The next two chapters are about probably lesser known forms of polyhedrism. The first is polyhedral marquetry often for furniture in the form of cabinets. There is an exquisite example of such a cabinet in the Museum for Applied Arts in Cologne, richly decorated with polyhedral objects. The little known, somewhat enigmatic artist Lorenz Stöer was a very popular inspiration for Augsburger cabinets. He produced eleven surrealistic colour graphics showing deserted ruins of cities populated by all kinds of polyhedral structures. The last chapter is about the invention of sophisticated lathes or turn tables used by master turners to make ivory columns (*Säulen*) which had on top, or sometimes half way, some sphere-like dodecahedron or another solid with opened faces so that inside you can find another smaller one that had yet another one inside, like Russian matryoshkas. These contrefait spheres were fashionable at the Saxonian court in Dresden. Egidius Lobenigk and Georg & Hans Wecker were famous master turners.

In an epilogue, Andrews reflects on the role played by the polyhedrists. The popularisation did come from the application, and not from the theory. It was only by the end of the 16th century that gradually the subject became again absorbed by the academia. This movement with popular *Lehrbücher* has awakened geometry from almost two thousand year of frozen knowledge. Artists started to think outside the plane and the generalisation and endless variation of the classical solids made it possible for geometry to break loose from its static immobility and an incentive was given for a more evolved geometry as developed by Kepler, Monge, Descartes, and at a larger scale even Einstein.

This is a nicely illustrated and easy to read book on some less-known historical aspects of applied geometry. Most of the material covered is restricted to Southern Germany and its content is mostly historical and cultural, with little mathematics. It is however interesting to learn how this specific geometrical topic became aesthetically fashionable and how it evolved outside the universities, yet undoubtedly had an impact on the development of geometry.

Noam Andrews, *The Polyhedrists.* MIT Press, 2022, 316 pages, Paperback ISBN 978-0-2620-4604-6.

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Adhemar Bultheel is emeritus professor at the Department of Computer Science of the KU Leuven (Belgium). He has been teaching mainly undergraduate courses in analysis, algebra, and numerical mathematics.

Adhemar.Bultheel@cs.kuleuven.be

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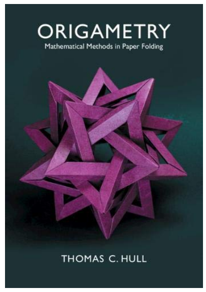
Origametry – Mathematical Methods in Paper Folding
by Thomas C. Hull

Reviewed by Ana Rita Pires



Origami is the art of paper folding, with ancient origins: the classic Japanese paper crane was supposedly devised in the 6th century. The last hundred years have brought new interest in this art, with the creation of increasingly complex and beautiful origami models (such as the five intersecting tetrahedra on the cover of the book, created by the author Tom Hull and voted by the British Origami Society as one of the top ten origami models of all time), and also with the appearance of applications ranging from nano-robots for medical use to solar arrays for spacecraft. In parallel, rich mathematical theories related to origami were developed, at a particularly rapid pace in the last decade.

"Origametry" is the most comprehensive reference book on the connections between origami and mathematics. Its author, Tom Hull, is an associate professor of mathematics at Western New England University who has been studying the mathematics of origami for decades. He compiles and describes in one volume a truly impressive amount of material created by numerous researchers on a diverse array of the mathematical aspects of paper folding.

The book is divided into four parts.

Part I describes *Geometric Constructions*. It introduces the basic origami operations and shows how they can be used to trisect an angle, construct a regular heptagon, and more generally solve any cubic equation – all of which are famously impossible to achieve using a straightedge and compass. A complete classification of what constructions are possible with these basic origami operations is achieved by determining the field of origami numbers using Galois theory. Further avenues of research in this direction concern geometric constructions that can be achieved using multifolds (in which the paper is folded in a way that creates more than one crease at once) or curved creases.

If you unfold an origami model, you get a crease pattern, a pattern of line segments that represent valley folds and mountain folds and intersect at definite angles. The main question in Part II of this book, titled *The Combinatorial Geometry of Flat Origami*, is whether a crease pattern can be flat-folded, that is, folded into an origami model that lies flat in a plane once all the creases are folded (such as a paper crane before pulling out the wing flaps to make it three-dimensional). Maekawa's and Kawasaki's Theorems give conditions for a crease pattern to be locally flat-foldable around each of its vertices. Both results are easy to state and have short proofs. The first gives the necessary condition that the number of valley folds and the number of mountain folds at each vertex differs by two, and the second gives the necessary and sufficient condition that the alternating sum of consecutive angles at each vertex is zero. It turns out that the question of whether a crease pattern is globally flat-foldable is much harder; it is in fact NP-hard. The proof involves reducing this problem to the not-all-equal 3-satisfiability problem, an NP-complete version of the Boolean satisfiability problem, by creating origami "gadgets" whose flat-foldability requirements mimic the Boolean values of the variables and the clauses. This second part of the book also contains a variety of other foldability questions, of which two examples are the fold-and-cut problem (Given a two-dimensional shape, can you fold a piece of paper so that applying a single straight cut will produce that shape? Yes, for any shape.) and Arnold's rumpled rouble problem (Is it possible to increase the perimeter of a rectangle by folding it into a different shape? Yes, as much as one wishes.).

