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A nonintersection property for extremals of variational problems with vector-valued functions

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Abstract

In this work we study the structure of extremals of variational problems with vector-valued functions on $[0, \infty)$. We show that if an extremal is not periodic, then the corresponding curve in the phase space does not intersect itself.

MSC: 49J99

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1. Introduction

In this paper we analyze the structure of extremals of infinite horizon variational problems associated with the functional

$$\int_{T_1}^{T_2} f\big(z(t), z'(t)\big)\,\mathrm{d}t,$$

where $T_1 \geqslant 0$, $T_2 > T_1$, $z:[T_1, T_2] \to \mathbb{R}^n$ is an absolutely continuous function and $f:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1$ belongs to a space of integrands described below.

Denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$. Let a be a positive constant and let $\psi:[0,\infty) \to [0,\infty)$ be an increasing function such that $\psi(t) \to +\infty$ as $t \to \infty$. Denote by $\mathcal{A}$ the set of all continuous functions $f:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1$ which satisfy the following assumptions:

A(i) for each $x \in \mathbb{R}^n$ the function $f(x,\cdot):\mathbb{R}^n \to \mathbb{R}^1$ is convex;
A(ii) $f(x,u) \geqslant \max\{\psi(|x|), \psi(|u|)|u|\} - a$ for each $(x,u) \in \mathbb{R}^n \times \mathbb{R}^n$;
A(iii) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$\big|f(x_1,u_1) - f(x_2,u_2)\big| \leqslant \epsilon \max\big\{f(x_1,u_1), f(x_2,u_2)\big\}$$

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for each $u_1, u_2, x_1, x_2 \in \mathbb{R}^n$ which satisfy

$$|x_i| \leqslant M, \quad i = 1, 2, \quad |u_i| \geqslant \Gamma, \quad i = 1, 2, \quad |x_1 - x_2|, |u_1 - u_2| \leqslant \delta.$$

It is easy to show that an integrand $f = f(x, u) \in C^1(\mathbb{R}^{2n})$ belongs to $\mathcal{A}$ if f satisfies assumptions A(i), A(ii) and if there exists an increasing function $\psi_0 : [0, \infty) \to [0, \infty)$ such that

$$\max\{|\partial f/\partial x(x, u)|, |\partial f/\partial u(x, u)|\} \leqslant \psi_0(|x|)\bigl(1 + \psi(|u|)|u|\bigr)$$

for each $x, u \in \mathbb{R}^n$.

We consider functionals of the form

$$I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f\bigl(x(t), x'(t)\bigr)\,\mathrm{d}t, \tag{1.1}$$

where $f \in \mathcal{A}$, $-\infty < T_1 < T_2 < \infty$ and $x : [T_1, T_2] \to \mathbb{R}^n$ is an absolutely continuous (a.c.) function.

For $f \in \mathcal{A}$, $y, z \in \mathbb{R}^n$ and real numbers T_1, T_2 satisfying $T_1 < T_2$ we set

$$\begin{aligned} &U^f(T_1, T_2, y, z) \\ &\quad = \inf\bigl\{I^f(T_1, T_2, x):\ x : [T_1, T_2] \to \mathbb{R}^n \text{ is an a.c. function satisfying } x(T_1) = y,\ x(T_2) = z\bigr\}. \end{aligned} \tag{1.2}$$

It is easy to see that $-\infty < U^f(T_1, T_2, y, z) < +\infty$ for each $f \in \mathcal{A}$, each $y, z \in \mathbb{R}^n$ and all numbers T_1, T_2 satisfying $T_1 < T_2$.

Let $f \in \mathcal{A}$. For any a.c. function $x : [0, \infty) \to \mathbb{R}^n$ we set

$$J(x) = \liminf_{T \to \infty} T^{-1} I^f(0, T, x). \tag{1.3}$$

Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\bigl\{J(x):\ x : [0, \infty) \to \mathbb{R}^n \text{ is an a.c. function}\bigr\}. \tag{1.4}$$

Clearly $-\infty < \mu(f) < \infty$.

Here we follow [4,7] in defining good functions for variational problems.

An a.c. function $x : [0, \infty) \to \mathbb{R}^n$ is called an (f)-good function if the function

$$T \to I^f(0, T, x) - \mu(f)T, \quad T \in (0, \infty),$$

is bounded.

In [16, Theorem 1.1, Proposition 1.1] we showed that for each $f \in \mathcal{A}$ and each $z \in \mathbb{R}^n$ there exists an (f)-good function $v : [0, \infty) \to \mathbb{R}^n$ satisfying $v(0) = z$.

Propositions 1.1 and 3.1 in [16] imply the following result.

Proposition 1.1. *For any a.c. function $x : [0, \infty) \to \mathbb{R}^n$ either $I^f(0, T, x) - T\mu(f) \to \infty$ as $T \to \infty$ or*

$$\sup\bigl\{|I^f(0, T, x) - T\mu(f)|:\ T \in (0, \infty)\bigr\} < \infty.$$

Moreover any (f)-good function $x : [0, \infty) \to \mathbb{R}^n$ is bounded.

We follow [9] in defining c-optimal functions.

An a.c. function $v : [0, \infty) \to \mathbb{R}^n$ is called c-optimal with respect to f (or just c-optimal if the function f is understood) if $\sup\{|v(t)|:\ t \in [0, \infty)\} < \infty$ and if for each $T > 0$ the equality

$$I^f(0, T, v) = U^f\bigl(0, T, v(0), v(T)\bigr)$$

holds.

Note that any c-optimal with respect to f function is (f)-good (see Proposition 2.4).

In [17, Theorem 1.1] it was proved the following result.

Proposition 1.2. *For any $z \in \mathbb{R}^n$ there exists a c-optimal with respect to f function $v : [0, \infty) \to \mathbb{R}^n$ such that $v(0) = z$.*

The notion of c-optimality is a slight modification of the notion of minimality introduced in [5] and discussed in [2,12–14]. The difference is that in our paper c-optimal solutions are bounded and defined on the interval $[0,\infty)$ while in [2,12–14] minimal solutions are defined on the whole space $\mathbb{R}^n$ and the boundedness is not assumed. Note that an analogous notion of minimality was used in the study of geodesics (see, for example, [1,6,11]).

Denote by $\mathcal{M}$ the set of all functions $f\in C^2(\mathbb{R}^{2n})$ which satisfy the following assumptions:

$$\partial f/\partial u_i(x,u)\in C^2\big(\mathbb{R}^{2n}\big)\quad \text{for all } (x,u)\in\mathbb{R}^n\times\mathbb{R}^n \text{ and } i=1,\dots,n;$$

the matrix $(\partial^2 f/\partial u_i\partial u_j)(x,u)$, $i,j=1,\dots,n$, is positive definite for all $(x,u)\in\mathbb{R}^{2n}$;

$$f(x,u)\geqslant \max\big\{\psi\big(|x|\big),\psi\big(|u|\big)|u|\big\}-a\quad \text{for all } (x,u)\in\mathbb{R}^n\times\mathbb{R}^n;$$

there exist a number $c_0>1$ and monotone increasing functions $\phi_i:[0,\infty)\to[0,\infty)$, $i=0,1,2$, such that

$$\phi_0(t)/t\to\infty\quad \text{as } t\to\infty,$$
$$f(x,u)\geqslant \phi_0\big(c_0|u|\big)-\phi_1\big(|x|\big),\quad x,u\in\mathbb{R}^n,$$
$$\max\big\{\big|\partial f/\partial x_i(x,u)\big|,\big|\partial f/\partial u_i(x,u)\big|\big\}\leqslant \phi_2\big(|x|\big)\big(1+\phi_0\big(|u|\big)\big),\quad x,u\in\mathbb{R}^n,\ i=1,\dots,n.$$

It is easy to see that $\mathcal{M}\subset\mathcal{A}$.

The following two theorems are the main results of the paper.

Theorem 1.1. *Let $f\in\mathcal{M}$ and let $v:[0,\infty)\to\mathbb{R}^n$ be a c-optimal function with respect to f. If there exist numbers $T_2>T_1\geqslant 0$ such that $v(T_1)=v(T_2)$, then $v(t+T_2-T_1)=v(t)$ for all $t\geqslant 0$.*

Theorem 1.2. *Let $f\in\mathcal{M}$ and $v_1,v_2:[0,\infty)\to\mathbb{R}^n$ be c-optimal functions with respect to f such that $v_1(0)=v_2(0)$. If there exist $t_1,t_2\in[0,\infty)$ such that $(t_1,t_2)\neq(0,0)$ and $v_1(t_1)=v_2(t_2)$, then $v_1(t)=v_2(t)$ for all $t\in[0,\infty)$.*

It should be mentioned that one-dimensional analogs of Theorems 1.1 and 1.2 were obtained in [10, Theorem 1.1] for c-optimal extremals of variational problems with scalar-valued functions arising in continuum mechanics.

The infinite horizon variational problems considered in [10] are associated with the functional

$$\int_{T_1}^{T_2} f\big(z(t),z'(t),z''(t)\big)\,\mathrm{d}t,$$

where $T_1\geqslant 0$, $T_2>T_1$, $z\in W^{2,1}([T_1,T_2])$ and $f:\mathbb{R}^3\to\mathbb{R}^1$ belongs to a certain space of integrands. The main result of [10, Theorem 1.1] establishes that if a c-optimal function v is not periodic, then the corresponding curve $\{(v(t),v'(t)):\ t\in[0,\infty)\}$ in the phase plane does not intersect itself. Note that in [10] the proof of this result was strongly based on the fact that the curve $\{(v(t),v'(t)):\ t\in[0,\infty)\}$ is a subset of $\mathbb{R}^2$ and on the existence of c-optimal periodic functions established in [8,15]. In our case for the variational problems with vector-valued functions the existence of c-optimal periodic functions is not guaranteed and the situation becomes more difficult and less understood.

It is known that if a function $f\in\mathcal{M}$ is strictly convex and $\bar{y}\in\mathbb{R}^n$ is a unique solution of the minimization problem $f(z,0)\to\min$, $z\in\mathbb{R}^n$, then there exists a c-optimal periodic function which is equal to $\bar{y}$ for all $t\geqslant 0$ [20]. For a general $f\in\mathcal{M}$ the existence of c-optimal periodic functions is a difficult problem which is still open.

Note that in [19] we considered an integrand

$$f(x,u)=|x|^2|x-e|^2+|u|^2,\quad x,u\in\mathbb{R}^n,$$

where $e=(1,1,\dots,1)\in\mathbb{R}^n$ and constructed a c-optimal function which is not periodic.

Now we consider two examples of integrands belonging to the set $\mathcal{M}$. In the first example we construct $f\in\mathcal{M}$ such that there is a periodic c-optimal with respect to f function which is not constant.

