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A Classification of Factors, II

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Abstract

The algebraic invariant $r_\infty(M)$ of a factor M, introduced in an earlier paper and called the asymptotic ratio set, is shown to be closed for any factor M. As a consequence, this set must be one of the following sets: (i) the empty set, (ii) $\{0\}$, (iii) $\{1\}$, (iv) a one parameter family of sets $\{0, x^n; n=0, \pm 1, \cdots\}$, $0<x<1$. (v) all non-negative reals, (vi) $\{0. 1\}$.

§1. Introduction

In an earlier paper [1], we introduced an algebraic invariant $r_\infty(M)$ for a factor M. It is the set of all x, $0\leqq x<\infty$, such that M is algebraically isomorphic to $M\otimes R_x$. Here R_0 is the type I_∞ factor, R_1 is the hyperfinite type II_1 factor, and $R_x=R_{x^{-1}}$ for $0<x<1$ is a type III factor given by definition 3.10 of [1].

In [1], it is shown that $r_\infty(M)-\{0\}$ is either empty or a multiplicative group. Furthermore, for the case where M is an infinite tensor product of type I factors, $r_\infty(M)$ is shown to be closed. However, this was not known in [1] for arbitrary M.

In this note, we show that $r_\infty(M)$ is closed for any factor M. The method of proof is already indicated in section 6 of [1], but new additional technique here is the use of weak clustering property, which is obtained by the crucial lemma 2.4.

§2. Lemmas

Lemma 2.1. Let R_ι be mutually commuting factors such that

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$R=(\bigcup_i R_i)''$ is a factor. Let D be a finite set of unit vectors. Given ϵ, there exists an N such that

$$(2.1) \qquad |(\Psi, Q\Phi)-(\Psi, \Phi)(\Phi, Q\Phi)|<\epsilon$$

for any $i>N$, $Q\in R_i$, $\|Q\|=1$, $\Psi\in D$, $\Phi\in D$.

Proof. (cf. [2]) Since R is a factor, the von Neumann algebra generated by $\bigcup_i R_i$ and R' is the set of all bounded operators. Thus the self adjoint elements of the $*$ algebra generated by $\bigcup_i R_i$ and R' are strongly dense among all self adjoint operators. In particular for the one dimensional projection $\mathrm{P}(\Phi)$ associated with a vector Φ, there exists a self adjoint P' in $\{(\bigcup_{i=1}^{N} R_i)\cup R'\}''$ for some finite N such that P' is in the following strong neighbourhood of $\mathrm{P}(\Phi)$:

$$(2.2) \qquad \{A;\ \|\{\mathrm{P}(\Phi)-A\}\Psi\|<\epsilon/2,\ \forall\Psi\in D\}.$$

Then for any $Q\in R_i$, $i>N$, $\|Q\|=1$, we have $[Q, P']=0$ and

$$\begin{aligned} &|(\Psi, Q\Phi)-(\Psi, \Phi)(\Phi, Q\Phi)| \\ &=|(\Psi, Q\{\mathrm{P}(\Phi)-P'\}\Phi)|+|(\{P'-\mathrm{P}(\Phi)\}\Psi, Q\Phi)|<\epsilon. \end{aligned}$$

Definition 2.2. Let M be a type I_n factor with a matrix unit u_{kl}, $k, l=1, \cdots, n$ and R be a factor containing M. For any $Q\in R$, define

$$(2.3) \qquad \tau_{kl}(M)Q=\sum_{j=1}^{n} u_{jk}Qu_{lj}.$$

Lemma 2.3. Let R be a factor and M be a type I_n factor in R'. For $Q\in(M\cup R)''$, $\tau_{kl}(M)Q$ is in R, $\|\tau_{kl}(M)Q\|\leqq\|Q\|$ and

$$(2.4) \qquad Q=\sum_{k,l} u_{kl}(\tau_{kl}(M)Q).$$

Futhermore, let ψ be a unit vector and

$$(2.5) \qquad Q'=\sum_{k,l} u_{kl}(\psi, \tau_{kl}(M)Q\psi)\in M.$$

Then $\|Q'\|\leqq\|Q\|$.

