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A Strong Form of Yamaguti and Nogi's Stability Theorem for Friedrichs' Scheme*

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In this paper we derive the Lax-Nirenberg Theorem [3–4] for Yamaguti and Nogi's pseudo-difference schemes of [7] and, as a corollary, we obtain a strong form of Yamaguti and Nogi's Stability Theorem for Friedrichs' scheme for regularly hyperbolic systems. These are new results.

As in [2] and [7] let $\mathcal{K}$ denote the class of $p\times p$ matrices $k(x,\xi)\in C^{\infty}(R^n_x\times R^n_\xi-\{0\})$, independent of x for $|x|>R$ fixed, and homogeneous of degree zero in ξ.

Lemma (Lax [2]). *Every $k\in\mathcal{K}$ can be expanded in a series*

$$k(x,\xi)=\sum_{\alpha}k_{\alpha}(x)e^{i\alpha\cdot\xi/|\xi|}, \tag{1.1}$$

where α varies over all multi-indices of integers. The series, and the differentiated series, with respect to x or ξ, converge uniformly for all x, and $|\xi|=1$.

The h-family of operators

$$K_h u(x)=\int e^{ix\cdot\xi}k(x,\lambda(h\xi))\hat{u}(\xi)d\xi$$

is associated with the *symbol* $k(x,\lambda(\xi))$ while the Fourier Transform of the adjoint family,

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$$\widehat{K_h^* u}(\xi) = (2\pi)^{-n} \int e^{-ix\cdot\xi} k^*(x, \lambda(h\xi)) u(x) dx$$

has the symbol $k^*(x_1, \lambda(\xi_2))$. Here the subscripts indicate that the multiplication by the variable x is performed before the differentiation corresponding to the co-variable ξ. We write also $k^{*R}(x, \lambda(\xi))$ for $k^*(x_1, \lambda(\xi_2))$.

The Fourier Transform

$$\hat{k}(\chi, \xi) = (2\pi)^{-n} \int e^{-i\chi\cdot x} k(x, \xi) dx$$

of $k \in \mathcal{K}$ exists and has finite L^1-maximum norm

$$||\hat{k}|| = \int [\sup_\xi |\hat{k}(\chi, \xi)|] d\chi < \infty$$

and the L^2 operator norm of K_h satisfies the inequality

$$||K_h|| \leq ||\hat{k}||,$$

provided we let

$$\hat{k}(\chi, \xi) = \delta(\chi) k(\xi) \qquad \text{and} \qquad ||\hat{k}|| = \sup_\xi |k(\xi)|$$

for a function $k(\xi)$ independent of x. Now, we represent the Fourier Transform of $K_h u$ and $K_h^* u$ in terms of $\hat{k}$, $\widehat{k^*}$ and $\hat{u}$:

$$\widehat{K_h u}(\xi) = \int \hat{k}(\xi - \xi', \lambda(h\xi')) \hat{u}(\xi') d\xi'$$

$$\widehat{K_h^* u}(\xi) = \int \widehat{k^*}, (\xi - \xi', \lambda(h\xi)) \hat{u}(\xi') d\xi'.$$

One sees that K_h is associated with the Fourier kernel $\hat{k} = \hat{k}(\chi, \lambda(\xi))$ while K_h^* is associated with $\widehat{k^{*R}} = \widehat{k^*}(\chi, \lambda(\xi + \chi))$. Clearly

$$||\widehat{k^{*R}}|| = ||\hat{k}||.$$

If k is hermitian, $k = k^*$, then K_h^* is associated with $\hat{k}^R = \widehat{k^{*R}}$.

Theorem [5]. *Suppose that* $p \in \mathcal{K}$, $p(x, \xi) \geq 0$. *If* $\lambda(\xi) \in C^2$, $\lambda(0) = 0$, *and* λ, λ_ξ *and* $\lambda_{\xi\xi}$ *are bounded, then*

$$Re\langle P_h \Lambda_h^2 u\rangle \geq -Kh ||u||^2, \; u \in L_m^2,$$

for all h *and some constant* K.

This particular form of the Lax-Nirenberg theorem is a sharp form of Theorem 3, [7], p. 159; in fact the proof of the latter requires that p be positive definite, $p>0$.

Corollary [5]. *If* $\lambda=k/h$ *satisfies*

$$\lambda\leq\frac{1}{\sqrt{n}\mu_0}, \tag{1.2}$$

then Friedrichs' scheme

$$S_h u = \sum_{j=1}^{n}\left\{\frac{u(x+\delta_j,\ t)+u(x-\delta_j,\ t)}{2n}+\lambda A_j(x)\frac{u(x+\delta_j,\ t)-u(x-\delta_j,\ t)}{2}\right\}$$

is stable in the sense of Lax-Richtmyer.

Here μ_0 is the supremum of the spectral radius of the regularly hyperbolic matrix $\sum a_j(x)\xi_j$ over $|\xi|=1$ and all $x\in R^n_x$.

This corollary is a strong form of Theorem[1] 4, [7], p. 162. The latter had only strict inequality in (1.2).

Our corollary follows from the proof of Theorem 4 [7], pp. 162-165, if in the last step of the proof one applies our theorem instead of Theorem 3 [7]. Therefore we need only prove our theorem.

We adapt to the case at hand Friedrichs' proof [1, 6] of the Lax-Nirenberg theorem for pseudo-differential operators.

