteq \|\tilde\mu\| + \mu_\infty \geqslant P(\nabla v) + S\|\nu\|^{\frac{p_+}{p^*}} + \mu_\infty$.

Proof. From Lemma 12, since $\phi_\varepsilon^{p_i} \leqslant \phi_\varepsilon$ and

$$\overline{\lim_{n}} \sum_{i=1}^N \frac{1}{p_i} \int \phi_\varepsilon^{p_i} \left|\frac{\partial v_n}{\partial x_i}\right|^{p_i}\mathrm{d}x \leqslant \int \phi_\varepsilon\,\mathrm{d}\tilde\mu,$$

one obtains

$$S a_j^{\frac{p_+}{p^*}} \leqslant \overline{\lim_{\varepsilon\to 0}} \int \phi_\varepsilon\,\mathrm{d}\tilde\mu \leqslant \lim_{\varepsilon\to 0} \tilde\mu\Big(B\Big(x_j;\ \max_{1\leqslant i\leqslant N} \varepsilon^{\frac{1}{q_i}}\Big)\Big). \tag{22}$$

This shows that $\{x_j\}_{j\in J}$ are all atomic points of $\tilde\mu$ and since $\sum_{i=1}^N \frac{1}{p_i}|\frac{\partial v}{\partial x_i}|^{p_i}$ is orthogonal to the atomic part of $\tilde\mu$, one deduces from relation (22) that

$$\tilde\mu \geqslant S\sum_{j\in J} a_j^{\frac{p_+}{p^*}} \delta_{x_j} + \sum_{i=1}^N \frac{1}{p_i}\left|\frac{\partial v}{\partial x_i}\right|^{p_i}. \tag{23}$$

This implies in particular that:

$$\|\tilde\mu\| \geqslant S\sum_{j\in J} a_j^{\frac{p_+}{p^*}} + P(\nabla v). \tag{24}$$

Since $\frac{p_+}{p^*} < 1$ one has

$$\Big(\sum_{j\in J} a_j\Big)^{\frac{p_+}{p^*}} \leqslant \sum_{j\in J} a_j^{\frac{p_+}{p^*}}. \tag{25}$$

As $\nu = \sum_{j\in J} a_j \delta_{x_j}$, it holds

$$\|\nu\| = \sum_{j\in J} a_j, \tag{26}$$

which means, combining relations (24) to (26), that:

$$\|\tilde{\mu}\| \geqslant S\|\nu\|^{\frac{p_+}{p^*}} + P(\nabla v).$$

For the last statement, we argue as before:

$$\begin{aligned} S &= \lim_n P(\nabla v_n) \\ &= \lim_{R\to+\infty} \lim_n \int_{\mathbb{R}^N} (1-\psi_R) \sum_{i=1}^N \frac{1}{p_i}\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i} \mathrm{d}x + \lim_{R\to+\infty} \overline{\lim_n} \int \psi_R \sum_{i=1}^N \frac{1}{p_i}\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i} \mathrm{d}x, \end{aligned}$$

where $\psi_R = 1$ on $|x| > R+1$, $0 \leqslant \psi_R \leqslant 1$, $\psi_R = 0$ if $|x| < R$, $\psi_R \in C(\mathbb{R})$.

By the definition of $\tilde{\mu}$, one has:

$$\lim_{R\to+\infty} \lim_n \int (1-\psi_R) \sum_{i=1}^N \frac{1}{p_i}\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i} \mathrm{d}x = \lim_R \int (1-\psi_R)\,\mathrm{d}\tilde{\mu} = \|\tilde{\mu}\|,$$

and (see Lemma 8):

$$\lim_{R\to+\infty} \overline{\lim_n} \int \psi_R \sum_{i=1}^N \frac{1}{p_i}\left|\frac{\partial v_n}{\partial x_i}\right|^{p_i} \mathrm{d}x = \mu_\infty,$$

thus, by the preceding statements:

$$S = \|\tilde{\mu}\| + \mu_\infty \geqslant P(\nabla v) + S\|\nu\|^{\frac{p_+}{p^*}} + \mu_\infty. \qquad \square$$

Lemma 14. *If* $\|v\|_{p^*} < 1$ *then* $\|\nu\| = 1$, $\nu_\infty = 0$ *and* $v = 0$.

