Solved and unsolved problems

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The present column is devoted to Differential Equations.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

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Let $f\colon [0,\infty) \to \mathbb{R}$ be a C^1-differentiable and convex function with $f(0) = 0$.

(i) Prove that, for every $x \in [0,\infty)$, the following inequality holds:

$$\int_0^x f(t)\,dt \le \frac{x^2}{2} f'(x).$$

(ii) Determine all functions f for which we have equality.

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Let $y(x)$ be the unknown function of the following fractional-order derivative Cauchy problem:

$$\begin{cases} D^\alpha y = f(x,y), \quad 0 < \alpha < 1, \\ y(0) = y^*. \end{cases}$$

Find the solution of this problem by solving an equivalent first-order ordinary Cauchy problem, with a solution independent on the kernel of the fractional operator.

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Let $y(x)$ be the unknown function of the following Bernoulli fractional-order Cauchy problem:

$$\begin{cases} D^\alpha y = g(x) y^\beta, \quad 0 < \alpha < 1,\ \beta \neq 0, 1, \\ y(0) = y^*, \end{cases}$$

where $g(x)$ is a continuous function in the interval $I = [0,\infty)$.

Find the solution of this problem by solving an equivalent first-order ordinary Cauchy problem, with a solution independent on the kernel of the fractional operator.

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Let g be a real-valued C^2-function defined on $(0,\infty)$, strictly increasing, such that $g(x) > 1$ for all $x \in (0,\infty)$ and $g(2) < 4$. Consider the boundary value problem

$$y'' = -g(x) y, \quad y(0) = 1, \quad y'(0) = 0.$$

Prove that the solution y has exactly one zero in $(0, \pi/2)$, i.e., there exists a unique point $x_0 \in (0,\pi/2)$ such that $y(x_0) = 0$, and give a positive lower bound for x_0.

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We propose an interesting stochastic-source scattering problem in acoustics. The stochastic nature for such problems forces us to deal with stochastic partial differential equations (SPDEs), rather than the partial differential equations (PDEs) which hold for the corresponding deterministic counterparts. In particular, we provide the appropriate variational formulation for the stochastic-source Helmholtz equation.

We consider the following boundary value problem (BVP) for the Helmholtz equation with a stochastic source:

$$\begin{cases} \Delta u + k^2 u = f & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1}$$

where $f = \sum_a f_a H_a$ is a generalized stochastic source and

$$H_a(\omega) = \prod_{i=1}^{\infty} h_{a_i}(\langle \omega, \xi_{\delta^i} \rangle)$$

are stochastic Hermite polynomials with $\omega \in \Omega$, Ω being a probability space. The Hermite polynomials are denoted by h_{a_i}, whereas the tensor product is denoted by ξ_{d^j}. We also define the Hermite functions $\xi_n(x)$ as follows:

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{n-1}(x), \quad n = 1, 2, 3, \ldots,$$

and we set $d^j \coloneqq (d_1^j, d_2^j, \ldots, d_m^j)$, where $d_i^j \in \mathbb{N}$ is related to the following tensor products:

$$\xi_{d^j} \coloneqq \xi_{d_1^j} \otimes \xi_{d_2^j} \otimes \cdots \otimes \xi_{d_m^j}, \quad j = 1, 2, 3, \ldots,$$

with $i < j \Leftrightarrow d_1^i + d_2^i + \cdots + d_m^i \le d_1^j + d_2^j + \cdots + d_m^j$ and $|d_j| = d_1^j + d_2^j + \cdots + d_m^j$. In addition, we employ the countable index $I = \{a = (a_1, a_2, \ldots) \mid a_i \in \mathbb{N} \cup \{0\}\}$, and there only finitely many $a_i \neq 0$.

