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The Dirichlet problem for harmonic maps from the disk into the euclidean n-sphere

by

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ABSTRACT. — Let
$\Omega = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \}, \qquad S^n = \{ v \in \mathbb{R}^{n+1} \mid |v| = 1 \} \quad (n \geqslant 2),$
and let $\gamma \in C^{2,\delta}(\partial\Omega; S^n)$. We study the following problem

$$(*) \qquad \begin{cases} u \in C^2(\Omega; S^n) \cap C^0(\overline{\Omega}; S^n) \\ -\Delta u = u \mid \nabla u \mid^2 \\ u = \gamma \quad \text{on} \quad \partial\Omega . \end{cases}$$

Problem (*) is the « Dirichlet » problem for a harmonic function u which takes its values in S^n. We prove that, if γ is not constant, then (*) has at least two distinct solutions.

Key Words: Dirichlet problem, harmonic map, conformal transformation, critical point, minimax principle.

RÉSUMÉ. — Soit γ une application du bord d'un disque de $\mathbb{R}^2$ à valeurs dans une sphère euclidienne de dimension n. On montre que, si γ n'est pas une application constante, il existe au moins deux applications harmoniques du disque dans la sphère égales à γ sur le bord du disque.

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1. INTRODUCTION

Let

$$\Omega = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1 \}$$

and

$$S^n = \{ v \in \mathbb{R}^{n+1} \mid |v| = 1 \} \quad n \geq 2 .$$

Let γ be a map from $\partial\Omega$ into S^n. We seek functions u in $C^2(\Omega; S^n) \cap C^0(\overline{\Omega}; S^n)$ such that:

$$(1.1) \qquad -\Delta u = u\,|\nabla u|^2$$

$$(1.2) \qquad u = \gamma \quad \text{on} \quad \partial\Omega .$$

We shall assume that

$$(1.3) \qquad \gamma \in C^{2,\delta}(\partial\Omega) \quad \text{with} \quad 0 < \delta < 1$$

which means that $\gamma \in C^2(\partial\Omega)$ and that the second derivative of γ is Hölder continuous with exponent δ.

The existence of at least one solution is obvious. To see this let

$$\mathscr{E} = \{ u \in H^1(\Omega; \mathbb{R}^{n+1}) \mid u|_{\partial\Omega} = \gamma, |u| = 1 \text{ a.e.} \}$$

where $H^1(\Omega; \mathbb{R}^{n+1})$ is the usual Sobolev space. Using (1.3) it is easy to see that $\mathscr{E}$ is non void. On $\mathscr{E}$ we consider the functional

$$E(u) = \int_\Omega |\nabla u|^2 .$$

Clearly there exists some $\underline{u}$ in $\mathscr{E}$ such that

$$(1.4) \qquad E(\underline{u}) = \operatorname*{Inf}_{\mathscr{E}} E = m .$$

$\underline{u}$ is a solution of (1) and (2) and thanks to a result of Morrey [M_2]

$$\underline{u} \in C^\infty(\Omega; S^n) \cap C^{2,\delta}(\overline{\Omega}; S^n) .$$

Our main result is:

THEOREM 1.1. — If γ is not constant then there exist at least two functions in $C^{2,\delta}(\overline{\Omega}; S^n)$ which are solutions of (1.1)-(1.2).

Remarks. — 1) If $u \in C^0(\overline{\Omega}; S^n) \cap H^1(\Omega; \mathbb{R}^{n+1})$ satisfies (1.1) ,u is harmonic; moreover it is well known (see [LU_2], [HW], [Wi]) that $u \in C^\infty(\Omega; S^n)$ and if $u|_{\partial\Omega} \in C^{k,\alpha}(\partial\Omega; S^n)$, (with $0 < \alpha < 1$) $u \in C^{k,\alpha}(\overline{\Omega}; S^n)$. In particular, in our case, if $u \in C^0(\overline{\Omega}; S^n) \cap H^1(\Omega; \mathbb{R}^{n+1})$ is a solution of (1.1)-(1.2) then $u \in C^{2,\delta}(\overline{\Omega}; S^n)$.

2) In the case $n = 2$ theorem 1 has been proved before by H. Brezis-J. M. Coron [BC_2] and J. Jost [J] independently.

In this case, it is possible to assume less regularity on γ; for example $\mathscr{E} \neq \phi$

is sufficient to guarantee at least two solutions in $H^1(\Omega; S^n)$; we do not know if this is the case for $n \geqslant 3$. The difference between $n = 2$ and $n \geqslant 3$ is that $\mathscr{E}$ is not connected when $n = 2$ and connected when $n \geqslant 3$. (To see that $\mathscr{E}$ is connected when $n \geqslant 3$, use the density result due to R. Schoen-K. Uhlenbeck [SU$_2$].)

3) When γ is constant it has been proved by L. Lemaire [LM] that, if $u \in C^0(\overline{\Omega}; S^n) \cap H^1(\Omega; \mathbb{R}^{n+1})$ is a solution of (1.1)-(1.2), then u is identically equal to the same constant.

In order to prove theorem 1.1 we introduce

$$(1.5) \quad \Sigma_p = \{ \sigma \mid \sigma \in C^0(S^{n-2}; W_\gamma^{1,p}(\Omega; S^n)), \sigma \text{ is not homotopic to a constant} \}$$

where $p > 2$,

$$W_\gamma^{1,p}(\Omega; S^n) = \{ u \mid u \in W^{1,p}(\Omega; S^n),\ u = \gamma \quad \text{on} \quad \partial\Omega \}$$

and $C^0(S^{n-1}; W_\gamma^{1,p}(\Omega; S^n))$ is the set of continuous functions from S^{n-2} into $W_\gamma^{1,p}(\Omega; S^n)$. Let

$$(1.6) \qquad \Sigma = \bigcup_{p>2} \Sigma_p$$

and

$$(1.7) \qquad c = \operatorname*{Inf}_{\sigma \in \Sigma} \operatorname*{Max}_{s \in S^{n-2}} E(\sigma(s)).$$

The main result of the paper is the following theorem:

THEOREM 1.2. — Suppose that $\gamma \in C^{2,\delta}(\partial\Omega; S^n)(n \geqslant 2)$ is not constant. Then problem (1.1), (1.2) has at least one solution $u \in C^{2,\delta}(\overline{\Omega}; S^n)$ such that $E(u) = c$; moreover if $c = m$, problem (1.1), (1.2) has infinitely many solutions when $n \geqslant 3$ (and at least two solutions when $n = 2$).

Clearly theorem 1.1 follows from theorem 1.2.

The main difficulty in proving theorem 1.2 comes from a lack of compactness. For this reason we are not able to prove directly that c, defined by (1.7) is a critical value of E (i. e. that there exists u solution of (1.1), (1.2) such that $E(u) = c$). For this reason, following an idea of J. Sacks and K. Uhlenbeck [SU$_1$] we study an approximate problem, i.e. the critical points of the functional

$$(1.8) \qquad E_\alpha(u) = \int_\Omega [(1 + |\nabla u|^2)^\alpha - 1]dx,\ u \in W_\gamma^{1,2\alpha},\ \alpha > 1.$$

This functional satisfies the Palais-Smale condition. Let

$$(1.9) \qquad c_\alpha = \operatorname*{inf}_{\sigma \in \Sigma_{2\alpha}} \operatorname*{Max}_{s \in S^{n-2}} E_\alpha(\sigma(s)).$$

We prove that c_α is a critical value of E_α larger than c and that

$$\lim_{\substack{\alpha \to 1 \\ \alpha > 1}} c_\alpha = c.$$

Just to explain the difficulty let us assume for the moment being that $c > m$. There exists u_α such that

$$E'_\alpha(u_\alpha) = 0$$

and

$$E(u_\alpha) = c_\alpha .$$

Obviously u_α is bounded in H^1 (1) and therefore we can extract a subsequence u_{α_n} which converges weakly in H^1 to some u; u satisfies (1.1)-(1.2) (see [SU$_1$]) and the key point is to prove that $u \neq \underline{u}$. In fact we shall prove that u_{α_n} tends strongly to u and then $E(u) = c > E(\underline{u})$. The proof of the strong convergence relies on some ideas used in [BC$_2$]. We prove the crucial strict inequality

$$c < m + 8\pi$$

then, using a theorem of E. Calabi [C] and arguments involved in J. Sacks-K. Uhlenbeck [SU$_1$] we prove the strong convergence.