Proof. From Lemma 10, we know that

$$S\nu_\infty^{\frac{p_+}{p^*}} \leqslant \mu_\infty.$$

And by Corollary 4 of Lemma 3, we have

$$S\|v\|_{p^*}^{p_+} \leqslant P(\nabla v).$$

From the last statement of Lemma 13 and the above inequalities we deduce that:

$$S \geqslant S\Big(\big(\|v\|_{p^*}^{p^*}\big)^{\frac{p_+}{p^*}} + \|\nu\|^{\frac{p_+}{p^*}} + \nu_\infty^{\frac{p_+}{p^*}}\Big).$$

Thus we obtain, due to Lemma 11, that

$$\Big(\big(\|v\|_{p^*}^{p^*}\big)^{\frac{p_+}{p^*}} + \|\nu\|^{\frac{p_+}{p^*}} + \nu_\infty^{\frac{p_+}{p^*}}\Big) \leqslant 1 = \big(\|v\|_{p^*}^{p^*} + \|\nu\| + \nu_\infty\big)^{\frac{p_+}{p^*}}.$$

Using the inequality

$$\big(\|v\|_{p^*}^{p^*} + \|\nu\| + \nu_\infty\big)^{\frac{p_+}{p^*}} \leqslant \|v\|_{p^*}^{\frac{p_+}{p^*}} + \|\nu\|^{\frac{p_+}{p^*}} + \nu_\infty^{\frac{p_+}{p^*}},$$

we get

$$\|v\|_{p^*}^{\frac{p_+}{p^*}} + \|\nu\|^{\frac{p_+}{p^*}} + \nu_\infty^{\frac{p_+}{p^*}} = \left(\|v\|_{p^*}^{p^*} + \|\nu\| + \nu_\infty\right)^{\frac{p_+}{p^*}}.$$

It follows that $\|v\|_{p^*}^{p^*}$, $\|\nu\|$ and ν_∞ are equal either to 0 or to 1. But using the fact that $\nu_\infty \leqslant \frac{1}{2}$, since $\int_{B(0,1)} |v_n|^{p^*}\,dx = \frac{1}{2}$, we conclude that $\nu_\infty = 0$, $\|v\|_{p^*} < 1$ (by our assumption) so that $v = 0$ and thus $\|\nu\| = 1$. □

Lemma 15. *If $\|v\|_{p^*} < 1$ then the measure ν is concentrated at a single point $z = x_{i_0}$.*

Proof. Since

$$S = \|\tilde{\mu}\| + \mu_\infty \geqslant S \sum_{j\in J} a_j^{\frac{p_+}{p^*}},$$

(see relation (24)) and $1 = \|\nu\| = \sum_{j\in J} a_j$, we then have:

$$\left(\sum_{j\in J} a_j\right)^{\frac{p_+}{p^*}} \geqslant \sum_{j\in J} a_j^{\frac{p_+}{p^*}} \geqslant \left(\sum_{j\in J} a_j\right)^{\frac{p_+}{p^*}}.$$

Thus the a_j are equal either to zero or to 1 that is, there is only one index i_0 such that $a_{i_0} = 1$ and $a_j = 0$ for $j \neq i_0$: $\nu = a_{i_0}\delta_{x_{i_0}}$. □

End of the proof of the main Lemma. If $\|v\|_{p^*} < 1$ thus ν concentrates at x_{i_0} and $\|\nu\| = 1$. On the other hand, we have

$$\frac{1}{2} = \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |v_n|^{p^*} \geqslant \int_{B(x_{i_0},1)} |v_n|^{p^*}\,dx \to \|\nu\| = 1,$$

which is impossible, we conclude then that $\|v\|_{p^*} = 1$. □

Consequently, the function v is a (nontrivial) extremal function that can be chosen nonnegative (replacing v by $|v|$).

End of the proof of Theorem 1. From usual Lagrange multiplier rule, there is $\lambda_0 > 0$, such that:

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p_i-2} \frac{\partial v}{\partial x_i}\right) = \lambda_0 v^{p^*-1} \quad \text{in } \mathcal{D}^{1,\vec{p}}\left(\mathbb{R}^N\right)'.$$

A similar rescaling argument used above (say $v(\lambda_0^{-\frac{1}{p_1}} x_1, \dots, \lambda_0^{-\frac{1}{p_N}} x_N)$) gives the result. □

The multiplicity of solutions comes directly from Lemma 2, that is:

Lemma 16. *Let $\alpha \in \mathbb{R}$, $\alpha_i = \alpha \frac{p^*}{p_i} - \alpha$, $i = 1, \dots, N$ and $u \in \mathcal{S}$. Then, for all $\lambda \in \mathbb{R}_+^*$ for all $z = (z_1, \dots, z_N) \in \mathbb{R}^N$, the function defined by*

$$u^{\lambda,z}(x) = \lambda^\alpha u\left(\lambda^{\alpha_1} x_1 + z_1, \dots, \lambda^{\alpha_N} x_N + z_N\right),$$

with $x = (x_1, \dots, x_N)$ belongs to $\mathcal{S}$.