For the stochastic problem (1), we use the expansions

$$u = \sum_{a \in I} u_a H_a \quad \text{and} \quad f = \sum_{a \in I} f_a H_a$$

to get a hierarchy of deterministic BVPs

$$\begin{cases} \Delta u_a + k^2 u_a = f_a & \text{in } D, \\ u_a = 0 & \text{on } \partial D. \end{cases} \tag{2}$$

Assume that $u_a \in H_0^1(D)$ solves problem (2). Then prove that, for every $v \in H_0^1(D)$, the solution u_a satisfies

$$-\int_D \nabla u_a \cdot \nabla v \, dx + \int_D k^2 u_a v \, dx = \int_D f_a v \, dx.$$

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For a Newtonian incompressible fluid, the Navier–Stokes momentum equation, in vector form, reads [3]

$$\begin{aligned} \rho\Big(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\Big) &= -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}, \\ \mathbf{u} = \mathbf{u}(\mathbf{x}, t), &\quad \mathbf{u} \colon \mathbf{R}^n \times (0, \infty) \to \mathbf{R}^n. \end{aligned} \tag{1}$$

Here, ρ is the fluid density, $\mathbf{u}$ is the velocity vector field, p is the pressure, μ is the viscosity, and $\mathbf{F}$ is an external force field.

(i) Assuming that both the pressure drop ∇p and the external field $\mathbf{F}$ are negligible, it is easy to show that equation (1) reduces to

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u},$$

and finally to equation (2), where $\nu = \frac{\mu}{\rho}$ is the so-called kinematic viscosity [4].

(ii) Regarding the one-dimensional viscous Burgers equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad u = u(x, t), \tag{2}$$

prove that an analytical solution can be obtained by means of the Tanh Method [1, 2, 4] as

$$u(x, t) = \lambda\Big[1 - \tanh\Big(\frac{\lambda}{2\nu}(x - \lambda t)\Big)\Big], \quad \lambda > 0.$$

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References

[1] W. Malfliet and W. Hereman, The tanh method. I. Exact solutions of nonlinear evolution and wave equations. *Phys. Scripta* **54**, 563–568 (1996)

[2] A.-M. Wazwaz, The tanh method for traveling wave solutions of nonlinear equations. *Appl. Math. Comput.* **154**, 713–723 (2004)

[3] M. A. Xenos, An Euler–Lagrange approach for studying blood flow in an aneurysmal geometry. *Proc. A.* **473**, 20160774 (2017)

[4] M. A. Xenos and A. C. Felias, Nonlinear dynamics of the KdV-B equation and its biomedical applications. In *Nonlinear Analysis, Differential Equations, and Applications*, Springer Optim. Appl. 173, Springer, Cham, 765–793 (2021)

II Open problems

(A) *Uniqueness of positive steady states for KPP equations in general domains*

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Reaction-diffusion equations

These arise ubiquitously in the modelling of population dynamics, and more generally in biology and ecology. Remarkably, various fields converge on these equations. In addition to modelling in the life sciences and, of course, nonlinear partial differential equations, they arise in probability theory (via branching particle systems) and statistical physics. These equations have witnessed remarkable progress in recent years. Yet, many basic problems remain open. The object of this note is to present a couple of such questions that are simple to formulate.

Reaction-diffusion equations of *homogeneous type* read in general as $\partial_t u - \Delta u = f(u)$ in $\mathbb{R}^N$. The nonlinear term f is called the reaction term and the Laplacian operator is associated with diffusion. This equation is termed *homogeneous* because it does not involve explicitly the location x (or time t) and also because it is set in all of space. The Fisher–KPP case (or strong KPP case) refers to the class of nonlinear terms f of class C^1 that satisfy

$$f(0) = f(1) = 0, \quad \text{and the function } s \mapsto \frac{f(s)}{s} \text{ is decreasing on } (0,1]. \tag{1}$$

The archetypal example is $f(u) = u(1-u)$. These reactions terms were introduced and first studied by Fisher [9] and Kolmogorov, Petrovsky and Piskunov (KPP) [10]. I will discuss some questions related to the uniqueness of bounded positive stationary solutions, that is, bounded positive solutions of the semilinear elliptic equation $-\Delta u = f(u)$ with boundary conditions.

Heterogeneous equations

In recent years, many works have addressed *heterogeneous* versions of the equations introduced above. These arise in various guises. First, the reaction term f is allowed to vary in space and time: $f = f(t,x,u)$. Likewise, in various models, one wishes to consider more general second-order elliptic operators than the Laplacian:

$$\sum_{ij} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial u}{\partial x_i}.$$

My works with Hamel and Rossi [6] and Hamel and Nadin [4] are devoted precisely to this type of question. The interested reader will find in or infer from these papers open problems analogous to several that I describe here.