Proof. Since M is type I_n, it is possible to identify the Hilbert space H with a tensor product $H_1\otimes H_2$, M with $\mathcal{B}(H_1)\otimes\mathbf{1}$ and u_{kl} with $\hat{u}_{kl}\otimes\mathbf{1}$, where H_1 is spanned by an orthonormal basis $\varphi_1, \cdots, \varphi_n$,

$\hat{u}_{kl}\varphi_j=\delta_{lj}\varphi_k$ and $\mathscr{B}(H_1)$ denotes the set of all bounded operators on H_1. M' is then $1\otimes\mathscr{B}(H_2)$, in which R is contained.

The equality (2.4) follows from (2.3) and $u_{kl}u_{ij}=\delta_{li}u_{kj}$, $\sum u_{kk}=1$. If Q is in the $*$ algebra generated by M and R, then it is of the form (2.4) where $\tau_{kl}(M)Q$ is in R. Therefore $\tau_{kl}(M)Q\in R$ holds also for the weak closure of such Q, namely for all Q in $(M\cup R)''$. The norm of $\tau_{kl}(M)Q$ can be estimated by

$$\|\tau_{kl}(M)Q\|\leqq\sup_{\|\psi^i\|=1}|(\varphi_1\otimes\psi^1, \tau_{kl}(M)Q\{\varphi_1\otimes\psi^2\})|$$

because $\tau_{kl}(M)Q\in 1\otimes\mathscr{B}(H_2)$. The right hand side is majorized by

$$\sup_{\|\psi^i\|=1}|(\varphi_1\otimes\psi^1, u_{1k}Qu_{l1}\{\varphi_1\otimes\psi^2\})|\leqq\|u_{1k}Qu_{l1}\|\leqq\|Q\|.$$

A unit vector ψ defines a density matrix ρ in $\mathscr{B}(H_2)$ ($\rho\geqq 0$, $\operatorname{tr}\rho=1$) through the relation

$$(\psi, \{1\otimes\widehat{Q}\}\psi)=\operatorname{tr}\rho\widehat{Q}, \qquad \widehat{Q}\in\mathscr{B}(H_2).$$

For any unit vector Φ_1 and Φ_2 in H_1, we have

$$|(\Phi_1, Q'\Phi_2)|=|\operatorname{tr}\{(\rho_1\otimes\rho)(u\otimes 1)Q\}|\leqq\|u\|\|Q\|=\|Q\|$$

where ρ_1 is the one dimensional projection on Φ_2 and u is an isometric operator with one dimensional range, bringing Φ_1 onto Φ_2. Therefore $\|Q'\|\leqq\|Q\|$.

Lemma 2.4. Let R_i be mutually commuting factors such that $R=(\bigcup_i R_i)''$ is a factor. Let M be a type I_n ($n<\infty$) factor contained in R'. Let D be a finite sets of unit vectors such that the inequality (2.1) holds for any $Q\in M$, $\|Q\|=1$, $\Psi\in D$, $\Phi\in D$. Given $\epsilon'>0$. Then there exists an N such that

$$(2.6)\qquad |(\Psi, Q\Phi)-(\Psi, \Phi)(\Phi, Q\Phi)|<\epsilon+\epsilon'$$

for any $i>N$, $Q\in(M\cup R_i)''$, $\|Q\|=1$, $\Psi\in D$, $\Phi\in D$.