Proof. It is the same as for Lemma 2 using a direct computation. □

3. Some properties of the solutions of (1)

We want to show first the:

Proposition 1. *Any nonnegative solution u being in $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ of (1) belongs to $L^q(\mathbb{R}^N)$ for all $p^* \leqslant q < +\infty$.*

Proof. We follow the proof of [4]. Let $a > 0$. Let j be fixed in $\{1, \ldots, N\}$, for $L > 0$ (large) we define $\varphi_{j,L} \doteq u \min[u^{ap_j}, L^{p_j}] \in \mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ and for all i

$$|\partial_i u|^{p_j-2}\partial_i u \partial_i \varphi_{j,L} \geqslant \min\bigl[u^{ap_j}, L^{p_j}\bigr]|\partial_i u|^{p_j} \quad \text{a.e.,} \tag{27}$$

and

$$\bigl|\partial_i\bigl(u \cdot \min\bigl[u^a, L\bigr]\bigr)\bigr|^{p_j} \leqslant (a+1)^{p_j} \min\bigl[u^{ap_j}, L^{p_j}\bigr]|\partial_i u|^{p_j} \quad \text{a.e.} \tag{28}$$

Choosing $\varphi_{j,L}$ as a test function, one has:

$$\int_{\mathbb{R}^N} \min\bigl[u^{ap_j}, L^{p_j}\bigr]|\partial_j u|^{p_j}\,\mathrm{d}x \leqslant \sum_{i=1}^N \int_{\mathbb{R}^N} |\partial_i u|^{p_i-2}\partial_i u \partial_i \varphi_{j,L}\,\mathrm{d}x = \int_{\mathbb{R}^N} u^{p^*} \min\bigl[u^{ap_j}, L^{p_j}\bigr]\,\mathrm{d}x. \tag{29}$$

Introducing $k > 0$, one has:

$$\int_{\mathbb{R}^N} u^{p^*} \min\bigl[u^{ap_j}, L^{p_j}\bigr]\,\mathrm{d}x \leqslant k^{ap_j} \int_{\mathbb{R}^N} u^{p^*}\,\mathrm{d}x + \int_{u \geqslant k} u^{p^*} \min\bigl[u^{ap_j}, L^{p_j}\bigr]\,\mathrm{d}x. \tag{30}$$

Writing that:

$$\int_{u\geqslant k} u^{p^*} \min\bigl[u^{ap_j}, L^{p_j}\bigr]\,\mathrm{d}x = \int_{u\geqslant k} u^{p^*-p_j} u^{p_j}\bigl(\min\bigl[u^a, L\bigr]\bigr)^{p_j}\,\mathrm{d}x. \tag{31}$$

The Hölder inequality applied to the right-hand side of relation (31) shows that:

$$\int_{u\geqslant k} u^{p^*} \min\bigl[u^{ap_j}, L^{p_j}\bigr]\,\mathrm{d}x \leqslant \Bigl(\int_{u\geqslant k} u^{p^*}\,\mathrm{d}x\Bigr)^{1-\frac{p_j}{p^*}} \Bigl(\int_{\mathbb{R}^N} \bigl(u \min\bigl[u^a, L\bigr]\bigr)^{p^*}\Bigr)^{\frac{p_j}{p^*}}. \tag{32}$$

By the Troisi's inequality (see Lemma 1)

$$\Bigl(\int_{\mathbb{R}^N} \bigl(u \min\bigl[u^a, L\bigr]\bigr)^{p^*}\Bigr)^{\frac{1}{p^*}} \leqslant c \sum_{i=1}^N \Bigl(\int_{\mathbb{R}^N} \bigl|\partial_i\bigl(u \min\bigl[u^a, L\bigr]\bigr)\bigr|^{p_i}\Bigr)^{\frac{1}{p_i}}. \tag{33}$$