Choose a smooth *even* mollifier $q^2(\sigma)$ with support in the unit sphere, $|\sigma|\leq 1$, and integral 1,

$$\int q^2(\sigma)d\sigma = 1\,. \tag{1.3}$$

Convolve

$$g(x,\ \xi) = p(x,\ \lambda(\xi))|\lambda(\xi)|^2 \tag{1.4}$$

with q^2 to obtain the mollified symbol

$$a(x,\ \xi) = \int g(x,\ \xi-h^{1/2}\sigma)q^2(\sigma)d\sigma\ , \tag{1.5}$$

which, after the change of variable

$$\zeta = \xi-h^{1/2}\sigma\ , \tag{1.6}$$

1) Theorem "3" is a misprint in [7], p. 162.

becomes

(1.7) $$a(x, \xi) = h^{-n/2}\int g(x, \zeta)q^2(h^{-1/2}[\xi-\zeta])d\zeta\,.$$

Rearrange (1.7) into the double symbol

(1.8) $$b(\xi_2, x_1, \xi_1) = h^{-n/2}\int q(h^{-1/2}[\xi_2-\zeta])g(x_1, \zeta)\cdot q(h^{-1/2}[\xi_1-\zeta])d\zeta.$$

Obviously b generates the symmetric operator

$$\widehat{B_h u}(\xi) = \int \hat{b}(h\xi, \xi-\xi', h\xi')\hat{u}(\xi')d\xi'\,.$$

We complete the proof by means of three lemmas.

Lemma 1.1. $\langle B_h u, u\rangle \geq 0,\ u\in L^2_x$.

Lemma 1.2. $||A_h - G_h|| = 0(h)$.

Lemma 1.3. $||B_h - SyA_h|| = 0(h)$.

This yields the desired result:

$$-Re\langle P_h\Lambda^2 u, u\rangle \leq \langle (B_h - SyG_h)u, u\rangle \leq [||B_h - SyA_h|| + ||A_h - G_h||]||u||^2 \leq 0(h)||u||^2\,.$$

Proof of Lemma 1.1. Since $b(h\xi_2, x_0, h\xi_1)$ is a non-negative symmetric kernel for each value x_0 of x:

$$\iint \bar{\hat{u}}(\xi_2)e^{i\xi_2\cdot x_0}b(h\xi_2, x_0, h\xi_1)\hat{u}(\xi_1)e^{-i\xi_1\cdot x_0}d\xi_1 d\xi_2 \geq 0\,,$$

integrate with respect to x_0, change the order of integration and apply Parserval's relation to get

$$0\leq \iint \bar{\hat{u}}(\xi_2)\hat{b}(h\xi_2, \xi_2-\xi_1, h\xi_1)\hat{u}(\xi_1)d\xi_1 d\xi_2 = \langle B_h u, u\rangle\,.$$

Proof of Lemma 1.2. By (1.3) and (1.5),

$$\hat{a}(\chi, \xi) - \hat{g}(\chi, \xi) = \int [\hat{g}(\chi, \xi-h^{1/2}\sigma) - \hat{g}(\chi, \xi)]q^2(\sigma)d\sigma\,.$$

To find a bound for $|\hat{a}-\hat{g}|$ note that $\hat{g}_{\xi_\mu}(\chi, \xi)$ exists and is uniformly Lipschitz continuous in ξ with Lipschitz bound $\hat{k}_\mu(\chi)\in L^1$. This follows from the representation (1.1) for $p(\chi, \xi)$ and the conditions on λ. A Taylor expansion of $\hat{g}(\chi, \xi-h^{1/2}\sigma)-\hat{g}(\chi, \xi)$ in $h^{1/2}\sigma$ yields the estimate

$$|\hat{a}(\chi, \xi)-\hat{g}(\chi, \xi)| \leq h^{1/2}|\sum_{\mu} \hat{g}_{\xi\mu}(\chi, \xi) \int \sigma_{\mu} q^2(\sigma) d\sigma|$$
$$+h \sum_{\mu} \hat{k}_{\mu}(\chi) \int |\sigma|^2 q^2(\sigma) d\sigma .$$

The term involving $h^{1/2}$ is zero since q^2 is even. Since $\int |\sigma|^2 q^2(\sigma) d\sigma \leq 1$, and $\sum \int \widehat{k_{\mu}}(\chi) d\chi < \infty$, Lemma 1.2 follows.

Proof of Lemma 1.3. By (1.7), (1.8) and (1.6) $A_h+A_h^*-2B_h$ is associated with

$$\hat{a}(\chi, h\xi)+\hat{a}(\chi, h\xi+h\chi)-2\hat{b}(h\xi+h\chi, \chi, h\xi)$$
$$=\int \hat{g}(\chi, h\xi-h^{1/2}\sigma)[q(\sigma+h^{1/2}\chi)-q(\sigma)]^2 d\sigma .$$

Thus,

$$|\hat{a}+\hat{a}^R-2\hat{b}| \leq \sup_{\xi} |\hat{g}(\chi, \xi)| |\chi|^2 h [\sum_{\mu} \iint_0^1 |\partial_{\sigma_{\mu}} q(\sigma+\beta h^{1/2}\chi)| d\beta d\sigma]^2$$
$$= Ch \sup_{\xi} |\hat{g}(\chi, \xi)| |\chi|^2 .$$

Integration with respect to χ yields Lemma 1.3:

$$\|\hat{a}+\hat{a}^R-2\hat{b}\| = O(h) .$$

This completes the proof of the theorem.

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