Proof. Let u_{kl}, $k, l=1, \cdots, n$ be a matrix unit for M. Let $\mathrm{P}(\Phi)$ be the one dimensional projection associated with each $\Phi\in D$. Find sufficiently large $\mathrm{N}(\Phi)$ for each $\Phi\in D$ such that there exists a selfadjoint $\mathrm{P}'(\Phi)$ belonging to

$$\{(\bigcup_{i=1}^{N(\Phi)} R_i) \cup R'\}''$$

and satisfying

$$\|\{P'(\Phi)-P(\Phi)\}\Psi\| < \epsilon'' \tag{2.7}$$

$$\|\{P'(\Phi)-P(\Phi)\}u_{lk}\Psi\| < \epsilon'' \tag{2.8}$$

for all $\Psi \in D$ and $l, k=1, \cdots, n$. Here

$$\epsilon'' = 2^{-1}(1+n^2)^{-1}\epsilon'. \tag{2.9}$$

Let $N=\max_{\Phi\in D} N(\Phi)$ and $Q\in(M\cup R_i)''$, $i>N$, $\|Q\|=1$. We then have the following inequalities, which proves (2.6):

$$\begin{aligned}
&|(\Psi, Q\Phi)-(\Psi, \Phi)(\Phi, Q\Phi)| \\
&\leqq |(\Psi, Q\{P(\Phi)-P'(\Phi)\}\Phi)| \\
&\quad + \sum_{k,l} |(\{P'(\Phi)-P(\Phi)\}u_{lk}\Psi, Q_{kl}\Phi)| \\
&\quad + |(\Psi, Q'\Phi)-(\Psi, \Phi)(\Phi, Q'\Phi)| \\
&\quad + \sum_{k,l} |(\{P(\Phi)-P'(\Phi)\}u_{lk}\Phi, Q_{kl}\Phi)(\Psi, \Phi)| \\
&\quad + |(\Phi, Q\{P'(\Phi)-P(\Phi)\}\Phi)(\Psi, \Phi)| \\
&< \|Q\|\epsilon'' + \sum_{k,l}\|Q_{kl}\|\epsilon'' + \|Q'\|\epsilon + \sum_{k,l}\|Q_{kl}\|\epsilon'' + \|Q\|\epsilon'' \\
&\leqq 2(1+n^2)\epsilon'' + \epsilon = \epsilon' + \epsilon.
\end{aligned}$$

Here we have used the notation and result of the previous lemma in which $\psi=\Phi$ and denoted $\tau_{kl}(M)Q$ simply by Q_{kl}.

Definition 2.5. A unit vector Ψ is *pure* for a type I factor M if $\varphi_\Psi(Q)=(\Psi, Q\Psi)$, $Q\in M$ is a pure state on M.

If $H=H_1\otimes H_2$, $M=\mathcal{B}(H_1)\otimes \mathbf{1}$, then Ψ is pure if and only if $\Psi=\Psi_1\otimes\Psi_2$ for some $\Psi_1\in H_1$, $\Psi_2\in H_2$.

Lemma 2.6. Let $H=H_a\otimes H_b$, $H_b=\otimes(H_\nu, \Omega_\nu)$, $R_\nu=\mathbf{1}_a\otimes\{\mathcal{B}(H_\nu)\otimes(\bigotimes_{\mu\neq\nu}\mathbf{1}_\mu)\}$. Let M be a type I factor in $\mathcal{B}(H_a)\otimes\mathbf{1}$ and let a unit vector Ψ be pure for M. Given $\epsilon>0$. Then there exists an N and a unit vector Ψ_ϵ such that Ψ_ϵ is pure for M as well as for R_ν for any $\nu>N$, $\|\Psi-\Psi_\epsilon\|<\epsilon$ and φ_{Ψ_ϵ} is the same as the vector state corresponding to Ω_ν for each R_ν, $\nu>N$.