Setting

$$I_i = \Bigl(\int \bigl|\partial_i\bigl(u \min\bigl[u^a, L\bigr]\bigr)\bigr|^{p_i}\Bigr)^{\frac{1}{p_i}}, \qquad \varepsilon_k = \int_{u\geqslant k} u^{p^*}\,\mathrm{d}x,$$

relations (28) to (33), lead to:

$$\begin{aligned}\int \bigl|\partial_j\bigl(u \cdot \min\bigl[u^a, L\bigr]\bigr)\bigr|^{p_j}\,\mathrm{d}x &\leqslant (a+1)^{p_j} \int \min\bigl[u^{ap_j}, L^{p_j}\bigr]|\partial_j u|^{p_j}\,\mathrm{d}x \\ &\leqslant (a+1)^{p_j} k^{ap_j}\Bigl(\int u^{p^*}\,\mathrm{d}x\Bigr) \\ &\quad + c(a+1)^{p_j}\varepsilon_k^{1-\frac{p_j}{p^*}}\Biggl[\sum_{i=1}^N \Bigl(\int \bigl|\partial_i\bigl(u \min\bigl[u^a, L\bigr]\bigr)\bigr|^{p_i}\Bigr)^{\frac{1}{p_i}}\Biggr]^{p_j}.\end{aligned}$$

Thus, for all j:

$$I_j \leqslant (a+1)k^a\left(\int u^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p_j}} + c(a+1)\varepsilon_k^{\frac{1}{p_j}-\frac{1}{p^*}}\left(\sum_{i=1}^N I_i\right). \tag{34}$$

The relation (34) infers:

$$\sum_{j=1}^N I_j \leqslant (a+1)k^a\left(\sum_{j=1}^N \|u\|_{p^*}^{\frac{p^*}{p_j}}\right) + c(a+1)\left(\sum_{j=1}^N \varepsilon_k^{\frac{1}{p_j}-\frac{1}{p^*}}\right)\left(\sum_{i=1}^N I_i\right). \tag{35}$$

Since $\lim_{k\to+\infty}\sum_{j=1}^N \varepsilon_k^{\frac{1}{p_j}-\frac{1}{p^*}} = 0$, there exists $k_a > 0$ such that for all $k \geqslant k_a$, such that $c(a+1)\sum_{j=1}^N \varepsilon_k^{\frac{1}{p_j}-\frac{1}{p^*}} \leqslant \frac{1}{2}$. Thus relation (35) infers then

$$\sum_{i=1}^N I_j \leqslant 2(a+1)k^a\sum_{j=1}^N \|u\|_{p^*}^{\frac{p^*}{p_j}}, \quad \text{for } k \geqslant k_a.$$

By the Troisi's inequality, one has:

$$\big\|u\cdot\min\big[u^a, L\big]\big\|_{L^{p^*}} \leqslant c\sum_{j=1}^N I_j \leqslant 2c(a+1)k^a\sum_{j=1}^N \|u\|_{p^*}^{\frac{p^*}{p_j}}.$$

Letting $L \to +\infty$, one has:

$$\big\|u^{a+1}\big\|_{L^{p^*}} \leqslant 2c(a+1)k^a\sum_{j=1}^N \|u\|_{p^*}^{\frac{p^*}{p_j}}.$$

Let $q = (a+1)p^*$, then we obtain the result. □

Proposition 2. *Any nonnegative solution u being in $\mathcal{D}^{1,\vec{p}}(\mathbb{R}^N)$ of* (1) *belongs to $L^\infty(\mathbb{R}^N)$. Moreover, there exists a number τ_0 depending only on p_j, N such that*

$$\|u\|_{p^*} \geqslant \tau_0 > 0, \quad \textit{for } u \textit{ nontrivial}.$$

Proof. For $u \geqslant 0$ solution of (1), we set $A_\tau = \{x \in \mathbb{R}^N,\ u(x) \geqslant \tau\}$ and $|A_\tau|$ its Lebesgue measure. Since $p^* > p_+$, one can choose $q > p^*$ so that

$$\varepsilon \doteq -\frac{1}{p^*} + \left(1 - \frac{p^*}{q}\right)\left(1 - \frac{1}{p^*}\right)\frac{1}{p_+ - 1} > 0.$$

Let $\varphi_k = (u-k)_+$, for $k > 0$ fixed. Choosing this function as a test function and using Proposition 1, one has:

$$\sum_{i=1}^N \left\|\frac{\partial\varphi_k}{\partial x_i}\right\|_{p_i}^{p_i} = \int u^{p^*-1}(u-k)_+ \leqslant c_1|A_k|^{(1-\frac{p^*}{q})(1-\frac{1}{p^*})}\|\varphi_k\|_{p^*}, \tag{36}$$

with $c_1 = \|u\|_q^{p^*-1}$.