Example 1. Let $n=2$. Define a function $f:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^1$ by

$$f(x_1,x_2,u_1,u_2)=\big(x_1^2+x_2^2-1\big)^2+(u_1+x_2)^2+(u_2-x_1)^2,\quad (x_1,x_2),(u_1,u_2)\in\mathbb{R}^2.$$

It is not difficult to see that $f \in \mathcal{M}$ under an appropriate choice of ψ and a. Clearly $f(x,u) \geqslant 0$ for all $x, u \in \mathbb{R}^2$. Set $w(t) = (\cos(t), \sin(t))$ for all $t \in [0,\infty)$. It is clear that $f(w(t), w'(t)) = 0$ for all $t \in [0,\infty)$. This implies that $\mu(f) = 0$. Now it is easy to see that w is a periodic c-optimal with respect to f function.

In our second example we construct a function $f \in \mathcal{M}$ such that any c-optimal with respect to f function is not periodic.

Example 2. Let $n = 4$. Define a function $f : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^1$ by

$$f(x_1,x_2,x_3,x_4,u_1,u_2,u_3,u_4) = \left(x_1^2 + x_2^2 - 1\right)^2 + (u_1 + x_2)^2 + (u_2 - x_1)^2 + \left(x_3^2 + x_4^2 - 1\right)^2 \\ + (u_3 + \pi x_4)^2 + (u_4 - \pi x_3)^2, \quad (x_1,x_2,x_3,x_4), (u_1,u_2,u_3,u_4) \in \mathbb{R}^4.$$

It is not difficult to see that $f \in \mathcal{M}$ under an appropriate choice of ψ and a. Clearly $f(x,u) \geqslant 0$ for all $x, u \in \mathbb{R}^4$. Set

$$w(t) = \left(\cos(t), \sin(t), \cos(\pi t), \sin(\pi t)\right)$$

for all $t \in \mathbb{R}^1$. It is clear that $f(w(t), w'(t)) = 0$ for all $t \in \mathbb{R}^1$. This implies that

$$\mu(f) = 0. \tag{1.5}$$

Let $v : [0,\infty) \to \mathbb{R}^4$ be a c-optimal with respect to f function. We show that v is not periodic. Let us assume the converse. Then there exists a real number $T > 0$ such that

$$v(t+T) = v(t) \quad \text{for all } t \in [0,\infty). \tag{1.6}$$

We have already mentioned that any c-optimal with respect to f function is (f)-good (see Proposition 2.4). Therefore v is an (f)-good function. Since the function f is nonnegative relations (1.5) and (1.6) imply that

$$\int_0^T f\left(v(t), v'(t)\right) \mathrm{d}t = 0.$$

Since $f \in \mathcal{M}$ we have $v \in C^2([0,\infty))$ (see Proposition 5.1), $f(v(t), v'(t)) = 0$ for all $t \in [0,T]$ and, in particular,

$$f\left(v(0), v'(0)\right) = 0. \tag{1.7}$$

It follows from (1.7) and the definition of f that there exist $s_1 \in [0, 2\pi)$ and $s_2 \in [0,2)$ such that

$$v(0) = \left(\cos(s_1), \sin(s_1), \cos(\pi s_2), \sin(\pi s_2)\right). \tag{1.8}$$

For all $t \geqslant 0$ define

$$u(t) = \left(\cos(s_1 + t), \sin(s_1 + t), \cos\left(\pi(s_2 + t)\right), \sin\left(\pi(s_2 + t)\right)\right).$$

By the definition of f and u the equality $f(u(t), u'(t)) = 0$ holds for all $t \geqslant 0$. Since the function f is nonnegative this implies that u is a c-optimal with respect to f function. In view of (1.6), (1.8) and the definition of u

$$v(T) = v(0) = u(0).$$

Together with Theorem 1.2 this equality implies that $v(t) = u(t)$ for all $t \in [0,\infty)$. It follows from this equality, (1.6) and the definition of u that for all $t \geqslant 0$

$$\left(\cos(s_1 + t + T), \sin(s_1 + t + T), \cos\left(\pi(s_2 + t + T)\right), \sin\left(\pi(s_2 + t + T)\right)\right) \\ = \left(\cos(s_1 + t), \sin(s_1 + t), \cos\left(\pi(s_2 + t)\right), \sin\left(\pi(s_2 + t)\right)\right).$$

Since the equality above holds for all $t \geqslant 0$ we obtain that $(2\pi)^{-1}T$ and $T/2$ are integers. The contradiction we have reached proves that v is not periodic.

Clearly, Theorem 1.1 is a particular case of Theorem 1.2. We state them as two separate results because in the paper we will only prove Theorem 1.1 and provide explanations concerning the proof of Theorem 1.2.

The paper is organized as follows. In Section 2 we explain the main ideas of the proofs of Theorems 1.1 and 1.2 and compare them with the proof of Theorem 1.1 of [10]. Section 2 also contains several auxiliary results. An important notion of a minimal limiting set is introduced and studied in Section 3. A basic lemma for the proofs of Theorems 1.1 and 1.2 is proved in Section 4. Theorem 1.1 is proved in Section 5.

2. Preliminaries

By a simple modification of the proof of Proposition 4.4 in [8] (see also [16]) we obtain the following proposition.

Proposition 2.1. *Let $f \in \mathcal{A}$. Then for each $T > 0$ and each $x, y \in \mathbb{R}^n$*

$$U^f(0, T, x, y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta_T^f(x, y), \tag{2.1}$$

where $\pi^f : \mathbb{R}^n \to \mathbb{R}^1$ is a continuous function defined by

$$\pi^f(x) = \inf\Big\{\liminf_{T\to\infty}\big[I^f(0, T, v) - \mu(f)T\big]\colon\ v : [0, \infty) \to \mathbb{R}^n \text{ is an a.c. function satisfying } v(0) = x\Big\},\quad x \in \mathbb{R}^n, \tag{2.2}$$

and $(T, x, y) \to \theta_T^f(x, y) \in \mathbb{R}^1$, $(T, x, y) \in (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$ is a continuous nonnegative function which satisfies the following condition:

for every $T > 0$ and every $x \in \mathbb{R}^n$ there is $y \in \mathbb{R}^n$ for which $\theta_T^f(x, y) = 0$.

We denote $d(x, B) = \inf\{|x - y|\colon\ y \in B\}$ for $x \in \mathbb{R}^n$, $B \subset \mathbb{R}^n$ and denote by $\mathrm{dist}(A, B)$ the Hausdorff metric for two sets $A, B \subset \mathbb{R}^n$. For every bounded a.c. function $x : [0, \infty) \to \mathbb{R}^n$ define

$$\Omega(x) = \big\{y \in \mathbb{R}^n\colon\ \text{there exists a sequence } \{t_i\}_{i=0}^{\infty} \subset (0, \infty) \text{ for which } t_i \to \infty,\ x(t_i) \to y \text{ as } i \to \infty\big\} \tag{2.3}$$

which is called a limiting set of x.

Let $f \in \mathcal{A}$. For each $\tau_1 \in \mathbb{R}^1$, $\tau_2 > \tau_1$, each $r_1, r_2 \in [\tau_1, \tau_2]$ satisfying $r_1 < r_2$ and each a.c. function $u : [\tau_1, \tau_2] \to \mathbb{R}^n$ set

$$\Gamma^f(r_1, r_2, u) = I^f(r_1, r_2, u) - \pi^f\big(u(r_1)\big) + \pi^f\big(u(r_2)\big) - (r_2 - r_1)\mu(f). \tag{2.4}$$

In view of Proposition 2.1

$$\Gamma^f(r_1, r_2, u) \geqslant 0 \quad \text{for each } \tau_1 \in \mathbb{R}^1,\ \tau_2 > \tau_1,\ \text{each } r_1, r_2 \in [\tau_1, \tau_2] \text{ satisfying } r_1 < r_2 \text{ and each a.c. function } u : [\tau_1, \tau_2] \to \mathbb{R}^n. \tag{2.5}$$

Let $T_1 \in \mathbb{R}^1$ and $T_2 > T_1$. It is clear that for each pair of a.c. functions $v_1, v_2 : [T_1, T_2] \to \mathbb{R}^n$ satisfying $v_1(T_i) = v_2(T_i)$, $i = 1, 2$, the following equality holds:

$$I^f(T_1, T_2, v_1) - I^f(T_1, T_2, v_2) = \Gamma^f(T_1, T_2, v_1) - \Gamma^f(T_1, T_2, v_2).$$

Hence for each $y, z \in \mathbb{R}^n$ the following two variational problems are equivalent:

$$I^f(T_1, T_2, v) \to \min,$$
$$v : [T_1, T_2] \to \mathbb{R}^n \text{ is an a.c. function such that } v(T_1) = y,\ v(T_2) = z$$

and

$$\Gamma^f(T_1, T_2, v) \to \min,$$
$$v : [T_1, T_2] \to \mathbb{R}^n \text{ is an a.c. function such that } v(T_1) = y,\ v(T_2) = z.$$

In the sequel we prefer to minimize the functional $\Gamma^f(\cdot, \cdot, \cdot)$ because it is always nonnegative by Proposition 2.1 and has other useful properties. For example, in view of Proposition 1.1 for any (f)-good function v we have $\sup\{\Gamma^f(0, T, v)\colon\ T \in [0, \infty)\} < \infty$. In [16, Theorem 8.3] we proved that for any $x \in \mathbb{R}^n$ there exists an (f)-good function v such that $v(0) = x$ and $\Gamma^f(0, T, v) = 0$ for all $T > 0$. In this paper we need to deal with the following question:

Is it possible for given $y, z \in \mathbb{R}^n$ to find $q > 0$ and an a.c. function $v : [0, q] \to \mathbb{R}^n$ such that $v(0) = y$, $v(q) = z$ and that $\Gamma^f(0, q, v)$ is small?

In general the existence of such q and v is not guaranteed but they do exist if y, z belong to certain subsets of $\mathbb{R}^n$. These subsets play an important role in our study.

Note that analogs of the notion of a limiting set and the functional $\Gamma^f(\cdot,\cdot,\cdot)$ were used in [10] for the class of variational problems studied there. As we have mentioned before the existence of a c-optimal periodic extremal plays an important role in the proof of Theorem 1.1 of [10]. The following two properties are the most important ingredients in the proof of Theorem 1.1 of [10]:

(a1) For any c-optimal extremal v there exists a c-optimal periodic extremal u such that $\{(u(t),u'(t)):\ t\in[0,\infty)\}$ is contained in the limiting set of the curve $\{(v(t),v'(t)):\ t\in[0,\infty)\}$;

(a2) Let u be a c-optimal periodic extremal and $\epsilon>0$. Then there exist numbers $q,\delta>0$ such that for each $h_1,h_2\in\mathbb{R}^2$ satisfying $d(h_i,u([0,\infty)))\leqslant\delta$, $i=1,2$, and each $T\geqslant q$ there exists $v\in W^{2,1}([0,T])$ which satisfies

$$\bigl(v(0),v'(0)\bigr)=h_1,\quad \bigl(v(T),v'(T)\bigr)=h_2,\quad \Gamma^f(0,T,v)\leqslant\epsilon.$$

For the variational problems with vector-valued functions considered in this paper the existence of c-optimal periodic functions is not guaranteed and the situation becomes more difficult. In order to overcome this difficulty we consider the collection of all limiting sets ordering by inclusion and show that the following properties hold:

(b1) Let $v:[0,\infty)\to\mathbb{R}^n$ be an (f)-good function. Then there is a minimal limiting set which is contained in the limiting set $\Omega(v)$ of v (see Lemma 3.2);

(b2) Let D be a minimal limiting set and let $\epsilon>0$. Then there exist numbers $q,\delta>0$ such that for each $h_1,h_2\in\mathbb{R}^n$ satisfying $d(h_i,D)\leqslant\delta$, $i=1,2$, and each $T\geqslant q$ there exists an a.c. function $v:[0,T]\to\mathbb{R}^n$ which satisfies

$$v(0)=h_1,\quad v(T)=h_2,\quad \Gamma^f(0,T,v)\leqslant\epsilon$$

(see Lemma 4.2).