Proof. Since M is-type I, we may identify H_a with $H_{a1}\otimes H_{a2}$, M with $\mathcal{B}(H_{a1})\otimes 1$. Since Ψ is pure for M, it can be identified with $\Psi_{a1}\otimes\Psi'$ where $\Psi'\in H_{a2}\otimes H_b$. For given $\epsilon>0$, there exists Ψ'_ϵ of the form $\sum_{i=1}^{k}\psi_i\otimes\psi'_i$, $\psi_i\in H_{a2}$, $(\psi_i, \psi_j)=\delta_{ij}$, $\sum\|\psi'_i\|^2=1$ such that

$$\|\Psi'-\Psi'_\epsilon\|<\epsilon/2. \tag{2.10}$$

By lemma 2.7 of [1], there exists an N and ψ''_i for each i such that

$$\psi''_i=\psi'''_i\otimes(\bigotimes_{\nu>N}\Omega_\nu), \tag{2.11}$$

$$\|\psi'_i-\psi''_i\|<\epsilon/(2k), \tag{2.12}$$

$$\|\psi''_i\|=\|\psi'_i\|. \tag{2.13}$$

Then the vector

$$\Psi_\epsilon=\sum_{i=1}^{k}\Psi_{a1}\otimes\psi_i\otimes\psi'''_i\otimes(\bigotimes_{\nu>N}\Omega_\nu) \tag{2.14}$$

has all the required properties.

§3. Theorem

Theorem 3.1. The asymptotic ratio set $\mathrm{r}_\infty(M)$ for any factor M is closed.

Proof. If $x\neq 0$ and $\neq 1$ is in $\mathrm{r}_\infty(M)$, then $R_x\sim R_x\otimes R_0\sim R_x\otimes R_1$ shows that $0\in\mathrm{r}_\infty(M)$ and $1\in\mathrm{r}_\infty(M)$. Thus we consider the case where $x_n\in\mathrm{r}_\infty(M)$, $\lim x_n=x$, $0<x_n<1$, $0<x<1$ and prove that $x\in\mathrm{r}_\infty(M)$; i.e. $M\sim M\otimes R_x$.

First fix a countable sequence of unit vectors Ψ_n, $n=1, 2, \cdots$ which are dense in the unit sphere of H and let $D_n=\{\Psi_1, \cdots, \Psi_n\}$. Let $\epsilon_n>0$, $\epsilon\equiv\sum\epsilon_n<\infty$. We shall now construct by a mathematical induction on n a sequence of mutually commuting type I_2 factors M_n in R, and N_n in R', and a sequence of unit vectors χ_n, $n=1, 2, \cdots$, such that (1) χ_n is pure for each $(M_m\cup N_m)''$, $m\leqq n$, (2) the vector state φ_{χ_n} for each M_m, $m\leqq n$ has a spectrum $((1+x_m)^{-1}, x_m(1+x_m)^{-1})$, (3) $\|\chi_n-\chi_{n-1}\|<\epsilon_n$ $(n\geqq 2)$ and (4)

$$|(\Psi, Q\Phi)-(\Psi, \Phi)(\Phi, Q\Phi)|<\sum_{\alpha=0}^{n-m}\epsilon_{m+\alpha} \tag{3.1}$$

for any Q in $\{\bigcup_{\alpha=0}^{n-m}(M_{m+\alpha}\cup N_{m+\alpha})\}''$, $\|Q\|=1$, $\Psi\in D_m$, $\Phi\in D_m$, $m\leqq n$.

For $n=0$, we do not have any object to construct. Now suppose M_n, N_n and χ_n are constructed for $n<k$ satisfying all the requirements related to M_n, N_n, χ_n $n<k$. We then want to construct M_k, N_k and χ_k.