Since $\|\varphi_k\|_{p^*} \leqslant \|u\|_{p^*}$, thus the Corollary 4 of Lemma 3 and relation (36) imply:

$$\|\varphi_k\|_{p^*}^{p_+} \leqslant c_2\sum_{i=1}^N \left\|\frac{\partial\varphi_k}{\partial x_i}\right\|_{p_i}^{p_i} \leqslant c_3|A_k|^{(1-\frac{p^*}{q})(1-\frac{1}{p^*})}\|\varphi_k\|_{p^*} \tag{37}$$

with

$$c_2 = \frac{1}{S\cdot p_-}\operatorname*{Max}_{1\leqslant j\leqslant N}\left(\|u\|_{p^*}^{p_+-p_j}\right), \qquad c_3 = c_1c_2.$$

Thus,

$$\|\varphi_k\|_{p^*} \leqslant c_4 |A_k|^{\frac{1}{p_+-1}(1-\frac{p^*}{q})(1-\frac{1}{p^*})} \tag{38}$$

with $c_4 = c_3^{\frac{1}{p_+-1}}$. By Cavalieri's principle, Hölder inequality and relation (38), one has, for all $k > 0$:

$$\int_k^{+\infty} |A_\tau| \, \mathrm{d}\tau = \int_{\mathbb{R}^N} (u-k)_+(x) \, \mathrm{d}x \leqslant |A_k|^{1-\frac{1}{p^*}} \|\varphi_k\|_{p^*} \leqslant c_4 |A_k|^{1+\varepsilon}. \tag{39}$$

This last relation is a Gronwall inequality, which shows that $\forall k > 0$

$$\|u\|_\infty \leqslant k + \frac{1+\varepsilon}{\varepsilon} \big\|(u-k)_+\big\|_1^{\frac{\varepsilon}{1+\varepsilon}} c_4^{\frac{1}{1+\varepsilon}}. \tag{40}$$

Setting

$$\gamma = (p^*-1)\frac{\varepsilon}{1+\varepsilon}, \qquad b_0 = \frac{1+\varepsilon}{\varepsilon} \|u\|_{p^*}^{\frac{\varepsilon p^*}{1+\varepsilon}} c_4^{\frac{1}{1+\varepsilon}},$$

and noticing that

$$\big\|(u-k)_+\big\|_1 \leqslant \frac{\|u\|_{p^*}^{p^*}}{k^{p^*-1}},$$

thus relation (40) becomes:

$$\|u\|_\infty \leqslant \operatorname*{Inf}_{k>0}\left[k + \frac{b_0}{k^\gamma}\right] = (\gamma+1)\gamma^{-\frac{\gamma}{\gamma+1}} b_0^{\frac{1}{1+\gamma}}. \tag{41}$$

Separating the contribution of $\|u\|_q$ and $\|u\|_{p^*}$, we have a continuous map $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ and constants $c_5 > 0$ and β depending only on p_+, p_* so that

$$\|u\|_\infty \leqslant c_5 \|u\|_q^\beta \Lambda\big(\|u\|_{p^*}\big), \tag{42}$$

with

$$\beta = \frac{p^*-1}{(p_+-1)(1+\varepsilon)(1+\gamma)}, \qquad \Lambda(\sigma) = \Big[\sigma^{\varepsilon p^*} \operatorname*{Max}_{1\leqslant j\leqslant N} (\sigma^{p_+-p_j})\Big]^{\frac{1}{(1+\varepsilon)(1+\gamma)}}.$$

Thus, from relation (42), we deduce

$$\|u\|_\infty^{1-\beta(1-\frac{p^*}{q})} \leqslant c_5 \|u\|_{p^*}^{\beta\frac{p^*}{q}} \Lambda\big(\|u\|_{p^*}\big) \ for\ u \not\equiv 0. \tag{43}$$

But the number $\kappa \doteq 1 - \beta(1-\frac{p^*}{q}) = 0$, so relation (43) implies that there is a number $\tau_0 > 0$ depending only p_j, p^* such that $\|u\|_{p^*} \geqslant \tau_0 > 0$. □

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