Remark. — Similar difficulties and methods also occur in [A], [BC$_1$], [BN], [J], [LB], [LN], [ST], [T] and [W$_2$].

2. A TOPOLOGICAL RESULT

In this section we shall prove a topological result which will be used in the proof of theorem 1.2.

Let $\Omega = \{ x \in \mathbb{R}^2 \mid |x| < 1 \}$ and let M be a C^2-manifold sitting in $\mathbb{R}^k$. Suppose that $\gamma \in C^1(\partial\Omega; M)$ is homotopic to a constant. We set

$$H^1_\gamma(\Omega; M) = \{ u \in H^1(\Omega; \mathbb{R}^k) \mid u|_{\partial\Omega} = \gamma \quad \text{and} \quad u(x) \in M \quad \text{for a.e.} \quad x \in \Omega \}$$

$$C^1_\gamma(\overline{\Omega}; M) = \{ u \in C^1(\overline{\Omega}; M) \mid u|_{\partial\Omega} = \gamma \} .$$

For $w \in H^1_\gamma(\Omega; M)$ we set

$$A_\delta(w) = \{ u \in H^1(\Omega; M) \mid \| u - w \|_{H^1} < \delta \quad \text{and} \quad u = w \quad \text{on} \quad \partial\Omega \} .$$

THEOREM 2.1. — For every $w \in H^1_\gamma(\Omega; M)$ there exist δ, $\varepsilon_0 > 0$ and a continuous map

$$T : [0, \varepsilon_0] \times A_\delta(w) \to H^1_\gamma(\Omega, M)$$

such that

i) $\quad T_0 u = u \quad$ for every $\quad u \in A_\delta(w)$

ii) $\quad T_{\varepsilon_0} u \in C^1_\gamma(\overline{\Omega}; M) \quad$ for every $\quad u \in A_\delta(w)$

iii) $\quad T_{\varepsilon_0} : A_\delta(w) \to C^1_\gamma(\overline{\Omega}; M) \quad$ is continuous

(1) For simplicity we write H^1 instead of $H^1(\Omega; R^{n+1})$.

iv) $\quad T : [0, \varepsilon_0] \times \left[W^{1,p}_\gamma(\Omega ; M) \cap A_\delta(w)\right] \to W^{1,p}_\gamma(\Omega ; M)$
is continuous for every $p \geqslant 2$.

First we shall prove theorem 2.1 in the case in which γ is identically equal to a constant c.

LEMMA 2.2. — If $\gamma \equiv c$ (c is a constant) then the conclusion of theorem 2.1 holds.

Proof. — We extend every map $u \in H^1_c(\Omega ; M)$ to $\mathbb{R}^2$ taking $u(x) \equiv c$ for $x \in \mathbb{R}^2 - \Omega$. We shall denote u and its extension by the same letter.

Let $\phi \in C^\infty(\mathbb{R}^2, [0, +\infty))$ with $\int_{\mathbb{R}^2} \phi = 1$ and

$$\phi(x) = 0 \quad \text{if} \quad x \notin \Omega . \tag{2.1}$$

We set

$$\phi_\varepsilon(x) = \varepsilon^{-2}\phi\left(\frac{|x|}{\varepsilon}\right)$$

and

$$u_\varepsilon(x) = (J_\varepsilon u)(x) = \int \phi_\varepsilon(x - y)u(y)dy . \tag{2.2}$$

We have the following inequality which is due to R. Schoen and K. Uhlenbeck [SU_2]: there exists $c_3 > 0$ such that $\forall \delta > 0 \exists \varepsilon_0 > 0$ such that

$$\text{dist}\,(u_\varepsilon(x), M) \leqslant c_3\delta \quad \text{for every} \quad u \in A_\delta(w) \quad \text{for every} \quad x \in \mathbb{R}^2, \quad \text{for every} \quad \varepsilon \in [0, \varepsilon_0] . \tag{2.3}$$

For the convenience of the reader we recall the proof. In fact, since $u(y) \in M$ for a.e. $y \in \mathbb{R}^2$ we have

$$\text{dist}\,(u_\varepsilon(x), M) \leqslant | u_\varepsilon(x) - u(y) | .$$

By the above formula, for $x \in \mathbb{R}^2$ we get

$$\begin{aligned} \pi\varepsilon^2\, \text{dist}\,(u_\varepsilon(x), M) &\leqslant \int_{|x-y|<\varepsilon} | u_\varepsilon(x) - u(y) |\, dy \\ &\leqslant c_1\varepsilon^2\left[\int_{|x-y|<\varepsilon} | \nabla u(y) |^2 dy\right]^{1/2} \quad \text{(by the Poincaré inequality)} \\ &\leqslant c_1\varepsilon^2\left(\int_{|x-y|<\varepsilon} | \nabla u(y) - \nabla w(y) |^2 dy + \int_{|x-y|<\varepsilon} | \nabla w(y) |^2 dy\right)^{1/2} \\ &\leqslant c_1\varepsilon^2\left(\| u - w \|^2_{H^1(\Omega)} + \int_{|x-y|<\varepsilon} | \nabla w(y) |^2 dy\right)^{1/2} . \end{aligned} \tag{2.4}$$

Since $| \nabla w |^2 \in L^1(\mathbb{R}^2)$, we can choose ε so small that

$$\int_{|x-y|<\varepsilon} | \nabla w(y) |^2 dy \leqslant \delta^2 \quad \text{for every} \quad x \in \mathbb{R}^2 .$$

So by (2.4) and the above inequalities we get

(2.5) $\operatorname{dist}(u_\varepsilon(x), M) \leqslant c_3\delta$ for every $u \in A_\delta(w)$, for $x \in \mathbb{R}^2$

and ε sufficiently small where c_3 is a suitable constant which depends only on the Poincaré constant c_1.

Now let d be a constant such that the projection map

$$P : N_d(M) \to M$$

is well defined. Here $N_d(M) = \{ x \in \mathbb{R}^k \mid \operatorname{dist}(x, M) < \delta \}$.

Now fix $\delta < \dfrac{d}{2c_3}$ and ε_0 small enough in order that (2.3) holds for every $\varepsilon \in (0, \varepsilon_0]$ (and every $x \in \mathbb{R}^k$, every $u \in A_\delta(w)$. Thus the map

$$P \circ J_\varepsilon : A_\delta(w) \to C^1(\mathbb{R}^2, M) \qquad \varepsilon \in (0, \varepsilon_0]$$

is well defined and continuous.

Now consider the map

$$R_\varepsilon : C^1(\mathbb{R}^2, M) \to C^1(\overline{\Omega}, M)$$

defined by

$$(R_\varepsilon u)(x) = u\left(\frac{x}{1+\varepsilon}\right).$$

Clearly R_ε is continuous in u and ε. Moreover, if $u \in P \circ J_\varepsilon(A_\delta(w))$ $(\varepsilon \leqslant \varepsilon_0)$ it is easy to see that $(R_\varepsilon u)|_{\partial\Omega} = c$. Therefore the map

$$T : [0, \varepsilon_0] \times A_\delta(w) \to H^1_\gamma(\Omega; M)$$
$$T_0 = \mathrm{Id}$$
$$T_\varepsilon = R_\varepsilon \circ P \circ J_\varepsilon$$

satisfies the requirements (*i*), (*ii*) and (*iii*).

