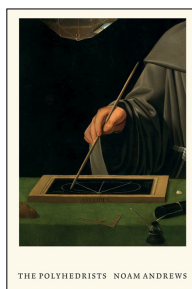# Book reviews

The Polyhedrists by Noam Andrews

Reviewed by Adhemar Bultheel

Euclid's *Elements* has been the book of choice for centuries to teach geometry. Three-dimensional geometry was only considered near the end in Book XI, and the regular or Platonic solids (tetrahedron, cube, octahedron, dodecahedron and icosahedron) appeared in the very last Book XIII. This was not on the regular curriculum of the quadrivium. Besides these five regular solids and the thirteen Archimedean solids that can be seen as semi-regular, it became fashionable among artists and craftsmen of the 15–17th century to use or depict these solids in their work. Soon several truncations, indentations, and stellations generated a plethora of so-called irregular solids. Irregular is a misleading term, since they are still very symmetric, but they do not satisfy the classical definition of the Platonic or Archimedean corpora. Note that it were not the mathematicians who published theory-free texts with plots and maps useful as manuals to produce these solids. With the invention of perspective, showing these objects in their pictures was a token of craftsmanship and it illustrated the supposed mathematical, or at least intellectual skills of the person being portrayed. With many examples and beautiful illustrations, the historian and architect Noam Andrews tells the story of this evolution in the wake of the Renaissance period.

The most iconic pictures involving nested polyhedra can be linked to Kepler, who associated the Platonic solids with the solar system. Also well known is Dürer's *Melencolia I*, featuring a polyhedron and a magic square. Some may know the portrait of Luca Pacioli attributed to Jacopo de' Barban that has a water-filled glass polyhedron hanging on a string and a solid polyhedron on the table. Pacioli is the author of *Divina proportione* (1509), a book illustrated by Leonardo Da Vinci in which he discusses the golden ratio and the use of perspective. The latter was invented less than a century ago and that was brought to a conclusion by Da Vinci.

Andrews explains how the polyhedral ideas evolved mainly throughout the 16th century. It all starts with Erhard Ratdolt's edition of Euclid's *Elements* (1482) that had some graphical illustrations. These illustrations are natural for us, but they were new then, in the early days of book printing. Graphics helped to understand the proofs, but it was a completely different matter to paint, etch or draw the three-dimensional geometric solids in proper perspective. At the universities, the study of perspective was part of optics. Drawing and constructing the solids was the concern of an artist and not so much of mathematical interest.

In the next chapters, the 16th century Bavarian city of Nuremberg is the historical scenery. This is where Dürer had access to the Regiomontanus-Walther library and where he published his book on geometry *Underweysung der Vermessung* (1525) [Instruction on measurement] treating among other things the Platonic, Archimedean, and other solids. That book contained a lot of accumulated knowledge, not only for painters and sculptors, but also for stonemasons, carpenters and goldsmiths. He places visualisation central, which was new for mathematical texts. Nets of planar developments of polyhedra could be cut out and folded and glued together to form a paper 3D version of the solid and vice versa, the solids could be developed into a printable 2D net.

Now *Lehrbücher* started circulating in Southern Germany. These were like very graphical instruction manuals, that were not generated at a university. They were manuals for artists and craftsmen that were devoid of theory and theorems. For example, Augustin Hirshvogel's *Geometria* (1543) was a very popular one. The 3D versions could be used as art objects but also as a teaching object to practice perspective. Placed in front of the student, mechanical devices were constructed that allowed to transfer the view of a solid to a planar perspective view. Theory and theorems were not involved.

Wenzel Jamnitzer was a famous goldsmith/mathematician who worked for the Holy Roman Emperors. He also came to Nuremberg where he published his *Perspectiva corporum regularium* (1568) providing 24 variations (6 per page) for each of the 5 Platonic solids, starting from the simple solid, modified by increasingly more complicate truncations, indentations, or stellations.