Let $M^{(k1)}\equiv(\bigcup_{n<k}M_n)''$, $M^{(k2)}\equiv\{M^{(k1)}\}'\cap M$, $N^{(k1)}\equiv(\bigcup_{n<k}N_n)''$, $N^{(k2)}\equiv\{N^{(k1)}\}'\cap M'$. Since M_n and N_n are finite type I factors, we may identify H with $H^{(k1)}\otimes H^{(k2)}$; $M^{(k1)}$, $M^{(k2)}$, $N^{(k1)}$, $N^{(k2)}$ with $\widehat{M}^{(k1)}\otimes\mathbf{1}$, $\mathbf{1}\otimes\widehat{M}^{(k2)}$, $\widehat{N}^{(k1)}\otimes\mathbf{1}$, $\mathbf{1}\otimes\widehat{N}^{(k2)}$; and $(M^{(k1)}\cup N^{(k1)})''$ with $\mathcal{B}(H^{(k1)})\otimes\mathbf{1}$. By using (2.4), it is easily shown that M and M' are identified with $\widehat{M}^{(k1)}\otimes\widehat{M}^{(k2)}$ and $\widehat{N}^{(k1)}\otimes\widehat{N}^{(k2)}$. Since M is type III ($x_n\neq1$, 0 is in $\mathrm{r}_\infty(M)$), M is spatially isomorphic to $\widehat{M}^{(k2)}$. Since χ_{k-1} is pure for each $M_n\cup N_n$, $n<k$, it is pure for $\widehat{M}^{(k1)}\otimes\widehat{N}^{(k1)}$ and can be identified with $\psi^{(k1)}\otimes\psi^{(k2)}$, $\|\psi^{(k1)}\|=\|\psi^{(k2)}\|=1$.

We now use the information that M is isomorphic to $M\otimes R_{x_k}$ where $R_{x_k}=\bigotimes_\nu\widehat{R}_k^\nu$ on $H_k=\bigotimes(H_k^\nu,\Omega_k^\nu)$. Let R_k^ν be $\mathbf{1}\otimes\widehat{R}_k^\nu\otimes(\bigotimes_{\mu\neq\nu}\mathbf{1}_\mu)$ and S_k^ν be $\mathbf{1}\otimes(\widehat{R}_k^\nu)'\otimes(\bigotimes_{\eta\neq\nu}\mathbf{1}_\mu)$. By lemma 2.6, there exist an N, and a unit vector $\psi^{(k3)}$ on $H^{(k2)}$ such that $\|\psi^{(k2)}-\psi^{(k3)}\|<\epsilon_k$, $\psi^{(k3)}$ is pure for every $(R_k^\nu\cup S_k^\nu)''$, with $\nu>N_1$, and the vector state $\varphi_{\psi^{(k3)}}$ for $(R_k^\nu\cup S_k^\nu)''$, $\nu>N_1$ is the same as $\varphi_{\Omega_k^\nu}$ for $(\widehat{R}_k^\nu\cup(\widehat{R}_k^\nu)')''$. We then set $\chi_k=\psi^{(k1)}\otimes\psi^{(k3)}$. (If $k=1$, take $\chi_k=\psi^{(k2)}=\psi\otimes(\bigotimes_\nu\Omega_1^\nu)$ for any $\|\psi\|=1$.) The conditions (1), (2), (3) are automatically satisfied for $M_k=R_k^\nu$, $N_k=S_k^\nu$, any $\nu>N_1$.

By lemma 2.1, there exists an N_2 such that

$$|(\Psi,Q\Phi)-(\Psi,\Phi)(\Phi,Q\Phi)|<\epsilon_k \tag{3.2}$$

for any $Q\subset R_k^\nu$, $\nu>N_2$, $\|Q\|=1$, $\Psi\in D_k$, $\Phi\in D_k$.

By lemma 2.4, there exists an N_3^n for each $n<k$ such that

$$|(\Psi,Q\Phi)-(\Psi,\Phi)(\Phi,Q\Phi)|<\sum_{\alpha=n}^{k-1}\epsilon_\alpha+\epsilon_k \tag{3.3}$$

for any $\nu>N_3^n$, $Q\in[\{\bigcup_{\alpha=n}^{k-1}(M_\alpha\cup N_\alpha)\}\cup(R_k^\nu\cup S_k^\nu)]''$, $\|Q\|=1$, $\Psi\in D_n$, $\Phi\in D_n$.