Moreover one can easily check that T is continuous and moreover satisfy (*iv*). □

Now we shall consider the case in which γ is not constant. Since we have assumed that γ is homotopic to a constant, there exists a homotopy $h \in C^0(I \times \partial\Omega; M)$ such that

$$(2.6) \qquad \begin{cases} (a) & h_0(x) = \gamma(x) \quad \forall x \in \partial\Omega \\ (b) & h_1(x) = c \quad \forall x \in \partial\Omega \quad (c \text{ is a constant}). \end{cases}$$

Since we have assumed γ to be of class C^1, we can suppose that also h is of class C^1.

LEMMA 2.3. — Under our assumptions there exist two continuous functions

$$H : I \times H^1_\gamma(\Omega; M) \to H^1(\Omega; M) \quad \text{with} \quad H_\lambda(u)|_{\partial\Omega} = h_\lambda(\gamma)$$

and

$$K : \{ (\lambda, u) \in I \times H^1(\Omega; M) \mid u|_{\partial\Omega} = h_\lambda(\gamma) \} \to H^1_\gamma(\Omega; M)$$

such that

$$H_0 = K_0 = \text{identity} \quad \text{in} \quad H^1_\gamma(\Omega; M).$$

Moreover H and K are continuous also in the $W^{1,p}(\Omega; M)$ topology.

Proof. — For $u \in H^1_\gamma(\Omega; M)$ set

$$\tilde{u}(x) = \begin{cases} u(x) & \text{for} \quad |x| \leqslant 1 \\ h_{|x|-1}\left(\dfrac{x}{|x|}\right) & \text{for} \quad 1 \leqslant |x| \leqslant 2 . \end{cases}$$

By virtue of (2.6) (*a*) $\tilde{u} \in H^1_c(\Omega_1; M)$ where $\Omega_1 = \{ x \in \mathbb{R}^2 \mid |x| < 2 \}$ and of course it depends continuously on $u \in H^1_\gamma(\Omega; M)$.

For $v \in H^1_{h_\lambda(\gamma)}(\Omega; M)$ we set

$$\tilde{v}_\lambda = \begin{cases} v(x) & \text{for} \quad |x| \leqslant 1 \\ h_{\lambda(2-|x|)}\left(\dfrac{x}{|x|}\right) & \text{for} \quad 1 \leqslant |x| \leqslant 2 . \end{cases}$$

Clearly $\tilde{v} \in H^1(\Omega; M)$.

Finally for $x \in \Omega$ set

$$(H_\lambda u)(x) = \tilde{u}((1 + \lambda)x) \qquad u \in H^1_\gamma(\Omega; M)$$

$$(K_\lambda v)(x) = \tilde{v}_\lambda((1 + \lambda)x) \qquad v \in H^1_{h_\lambda(\gamma)}(\Omega; M).$$

It is easy to check that H_λ and K_λ satisfy the required conditions.

Proof of theorem 2.1. — Let H be the map defined in lemma 2.3. Then $H_1(w) \in H^1_c(\Omega; M)$.

By lemma 2.2, there exists $\tilde{\delta}$, $\tilde{\varepsilon}_0 > 0$ and a continuous map

$$\tilde{T} : [0, \tilde{\varepsilon}_0] \times A_{\tilde{\delta}}(H_1(w)) \to H^1_c(\Omega; M)$$

which satisfies (*i*), (*ii*), (*iii*) and (*iv*) of theorem 2.1.

Since $H_1 : H^1_\gamma(\Omega; M) \to H^1_c(\Omega; M)$ is continuous, there exists $\delta > 0$ such that

$$H_1(A_\delta(w)) \subset A_{\tilde{\delta}}(H_1(w)).$$

Therefore it makes sense to define a map $T: [0, 1+\varepsilon_0] \times A_\delta(w) \to H^1_\gamma(\Omega, M)$ as follows

$$T_\lambda(u) \begin{cases} K_\lambda \circ H_\lambda(u) & \text{for} \quad \lambda \in [0, 1] \\ K_1 \circ \tilde{T}_{\lambda-1} \circ H_1(u) & \text{for} \quad \lambda \in [1, 1 + \tilde{\varepsilon}_0]. \end{cases}$$

Such a map satisfies (*i*), (*ii*), (*iii*) and (*iv*) of Theorem 2.1 with $\varepsilon_0 = 1 + \tilde{\varepsilon}_0$. □

LEMMA 2.3. — Let $z \in C^1_\gamma(\overline{\Omega}; M)$ and set

$$N_\eta(z) = \{ u \in C^1_\gamma(\overline{\Omega}; M) \mid \| z - u \|_{C^1} < \eta \}.$$

Then if η is sufficiently small, $N_\eta(z)$ is a strong deformation retract of $\{ z \}$ for every $z \in C^1_\gamma(\overline{\Omega}; M)$.

Proof. — Choose η small enough in order that $B_\eta(y) \cap M$ is geodesically convex in M for every $y \in M$; ($B_r(y) = \{ x \in \mathbb{R}^k \mid | s - y | < r \}$). Then for $x \in B_\eta(y)$ we define:

$$h_t(y, x) = \beta(t)$$

where $\beta(t)$ is the (unique) geodesic on M parametrized with the arc length such that

$$\beta(0) = y \qquad \text{and} \qquad \beta(1) = x.$$

So if M is a smooth manifold h is smooth.

For $u \in N_\eta(z)$ we set

$$S_t(u)(x) = h_t(z(x), u(x)).$$

Clearly $S : I \times N_\eta(z) \to C^1_\gamma(\overline{\Omega}: M)$ is continuous, $S_0 \equiv \mathrm{Id}_{N_\eta(z)}$; $S_1(u) = z$ for every $u \in N_\eta(z)$ and $S_t(z) = z$ for every $t \in [0, 1]$. □

By theorem 2.1 and lemma 2.3 the following Corollary follows which will be used in the proof of our main theorem.

COROLLARY 2.3. — For every $w \in H^1_\gamma(\Omega; M)$ there is a constant $\theta > 0$ such that $A_\theta(w) \cap W^{1,p}(\Omega; M)$ is contractible to a point in $W^{1,p}(\Omega; M)$, $p \geqslant 2$.

Proof. — By theorem 2.1 there exists a continuous map T_{ε_0} : $A_\delta(w) \to C^1_\gamma(\Omega; M)$. So given η as in lemma 2.3, there exists $\theta \in [0, \delta]$ such that $T_{\varepsilon_0}(A_\theta(w)) \subset N_\eta(T_{\varepsilon_0}(w))$.

By lemma 2.3, $N_\eta(T_{\varepsilon_0}(w))$ is contractible, then also $A_\theta(w) \cap W^{1,p}(\Omega; M)$ is contractible to a point in $W^{1,p}(\Omega; M)$. □

3. A CONVERGENCE THEOREM

In order to approximate the solutions of problem (1.1), (1.2) by the critical points of the functional (1.8) we need the following theorem which has been inspired by J. Sacks and K. Uhlenbeck [SU₁].

THEOREM 3.1. — For every $\alpha > 1$ let $u_\alpha \in \mathscr{E}_\alpha$ be a solution of

$$E'_\alpha(u_\alpha) = 0 \tag{3.1}$$

and suppose that

$$\lim_{\alpha \downarrow 1} E(u_\alpha) < m + 8\pi. \tag{3.2}$$

Then u_α has a subsequence $u_{\alpha_k} \to u$ in $C^1(\overline{\Omega}; S^n)$ and u is a solution of (1.1).

In order to prove theorem 3.1 we need the following proposition due to J. Sacks and K. Uhlenbeck [SU_1].

PROPOSITION 3.1. — There exist $\alpha_0 > 1$ such that if $u \in \mathscr{E}_\alpha$ with $1 \leqslant \alpha < \alpha_0$ and $E'_\alpha(u) = 0$ then $u \in C^{2,\delta}(\overline{\Omega})$.