After looking at the geometry and combinatorics of flat origami, the book turns in Part III to connections with other branches of mathematics, namely *Algebra, Topology, and Analysis in Origami*. For algebra, group theory is used to relate the symmetries of a crease pattern with the symmetries of its flat-folded model. For topology, the notion of folding along straight lines on (a subset of) the Euclidean plane is extended not just to folding along geodesics on Riemannian surfaces, but further to "isometric foldings" of Riemannian manifolds in arbitrary dimension. An isometric folding is a continuous map from the crease pattern manifold to the origami model manifold that sends piecewise geodesic segments to piecewise geodesic segments. It turns out that even in this setting, suitable generalizations of Maekawa's and Kawasaki's Theorems exist. For analysis, it examines the problem of finding an isometric folding on Euclidean space that satisfies a given differential equation and boundary condition – a Dirichlet problem.

Part IV of the book is titled *Non-Flat Folding* and mostly examines the mathematical underpinnings of rigid origami, that is, three-dimensional origami models made of flat polygonal faces which remain rigid during the folding process. Rigid origami is the natural setting for applications in engineering, with objects whose faces are made of a rigid material such as metal or glass and are joined by hinges. This is an active area of research, with practical problems often driving the mathematical research. For example: a question coming from mechanics and robotics is whether a certain crease pattern will self-fold to its desired final state by applying forces in certain hinges.

This is a true maths book: with theorems, proofs, definitions, and examples. It also contains historical remarks, open problems, and diversions, which range from interesting and fun exercises to explore to straightforward parts of proofs that the reader is invited to complete. Between the diversions and the open problems, this book is bound to inspire several undergraduate, master's, and even PhD theses. It is a delightful and informative read for mathematicians curious about the mathematics behind origami, essential for researchers starting out in this area, and handy for educators searching for ideas in topics connecting mathematics, origami, and its applications. Even though it is not written with that goal specifically in mind, it could be used as a textbook for a graduate course or a reading course.

A final word of advice: have some paper at the ready, it is difficult to resist folding along while reading!

Thomas C. Hull, *Origametry – Mathematical Methods in Paper Folding.* Cambridge University Press, 2020, 342 pages, Paperback ISBN 978-1-1087-4611-3.

Ana Rita Pires is a symplectic geometer and lecturer in mathematics at the University of Edinburgh (Scotland). She did her undergraduate studies at Instituto Superior Técnico in Portugal and received her PhD at the Massachusetts Institute of Technology in the US. She went on to work at Cornell University, Institute for Advanced Study, and Fordham University in the US, and then at Murray Edwards College in Cambridge UK before moving up to Scotland. She has done some unusual teaching and outreach, with audiences ranging from very young children to incarcerated people, and from maths teachers to strangers at a bar.

apires@ed.ac.uk

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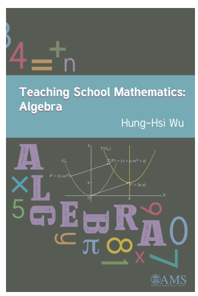
Teaching School Mathematics: Algebra by Hung-Hsi Wu

Reviewed by António de Bivar Weinholtz



This is the third book in a series of six covering the K-12 curriculum, as an instrument for the mathematical education of schoolteachers. It follows two volumes entitled "Understanding Numbers in Elementary School Mathematics" and "Teaching School Mathematics: Pre-Algebra", and it completes the presentation of the mathematical topics included in the K-8 curriculum. With numbers and operations, finite probability and an introduction to geometry and geometric measurement covered in the previous two volumes, here the author deals with the topics that can be found in any middle school or high school introductory course on algebra: linear equations in one and two variables, linear inequalities in one and two variables, simultaneous linear equations, the concepts of a function, polynomial functions, exponents, and a detailed study of linear and quadratic functions. The volume also contains a very helpful appendix with a list of assumptions, definitions, theorems, and lemmas from the previous pre-algebra volume.