The property (b1) is established in Section 3 while the property (b2) is established in Section 4. Then arguing as in the proof of Theorem 1.1 of [10] and replacing (a1) and (a2) by (b1) and (b2) we can complete the proofs of Theorems 1.1 and 1.2. Note that the proof of the property (b2) is more complicated than the proof of its analog (a2) because in (a2) we have periodicity of u.

In the sequel we use the following auxiliary results.

Proposition 2.2. *[16, Proposition 5.1] Let $g\in\mathcal{A}$, $y:[0,\infty)\to\mathbb{R}^n$ be a (g)-good function and let $\epsilon>0$. Then there exists $T_0>0$ such that for each $T\geqslant T_0$, $\overline{T}>T$*

$$I^g\bigl(T,\overline{T},y\bigr)\leqslant U^g\bigl(T,\overline{T},y(T),y\bigl(\overline{T}\bigr)\bigr)+\epsilon.$$

Proposition 2.3. *[16, Theorem 6.1] Assume that $f\in\mathcal{A}$. Then the mapping $(T_1,T_2,x,y)\to U^f(T_1,T_2,x,y)$ is continuous for $T_1\in(0,\infty)$, $T_2\in(T_1,\infty)$, $x,y\in\mathbb{R}^n$.*

Proposition 2.4. *[16, Proposition 5.2] Let $f\in\mathcal{A}$, $S_0,S_1>0$ and let $x:[0,\infty)\to\mathbb{R}^n$ be an a.c. function such that $|x(t)|\leqslant S_0$ for all $t\in[0,\infty)$ and $I^f(0,i,x)\leqslant U^f(0,i,x(0),x(i))+S_1$, $i=1,2,\ldots$. Then x is an (f)-good function.*

Proposition 2.5. *[3] Assume that $f\in\mathcal{A}$, $M_1>0$, $0\leqslant T_1<T_2$, $x_i:[T_1,T_2]\to\mathbb{R}^n$, $i=1,2,\ldots$, is a sequence of a.c. functions such that $I^f(T_1,T_2,x_i)\leqslant M_1$, $i=1,2,\ldots$. Then there exist a subsequence $\{x_{i_k}\}_{k=1}^\infty$ and an a.c. function $x:[T_1,T_2]\to\mathbb{R}^n$ such that $I^f(T_1,T_2,x)\leqslant M_1$, $x_{i_k}(t)\to x(t)$ as $k\to\infty$ uniformly on $[T_1,T_2]$ and $x'_{i_k}\to x'$ as $k\to\infty$ weakly on $L^1(\mathbb{R}^n;(T_1,T_2))$.*

Lemma 2.1. *Let $M,\epsilon>0$.Then there exist $\Gamma,\delta>0$ such that*

$$\bigl|f(x_1,u_1)-f(x_2,u_2)\bigr|\leqslant\epsilon\min\bigl\{f(x_1,u_1),f(x_2,u_2)\bigr\}$$

for each $u_1,u_2,x_1,x_2\in\mathbb{R}^n$ which satisfy

$$|x_i|\leqslant M,\quad |u_i|\geqslant\Gamma,\quad i=1,2,\quad |x_1-x_2|,|u_1-u_2|\leqslant\delta. \tag{2.6}$$

Proof. Choose $\epsilon_0 \in (0, 1)$ such that

$$(1-\epsilon_0)^{-1}\epsilon_0 < \epsilon. \tag{2.7}$$

By A(iii) there are $\Gamma, \delta > 0$ such that

$$\bigl|f(x_1,u_1) - f(x_2,u_2)\bigr| \leqslant \epsilon_0 \max\bigl\{f(x_1,u_1), f(x_2,u_2)\bigr\} \tag{2.8}$$

for each $u_1, u_2, x_1, x_2 \in \mathbb{R}^n$ satisfying (2.6).

Assume that $u_1, u_2, x_1, x_2 \in \mathbb{R}^n$ satisfy (2.6). Then (2.8) holds. We may assume without loss of generality that

$$f(x_2,u_2) \geqslant f(x_1,u_1). \tag{2.9}$$

Relations (2.8) and (2.9) imply that

$$f(x_2,u_2) - f(x_1,u_1) \leqslant \epsilon_0 f(x_2,u_2) \tag{2.10}$$

and

$$(1-\epsilon_0) f(x_2,u_2) \leqslant f(x_1,u_1). \tag{2.11}$$

By (2.9), (2.10), (2.7) and (2.11)

$$\begin{aligned}\bigl|f(x_2,u_2) - f(x_1,u_1)\bigr| &\leqslant \epsilon_0 f(x_2,u_2) \leqslant \epsilon_0(1-\epsilon_0)^{-1} f(x_1,u_1)\\ &\leqslant \epsilon f(x_1,u_1) = \epsilon \min\bigl\{f(x_1,u_1), f(x_2,u_2)\bigr\}.\end{aligned}$$

Lemma 2.1 is proved. □

Lemma 2.1 implies the following auxiliary result.

Lemma 2.2. *Let $M, \epsilon > 0$. Then there exist $\Gamma, \delta > 0$ such that*

$$\bigl|f(x_1,u_1) - f(x_2,u_2)\bigr| \leqslant \epsilon \min\bigl\{f(x_1,u_1), f(x_2,u_2)\bigr\}$$

for each $u_1, u_2, x_1, x_2 \in \mathbb{R}^n$ which satisfies

$$|x_i| \leqslant M, \quad i = 1, 2, \quad |u_1| \geqslant \Gamma, \quad |x_1 - x_2|, |u_1 - u_2| \leqslant \delta.$$

Lemma 2.3. *Let $M_0, M_1 > 0$. Then there exists a positive number L such that for each $x_1, x_2, u_1, u_2 \in \mathbb{R}^n$ satisfying*

$$|x_1| \leqslant M_0, \qquad |x_1 - x_2|, |u_1 - u_2| \leqslant M_1 \tag{2.12}$$

the following inequality holds:

$$\bigl|f(x_1,u_1) - f(x_2,u_2)\bigr| \leqslant L^2 + L \min\bigl\{f(x_1,u_1), f(x_2,u_2)\bigr\}. \tag{2.13}$$

Proof. By Lemma 2.2 there are δ_0, Γ_0 such that

$$\bigl|f(x_1,u_1) - f(x_2,u_2)\bigr| \leqslant 2^{-1} \min\bigl\{f(x_1,u_1), f(x_2,u_2)\bigr\} \tag{2.14}$$

for each $x_1, x_2, u_1, u_2 \in \mathbb{R}^n$ satisfying

$$|x_1|, |x_2| \leqslant M_0 + M_1 + 2, \quad |u_1| \geqslant \Gamma_0, \quad |x_1 - x_2|, |u_1 - u_2| \leqslant \delta_0. \tag{2.15}$$

Assume that $x_1, x_2, u_1, u_2 \in \mathbb{R}^n$ satisfy (2.15). Then (2.14) holds and

$$f(x_1,u_1), f(x_2,u_2) \geqslant 0. \tag{2.16}$$

Inequality (2.14) implies that

$$f(x_1,u_1) \leqslant (3/2) f(x_2,u_2), \qquad f(x_2,u_2) \leqslant (3/2) f(x_1,u_1). \tag{2.17}$$

Choose a natural number q such that

$$(M_1 + 1)q^{-1} < \delta_0. \tag{2.18}$$

Assume that $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ satisfy

$$|x_1 - x_2|, |y_1 - y_2| \leqslant M_1, \quad |x_1|, |x_2| \leqslant M_0 + M_1 + 2, \quad |y_1| \geqslant \Gamma_0 + M_1 + 1. \tag{2.19}$$

For $i = 0, \ldots, q$ define

$$\left(x^{(i)}, y^{(i)}\right) = (x_1, y_1) + iq^{-1}(x_2 - x_1, y_2 - y_1). \tag{2.20}$$

Clearly,

$$\left(x^{(0)}, y^{(0)}\right) = (x_1, y_1), \qquad \left(x^{(q)}, y^{(q)}\right) = (x_2, y_2). \tag{2.21}$$

Let $i \in \{0, \ldots, q-1\}$. By (2.20), (2.19) and (2.18)

$$\left|x^{(i)} - x^{(i+1)}\right| \leqslant q^{-1}|x_2 - x_1| \leqslant M_1 q^{-1} < \delta_0,$$
$$\left|y^{(i)} - y^{(i+1)}\right| \leqslant q^{-1}|y_2 - y_1| \leqslant M_1 q^{-1} < \delta_0,$$
$$\left|x^{(i)}\right|, \left|x^{(i+1)}\right| \leqslant M_0 + M_1 + 2, \qquad \left|y^{(i)}\right|, \left|y^{(i+1)}\right| \geqslant \Gamma_0.$$

It follows from these inequalities and the choice of δ_0, Γ_0 (see (2.14), (2.15)) that

$$\left|f\left(x^{(i)}, y^{(i)}\right) - f\left(x^{(i+1)}, y^{(i+1)}\right)\right| \leqslant 2^{-1} \min\left\{f\left(x^{(i)}, y^{(i)}\right), f\left(x^{(i+1)}, y^{(i+1)}\right)\right\}$$

and

$$0 \leqslant f\left(x^{(i)}, y^{(i)}\right) \leqslant 2f\left(x^{(i+1)}, y^{(i+1)}\right) \leqslant 4f\left(x^{(i)}, y^{(i)}\right).$$

These inequalities imply that

$$\max\left\{f\left(x^{(i)}, y^{(i)}\right):\ i = 0, \ldots, q\right\} \leqslant 2^q \min\left\{f(x_1, y_1), f(x_2, y_2)\right\}.$$

By this inequality and (2.21)

$$\begin{aligned}\left|f(x_2, y_2) - f(x_1, y_1)\right| &= \max\left\{f\left(x^{(0)}, y^{(0)}\right), f\left(x^{(q)}, y^{(q)}\right)\right\} - \min\left\{f\left(x^{(0)}, y^{(0)}\right), f\left(x^{(q)}, y^{(q)}\right)\right\} \\ &\leqslant 2^q \min\left\{f(x_1, y_1), f(x_2, y_2)\right\} - \min\left\{f(x_1, y_1), f(x_2, y_2)\right\} \\ &\leqslant 2^q \min\left\{f(x_1, y_1), f(x_2, y_2)\right\}.\end{aligned}$$

Thus we have shown that for each $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ satisfying (2.19) the following relation holds:

$$\left|f(x_2, y_2) - f(x_1, y_1)\right| \leqslant 2^q \min\left\{f(x_1, y_1), f(x_2, y_2)\right\}. \tag{2.22}$$

Since f is a continuous function there is a number $q_1 > 0$ such that for each $x_1, x_2, u_1, u_2 \in \mathbb{R}^n$ satisfying

$$|x_i|, |u_i| \leqslant 2M_0 + 2M_1 + 2\Gamma_0 + 2, \quad i = 1, 2,$$

the following inequality holds:

$$\left|f(x_1, u_1) - f(x_2, u_2)\right| \leqslant q_1. \tag{2.23}$$

Choose a number

$$L > 2^q + (a+1)q_1 + 1 + a. \tag{2.24}$$

Assume that $x_1, x_2, u_1, u_2 \in \mathbb{R}^n$ satisfy (2.12). There are two cases:

$$|u_1| \geqslant \Gamma_0 + M_1 + 1 \tag{2.25}$$

and

$$|u_1| < \Gamma_0 + M_1 + 1. \tag{2.26}$$

Assume that (2.25) holds. Then it follows from (2.24), (2.25), (2.12), (2.19) and (2.22) that

$$\left|f(x_2, u_2) - f(x_1, u_1)\right| \leqslant 2^q \min\left\{f(x_1, u_1), f(x_2, u_2)\right\} < L \min\left\{f(x_1, u_1), f(x_2, u_2)\right\}$$

and (2.13) is true.