Proof. — See the proof of proposition 2.3 in [SU_1]. In fact in [SU_1] only the interior regularity is proved. But the theorem 1.11. 1′ of Morrey [M_2] which is used in [SU_1] is still valid up to the boundary if z is assumed to be in H_0^1 (see p. 38 in [M_2]). Therefore we may apply this theorem to $z = u - \phi$ where $\phi \in C^{2,\delta}(\overline{\Omega})$ with $\phi = \gamma$ on $\partial\Omega$. We conclude that $\nabla u \in H^1$. The conclusion of the proof is an easy adaptation of the proof in [SU_1]. □

Proof of theorem 3.1. — In what follows we will always assume that $1 < \alpha \leqslant \alpha_0$. Since u_α is bounded in L^∞ and $E(u_\alpha)$ is bounded, u_α is bounded in H^1. Therefore there exist a sequence $(\alpha_k)_{k\in\mathbb{N}}$ such that u_{α_k} tends weakly in H^1 to some u. For simplicity we shall write u_k instead of u_{α_k}. Using (3.1) (and Proposition (3.1)) we have

$$(3.3) \qquad -\Delta u_k - 2\frac{\alpha_k - 1}{(1 + |\nabla u_k|^2)}(\nabla u_k, \nabla u_k, \nabla^2 u_k) = u_k |\nabla u_k|^2$$

where

$$(\nabla u_k, \nabla u_k, \nabla^2 u_k) = \sum_{\substack{1 \leqslant i \leqslant 2 \\ 1 \leqslant j \leqslant 2 \\ 1 \leqslant p \leqslant n+1 \\ 1 \leqslant q \leqslant n+1}} \frac{\partial^2 u_k^p}{\partial x_i \partial x_j} \frac{\partial u_k^p}{\partial x_i} \frac{\partial u_k^q}{\partial x_j} e_q$$

and

$$u_k = (u_k^1, \ldots, u_k^q, \ldots, u_k^{n+1}) = \sum_{q=1}^{n+1} u_k^q e_q .$$

Let

$$\theta_k = \operatorname*{Max}_{x \in \overline{\Omega}} |\nabla u_k(x)| .$$

First let us assume that θ_k is bounded.

We are going to prove that in this case u_k tends to u in $C^1(\overline{\Omega})$ and that:

$$-\Delta u = u |\nabla u|^2 .$$

Using (3.3) we have:

$$(3.4) \quad -\Delta u_k^p + (\alpha_k - 1) \sum_{\substack{1 \leqslant i \leqslant 2 \\ 1 \leqslant j \leqslant 2 \\ 1 \leqslant q \leqslant n+1}} A_{ijk}^{pq} \frac{\partial^2 u_k^q}{\partial x_i \partial x_j} = u_k^p |\nabla u_k|^2 \quad 1 \leqslant p \leqslant n+1$$

with

$$(3.5) \qquad \| A_{ijk}^{pq} \|_{C^0(\overline{\Omega})} \leqslant C .$$

Since θ_k is bounded we have:

$$\| u_k \mid \nabla u_k \mid^2 \|_{C^0(\overline{\Omega})} \leqslant C. \tag{3.6}$$

It follows from (3.4), (3.5), (3.6) and a theorem of Morrey [M_1] (see also [N]) that:

$$\exists \gamma > 0 \quad \text{such that} \quad \| u_k \|_{C^{1,\gamma}(\overline{\Omega})} \leqslant C. \tag{3.7}$$

(Actually in [M_1] and [N] the theorems are stated for one equation and not for a system. But the proofs can be easily adapted to the system (3.4).) It follows from (3.7) that u_k tends to u in $C^1(\overline{\Omega})$. Moreover (3.3) may be written in the following divergence form:

$$-\frac{\partial}{\partial x_i}\left((1 + \mid \nabla u_k \mid^2)^{\alpha_k - 1}\frac{\partial u_k}{\partial x_i}\right) = u_k \mid \nabla u_k \mid^2 (1 + \mid \nabla u_k \mid^2)^{\alpha_k - 1}, \qquad i = 1, 2.$$

Using the convergence of u_k to u in $C^1(\overline{\Omega})$ we have

$$-\Delta u = u \mid \nabla u \mid^2. \tag{3.8}$$

Now we want to show that

$$\lim_{k \to +\infty} \theta_k = +\infty. \tag{3.9}$$

is not possible. We argue indirectly and suppose that (3.9) holds. Let $a_k \in \overline{\Omega}$ such that

$$\theta_k = \mid \nabla u_k(a_k) \mid.$$

After extracting a subsequence we may assume that either

$$\lim_{k \to +\infty} \theta_k d(a_k, \partial\Omega) = +\infty \tag{3.10}$$

or

$$\lim_{k \to +\infty} \theta_k d(a_k, \partial\Omega) = \rho < +\infty \tag{3.11}$$

where $d(a_k, \partial\Omega)$ is the distance from a_k to $\partial\Omega$.

First let us assume that (3.10) holds. Then, like in [SU_1], we define

$$v_k(x) = u_k\left(\frac{x}{\theta_k} + a_k\right).$$

v_k is defined on $\overline{\Omega}_k$ where

$$\Omega_k = \{ \theta_k(y - a_k) \mid y \in \Omega \}.$$

Using (3.10) it is easy to see that

$$\forall R > 0 \quad \exists k(R) \quad \text{such that} \quad k \geqslant k(R) \Rightarrow B(0, R) \subset \Omega_k \tag{3.12}$$

where $B(0, R) = \{ x \in \mathbb{R}^2 \mid \mid x \mid \leqslant R \}$. Moreover it follows from (3.3) that, in Ω_k,

$$-\Delta v_k - 2\frac{\alpha_k - 1}{(\theta_k^{-2} + \mid \nabla v_k \mid^2)}(\nabla v_k, \nabla v_k, \nabla^2 v_k) = v_k \mid \nabla v_k \mid^2. \tag{3.13}$$

We have

$$\| \nabla v_k \|_{C^{\alpha}(\bar{\Omega}_k)} \leqslant 1 . \tag{3.14}$$

As before it follows from (3.12), (3.13), (3.14) and [M_1] (or [N]) that there exists $\gamma > 0$ such that $\forall R > 0$ C(R) such that

$$\| v_k \|_{C^{1,\gamma}(B(0,R))} \leqslant C(R) \ \forall k . \tag{3.15}$$

Therefore (after extracting a subsequence) we have

$$v_k \to v \quad \text{in} \quad C^1(B(0, R)) \ \forall R \tag{3.16}$$

and in particular

$$| \nabla v(0) | = \lim_{k \to +\infty} | \nabla v_k(0) | = 1 . \tag{3.17}$$

We write (3.13) in a divergence form:

$$-\frac{\partial}{\partial x_i}\left((1+\theta_k^2 \,|\, \nabla v_k \,|^2)^{\alpha_k - 1} \frac{\partial v_k}{\partial x_i}\right) = v_k \,|\, \nabla v_k \,|^2 (1+\theta_k \,|\, v_k \,|^2)^{\alpha_k - 1} \quad i=1,2. \tag{3.18}$$

From (3.16) and (3.18) we get

$$- \Delta v = v \,|\, \nabla v \,|^2 . \tag{3.19}$$

Moreover

$$\int_{\Omega_k} | \nabla v_k |^2 = \int_{\Omega} | \nabla u_k |^2 \leqslant c ,$$

thus

$$\int_{\mathbb{R}^2} | \nabla v |^2 < + \infty . \tag{3.20}$$

From (3.19), (3.20) and [SU_1] (theorem 3.6) it follows that v can be extended to a regular harmonic map from $\mathbb{R}^2 \cup \{ \infty \} = S^2$ into S^n.

The following theorem is due to E. Calabi [C] (theorem 5.5):

THEOREM. — Let v be a harmonic map from S^2 into S^m whose image does not lie in any equatorial hyperplane of S^m then

i) the area $A(v)$ of $v(S^2)$ is an integer multiple of 2π

ii) m is even, and $A(v) \geqslant \dfrac{m(m-2)}{2}\pi$.

Remark. — In [C] v is assumed to be an immersion but the proof given in [C] works also if v is not an immersion (note that the points where v is not an immersion are isolated and branch points, see e. g. [GOR]).