We then set $M_k=R_k^\nu$, $N_k=S_k^\nu$ for some ν larger than max $(N_1, N_2, N_3^1, \cdots, N_3^{k-1})$. The required properties are now all satisfied.

By the property (3) and $\sum \epsilon_n < \infty$, the unit vectors χ_n form a Cauchy sequence. Let χ be its strong limit. Then χ is a unit vector, pure for each $(M_n \cup N_n)''$ and the vector state φ_χ on M_n has the spectrum $((1+x_n)^{-1}, x_n(1+x_n)^{-1})$. Let

$$(3.4) \qquad R = (\bigcup_n M_n)'', \quad S = (\bigcup_n N_n)'',$$

$$(3.5) \qquad H_0 = [(R \cup S)'' \chi]^w$$

where w denotes the closure. The properties of χ imply that the restrictions of R and S to H_0 and the space H_0 are unitarily equivalent to $\otimes R_n$, $\otimes R_n'$ and $\otimes (H_n, \Omega_n)$ where $\dim H_n = 4$, $\mathrm{Sp}(\Omega_n/R_n) = \mathrm{Sp}(\Omega_n/R_n') = ((1+x_n)^{-1}, x_n(1+x_n)^{-1})$. Thus $(R \mid H_0) \sim (S \mid H_0) \sim \otimes R_n$, where $R \mid H_0$ denotes the restriction of R to H_0.

Next we use the clustering property (4) to show that R, S and $(R \cup S)''$ are factors. Let Q be an operator in the center of either R, S or $(R \cup S)''$ and $\|Q\| = 1$. Then Q must commute with all $(M_n \cup N_n)''$, $n = 1, 2, 3, \cdots$ and hence it is in $\{\bigcup_{n>N} (M_n \cup N_n)\}''$ for any N. (Again use the fact that $\{\bigcup_{n<N} (M_n \cup N_n)\}''$ is a finite type I factor and (2.4).) Since the unit ball of $\bigcup_{m=1}^{\infty} (\bigcup_{n=N+1}^{N+m} [M_n \cup N_n])''$ is weakly dense in the unit ball of $(\bigcup_{n<N} [M_n \cup N_n])''$, we have

$$(3.6) \qquad |(\Psi, Q\Phi) - (\Psi, \Phi)(\Phi, Q\Phi)| < \sum_{n=N+1}^{\infty} \epsilon_n$$

for any $\Psi \in D_{N+1}$, $\Phi \in D_{N+1}$. Since N is arbitrary, we obtain in the limit of $N \to \infty$,

$$(3.7) \qquad (\Psi, Q\Phi) = (\Psi, \Phi)(\Phi, Q\Phi).$$

The same equation for Q^*, with Ψ and Φ interchanged implies that

$$(3.8) \qquad (\Psi, Q\Psi) = (\Phi, Q\Phi)$$

for $(\Psi, \Phi) \neq 0$. Since Ψ, Φ run over a set of unit vectors $\{\Psi_n\}$ which is dense in the set of all unit vectors, (3.8) and (3.7) imply that $Q = c1$. This proves that R, S and $(R \cup S)''$ are factors.

Since the projection on H_0 commutes with R, S and $(R \cup S)''$, the factors R, S and $(R \cup S)''$ are isomorphic to its restriction on H_0.

In particular, $R\sim R\otimes R_x$ and $(R\cup S)''$ is a type I factor.

The proof of the theorem can now be completed by

Lemma 3.2. Let $H=H_1\otimes H_2$, $\widehat{R}$ be an infinite tensor product of type I_2 factors on H_2, $R=\mathbf{1}\otimes\widehat{R}$, $S=\mathbf{1}\otimes\widehat{R}'$. Let M be a factor on H such that $M\supset R$, $M'\supset S$. Then $M=M_1\otimes R$ for some factor M_1 on H_1.