Assume that (2.26) is true. Then by (2.12), (2.26) and the choice of q_1 (2.23) is valid. Relations (2.24) and (2.23) imply (2.13). Thus (2.13) is true in both cases. Lemma 2.3 is proved. □

3. Minimal sets

Let $f \in \mathcal{A}$. Denote by $\mathcal{D}(f)$ the collection of all sets $\Omega(v)$ where $v:[0,\infty) \to \mathbb{R}^n$ is an (f)-good function.

Let $D_1, D_2 \in \mathcal{D}(f)$. We say that $D_1 \leqslant D_2$ if and only if $D_1 \subset D_2$.

A set $D_0 \in \mathcal{D}(f)$ is called minimal if for each $D \in \mathcal{D}(f)$ satisfying $D \leqslant D_0$ we have $D = D_0$.

Note that the following lemma holds.

Lemma 3.1. *For each $D \in \mathcal{D}(f)$ there exists a minimal element D_0 of $\mathcal{D}(f)$ such that $D_0 \leqslant D$.*

For the proof of Lemma 3.1 see Lemma 9.1 of [16].

It is easy to see that Lemma 3.1 implies the following auxiliary result.

Lemma 3.2. *Let $v:[0,\infty) \to \mathbb{R}^n$ be an (f)-good function. Then there is a minimal element D of $\mathcal{D}(f)$ such that $D \subset \Omega(v)$.*

Lemma 3.3. *Let $D \in \mathcal{D}(f)$ and $z \in D$. Then there exists an a.c. function $v:\mathbb{R}^1 \to D$ such that*

$$v(0) = z \quad \text{and} \quad \Gamma^f(-T, T, v) = 0 \quad \text{for all } T > 0.$$

Proof. There is an (f)-good function $u:[0,\infty) \to \mathbb{R}^n$ such that

$$\Omega(u) = D. \tag{3.1}$$

In view of Proposition 1.1

$$\sup\{|u(t)|:\ t \in [0,\infty)\} < \infty. \tag{3.2}$$

Since $z \in \Omega(u)$ there exists a sequence of positive numbers $\{t_i\}_{i=1}^\infty$ such that $t_i \to \infty$ as $i \to \infty$ and that

$$u(t_i) \to z \quad \text{as } i \to \infty. \tag{3.3}$$

By Proposition 2.2 the following property holds:

(a) For each $\epsilon > 0$ there is $T(\epsilon) > 0$ such that for each $T_1 \geqslant T(\epsilon)$, $T_2 > T_1$

$$I^f(T_1, T_2, u) \leqslant U^f(T_1, T_2, u(T_1), u(T_2)) + \epsilon.$$

By Proposition 1.1 and (2.4) $\sup\{\Gamma^f(0, T, u):\ T > 0\} < \infty$. In view of this inequality and (2.5) the following property holds:

(b) For each $\epsilon > 0$ there is $T(\epsilon) > 0$ such that for each $T_1 \geqslant T(\epsilon)$, $T_2 > T_1$

$$\Gamma^f(T_1, T_2, u) \leqslant \epsilon.$$

For every integer $i \geqslant 1$ set

$$v_i(t) = u(t + t_i), \quad t \in [-t_i, \infty). \tag{3.4}$$

It follows from (3.4), property (a), (3.2) and Proposition 2.3 that for each natural number k the sequence $\{I^f(-k, k, v_i):\ i \text{ is an integer and } t_i \geqslant k\}$ is bounded. By Proposition 2.5 there exist a subsequence $\{v_{i_q}\}_{q=1}^\infty$ of the sequence $\{v_i\}_{i=1}^\infty$ and an a.c. function $v:\mathbb{R}^1 \to \mathbb{R}^n$ such that for each natural number k

$$v_{i_q}(t) \to v(t) \quad \text{as } q \to \infty \text{ uniformly on } [-k, k], \tag{3.5}$$

$$I^f(-k, k, v) \leqslant \liminf_{q\to\infty} I^f(-k, k, v_{i_q}). \tag{3.6}$$

Relations (3.5), (3.4), (3.3) and (3.1) imply that

$$v(0) = \lim_{q\to\infty} v_{i_q}(0) = \lim_{q\to\infty} u(t_{i_q}) = z$$

and that for each $t \in \mathbb{R}^1$

$$v(t) = \lim_{q\to\infty} v_{i_q}(t) = \lim_{q\to\infty} u(t + t_{i_q}) \in \Omega(u) = D.$$

It follows from (2.4), (3.4)–(3.6), the continuity of the function π^f (see Proposition 2.1) and property (b) that for all integers $k \geqslant 1$

$$\begin{aligned}\Gamma^f(-k,k,v) &= I^f(-k,k,v) - \pi^f\big(v(-k)\big) + \pi^f\big(v(k)\big) - 2k\mu(f)\\ &\leqslant \liminf_{q\to\infty}\big[I^f(-k,k,v_{i_q}) - \pi^f\big(v_{i_q}(-k)\big) + \pi^f\big(v_{i_q}(k)\big) - 2k\mu(f)\big]\\ &= \liminf_{q\to\infty}\Gamma^f(-k,k,v_{i_q}) = \liminf_{q\to\infty}\Gamma^f(-k+t_{i_q},k+t_{i_q},u) = 0.\end{aligned}$$

Together with (2.5) this implies that $\Gamma^f(-k,k,v) = 0$ for all integers $k \geqslant 1$. Lemma 3.3 is proved. □

Lemma 3.4. *Let D be a minimal element of $\mathcal{D}(f)$, $v : [0,\infty) \to D$ be an (f)-good function and let $\epsilon > 0$. Then there is $L > 0$ such that* $\mathrm{dist}(\{v(t)\colon t \in [0,L]\}, D) \leqslant \epsilon$.

Proof. Clearly $\Omega(v) \subset D$. Since D is a minimal element of $\mathcal{D}(f)$ we have $\Omega(v) = D$. This equality implies the validity of Lemma 3.4. □

Lemma 3.5. *Let D be a minimal element of $\mathcal{D}(f)$ and let $\epsilon > 0$. Then there exists a natural number k such that for each a.c. function $v : [0,k] \to D$ satisfying $\Gamma^f(0,k,v) = 0$ the following inequality holds*:

$$\mathrm{dist}\big(D, \big\{v(t)\colon t \in [0,k]\big\}\big) \leqslant \epsilon.$$

Proof. Let us assume the converse. Then for each natural number k there exists an a.c. function $v_k : [0,k] \to D$ such that

$$\Gamma^f(0,k,v_k) = 0 \quad \text{and} \quad \mathrm{dist}\big(D, v_k\big([0,k]\big)\big) > \epsilon. \tag{3.7}$$

Let $i \geqslant 1$ be an integer. Since the set D is bounded and the function π^f is continuous it follows from (3.7) and (2.4) that the sequence $\{I^f(0,i,v_k)\}_{k=i}^{\infty}$ is bounded. Together with Proposition 2.5 this implies that there exist a subsequence $\{v_{k_j}\}_{j=1}^{\infty}$ and an a.c. function $v : [0,\infty) \to \mathbb{R}^n$ such that for each integer $i \geqslant 1$

$$v_{k_j}(t) \to v(t) \quad \text{as } j \to \infty \text{ uniformly on } [0,i], \tag{3.8}$$

$$I^f(0,i,v) \leqslant \liminf_{j\to\infty} I^f(0,i,v_{k_j}). \tag{3.9}$$

Relation (3.8) implies that

$$v(t) \in D \quad \text{for all } t \in [0,\infty). \tag{3.10}$$

It follows from (2.4), (2.5), (3.8), (3.9), the continuity of π^f (see Proposition 2.1) and (3.7) that for each integer $i \geqslant 1$

$$\begin{aligned}0 &\leqslant \Gamma^f(0,i,v) = I^f(0,i,v) - \pi^f\big(v(0)\big) + \pi^f\big(v(i)\big) - i\mu(f)\\ &\leqslant \liminf_{j\to\infty}\big[I^f(0,i,v_{k_j}) - \pi^f\big(v_{k_j}(0)\big) + \pi^f\big(v_{k_j}(i)\big) - i\mu(f)\big]\\ &= \liminf_{j\to\infty}\Gamma^f(0,i,v_{k_j}) = 0.\end{aligned}$$

Thus

$$\Gamma^f(0,i,v) = 0 \quad \text{for all integers } i \geqslant 1. \tag{3.11}$$

By (3.10), (3.11) and Lemma 3.4 there is a natural number L such that

$$\mathrm{dist}\big(v\big([0,L]\big), D\big) \leqslant \epsilon/4. \tag{3.12}$$

In view of (3.8) there exists a natural number $q > L$ such that

$$\left|v_q(t) - v(t)\right| \leqslant \epsilon/4 \quad \text{for all } t \in [0, L]. \tag{3.13}$$

Relations (3.13) and (3.12) imply that

$$\operatorname{dist}\left(v_q\left([0, L]\right), D\right) \leqslant \epsilon/2.$$

Since $q > L$ and $v_q([0, q]) \subset D$ we conclude that

$$\operatorname{dist}\left(v_q\left([0, q]\right), D\right) \leqslant \epsilon/2.$$

This inequality contradicts (3.7). The contradiction we have reached proves Lemma 3.5. □

4. A basic lemma

Assume that $f \in \mathcal{A} \cap C^1(\mathbb{R}^{2n})$ and satisfies the following assumptions:

A(iv) For each $M > 0$ there exists $c_0 > 0$ such that

$$\left|(\partial f/\partial x)(x, u)\right|, \left|(\partial f/\partial u)(x, u)\right| \leqslant c_0\left(f(x, u) + c_0\right)$$

for each $x, u \in \mathbb{R}^n$ satisfying $|x| \leqslant M$.