Another natural heterogeneity arises from the *geometry* of the domain of propagation when it is not the whole space. Given an open subset $\Omega \subset \mathbb{R}^N$ subject to Dirichlet boundary conditions, we are led to study the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ 0 < u \le 1 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Indeed, in many cases of interest, $f(s) < 0$ for all $s > 1$, and then one can show that any non-negative bounded solution (besides 0) satisfies $0 < u < 1$.

Existence

To discuss the existence of a positive solution of (2), we use the generalized principal Dirichlet eigenvalue in the domain Ω, defined as

$$\lambda(\Omega) := \inf_{\phi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla \phi|^2}{\int_\Omega \phi^2}.$$

This definition coincides in the present case with the notion of generalized principal eigenvalue introduced in [7] and applied to unbounded domains in [8]. We can then state the existence result in a more general framework of *weak KPP* class:

$$f(0) = f(1) = 0, \quad 0 < f(s) < f'(0)s \quad \text{for all } s \in (0,1). \tag{3}$$

Existence in (2) is conditioned by this eigenvalue.

Theorem 1. *Let f satisfy the weak KPP condition* (3)*. Then* (2) *admits a positive bounded solution if $\lambda(\Omega) < f'(0)$. Conversely, if $\lambda(\Omega) > f'(0)$,* (2) *has no positive bounded solution.*

This result from [2] is analogous to the one for variable-coefficient operators in $\mathbb{R}^d$ in [6] and is obtained with the same arguments.

Uniqueness. When the domain Ω is bounded and f satisfies the strong KPP assumption (1), the solution of (2) is unique when it exists [1]. This raises a natural question: is the same true in unbounded domains? Cole Graham and myself [2] have been working on this problem and our progress leads us to formulate the following.

Conjecture 2. *Consider an unbounded uniformly smooth (say $C^{2,\alpha}$) domain Ω. Under the strong KPP condition* (1)*, the solution of problem* (2) *is unique when it exists.*

Here, "uniformly smooth" means that there is a fixed $r > 0$ such that for any boundary point $p \in \partial\Omega$, its boundary neighbourhood $\partial\Omega \cap B_r(p)$ can be represented as the graph of some $C^{2,\alpha}$ function $\phi_p : D \to \mathbb{R}$, where D is the unit ball in $\mathbb{R}^{N-1}$ and $\|\phi_p\|_{C^{2,\alpha}}$ is bounded independently of the point p (see [5, Section 1.3]). One may be even more demanding and lift this uniform regularity condition.

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Open problem. *In a locally smooth domain Ω with f of strong KPP-type, is the solution of problem* (2) *unique when it exists?*

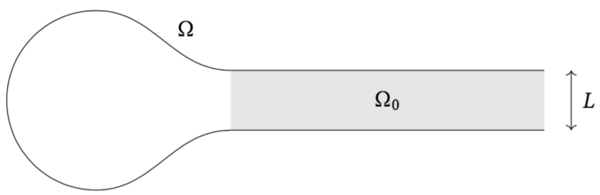
The conjecture in its full generality is open. In my work with Cole Graham [2], we prove uniqueness under a non-degeneracy condition. This result covers a large variety of cases and can be viewed as generic. Its statement requires the use of eigenvalues on various limits of Ω. We say that Ω^* is the *connected limit* of Ω along a sequence $(x_n)_{n\in\mathbb{N}} \subset \Omega$ if the following holds. There exists a uniformly $C^{2,\alpha}$ domain $\tilde{\Omega} \supset \Omega^*$ such that $\Omega - x_n \to \tilde{\Omega}$ locally uniformly in $C^{2,\alpha}$ as $n \to \infty$, and Ω^* is the connected component of $\tilde{\Omega}$ whose closure contains 0. We then define the *principal limit spectrum* as

$$\Sigma(\Omega) := \{\lambda(\Omega^*) \mid \Omega^* \text{ is a connected limit of } \Omega\},$$

and we let $\overline{\Sigma}(\Omega)$ denote its closure. We refer to the elements of Σ as *(principal) limit eigenvalues*. One of our main results in [2] is the following.

Theorem 3. *Suppose* Ω *is uniformly smooth, f satisfies* (1)*, and* $f'(0) \notin \overline{\Sigma}(\Omega)$*. Then the solution of* (2) *is unique when it exists.*

An example. To illustrate Conjecture 2 and Theorem 3, consider the following domain in $\mathbb{R}^2$ that we call the "infinite light bulb".