Proof of Theorem 3.1 continued. — Any harmonic map w from S^2 into S^2 which is not constant satisfies (see, for example [L] theorem (8.4))

$$E(w) \geqslant 8\pi .$$

Therefore if w is a harmonic map from S^2 into S^n which is not constant, using the Calabi theorem and an easy induction argument we have

$$E(w) \geqslant 8\pi .$$

(we recall that $E(w) \geqslant 2A(w)$).

Our map v is a harmonic map from S^2 into S^n and (see (3.17)) v is not constant. Therefore

$$E(v) \geqslant 8\pi . \tag{3.21}$$

We are going to prove (as in [SU$_1$]) that

$$\varliminf_{k\to+\infty} E(u_k) \geqslant E(u) + E(v) . \tag{3.22}$$

Since by definition of m (see (1.4))

$$E(u) \geqslant m \tag{3.23}$$

using (3.21), (3.22), (3.23) and (3.2) we obtain a contradiction.

We may assume that a_k tends to some a in $\overline{\Omega}$. Let $\varepsilon > 0$ and $r > 0$ such that

$$\int_{D(a,r)} |\nabla u|^2 \leqslant \varepsilon \tag{3.24}$$

where

$$D(a, r) = \{ x\in\Omega \mid |x - a| \leqslant r \} .$$

We have

$$\int_{D(a,r)} |\nabla u_k|^2 = \int_{C_k} |\nabla v_k|^2 \tag{3.25}$$

where

$$C_k = \{ \theta_k^{1/2}(y - a_k) \mid y \in D(a, r) \} .$$

Using (3.10) we have

$$\forall R > 0 \quad \exists k(R) \quad \text{such that} \quad k \geqslant k(R) \Rightarrow B(0, R) \subset C_k .$$

Therefore

$$\varliminf_{k\to+\infty} \int_{C_k} |\nabla v_k|^2 \geqslant E(v) . \tag{3.26}$$

From (3.24), (3.25) and (3.26) we have

$$\varliminf_{k\to+\infty} E(u_k) \geqslant E(u) + E(v) - \varepsilon \quad (\forall\varepsilon > 0)$$

which proves (3.22).

Now it remains to exclude (3.11). We assume that (3.11) holds. Now (3.12) is false.

We may assume that a_k tends to some a. Using (3.5) and (3.9) we see that $a \in \partial\Omega$; without loss of generality we may assume that

$$\lim_{k\to+\infty} a_k = (-1,0) = a .$$

Let $T : \mathbb{R}^2 - \{(1,0)\} \to \mathbb{R}^2$

(3.27) $$T(x_1, x_2) = \left(-\frac{x_1 - 1}{(x_1-1)^2 + x_2^2}, \frac{x_2}{(x_1 - 1)^2 + x_2^2} \right) = (\bar{x}_1, \bar{x}_2).$$

T is a conformal diffeomorphism between $\Omega - \{(1,0)\}$ and $\left]\frac{1}{2}, +\infty\right[\times \mathbb{R}$ and

$$T(\partial\Omega - \{(1,0)\}) = \left\{\frac{1}{2}\right\} \times \mathbb{R}$$

(3.28) $$T^{-1}((\bar{x}_1, \bar{x}_2)) = \left(1 - \frac{\bar{x}_1}{\bar{x}_1^2 + \bar{x}_2^2}, \frac{\bar{x}_2}{\bar{x}_1^2 + \bar{x}_2^2}\right).$$

Let $U = \left]\frac{1}{2}, +\infty\right[\times \mathbb{R}$ and let

$$\bar{u}_k = u_k \circ T^{-1}.$$

Clearly

$$\bar{u}_k \in C^1(\bar{U}),$$

and a straightforward computation yields

$$\Delta u_k(x) = |\bar{x}|^4 \Delta\bar{u}_k(\bar{x})$$
$$|\nabla u_k|^2(x) = |\bar{x}|^4 |\nabla\bar{u}_k|^2(\bar{x})$$

where $\bar{x} = Tx$.

In particular:

(3.29) $$\|\nabla\bar{u}_k\|_{C^0(\bar{U})} \leqslant 4\theta_k$$

and

(3.30) $$|\nabla\bar{u}_k(\bar{a}_k)| = \frac{\theta_k}{|\bar{a}_k|^2} \sim 4\theta_k \quad as \quad k \to \infty,$$

where $\bar{a}_k = Ta_k$.

Using (3.3) we find $(1 \leqslant p \leqslant n)$:

(3.31) $$-\Delta\bar{u}_k^p + (\alpha_k - 1) \sum_{\substack{1 \leqslant i \leqslant 2 \\ 1 \leqslant j \leqslant 2 \\ 1 \leqslant q \leqslant n+1}} B_{ijk}^{pq} \frac{\partial^2 \bar{u}_k^q}{\partial x_i \partial x_j} = \bar{u}_k^p |\nabla\bar{u}_k|^2 + (\alpha_k - 1) \sum_{\substack{1 \leqslant i \leqslant 2 \\ 1 \leqslant q \leqslant n+1}} C_{ik}^{pq} \frac{\partial \bar{u}_k^q}{\partial x_i}$$

where

$$B_{ijk}^{pq} \in C^0(\bar{U}), \qquad C_{ik}^{pq} \in C^0(\bar{U})$$

and

(3.32) $$\|B_{ijk}^{pq}\|_{C^0(\bar{U})} \leqslant C, \qquad \|C_{ik}^{pq}\|_{C^0(\bar{U})} \leqslant C.$$

We have

$$\overline{u}_k = \overline{\gamma} \quad \text{on} \quad \partial\Omega$$

with

$$\overline{\gamma}\left(\frac{1}{2}, t\right) = \gamma\left(\frac{4t^2 - 1}{4t^2 + 1}, \frac{4t}{4t^2 + 1}\right).$$

If $\overline{a}_k = (\overline{x}_k, \overline{y}_k)$ and $a_k = (x_k, y_k)$, using (3.28) we obtain:

$$\begin{aligned}\overline{x}_k - \frac{1}{2} &= \frac{1 - x_k}{(x_k - 1)^2 + y_k^2} - \frac{1}{2} \\ &= \frac{1 - (x_k^2 + y_k^2)}{2[(x_k - 1)^2 + y_k^2]}.\end{aligned}$$

Then, by (3.11), we have:

$$\overline{x}_k = \frac{1}{2} + \frac{\rho}{4\theta_k} + \frac{o(1)}{\theta_k}(k \to +\infty). \tag{3.33}$$

Let

$$\widetilde{u}_k(\widetilde{x}, \widetilde{y}) = \overline{u}_k\left(\frac{1}{2} + \frac{1}{\theta_k}\left(\widetilde{x} - \frac{1}{2}\right), \frac{\widetilde{y}}{\theta_k} + k\right).$$

We have $\widetilde{u}_k = \widetilde{\gamma}_k$ on $\partial\Omega$ with

$$\widetilde{\gamma}_k\left(\frac{1}{2}, t\right) = \overline{\gamma}\left(\frac{1}{2}, \frac{t}{\theta_k} + y_k\right)$$

and thus

$$\widetilde{\gamma}_k \to \overline{\gamma}\left(\frac{1}{2}, 0\right) \quad \text{in} \quad C^{2,\delta}(\partial U). \tag{3.34}$$

Using (3.29) we have

$$\|\nabla \widetilde{u}_k\|_{C^0(\overline{U})} \leqslant 4. \tag{3.35}$$

Using (3.31) and (3.32) we have (for $1 \leqslant p \leqslant n$):

$$\left\{\begin{aligned} &- \Delta\widetilde{u}_k^p + (\alpha_k - 1) \sum_{\substack{1 \leqslant i \leqslant 2 \\ 1 \leqslant j \leqslant 2 \\ 1 \leqslant q \leqslant n+1}} \widetilde{B}_{ijk}^{pq} \frac{\partial^2 \widetilde{u}_k^q}{\partial x_i \partial x_j} \\ &= \widetilde{u}_k^q |\nabla \widetilde{u}_k|^2 + \frac{1}{\theta_k}(\alpha_k - 1) \sum_{\substack{1 \leqslant i \leqslant 2 \\ 1 \leqslant q \leqslant n+1}} \widetilde{C}_{ik}^{pq} \frac{\partial \widetilde{u}_k^q}{\partial x_i}\end{aligned}\right. \tag{3.36}$$

with

$$\|\widetilde{B}_{ijk}^{pq}\|_{C^0(\overline{U})} \leqslant C, \qquad \|\widetilde{C}_{ik}^{pq}\|_{C^0(\overline{U})} \leqslant C. \tag{3.37}$$