In this section we prove the following auxiliary result.

Lemma 4.1. *Let D be a minimal element of $\mathcal{D}(f)$ and $\epsilon \in (0, 1)$. Then there exists a number $q > 0$ such that for each $h_1, h_2 \in D$ there exists an a.c. function $v : [0, q] \to \mathbb{R}^n$ which satisfies*

$$v(0) = h_1, \quad v(q) = h_2, \quad \Gamma^f(0, q, v) \leqslant \epsilon.$$

Proof. By Lemma 3.3 there exists an a.c. function $\tilde{v} : \mathbb{R}^1 \to D$ such that

$$\Gamma^f(-T, T, \tilde{v}) = 0 \quad \text{for all } T > 0. \tag{4.1}$$

Relations (4.1) and (2.4) imply that for each $T > 0$

$$\pi^f\left(\tilde{v}(T)\right) = -I^f(0, T, \tilde{v}) + \pi^f\left(\tilde{v}(0)\right) + \mu(f)T. \tag{4.2}$$

This equality implies that the function $\pi^f \circ \tilde{v} : [0, \infty) \to \mathbb{R}^n$ is absolutely continuous. Therefore there exists $r_0 > 0$ such that the functions $\tilde{v}, \pi^f \circ \tilde{v}$ are differentiable at r_0. Set

$$v_*(t) = \tilde{v}(t - 1 + r_0), \quad t \in \mathbb{R}^1. \tag{4.3}$$

Clearly

$$v_*\left(\mathbb{R}^1\right) \subset D. \tag{4.4}$$

In view of (4.1), (4.3) and (2.4)

$$\Gamma^f(-T, T, v_*) = 0 \quad \text{for all } T > 0. \tag{4.5}$$

By (4.3) v_* and $\pi^f \circ v_*$ are differentiable at 1. Choose

$$c_1 > \sup\left\{|z|: z \in D\right\} + 8. \tag{4.6}$$

For each $\tau \in [0, \infty)$ define

$$P_\tau(t) = t\left(v_*(\tau) - v_*(1)\right), \quad t \in \mathbb{R}^1, \quad z(\tau) = v_*(\tau) - v_*(1), \tag{4.7}$$

$$\psi(\tau) = I^f(0, 1, v_* + P_\tau). \tag{4.8}$$

Let $\tau \geqslant 0$. We show that $\psi(\tau)$ is finite. By A(ii) $\psi(\tau) > -\infty$. In view of (4.8) and (4.7)

$$\psi(\tau) = \int_0^1 f\big(v_*(t) + tz(\tau), v_*'(t) + z(\tau)\big)\,\mathrm{d}t. \tag{4.9}$$

It follows from the boundedness of v_* (see (4.4)), the inequality $I^f(0, 1, v_*) < \infty$ and Lemma 2.3 that $\psi(\tau) < \infty$. Hence $\psi(\tau)$ is finite for all $\tau \geqslant 0$.

We show that ψ is differentiable at $t = 1$. Let $h \geqslant 0$ and $h \neq 1$. By (4.9) and (4.7)

$$(h-1)^{-1}\big(\psi(h) - \psi(1)\big) = (h-1)^{-1} \int_0^1 \big[f\big(v_*(t) + tz(h), v_*'(t) + z(h)\big) - f\big(v_*(t), v_*'(t)\big)\big]\,\mathrm{d}t. \tag{4.10}$$

Set

$$\xi_h(t) = (h-1)^{-1}\big[f\big(v_*(t) + tz(h), v_*'(t) + z(h)\big) - f\big(v_*(t), v_*'(t)\big)\big], \quad t \in [0, 1]. \tag{4.11}$$

Clearly the function ξ_h is integrable. Denote by Ω the set of all points $t \in [0, 1]$ such that $v_*'(t)$ exists. It is clear that the Lebesgue measure of the set $[0, 1] \setminus \Omega$ is zero. By (4.11) and the mean value theorem for each $t \in \Omega$ there exists $\lambda_h(t) \in [0, 1]$ such that

$$\begin{aligned}\xi_h(t) = (h-1)^{-1}\big[&(\partial f/\partial x)\big(v_*(t) + \lambda_h(t)tz(h), v_*'(t) + \lambda_h(t)z(h)\big)tz(h) \\ &+ (\partial f/\partial u)\big(v_*(t) + \lambda_h(t)tz(h), v_*'(t) + \lambda_h(t)z(h)\big)z(h)\big].\end{aligned} \tag{4.12}$$

By A(iv) there is $L_0 > 0$ such that

$$\big|(\partial f/\partial x)(x, u)\big|, \big|(\partial f/\partial u)(x, u)\big| \leqslant L_0\big(f(x, u) + L_0\big) \tag{4.13}$$

for each $x, u \in \mathbb{R}^n$ satisfying $|x| \leqslant 4c_1 + 4$.

It follows from (4.12), (4.4), (4.7), (4.6) and (4.13) that for all $t \in \Omega$

$$\big|\xi_h(t)\big| \leqslant |h-1|^{-1}\big[\big(t\big|z(h)\big| + \big|z(h)\big|\big)L_0\big(L_0 + f\big(v_*(t) + \lambda_h(t)tz(h), v_*'(t) + \lambda_h(t)z(h)\big)\big)\big]. \tag{4.14}$$

In view of Lemma 2.3 there is $L_1 > 0$ such that for each $x_1, x_2, u_1, u_2 \in \mathbb{R}^n$ satisfying

$$|x_1| \leqslant 2c_1 + 2, \qquad |x_1 - x_2|, |u_1 - u_2| \leqslant 4c_1 + 4 \tag{4.15}$$

the following inequality holds:

$$\big|f(x_1, u_1) - f(x_2, u_2)\big| \leqslant L_1\big(L_1 + f(x_1, u_1)\big). \tag{4.16}$$

It follows from the choice of L_1 (see (4.15), (4.16)), (4.4), (4.6) and (4.7) that for all $t \in \Omega$

$$f\big(v_*(t) + \lambda_h(t)tz(h), v_*'(t) + \lambda_h(t)z(h)\big) \leqslant f\big(v_*(t), v_*'(t)\big) + L_1\big(L_1 + f\big(v_*(t), v_*'(t)\big)\big).$$

Together with (4.14) this inequality implies that for all $t \in \Omega$

$$\big|\xi_h(t)\big| \leqslant |h-1|^{-1}2\big|z(h)\big|\big[L_0^2 + L_0L_1^2 + L_0(L_1 + 1)f\big(v_*(t), v_*'(t)\big)\big]. \tag{4.17}$$

We have shown that (4.17) is valid for all $t \in \Omega$ and all $h \geqslant 0$ such that $h \neq 1$. Since v_* is differentiable at 1 (4.7) implies that there exists

$$\lim_{h \to 1}(h-1)^{-1}z(h) = v_*'(1) \in \mathbb{R}^n. \tag{4.18}$$

By (4.18) and (4.7) there exists $L_2 > 0$ such that

$$|h-1|^{-1}\big|z(h)\big| \leqslant L_2 \quad \text{for all } h \in \big[(3/4), (5/4)\big] \setminus \{1\}. \tag{4.19}$$

Combined with (4.17) this inequality implies that for all $h \in [(3/4), (5/4)] \setminus \{1\}$ and all $t \in \Omega$

$$\big|\xi_h(t)\big| \leqslant 2L_2\big[L_0^2 + L_0L_1^2 + L_0(L_1 + 1)f\big(v_*(t), v_*'(t)\big)\big]. \tag{4.20}$$

Relations (4.12), (4.8) and (4.18) imply that for all $t \in \Omega$ there exists

$$\lim_{h\to 1}\xi_h(t) = (\partial f/\partial x)\big(v_*(t), v_*'(t)\big)t\lim_{h\to 1}(h-1)^{-1}z(h) + (\partial f/\partial u)\big(v_*(t), v_*'(t)\big)\lim_{h\to 1}(h-1)^{-1}z(h)$$
$$= (\partial f/\partial x)\big(v_*(t), v_*'(t)\big)tv_*'(1) + (\partial f/\partial u)\big(v_*(t), v_*'(t)\big)v_*'(1). \tag{4.21}$$

It follows from (4.10), (4.11), (4.20), (4.21) and the Lebesgue theorem that there exists a finite limit

$$\lim_{h\to 1}(h-1)^{-1}\big(\psi(h)-\psi(1)\big) = \lim_{h\to 1}\int_0^1 \xi_h(t)\,\mathrm{d}t$$
$$= \int_0^1 \Big((\partial f/\partial x)\big(v_*(t), v_*'(t)\big)tv_*'(1) + (\partial f/\partial u)\big(v_*(t), v_*'(t)\big)v_*'(1)\Big)\,\mathrm{d}t.$$

Thus ψ is differentiable at $t=1$. Define a function $\phi:[0,\infty)\to\mathbb{R}^1$ by

$$\phi(t) = \psi(t) - \mu(f) - \pi^f\big(v_*(0)\big) + \pi^f\big(v_*(t)\big), \quad t\in[0,\infty). \tag{4.22}$$

Since $\psi, \pi^f\circ v_*$ are differentiable at 1 we have that ϕ is also differentiable at 1. By (4.22), (4.8), (4.7), (2.4) and (2.5) for each $t\geqslant 0$

$$\phi(t) = I^f(0,1,v_*+P_t) - \mu(f) - \pi^f\big(v_*(0)\big) + \pi^f\big(v_*(t)\big)$$
$$= I^f(0,1,v_*+P_t) - \mu(f) - \pi^f\big((v_*+P_t)(0)\big) + \pi^f\big((v_*+P_t)(1)\big) = \Gamma^f(0,1,v_*+P_t)\geqslant 0. \tag{4.23}$$

Thus

$$\phi(t)\geqslant 0 \quad \text{for all } t\geqslant 0. \tag{4.24}$$

In view of (4.23), (4.5) and (4.7)

$$\phi(1) = \Gamma^f(0,1,v_*) = 0. \tag{4.25}$$

Let us define a constant $q>0$. It follows from Proposition 2.3 and the continuity of the function π^f that there exists a sequence of positive numbers $\{\delta_i\}_{i=0}^\infty$ such that

$$\delta_0\in\big(0, 8^{-1}\epsilon\big), \quad \delta_{i+1}<\delta_i, \quad i=0,1,\ldots, \tag{4.26}$$

and that for each integer $i\geqslant 0$ and each $x_1, x_2, y_1, y_2\in\mathbb{R}^n$ which satisfy

$$d(x_j, D), d(y_j, D)\leqslant 8, \quad j=1,2, \quad |x_j-y_j|\leqslant\delta_i, \quad j=1,2, \tag{4.27}$$

the following inequalities hold:

$$\big|U^f(0,1,x_1,x_2) - U^f(0,1,y_1,y_2)\big|\leqslant 2^{-i-8}\epsilon, \tag{4.28}$$
$$\big|\pi^f(x_j) - \pi^f(y_j)\big|\leqslant 2^{-i-8}\epsilon, \quad j=1,2. \tag{4.29}$$