Proof. Let

$$H_2=\otimes(H_2^\nu,\,\Omega_\nu),\quad \widehat{R}=\otimes\widehat{R}_\nu, \tag{3.9}$$

$$\mathrm{D}(n)=H_1\otimes(\bigotimes_{\nu=1}^{n}H_2^\nu)\otimes(\bigotimes_{\nu>n}\Omega_\nu), \tag{3.10}$$

$$\mathrm{D}(n,n+k)=H_1\otimes(\bigotimes_{\nu=1}^{n}H_2^\nu)\otimes(\bigotimes_{l=1}^{k}\Omega_{n+l})\otimes(\bigotimes_{\nu>n+k}H_2^\nu). \tag{3.11}$$

Let u_{ij}^ν be a standard matrix unit of

$$R_\nu\equiv\mathbf{1}_1\otimes\{\widehat{R}_\nu\otimes(\bigotimes_{\mu\neq\nu}\mathbf{1}_\mu)\} \tag{3.12}$$

relative to Ω_ν, $\mathrm{Sp}(\Omega_\nu/\widehat{R}_\nu)$ be $(\lambda_\nu, 1-\lambda_\nu)$ and

$$\tau_\nu A=\lambda_\nu\tau_{11}(R_\nu)A+(1-\lambda_\nu)\tau_{22}(R_\nu)A, \tag{3.13}$$

$$\tau_{n,n+k}=\prod_{l=1}^{k}\tau_{n+l}\,. \tag{3.14}$$

Further let $[A]_n$ be the unique operator in $\mathscr{B}(H_1\otimes(\bigotimes_{\nu=1}^{n}H_2^\nu))\otimes(\bigotimes_{\nu>n}\mathbf{1}_\nu)$ satisfying

$$(\Phi_1,\,[A]_n\Phi_2)=(\Phi_1,\,A\Phi_2) \tag{3.15}$$

for all $\Phi_1,\Phi_2\in\mathrm{D}(n)$.

If $A\in M$, then $\tau_{n,n+k}A\in M$, $\|\tau_{n,n+k}A\|\leqq\|A\|$ and

$$(\Phi_1,\,(\tau_{n,n+k}A)\Phi_2)=(\Phi_1,\,A\Phi_2) \tag{3.16}$$

for all $\Phi_1,\Phi_2\in\mathrm{D}(n,n+k)$. Hence

$$(\Phi_1,\,(\tau_{n,n+k}A)\Phi_2)=(\Phi_1,\,[A]_n\Phi_2) \tag{3.17}$$

for $\Phi_1,\Phi_2\in\mathrm{D}(n+k)$. Since $\mathrm{D}(n)$ is an increasing sequence of sets with a dense union and $\|\tau_{n,n+k}A\|$ is bounded uniformly in k, $[A]_n$ is the weak limit of $\tau_{n,n+k}A$ as $k\to\infty$ and hence is in M. By definition, $[A]_n$ is then in

$$M^{(n)} \equiv M \cap (\bigcup_{\nu>n} R_\nu)'. \tag{3.18}$$

Because (3.15) holds for $\Phi_1, \Phi_2 \in D_n$, D_n is an increasing sequence of sets with a dense union and $\|[A]_n\|$ is uniformly bounded by $\|A\|$ (which immediately follows from (3.15)), A is the weak limit of $[A]_n$ as $n \to \infty$. Hence

$$M = (\bigcup_n M^{(n)})''. \tag{3.19}$$

Since $\bigcup_{\nu=1}^{n} R_\nu$ is a finite type I factor, $M^{(n)}$ is generated by $\bigcup_{\nu=1}^{n} R_\nu$ and $M^{(0)} = M \cap R'$. Hence

$$M = (M^{(0)} \cup R)'' = \{(M \cap R') \cup R\}''. \tag{3.20}$$

Since $M \cap R'$ commutes with R and S, it is isomorphic to $M_1 \otimes 1$ on $H_1 \otimes H_2$ for some M_1 and $M = M_1 \otimes \hat{R}$.

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