Let $R > 0$ and $U_R = U \cap \{ x \in \mathbb{R}^2 \mid |x| < R \}$. Using (3.34), (3.35), (3.36), (3.37) and the Morrey-Nirenberg estimate $[M_1]$, [N] we obtain:

$$(3.38) \quad \exists \alpha > 0 \quad \exists C(R) \quad \text{such that} \quad \| \tilde{u}_k \|_{C^{1,\alpha}(\overline{U}_R)} \leqslant C(R), \qquad \forall k.$$

Remark. — Actually in $[M_1]$ there is no estimate up to the boundary but such an estimate can be deduced from the interior estimate, see [GT] (p. 248-249). One can find estimate up to the boundary in $[LU_1]$ (p. 455-456) and [N]. In all these references the theorems are stated for only one equation but the proofs can be easily adapted to our system (3.36).

Proof of Theorem 3.1 concluded. — We may assume that for some $\tilde{u}$ in $C^{1,\alpha}(\overline{U})$:

$$(3.39) \qquad \lim_{k \to +\infty} \| \tilde{u}_k - \tilde{u} \|_{C^1(\overline{U}_R)} = 0 .$$

Moreover, using (3.36), (3.37), (3.35) it is easy to see that if ω is a bounded regular open set of U such that $\overline{\omega} \subset U$ then

$$\| \tilde{u}_k \|_{W^{2,2}(\omega)} \leqslant C(\omega) .$$

Therefore, using (3.36) we have:

$$(3.40) \qquad - \Delta \tilde{u} = \tilde{u} \, | \nabla \tilde{u} |^2 \quad \text{in} \quad U .$$

With (3.34) we get

$$(3.41) \qquad \tilde{u} = \overline{\gamma}\left(\frac{1}{2}, 0\right) \quad \text{on} \quad \partial U .$$

Moreover

$$\int_\Omega | \nabla u_k |^2 = \int_U | \nabla \overline{u}_k |^2 = \int_U | \nabla \tilde{u}_k |^2 ,$$

therefore:

$$(3.42) \qquad \int_U | \nabla \tilde{u} |^2 < + \infty .$$

We recall that $\tilde{u} \in C^0(\overline{U})$ (and even $\in C^{1,\alpha}(\overline{U})$). Then using (3.40), (3.41), (3.42) and a very slight modification of a theorem of L. Lemaire (see the appendix we have

$$(3.43) \qquad \tilde{u} \equiv \overline{\gamma}\left(\frac{1}{2}, 0\right).$$

But, using (3.30):

$$(3.44) \qquad \lim_{k \to +\infty} \left| \nabla \tilde{u}_k\left(\theta_k\left(\overline{x}_k - \frac{1}{2}\right) + \frac{1}{2}, 0\right) \right| = \sqrt{2}$$

and using (3.33):

$$(3.45) \qquad \lim_{k \to +\infty} \theta_k\left(\overline{x}_k - \frac{1}{2}\right) + \frac{1}{2} = \frac{1}{2} + \frac{\rho}{4}$$

and then using (3.39), (3.43), (3.44), (3.45) we get a contradiction. □

4. PROOF OF THEOREM 1.2

The proof of theorem 1.2 relies on several lemmas.

LEMMA 4.1. — Let m and c be the constants defined by (1.4) and (1.7) respectively. Then

$$c < m + 8\pi .$$

Proof. — We shall construct a map $\sigma_\varepsilon \in \Sigma_3$ such that

$$E(\sigma_\varepsilon(s)) < m + 8\pi . \tag{4.1}$$

Then the conclusion follows from the definition of c. The construction of such a map is an adaptation of the proof of lemma 2 in [BC_2].

Let $\underline{u} \in \mathscr{E}$ such that $E(\underline{u}) = m$. Thanks to Morrey's regularity result $\underline{u} \in C^\infty(\Omega ; \mathbb{R}^{n+1}) \cap C^{2,\varepsilon}(\overline{\Omega}, \mathbb{R}^{n+1})$. Since γ is not constant $\underline{u}$ is not constant and therefore $\nabla \underline{u}(x_0, y_0) \neq 0$ for some (x_0, y_0) in Ω; rotating coordinates in $\mathbb{R}^2$ we may always assume that

$$\underline{u}_x(x_0, y_0) . \underline{u}_y(x_0, y_0) = 0 .$$

Let $(e_i)_{1 \leqslant i \leqslant n+1}$ be an orthonormal basis in $\mathbb{R}^{n+1}$ such that:

$$\underline{u}_x(x_0, y_0) = ae_1$$
$$\underline{u}_y(x_0, y_0) = be_2$$
$$\underline{u}(x_0, y_0) = e_3$$

with $a \geqslant 0, b \geqslant 0, a+b > 0$.

We shall identify S^{n-2} with $S^n \cap \{ v \in S^n \mid v.e_1 = 0, v.e_2 = 0 \}$. Let r and θ be such that $x - x_0 = r \cos \theta$, $y - y_0 = r \sin \theta$. Let $\varepsilon > 0$ be small enough. Let $\lambda = \frac{1}{2} \varepsilon^2 \operatorname{Max}(a, b) > 0$.

We define a map $\sigma_\varepsilon \in C^0(S^{n-2} ; W^{1,3}_\gamma(\Omega ; S^n))$ in the following way (where $s \in S^{n-2}$):

if $\quad 2\varepsilon < r, \quad \sigma_\varepsilon(s)(x, y) = \underline{u}(x, y)$

if $\quad \lambda < r < \varepsilon, \quad \sigma_\varepsilon(s)(x, y) = \dfrac{2\lambda}{\lambda^2 + r^2}(x - x_0)e_1 + \dfrac{2\lambda}{\lambda^2 + r^2}(y - y_0)e_2 + \dfrac{r^2 - \lambda^2}{\lambda^2 + r^2} e_3$

if $\quad r < \lambda, \quad \sigma_\varepsilon(s)(x, y) = \dfrac{2\lambda}{\lambda^2 + r^2}(x - x_0)e_1 + \dfrac{2\lambda}{\lambda^2 + r^2}(y - y_0)e_2 + \dfrac{r^2 - \lambda^2}{\lambda^2 + r^2} s$

if $\quad \varepsilon < r < 2\varepsilon, \quad \sigma_\varepsilon(s)(x, y) = \displaystyle\sum_{\substack{i=1 \\ i \neq 3}}^{n+1} (A_i r + B_i)e_i + \left[1 - \sum_{i=1}^{n+1} (A_i r + B_i)^2 \right]^{1/2} e_3$

where A_i and B_i depend only on θ and ε and are such that $\sigma_\varepsilon(s)$ is continuous at $r = \varepsilon$ and $r = 2\varepsilon$ for each s. More precisely

$$\begin{aligned}
&2\varepsilon A_i + B_i = \underline{u}^i(x_0 + 2\varepsilon \cos \theta, y_0 + 2\varepsilon \sin \theta), \quad 1 \leqslant i \leqslant n + 1 \\
&\varepsilon A_1 + B_1 \frac{2\lambda\varepsilon}{\lambda^2 + \varepsilon^2} \cos \theta \\
&\varepsilon A_2 + B_2 = \frac{2\lambda\varepsilon}{\lambda^2 + \varepsilon^2} \sin \theta \\
&\varepsilon A_i + B_i = 0, \qquad 3 \leqslant i \leqslant n + 1 .
\end{aligned}$$

Since $\underline{u} \in W^{1,3}(\Omega ; S^n)$, $\sigma_\varepsilon \in C^0(S^{n-2}, W_\gamma^{1,3}(\Omega ; S^n))$. Moreover

$$E(\sigma_\varepsilon(s)) = E(\sigma_\varepsilon(e_3)) \qquad \text{for every} \quad s \in S^{n-2},$$

and a straightforward computation leads to

$$E(\sigma_\varepsilon(e_3)) = E(\underline{u}) + 8\pi - \nu\varepsilon^2 + o(\varepsilon^2), \quad (\varepsilon \to 0),$$

where $\nu > 0$ (see [BC_2]).