By Lemma 3.5, (4.4) and (4.5) there exists an integer $L\geqslant 10$ such that for all $T\geqslant 0$

$$\mathrm{dist}\big(D, \{v_*(t):\ t\in[T,T+L]\}\big)\leqslant 4^{-1}\delta_0. \tag{4.30}$$

In view of Lemma 3.5, (4.4) and (4.5) there exists a sequence of numbers $\{T_p\}_{p=1}^\infty$ such that

$$T_p\geqslant 2L+8, \quad \big|v_*(0) - v_*(T_p)\big|\leqslant 2^{-8}\delta_p, \quad p=1,2,\ldots. \tag{4.31}$$

Fix a positive number ϵ_0 for which

$$\epsilon_0 < 2^{-8}L^{-1}\epsilon. \tag{4.32}$$

Relations (4.24), (4.25) and the differentiability of ϕ at 1 imply that $\phi'(1)=0$ and that there exists a positive number Δ such that

$$\Delta < 2^{-8}, \tag{4.33}$$
$$\phi(t) = \phi(t) - \phi(1)\leqslant |t-1|2^{-1}\epsilon_0 \quad \text{for all } t\in[1-\Delta, 1+\Delta]. \tag{4.34}$$

Choose an integer

$$N > 64(L+1)\Delta^{-1} \tag{4.35}$$

and put

$$q = \sum_{i=1}^{N} T_i + 8L + 8. \tag{4.36}$$

Let $h_1, h_2 \in D$. We show that there exists an a.c. function $v : [0, q] \to \mathbb{R}^n$ such that

$$v(0) = h_1, \quad v(q) = h_2, \quad \Gamma^f(0, q, v) \leqslant \epsilon.$$

Relation (4.30) which holds for all $T \geqslant 0$ implies that there exist numbers t_1, t_2 such that

$$t_1 \in [0, L], t_2 \in [8, L+8], \quad \left|h_j - v_*(t_j)\right| \leqslant \delta_0/4, \quad j = 1, 2. \tag{4.37}$$

Set

$$\Delta_0 = (N-1)^{-1}\bigl(8L + 8 - (t_2 - t_1)\bigr). \tag{4.38}$$

In view of (4.38), (4.37), (4.35) and (4.33)

$$0 \leqslant \Delta_0 < \Delta. \tag{4.39}$$

It follows from (4.31), (4.37) and Proposition 2.5 that there exists an a.c. function $w_0 : [0, T_1 - t_1] \to \mathbb{R}^n$ such that

$$\begin{aligned}
&w_0(0) = h_1, \quad w_0(t) = v_*(t_1 + t), \quad t \in [1, T_1 - t_1 - 1], \\
&w_0(T_1 - t_1) = v_*(0), \\
&I^f(\tau, \tau + 1, w_0) = U^f\bigl(0, 1, w_0(\tau), w_0(\tau + 1)\bigr), \quad \tau = 0,\ T_1 - t_1 - 1.
\end{aligned} \tag{4.40}$$

We will estimate $\Gamma^f(0, T_1 - t_0, w_0)$. In view of (4.40), (2.4) and (4.5)

$$\begin{aligned}
\Gamma^f(0, T_1 - t_1, w_0) &= \Gamma^f(0, 1, w_0) + \Gamma^f(1, T_1 - t_1 - 1, w_0) + \Gamma^f(T_1 - t_1 - 1, T_1 - t_1, w_0) \\
&= \Gamma^f(0, 1, w_0) + \Gamma^f(T_1 - t_1 - 1, T_1 - t_1, w_0) \\
&= I^f(0, 1, w_0) - \pi^f\bigl(w_0(0)\bigr) + \pi^f\bigl(w_0(1)\bigr) - \mu(f) + I^f(T_1 - t_1 - 1, T_1 - t_1, w_0) \\
&\quad - \pi^f\bigl(w_0(T_1 - t_1 - 1)\bigr) + \pi^f\bigl(w_0(T_1 - t_1)\bigr) - \mu(f) \\
&= U^f\bigl(0, 1, h_1, v_*(t_1 + 1)\bigr) - \pi^f(h_1) + \pi^f\bigl(v_*(t_1 + 1)\bigr) - \mu(f) \\
&\quad + U^f\bigl(0, 1, v_*(T_1 - 1), v_*(0)\bigr) - \pi^f\bigl(v_*(T_1 - 1)\bigr) + \pi^f\bigl(v_*(0)\bigr) - \mu(f).
\end{aligned} \tag{4.41}$$

By the choice of δ_0 (see (4.26)–(4.29)), (4.4), (4.37) and (4.31)

$$\begin{aligned}
&\bigl|U^f\bigl(0, 1, h_1, v_*(t_1 + 1)\bigr) - U^f\bigl(0, 1, v_*(t_1), v_*(t_1 + 1)\bigr)\bigr| \leqslant 2^{-8}\epsilon, \\
&\bigl|U^f\bigl(0, 1, v_*(T_1 - 1), v_*(0)\bigr) - U^f\bigl(0, 1, v_*(T_1 - 1), v_*(T_1)\bigr)\bigr| \leqslant 2^{-8}\epsilon, \\
&\bigl|\pi^f(h_1) - \pi^f\bigl(v_*(t_1)\bigr)\bigr|, \bigl|\pi^f\bigl(v_*(0)\bigr) - \pi^f\bigl(v_*(T_1)\bigr)\bigr| \leqslant 2^{-8}\epsilon.
\end{aligned}$$

Together with (4.41) and (4.5) these inequalities imply that

$$\begin{aligned}
\Gamma^f(0, T_1 - t_1, w_0) &\leqslant U^f\bigl(0, 1, v_*(t_1), v_*(t_1 + 1)\bigr) - \pi^f\bigl(v_*(t_1)\bigr) + \pi^f\bigl(v_*(t_1 + 1)\bigr) - \mu(f) \\
&\quad + U^f\bigl(0, 1, v_*(T_1 - 1), v_*(T_1)\bigr) - \pi^f\bigl(v_*(T_1 - 1)\bigr) + \pi^f\bigl(v_*(T_1)\bigr) - \mu(f) + 4 \times 2^{-8}\epsilon \\
&\leqslant \Gamma^f(T_1 - 1, T_1, v_*) + \Gamma^f(t_1, t_1 + 1, v_*) + 2^{-6}\epsilon = 2^{-6}\epsilon.
\end{aligned} \tag{4.42}$$

Let $k \geqslant 1$ be an integer. It follows from (4.39), (4.33), (4.31), (4.7) and Proposition 2.5 that there exists an a.c. function $w_k : [0, \Delta_0 + T_{k+1}] \to \mathbb{R}^n$ such that

$$\begin{aligned} &w_k(t) = v_*(t) + P_{1-\Delta_0}(t), \quad t \in [0,1], \\ &w_k(t) = v_*(t - \Delta_0), \quad t \in [1, \Delta_0 + T_{k+1} - 1], \quad w_k(\Delta_0 + T_{k+1}) = v_*(0), \\ &I^f(\Delta_0 + T_{k+1} - 1, \Delta_0 + T_{k+1}, w_k) = U^f\big(0, 1, w_k(\Delta_0 + T_{k+1} - 1), w_k(\Delta_0 + T_{k+1})\big). \end{aligned} \tag{4.43}$$

Relations (4.43) and (4.7) imply that

$$w_k(0) = v_*(0). \tag{4.44}$$

We will show that

$$\Gamma^f(0, T_{k+1} + \Delta_0, w_k) \leqslant 2^{-1}\epsilon_0\Delta_0 + 2^{-k-8}\epsilon. \tag{4.45}$$

By (4.43) and (4.5)

$$\begin{aligned} \Gamma^f(0, T_{k+1} + \Delta_0, w_k) &= \Gamma^f(0, 1, w_k) + \Gamma^f(1, T_{k+1} + \Delta_0 - 1, w_k) + \Gamma^f(T_{k+1} + \Delta_0 - 1, T_{k+1} + \Delta_0, w_k) \\ &= \Gamma^f(0, 1, w_k) + \Gamma^f(T_{k+1} + \Delta_0 - 1, T_{k+1} + \Delta_0, w_k). \end{aligned} \tag{4.46}$$

In view of (4.43), (2.4), (4.8), (4.7) and (4.22)

$$\begin{aligned} \Gamma^f(0, 1, w_k) &= I^f(0, 1, w_k) - \pi^f\big(w_k(0)\big) + \pi^f\big(w_k(1)\big) - \mu(f) \\ &= I^f(0, 1, v_* + P_{1-\Delta_0}) - \pi^f\big(v_*(0) + P_{1-\Delta_0}(0)\big) + \pi^f\big(v_*(1) + P_{1-\Delta_0}(1)\big) - \mu(f) \\ &= \psi(1 - \Delta_0) + \pi^f\big(v_*(0)\big) + \pi^f\big(v_*(1 - \Delta_0)\big) - \mu(f) = \phi(1 - \Delta_0). \end{aligned} \tag{4.47}$$

Relations (4.47), (4.34) and (4.39) imply that

$$\Gamma^f(0, 1, w_k) = \phi(1 - \Delta_0) \leqslant 2^{-1}\epsilon_0\Delta_0. \tag{4.48}$$

It follows from (2.4) and (4.44) that

$$\begin{aligned} &\Gamma^f(T_{k+1} + \Delta_0 - 1, T_{k+1} + \Delta_0, w_k) \\ &\quad = I^f(T_{k+1} + \Delta_0 - 1, T_{k+1} + \Delta_0, w_k) - \pi^f\big(w_k(T_{k+1} + \Delta_0 - 1)\big) + \pi^f\big(w_k(T_{k+1} + \Delta_0)\big) - \mu(f) \\ &\quad = U^f\big(0, 1, w_k(\Delta_0 + T_{k+1} - 1), w_k(\Delta_0 + T_{k+1})\big) - \pi^f\big(v_*(T_{k+1} - 1)\big) + \pi^f\big(v_*(0)\big) - \mu(f) \\ &\quad = U^f\big(0, 1, v_*(T_{k+1} - 1), v_*(0)\big) - \pi^f\big(v_*(T_{k+1} - 1)\big) + \pi^f\big(v_*(0)\big) - \mu(f). \end{aligned} \tag{4.49}$$

By the choice of $\{\delta_i\}_{i=0}^{\infty}$ (see (4.26)–(4.29)), (4.4) and (4.3),

$$\begin{aligned} &\big|U^f\big(0, 1, v_*(T_{k+1} - 1), v_*(0)\big) - U^f\big(0, 1, v_*(T_{k+1} - 1), v_*(T_{k+1})\big)\big| \leqslant 2^{-k-8}\epsilon, \\ &\big|\pi^f\big(v_*(0)\big) - \pi^f\big(v_*(T_{k+1})\big)\big| \leqslant 2^{-k-9}\epsilon. \end{aligned}$$