The next two chapters are about probably lesser known forms of polyhedrism. The first is polyhedral marquetry often for furniture in the form of cabinets. There is an exquisite example of such a cabinet in the Museum for Applied Arts in Cologne, richly decorated with polyhedral objects. The little known, somewhat enigmatic artist Lorenz Stöer was a very popular inspiration for Augsburger cabinets. He produced eleven surrealistic colour graphics showing deserted ruins of cities populated by all kinds of polyhedral structures. The last chapter is about the invention of sophisticated lathes or turn tables used by master turners to make ivory columns (*Säulen*) which had on top, or sometimes half way, some sphere-like dodecahedron or another solid with opened faces so that inside you can find another smaller one that had yet another one inside, like Russian matryoshkas. These contrefait spheres were fashionable at the Saxonian court in Dresden. Egidius Lobenigk and Georg & Hans Wecker were famous master turners.

In an epilogue, Andrews reflects on the role played by the polyhedrists. The popularisation did come from the application, and not from the theory. It was only by the end of the 16th century that gradually the subject became again absorbed by the academia. This movement with popular *Lehrbücher* has awakened geometry from almost two thousand year of frozen knowledge. Artists started to think outside the plane and the generalisation and endless variation of the classical solids made it possible for geometry to break loose from its static immobility and an incentive was given for a more evolved geometry as developed by Kepler, Monge, Descartes, and at a larger scale even Einstein.

This is a nicely illustrated and easy to read book on some less-known historical aspects of applied geometry. Most of the material covered is restricted to Southern Germany and its content is mostly historical and cultural, with little mathematics. It is however interesting to learn how this specific geometrical topic became aesthetically fashionable and how it evolved outside the universities, yet undoubtedly had an impact on the development of geometry.

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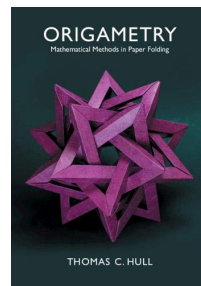
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Origametry – Mathematical Methods in Paper Folding
by Thomas C. Hull

Reviewed by Ana Rita Pires

Origami is the art of paper folding, with ancient origins: the classic Japanese paper crane was supposedly devised in the 6th century. The last hundred years have brought new interest in this art, with the creation of increasingly complex and beautiful origami models (such as the five intersecting tetrahedra on the cover of the book, created by the author Tom Hull and voted by the British Origami Society as one of the top ten origami models of all time), and also with the appearance of applications ranging from nano-robots for medical use to solar arrays for spacecraft. In parallel, rich mathematical theories related to origami were developed, at a particularly rapid pace in the last decade.

"Origametry" is the most comprehensive reference book on the connections between origami and mathematics. Its author, Tom Hull, is an associate professor of mathematics at Western New England University who has been studying the mathematics of origami for decades. He compiles and describes in one volume a truly impressive amount of material created by numerous researchers on a diverse array of the mathematical aspects of paper folding.

The book is divided into four parts.

Part I describes *Geometric Constructions*. It introduces the basic origami operations and shows how they can be used to trisect an angle, construct a regular heptagon, and more generally solve any cubic equation – all of which are famously impossible to achieve using a straightedge and compass. A complete classification of what constructions are possible with these basic origami operations is achieved by determining the field of origami numbers using Galois theory. Further avenues of research in this direction concern geometric constructions that can be achieved using multifolds (in which the paper is folded in a way that creates more than one crease at once) or curved creases.

If you unfold an origami model, you get a crease pattern, a pattern of line segments that represent valley folds and mountain folds and intersect at definite angles. The main question in Part II of this book, titled *The Combinatorial Geometry of Flat Origami*, is whether a crease pattern can be flat-folded, that is, folded into an origami model that lies flat in a plane once all the creases are folded (such as a paper crane before pulling out the wing flaps to make it three-dimensional). Maekawa's and Kawasaki's Theorems give conditions for a crease pattern to be locally flat-foldable around each of its vertices. Both results are easy to state and have short proofs. The first gives the necessary condition that the number of valley folds and the number of mountain folds at each vertex