Therefore we can fix ε small enough in order that

$$E(\sigma(s)) < E(\underline{u}) + 8\pi$$

where $\sigma = \sigma_\varepsilon$.

It remains to prove that $\sigma \in \Sigma_\alpha\left(1 < \alpha \leqslant \frac{3}{2}\right)$ i.e. that σ is an essential map. We argue indirectly. Suppose that σ is not essential. Then there exists a continuous map $\bar{\sigma}$

$$\bar{\sigma} : I \times S^{n-2} \to W_\gamma^{1,2\alpha}(\Omega ; S^n) \qquad (I = [0, 1])$$

such that

$$\begin{aligned}
&\bar{\sigma}(0, .) = \sigma(.) ; \\
&\bar{\sigma}(1, s) = u
\end{aligned}$$

for every $s \in S^{n-2}$ where $u \in W_\gamma^{1,2\alpha}(\Omega ; S^n)$.

Now we define $\eta : I \times \bar{\Omega} \times S^{n-2} \to S^n$ as follows:

$$\eta(t, x, y, s) = \bar{\sigma}(t, s)(x, y) .$$

Clearly η is continuous in all its variables and we have:

$$(4.2) \quad \begin{cases} a) \quad \eta(0, x, y, s) = \sigma(s)(x, y) \\ b) \quad \eta(1, x, y, s) = u(x, y) \\ c) \quad \eta(t, x, y, s) = \gamma(x, y) \quad \forall (x, y) \in \partial\Omega, \quad \forall t \in I, \quad \forall s \in S^{n-2} . \end{cases}$$

Our next step is to extend η to a map

$$\zeta : I \times \partial(\Omega \times B^{n-1}) \to S^n$$

as follows

$$\zeta(t, x, y, s) = \begin{cases} \eta(t, x, y, s) & \text{if} \quad (x, y) \in \Omega \quad \text{and} \quad s \in \partial B^{n-1} = S^{n-2} \\ \gamma(x, y) & \text{if} \quad (x, y) \in \partial\Omega \quad \text{and} \quad s \in B^{n-1}. \end{cases}$$

By (4.2) (*c*) it follows that ζ is continuous. Since $\partial(\Omega \times B^{n-1})$ is topologically equivalent to S^n the topological degree of $\zeta(t, .)$ is well defined for every $t \in I$. We shall compute it for $t = 0$ and $t = 1$. To this end we extend $\zeta(t, .)$ to a map

$$\theta(t, .) : \overline{\Omega} \times B^{n-1} \to R^{n+1}$$

since

$$(4.3) \qquad \deg(\zeta(t, .)) = \deg(\theta(t, .), \Omega \times B^{n-1}, w)$$

for every $w \in \operatorname{int}(B^{n+1})$. For $t = 1$ we set

$$\theta(1, x, y, z) = u(x, y).$$

Then by (4.3) it follows that

$$(4.4) \qquad \deg(\zeta(1, .)) = 0$$

since $\theta(1, x, y, z)$ is independent of z. For $t = 0$ we set

$$\theta(0, x, y, z) = \begin{cases} \eta(0, x, y, s_0) = \sigma(s_0)(x, y) \quad \text{if} \quad r \geqslant \lambda, \quad s_0 \in S^{n-2} \quad \text{fixed} \\ \dfrac{2\lambda}{\lambda^2 + r^2}(x - x_0)e_1 + \dfrac{2\lambda}{\lambda^2 + r^2} y - y_0 e_2 + \dfrac{r^2 - \lambda^2}{\lambda^2 + r^2} z \quad \text{if} \quad r < \lambda \end{cases}$$

where $r = [(x - x_0)^2 + (y - y_0)^2]^{1/2}$ and we shall compute

$$\deg(\theta(0, .), \Omega \times B^{n-1}, w) \qquad \text{with} \qquad w = 0$$

First notice that $|w| < 1$, so the degree is well defined and it is equal to the algebraic sum of the nondegenerate solutions of the equation

$$(4.5) \qquad \begin{cases} (x, y, z) \in \overline{\Omega} \times B^{n-1} \\ \theta(0, x, y, z) = w. \end{cases}$$

Since $|w| < 1$ and $|\theta(0, x, y, z)| = 1$ for $|(x, y)| \geqslant \lambda$ the solutions of (4.5) are the same that the solutions of the following equation

$$(4.6) \qquad \begin{cases} (x, y, z) \in \overline{\Omega} \times B^{n-1} \\ |(x, y)| \leqslant \lambda \\ \dfrac{2\lambda}{\lambda^2 + r^2}[(x - x_0)e_1 + (y - y_0)e_2] + \dfrac{r^2 - \lambda^2}{\lambda^2 + r^2} z = w. \end{cases}$$

By inspection we see that the only solution of (4.6) is $x = x_0$, $y = y_0$, $z = 0$, and that it is not degenerate. Therefore $\deg(\zeta(0, .)) = \pm 1$ and this contradicts (4.4). □

We now set

$$c_\alpha = \inf_{\sigma \in \Sigma_{2\alpha}} \sup_{s \in S^{n-2}} E_\alpha \circ \sigma(s) \tag{4.7}$$

where $\Sigma_{2\alpha}$ is defined by (1.5).

LEMMA 4.2. — For every $\alpha > 1$, the c_α's defined by (4.7) are critical values of E_α. Moreover $c_\alpha \to c$ for $\alpha \to 1$ and $c_\alpha \geqslant c$.

Proof. — It is straightforward to check that E_α satisfies the assumption (c) of Palais-Smale on $\mathscr{E}_\alpha$. Then by well known facts about the critical point theory the c_α's are critical values of E_α.

Now we shall prove the second statement. Since $E_\alpha(u) > E(u)$ for every $u \in \mathscr{E}_\alpha$, we have that

$$\begin{aligned} c_\alpha &\geqslant \inf_{\sigma \in \Sigma_{2\alpha}} \sup_{s \in S^{n-2}} E \circ \sigma(s) \\ &\geqslant \inf_{\sigma \in \Sigma} \sup_{s \in S^{n-2}} E \circ \sigma(s) = c \quad (\text{since } \Sigma_{2\alpha} \subset \Sigma). \end{aligned}$$

Thus $c_\alpha \geqslant c$ for every $\alpha > 1$.