Together with (4.49) and (4.5) these inequalities imply that

$$\begin{aligned} \Gamma^f(T_{k+1} + \Delta_0 - 1, T_{k+1} + \Delta_0, w_k) &\leqslant U^f\big(0, 1, v_*(T_{k+1} - 1), v_*(T_{k+1})\big) - \mu(f) \\ &\quad - \pi^f\big(v_*(T_{k+1} - 1)\big) + \pi^f\big(v_*(T_{k+1})\big) + 2^{-k-8}\epsilon \\ &\leqslant \Gamma^f(T_{k+1} - 1, T_{k+1}, v_*) + 2^{-k-8}\epsilon = 2^{-k-8}\epsilon. \end{aligned}$$

Combined with (4.46) and (4.48) this inequality implies that (4.45) holds. Proposition 2.5 implies that there exists an a.c. function $u_0 : [0, t_2] \to \mathbb{R}^n$ such that

$$\begin{aligned} &u_0(t) = v_*(t), \quad t \in [0, t_2 - 1], \quad u_0(t_2) = h_2, \\ &I^f(t_2 - 1, t_2, u_0) = U^f\big(0, 1, u_0(t_2 - 1), u_0(t_2)\big). \end{aligned} \tag{4.50}$$

In view of (4.50), (4.5) and (2.4)

$$\begin{aligned} \Gamma^f(0, t_2, u_0) &= \Gamma^f(0, t_2 - 1, u_0) + \Gamma^f(t_2 - 1, t_2, u_0) = \Gamma^f(t_2 - 1, t_2, u_0) \\ &= U^f\big(0, 1, v_*(t_2 - 1), h_2\big) + \pi^f\big(v_*(t_2 - 1)\big) + \pi^f(h_2) - \mu(f). \end{aligned} \tag{4.51}$$

By (4.4), the choice of $\{\delta_i\}_{i=0}^{\infty}$ (see (4.26)–(4.29)) and (4.37)

$$\big|U^f\big(0,1,v_*(t_2-1),h_2\big)-U^f\big(0,1,v_*(t_2-1),v_*(t_2)\big)\big|\leqslant 2^{-8}\epsilon,$$
$$\big|\pi^f\big(v_*(t_2)\big)-\pi^f(h_2)\big|\leqslant 2^{-8}\epsilon.$$

Together with (4.51), (4.4) and (4.5) these inequalities imply that

$$\begin{aligned}\Gamma^f(0,t_2,u_0)&\leqslant U^f\big(0,1,v_*(t_2-1),v_*(t_2)\big)-\pi^f\big(v_*(t_2-1)\big)+\pi^f\big(v_*(t_2)\big)+2^{-7}\epsilon-\mu(f)\\&\leqslant \Gamma^f(t_2-1,t_2,v_*)+2^{-7}=2^{-7}\epsilon.\end{aligned}\tag{4.52}$$

Relations (4.38) and (4.36) imply that

$$T_1-t_1+\sum_{k=1}^{N-1}(\Delta_0+T_{k+1})+t_2=T_1-t_1+8L+8-t_2+t_1+\sum_{k=1}^{N-1}T_{k+1}+t_2=\sum_{k=1}^{N}T_k+8L+8=q.\tag{4.53}$$

It follows from (4.53) (4.40), (4.43), (4.44) and (4.50) that there exists an a.c. function $v:[0,q]\to\mathbb{R}^n$ such that

$$\begin{aligned}&v(t)=w_0(t),\quad t\in[0,T_1-t_1],\quad v(t)=w_k\left(t-\left(\sum_{i=1}^{k}T_i+(k-1)\Delta_0-t_1\right)\right),\\&t\in\left[\sum_{i=1}^{k}T_i+(k-1)\Delta_0-t_1,\sum_{i=1}^{k+1}T_i+k\Delta_0-t_1\right],\quad k=1,\dots,N-1,\\&v(t)=u_0\left(t-\left(\sum_{i=1}^{N}T_i+(N-1)\Delta_0-t_1\right)\right),\quad t\in\left[\sum_{i=1}^{N}T_i+(N-1)\Delta_0-t_1,q\right].\end{aligned}\tag{4.54}$$

By (4.54), (4.40), (4.53) and (4.50)

$$v(0)=w_0(0)=h_1,\qquad v(q)=u_0(t_2)=h_2.\tag{4.55}$$

In view of (4.54), (4.42), (4.45), (4.52), (4.38) and (4.32)

$$\begin{aligned}\Gamma^f(0,q,v)&=\Gamma^f(0,T_1-t_1,w_0)+\sum_{k=1}^{N-1}\Gamma^f(0,T_{k+1}+\Delta_0,w_k)+\Gamma^f(0,t_2,u_0)\\&\leqslant 2^{-6}\epsilon+\sum_{k=1}^{N-1}\big(2^{-1}\epsilon_0\Delta_0+2^{-k-8}\epsilon\big)+2^{-7}\epsilon\\&\leqslant 2^{-5}\epsilon+(N-1)2^{-1}\epsilon_0\Delta_0\leqslant 2^{-5}\epsilon+2^{-1}\epsilon_0(9L+16)\leqslant 2^{-1}\epsilon.\end{aligned}$$

This completes the proof of Lemma 4.1. □

Lemma 4.2 *(Basic Lemma). Let D be a minimal element of $\mathcal{D}(f)$ and $\epsilon\in(0,1)$. Then there exist numbers $q,\delta>0$ such that for each $h_1,h_2\in\mathbb{R}^n$ satisfying $d(h_i,D)\leqslant\delta$, $i=1,2$, and each $T\geqslant q$ there exists an a.c. function $v:[0,T]\to\mathbb{R}^n$ which satisfies*

$$v(0)=h_1,\quad v(T)=h_2,\quad \Gamma^f(0,T,v)\leqslant\epsilon.$$

Proof. By Lemma 4.1 there exists a number $q>2$ such that for each $z_1,z_2\in D$ there exists an a.c. function $v:[0,q-2]\to\mathbb{R}^n$ such that

$$v(0)=z_1,\quad v(q-2)=z_2,\quad \Gamma^f(0,q-2,v)\leqslant\epsilon/4.\tag{4.56}$$

It follows from Proposition 2.3 and the continuity of the function π^f that there exists $\delta\in(0,\epsilon)$ such that for each $x_1,x_2,y_1,y_2\in\mathbb{R}^n$ which satisfy

$$d(x_j,D),d(y_j,D)\leqslant 8,\quad j=1,2,\quad |x_j-y_j|\leqslant\delta,\quad j=1,2,\tag{4.57}$$

the following inequalities hold:

$$\left|U^f(0,1,x_1,x_2) - U^f(0,1,y_1,y_2)\right| \leqslant \epsilon/16, \tag{4.58}$$

$$\left|\pi^f(x_j) - \pi^f(y_j)\right| \leqslant \epsilon/16, \quad j = 1,2. \tag{4.59}$$

Let $T \geqslant q$ and $h_1, h_2 \in \mathbb{R}^n$ satisfy

$$d(h_i, D) \leqslant \delta, \quad i = 1,2. \tag{4.60}$$

Clearly there exist vectors $\bar{h}_1, \bar{h}_2$ such that

$$\bar{h}_1, \bar{h}_2 \in D, \quad |h_i - \bar{h}_i| \leqslant \delta, \quad i = 1,2. \tag{4.61}$$

Since $D \in \mathcal{D}(f)$ it follows from (4.61) and Lemma 3.3 that there exist

$$v_i : \mathbb{R}^1 \to D, \quad i = 1,2, \tag{4.62}$$

such that

$$v_i(0) = \bar{h}_i, \quad i = 1,2, \quad \Gamma^f(-\tau, \tau, v_i) = 0 \quad \text{for all } \tau > 0 \text{ and } i = 1,2. \tag{4.63}$$

By (4.62) and the choice of q (see (4.56)) there exists an a.c. function $u : [0, q-2] \to \mathbb{R}^n$ such that

$$u(0) = v_1(1), \quad u(q-2) = v_2(q-1-T), \quad \Gamma^f(0, q-2, u) \leqslant \epsilon/4. \tag{4.64}$$

It follows from Proposition 2.5 and (4.64) that there exists an a.c. function $v : [0,T] \to \mathbb{R}^n$ such that

$$\begin{aligned} &v(0) = h_1, \quad v(t) = u(t-1), \quad t \in [1, q-1], \quad I^f(0,1,v) = U^f\big(0,1,h_1,u(0)\big), \\ &v(t) = v_2(t-T), \quad t \in [q-1, T-1], \quad v(T) = h_2, \quad I^f(T-1,T,v) = U^f\big(0,1,v_2(-1),h_2\big). \end{aligned} \tag{4.65}$$

Relations (4.65), (2.4), (4.64) and (4.63) imply that

$$\begin{aligned} \Gamma^f(0,T,v) &= \Gamma^f(0,1,v) + \Gamma^f(1,q-1,v) + \Gamma^f(q-1,T-1,v) + \Gamma^f(T-1,T,v) \\ &= \Gamma^f(0,1,v) + \Gamma^f(0,q-2,u) + \Gamma^f(q-1-T,-1,v_2) + \Gamma^f(T-1,T,v) \\ &\leqslant \Gamma^f(0,1,v) + \epsilon/4 + \Gamma^f(T-1,T,v). \end{aligned} \tag{4.66}$$

In view of (2.4), (4.65) and (4.64)

$$\begin{aligned} \Gamma^f(0,1,v) &= I^f(0,1,v) - \mu(f) + \pi^f\big(v(0)\big) + \pi^f\big(v(1)\big) \\ &= U^f\big(0,1,h_1,u(0)\big) - \mu(f) - \pi^f(h_1) + \pi^f\big(v_1(1)\big) \\ &= U^f\big(0,1,h_1,v_1(1)\big) - \mu(f) - \pi^f(h_1) + \pi^f\big(v_1(1)\big), \end{aligned} \tag{4.67}$$

$$\begin{aligned} \Gamma^f(T-1,T,v) &= I^f(T-1,T,v) - \mu(f) - \pi^f\big(v(T-1)\big) + \pi^f\big(v(T)\big) \\ &= U^f\big(0,1,v_2(-1),h_2\big) - \mu(f) - \pi\big(v_2(-1)\big) + \pi^f(h_2). \end{aligned} \tag{4.68}$$

By the choice of δ (see (4.57)–(4.59)), (4.62), (4.60), (4.63) and (4.61)

$$\begin{aligned} &\left|U^f\big(0,1,h_1,v_1(1)\big) - U^f\big(0,1,v_1(0),v_1(1)\big)\right| \leqslant \epsilon/16, \\ &\left|\pi^f(h_1) - \pi^f\big(v_1(0)\big)\right| \leqslant \epsilon/16, \end{aligned} \tag{4.69}$$

$$\begin{aligned} &\left|U^f\big(0,1,v_2(-1),h_2\big) - U^f\big(0,1,v_2(-1),v_2(0)\big)\right| \leqslant \epsilon/16, \\ &\left|\pi^f(h_2) - \pi^f\big(v_2(0)\big)\right| \leqslant \epsilon/16. \end{aligned} \tag{4.70}$$

Relations (4.67), (4.69) and (4.63) imply that

$$\begin{aligned} \Gamma^f(0,1,v) &\leqslant U^f\big(0,1,v_1(0),v_1(1)\big) - \mu(f) - \pi^f\big(v_1(0)\big) + \pi^f\big(v_1(1)\big) + \epsilon/8 \\ &\leqslant \Gamma^f(0,1,v_1) + \epsilon/8 = \epsilon/8. \end{aligned} \tag{4.71}$$

Relations (4.68), (4.70) and (4.63) imply that

$$\Gamma^f(T-1,T,v) \leqslant U^f\big(0,1,v_2(-1),v_2(0)\big)-\mu(f)-\pi^f\big(v_2(-1)\big)+\pi^f\big(v_2(0)\big)+\epsilon/8$$
$$\leqslant \Gamma^f(-1,0,v_2)+\epsilon/8=\epsilon/8.$$

Combined with (4.71) and (4.66) this inequality implies that

$$\Gamma^f(0,T,v) \leqslant \epsilon/8+\epsilon/4+\epsilon/8=\epsilon/2.$$

This completes the proof of Lemma 4.2. □

5. Proofs of Theorems 1.1 and 1.2

Let $f \in \mathcal{M} \subset \mathcal{A}$. Clearly f satisfies assumption A(iv).