Now let us prove that $c_\alpha \to c$. Choose $\varepsilon > 0$. Then there exists $p > 2$ and $\bar{\sigma} \in \Sigma_p$ such that

$$c + \varepsilon > \sup_{u \in \bar{\sigma}(S^{n-2})} E(u). \tag{4.8}$$

For $u \in \bar{\sigma}(S^{n-2}) \subset \mathscr{E}_\alpha$ with $\alpha < p/2$ we have

$$\frac{d}{d\alpha} E_\alpha(u) = \int_\Omega (1 + |\nabla u|^2)^\alpha \log (1 + |\nabla u|^2) dx .$$

In particular, if we fix $\alpha_0 < p/2$ we have that the function $(\alpha, s) \mapsto \dfrac{d}{d\alpha} E_\alpha(\bar{\sigma}(s))$ is bounded by a constant M in $[1, \alpha_0] \times S^{n-2}$. Thus, for $u \in \bar{\sigma}(S^{n-2})$ we have

$$E_\alpha(u) \leqslant E(u) + (\alpha - 1) \left| \frac{d}{d\alpha} E_\alpha(u) \right| \leqslant E(u) + (\alpha - 1) M .$$

We now choose $\bar{\alpha}$ such that $E_\alpha(u) \leqslant E(u) + \varepsilon \ \forall u \in \sigma(S^{n-2}) \ \forall \alpha < \bar{\alpha}$. Then by (4.8), $\forall \alpha < \bar{\alpha}$

$$\begin{aligned} c+\varepsilon &> \sup_{s \in S^{n-2}} (E_\alpha \circ \bar{\sigma}(s) - \varepsilon) = \sup_{s \in S^{n-2}} E_\alpha \circ \bar{\sigma}(s) - \varepsilon \geqslant \\ &\geqslant \inf_{\sigma \in \Sigma_{2\bar{\alpha}}} \sup_{s \in S^{n-2}} E_\alpha \circ \sigma(s) - \varepsilon = c_\alpha - \varepsilon, \quad \text{i.e.} \quad c_\alpha < c + 2\varepsilon \qquad \forall \alpha < \bar{\alpha} . \qquad \square \end{aligned}$$

We can now conclude with the proof of theorem 1.2.

Proof of theorem 1.2. — We consider two cases: $c > m$ and $c = m$.

I. CASE $c > m$. For $\alpha > 1$, let u_α be a solution of $E'_\alpha(u_\alpha) = 0$ which exists by lemma 4.7. Also by lemma 4.2 and 4.1, it follows that

$$\lim_{\substack{\alpha \to 1 \\ \alpha > 1}} E_\alpha(u_\alpha) = \lim_{\substack{\alpha \to 1 \\ \alpha > 1}} c_\alpha = c < m + 8\pi$$

and since $m \leqslant E(u_\alpha) \leqslant E_\alpha(u_\alpha)$ we have that

$$\lim_{\substack{\alpha \to 1 \\ \alpha > 1}} E(u_\alpha) = c < m + 8\pi .$$

Then the conclusion follows from theorem 3.1. □

II. CASE $c = m$. Choose $\varepsilon > 0$, then there exists $\sigma_\varepsilon \in \Sigma$ such that

$$\max_{s \in S^{n-2}} E \circ \sigma(s) < m + \varepsilon . \tag{4.9}$$

Let $\underline{u}_\varepsilon$ be such that $E(\underline{u}_\varepsilon) = \min_{s \in S^{n-2}} E \circ \sigma(s)$.

We consider a subsequence $\underline{u}_{\varepsilon_k}(\varepsilon_k \to 0)$ (which for simplicity will be denoted $\underline{u}_k$) which converges weakly to some $\underline{u}$. Since $\lim_{k \to +\infty} E(\underline{u}_k) = m$, and since E is weakly lower semicontinuous it follows that

$$\lim_{k \to +\infty} E(\underline{u}_k) = E(\underline{u}) .$$

The above equality and the weak convergence $\underline{u}_k \rightharpoonup \underline{u}$ imply that $\underline{u}_k \to u$ strongly in H^1. By Corollary 2.3 we can choose $\delta_0 > 0$ such that $A_{\delta_0,\gamma}(\underline{u})$ is contractible in $W^{1,2\alpha}_\gamma(\Omega ; M)$.

We claim that for every $\delta < \delta_0$ and ε_k small enough there is $u^\delta_k \in \sigma_{\varepsilon_k}(S^{n-2})$ such that

$$\| \underline{u}_k - u^\delta_k \|_{H^1} = \delta . \tag{4.10}$$

In fact, if the above equality does not hold, then

$$\sigma_{\varepsilon_k}(S^{n-2}) \subset A_{\delta_0,\gamma}(\underline{u})$$

and this is absurd since σ_{ε_k} is an essential map. Therefore, by (4.9) with $\varepsilon = \varepsilon_k$, we get

$$\lim_{k \to +\infty} E(u^\delta_k) = m$$

and since E is weakly lower semicontinuous we get that

$$\lim_{k \to +\infty} E(u^\delta_k) = E(u^\delta) \tag{4.11}$$

where u^δ it the weak limit of u^δ_k (may be after having taken a subsequence). By the weak convergence of u^δ_k and (4.11), it follows that $u^\delta_k \to u^\delta$ strongly in H^1. So taking the limit in (4.10) we get

$$\| \underline{u} - \underline{u}^\delta \| = \delta .$$

Thus, for any $\delta \in [0, \delta_0]$ we get at least one solution $\underline{u}^\delta$ of our problem. □

APPENDIX

Let $\omega = (0, +\infty) \times \mathbb{R}$ and $u \in C^0(\overline{\omega}; S^n)$ be such that

$$\nabla u \in L^2(\omega) \tag{A.1}$$

$$-\Delta u = u\,|\nabla u|^2 \tag{A.2}$$

$$\exists P \in S^n \quad \text{such that} \quad u = P \quad \text{on} \quad \partial\omega, \tag{A.3}$$

then

$$u \equiv P \quad \text{in} \quad \omega. \tag{A.4}$$

Remarks. — 1. When ω is a bounded contractible open set of $\mathbb{R}^2$ (A.4) is also true; this theorem is due to L. Lemaire [LM] (Théorème (3.2)). However, we cannot obtain (A.4) from the result of L. Lemaire and a conformal change of the variable. In fact consider a conformal diffeomorphism I between ω and Ω (the open unit disk of $\mathbb{R}^2$) such that (for example).

$$I(\partial\omega) = \partial\Omega - \{(0,1)\}.$$

Let

$$v = u.I^{-1}.$$

Clearly we have:

$$v \in C^0(\overline{\Omega} - \{(0,1)\}; S^n)$$

$$\int_\Omega |\nabla v|^2 < +\infty$$

$$-\Delta v = v\,|\nabla v|^2$$

$$v = P \quad \text{on} \quad \partial\Omega.$$

But we cannot apply directly the theorem of Lemaire since we do not know if $v \in C^0(\overline{\Omega}; S^n)$.

2. Thanks to a classical theorem (see, for example [HH], [LU_2], p. 485-493) using (A.1), (A.2) and $u \in C^0(\overline{\omega}, S^n)$ we know that u is analytic in $\overline{\Omega}$.

3. Our proof of (A.4) is inspired from H. Wente [W_1].

Proof of (A.4). — We may assume that $P = e_{n+1}$. Let w be the following function from $\mathbb{R}^2$ into S^n:

$$\text{if} \quad x \geqslant 0 \qquad w(x,y) = u(x,y)$$

$$\text{if} \quad x < 0 \quad \begin{cases} w^p(x,y) = -u^p(-x,y) & \text{for} \quad 1 \leqslant p \leqslant n \\ w^{n+1}(x,y) = u^{n+1}(-x,y). \end{cases}$$

Since $|u|^2 = 1$ and $u(0,y) = P\ \forall y \in \mathbb{R}$ we have:

$$\frac{\partial}{\partial x} w^{n+1}(0,y) = 0 \qquad \forall y \in \mathbb{R}. \tag{A.5}$$

Then, using (A.2), (A.3) and (A.5), it is easy to see that

$$-\Delta w = w\,|\nabla w|^2 \qquad \text{(in the distribution sense)}. \tag{A.6}$$

Moreover $w \in C^0(\mathbb{R}^2) \cap H^1_o(\mathbb{R}^2)$. Thus (see [$LU_2$], [Wi] or [HW]) w is analytic.

Let $\phi \in C^\infty(\mathbb{R}^2, \mathbb{C})$ be defined by:

$$\phi = w_x^2 - x_y^2 - 2iw_x . w_y .$$

Using (A.6) and $|w| = 1$ it is easy to see that ϕ is holomorphic. Moreover, by (A.1), we have $\phi \in L^1(\mathbb{R}^2)$ and, therefore $\phi = 0$. Hence

$$\nabla w = 0 \quad \text{on} \quad \{0\} \times \mathbb{R},$$

which implies

$$w \equiv P \quad \text{in} \quad \mathbb{R}^2.$$

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