We use the following result established in [18, Proposition 7.1].

Proposition 5.1. *Let $x, y \in \mathbb{R}^n$, $T_1 \geqslant 0$, $T_2 > T_1$ and let $w : [T_1, T_2] \to \mathbb{R}^n$ be an a.c. function such that*

$$w(T_1)=x, \quad w(T_2)=y, \quad I^f(T_1,T_2,w)=U^f(T_1,T_2,x,y).$$

Then $w \in C^2([T_1,T_2];\mathbb{R}^n)$ and for each $i \in \{1,\dots,n\}$ and each $t \in [T_1,T_2]$

$$\begin{aligned}(\partial f/\partial x_i)\big(w(t),w'(t)\big) &= (\mathrm{d}/\mathrm{d}t)(\partial f/\partial u_i)\big(w(t),w'(t)\big)\\ &= \sum_{j=1}^n\big(\partial^2 f/\partial u_i\partial x_j\big)\big(w(t),w'(t)\big)w_j'(t)+\sum_{j=1}^n\big(\partial^2 f/\partial u_i\partial u_j\big)\big(w(t),w'(t)\big)w_j''(t). \end{aligned} \tag{5.1}$$

(Here $w(t)=(w_1(t),\dots,w_n(t))$, $t \in [T_1,T_2]$.)

For each $x,u \in \mathbb{R}^n$ set

$$A(x,u)=\big(\big(\partial^2 f/\partial u_i\partial u_j\big)(x,u)\big)_{i,j=1}^n, \qquad B(x,u)=\big(\big(\partial^2 f/\partial u_i\partial x_j\big)(x,u)\big)_{i,j=1}^n$$

and by $C(x,u) \in \mathbb{R}^n$ denote the vector $((\partial f/\partial x_i)(x,u))_{i=1}^n$. Then the system of the differentiable equations (5.1) has the following equivalent form:

$$A\big(w(t),w'(t)\big)w''(t)+B\big(w(t),w'(t)\big)w'(t)=C\big(w(t),w'(t)\big), \quad t \in [T_1,T_2]. \tag{5.2}$$

Since the matrix $A(x,u)$ is positive definite for all $x,u \in \mathbb{R}^n$ (see the definition of $\mathcal{M}$) there exists $A^{-1}(x,u)$ for all $(x,u) \in \mathbb{R}^n$ and Eqs. (5.1) and (5.2) have the following equivalent form:

$$w''(t)=-\big(A\big(w(t),w'(t)\big)\big)^{-1}B\big(w(t),w'(t)\big)w'(t)+\big(A\big(w(t),w'(t)\big)\big)^{-1}C\big(w(t),w'(t)\big), \quad t \in [T_1,T_2]. \tag{5.3}$$

It follows from Proposition 5.1, the inclusions $A,B,C \in C^1$ (see the definition of $\mathcal{M}$) and our discussion above that the following lemma holds.

Lemma 5.1. *Let $v_1, v_2 : [0,\infty) \to \mathbb{R}^n$ be c-optimal functions with respect to f, $0 \leqslant T_1 < T_2$ and let $v_1(t)=v_2(t)$ for all $t \in [T_1,T_2]$. Then $v_1(t)=v_2(t)$ for all $t \in [0,\infty)$.*

Proof of Theorem 1.1. Assume that $0 \leqslant T_1 < T_2$ and

$$v(T_1)=v(T_2). \tag{5.4}$$

We show that $v(t+T_2-T_1)=v(t)$ for all $t \geqslant 0$. First we show that

$$\Gamma^f(T_1,T_2,v)=0. \tag{5.5}$$

Let us assume the converse. Then

$$\Gamma^f(T_1,T_2,v)>0. \tag{5.6}$$

Put

$$\lambda=\Gamma^f(T_1,T_2,v). \tag{5.7}$$

By Lemma 3.2 and Proposition 2.4 there exists a minimal element D of $\mathcal{D}(f)$ such that

$$D \subset \Omega(v). \tag{5.8}$$

By Lemma 4.2 there exist $q, \delta > 0$ such that the following property holds:

(P1) For each $h_1, h_2 \in \mathbb{R}^n$ satisfying $d(h_i, D) \leqslant \delta$, $i = 1, 2$, and each $T \geqslant q$ there exists an a.c. function $u : [0, T] \to \mathbb{R}^n$ such that

$$u(0) = h_1, \quad u(T) = h_2, \quad \Gamma^f(0, T, u) \leqslant \lambda/4.$$

In view of (5.8) there exist $t_1, t_2 \geqslant 0$ such that

$$t_1 > T_2 + 8, \quad t_2 > t_1 + q + (T_2 - T_1) + 8, \quad d\big(v(t_i), D\big) \leqslant \delta, \quad i = 1, 2. \tag{5.9}$$

It follows from (5.9) and property (P1) that there exists an a.c. function $u : [0, t_2 - t_1 + T_2 - T_1] \to \mathbb{R}^n$ such that

$$u(0) = v(t_1), \quad u(t_2 - t_1 + T_2 - T_1) = v(t_2), \quad \Gamma^f(0, t_2 - t_1 + T_2 - T_1, u) \leqslant \lambda/4. \tag{5.10}$$

Relations (5.4), (5.9) and (5.10) imply that there exists an a.c. function $\tilde{v} : [0, \infty) \to \mathbb{R}^n$ such that

$$\begin{aligned} &\tilde{v}(t) = v(t), \quad t \in [0, T_1], \quad \tilde{v}(t) = v(t + T_2 - T_1), \quad t \in [T_1, t_1 + T_1 - T_2], \\ &\tilde{v}(t) = u\big(t - (t_1 + T_1 - T_2)\big), \quad t \in [t_1 + T_1 - T_2, t_2], \quad \tilde{v}(t) = v(t), \quad t \in [t_2, \infty). \end{aligned} \tag{5.11}$$

In view of (5.11)

$$\tilde{v}(0) = v(0), \qquad \tilde{v}(t_2) = v(t_2). \tag{5.12}$$

Since v is c-optimal with respect to f (5.12) implies that

$$I^f(0, t_2, \tilde{v}) \geqslant I^f(0, t_2, v). \tag{5.13}$$

On the other hand it follows from (5.12), (2.4), (5.11), (5.9), (5.10), (5.7) and (5.6) that

$$\begin{aligned} I^f(0, t_2, \tilde{v}) - I^f(0, t_2, v) &= \Gamma^f(0, t_2, \tilde{v}) - \Gamma^f(0, t_2, v) \\ &= \Gamma^f(0, T_1, v) + \Gamma^f(T_2, t_1, v) + \Gamma^f(0, t_2 - t_1 + T_2 - T_1, u) - \Gamma^f(0, t_2, v) \\ &\leqslant \Gamma^f(0, t_2 - t_1 + T_2 - T_1, u) - \Gamma^f(T_1, T_2, v) - \Gamma^f(t_1, t_2, v) \\ &\leqslant \lambda/4 - \Gamma^f(T_1, T_2, v) \leqslant -(3/4)\lambda < 0. \end{aligned}$$

This contradicts (5.13). The contradiction we have reached proves (5.5). By (5.4) there exists an a.c. function $w : [0, \infty) \to \mathbb{R}^n$ such that

$$w(t) = v(t), \quad t \in [T_1, T_2], \quad w(t + T_2 - T_1) = w(t), \quad t \in [0, \infty). \tag{5.14}$$

In view of (5.14), (5.4), (2.4) and (2.5)

$$\Gamma^f(s_1, s_2, w) = 0 \quad \text{for each } s_1 \geqslant 0,\ s_2 > s_1.$$

Thus w is an c-optimal with respect to f. Together with (5.14) and Lemma 5.1 this implies that $w(t) = v(t)$ for all $t \in [0, \infty)$. Theorem 1.1 is proved. □

In order to prove Theorem 1.2 we need the following result.

Proposition 5.2. *Assume that an a.c. function $v : [0, \infty) \to \mathbb{R}^n$ is c-optimal with respect to f. Then for each $T, S > 0$*

$$U^f\big(0, T, v(0), v(T)\big) - T\mu(f) \leqslant U^f\big(0, S, v(0), v(T)\big) - S\mu(f). \tag{5.15}$$

Proof. Let us assume the converse. Then there exist $T_0, S_0 > 0$ such that

$$U^f\big(0, T_0, v(0), v(T_0)\big) - T_0\mu(f) - \big(U^f\big(0, S_0, v(0), v(T_0)\big) - S_0\mu(f)\big) := \lambda > 0.$$

By Lemma 3.2 there exists a minimal element D of $\mathcal{D}(f)$ such that $D \subset \Omega(v)$. By Lemma 4.2 there exist $q, \delta > 0$ such that the following property holds:

For each $h_1, h_2 \in \mathbb{R}^n$ satisfying $d(h_i, D) \leqslant \delta$, $i = 1, 2$, and each $\tau \geqslant q$ there exists an a.c. function $u : [0, \tau] \to \mathbb{R}^n$ such that

$$u(0) = h_1, \quad u(\tau) = h_2, \quad \Gamma^f(0, \tau, u) \leqslant \lambda/4.$$

The inclusion $D \subset \Omega(v)$ implies that there exist numbers τ_1, τ_2 such that

$$\tau_1 > T_0, \qquad \tau_2 > \tau_1 + S_0 + q,$$
$$d\big(v(\tau_i), D\big) < \delta, \quad i = 1, 2.$$

Now we can complete the proof of Proposition 5.2 arguing as in the proof of Proposition 4.2 of [10]. □

We can prove Theorem 1.2 arguing as in the proof of part (b) of Theorem 1.1 of [10] and using Proposition 5.2 instead of Proposition 4.2 of [10].

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