We assume that the round portion is sufficiently large that $\lambda_1(\Omega) < f'(0)$. Then, by Theorem 1, we know that (2) admits at least one solution. We can show that $\Sigma(\Omega) = \{\lambda(\Omega), \pi^2/L^2\}$. Thus Theorem 3 applies when $L \neq \pi/\sqrt{f'(0)}$. The critical case $L = \pi/\sqrt{f'(0)}$ is not covered by our result. Nonetheless, in [2], we exploit the explicit structure of Ω to prove that the solution of (2) is still unique in this case. This supports Conjecture 2.

Robin type conditions

Other types of boundary conditions are of interest as well. We can consider the Robin problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \quad 0 < u \leq 1, \\ -\dfrac{\partial u}{\partial \nu} = \gamma u & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where ν is the unit outward normal vector field on the boundary $\partial\Omega$ and $\gamma \geq 0$ is a constant. More generally, one might consider a function $\gamma(x) \geq 0$ that varies on $\partial\Omega$.

Conjecture 4. *In a uniformly smooth domain* Ω *with f of strong KPP-type, the solution of problem* (4) *is unique when it exists.*

In our forthcoming work [2], we establish an analogue of Theorem 3 in the Robin case. This requires a suitable notion of the generalized principal *Robin* eigenvalue.

General positive and other reaction terms

In [2], we also consider the more general class of *positive* nonlinear terms f. This class is defined by the conditions

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f(s) > 0 \quad \text{for all } s \in (0,1). \tag{5}$$

In all of space $\mathbb{R}^N$, uniqueness holds in the more general *positive* case. Indeed, under conditions (5), $u \equiv 1$ is the unique solution of (2) when $\Omega = \mathbb{R}^N$. For a proof, I refer the reader to the forthcoming book [3]. The presence of boundary changes matters significantly. In fact, in a proper subset $\Omega \subset \mathbb{R}^N$ with Dirichlet or Robin boundary, solutions of (2) (or (4)) need *not* be unique. However, uniqueness holds under *Neumann* boundary conditions.

Theorem 5. *In a uniformly smooth domain* Ω *with f of positive type* (5)*, the unique solution of the Neumann problem* (4) *with* $\gamma = 0$ *is* $u \equiv 1$.

This result is a generalization of one in my earlier work with Hamel and Nadirashvili [5]. This form is due to Rossi [11]. It naturally calls for the following.

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Problem. *Can the result of Theorem* 5 *be extended to locally smooth domains?*

References

[1] H. Berestycki, Le nombre de solutions de certains problèmes semi-linéaires elliptiques. *J. Funct. Anal.* **40**, 1–29 (1981)

[2] H. Berestycki and C. Graham, The steady states of strong-KPP reactions in general domains, in preparation (2022)

[3] H. Berestycki and F. Hamel, *Reaction-diffusion equations and propagation phenomena*, Springer (to appear)

[4] H. Berestycki, F. Hamel and G. Nadin, Asymptotic spreading in heterogeneous diffusive excitable media. *J. Funct. Anal.* **255**, 2146–2189 (2008)

[5] H. Berestycki, F. Hamel and N. Nadirashvili, The speed of propagation for KPP type problems. II. General domains. *J. Amer. Math. Soc.* **23**, 1–34 (2010)

[6] H. Berestycki, F. Hamel and L. Rossi, Liouville-type results for semilinear elliptic equations in unbounded domains. *Ann. Mat. Pura Appl. (4)* **186**, 469–507 (2007)

[7] H. Berestycki, L. Nirenberg and S. R. S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains. *Comm. Pure Appl. Math.* **47**, 47–92 (1994)

[8] H. Berestycki and L. Rossi, Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains. *Comm. Pure Appl. Math.* **68**, 1014–1065 (2015)

[9] R. A. Fisher, The wave of advance of advantageous genes. *Ann. Eugen.* **7**, 355–369 (1937)

[10] A. N. Kolmogorov, I. G. Petrovsky and N. S. Piskunov, Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bull. Univ. Moskow, Ser. Internat., Sec. A* **1**, 1–25 (1937)

[11] L. Rossi, Stability analysis for semilinear parabolic problems in general unbounded domains. *J. Funct. Anal.* **279**, Article ID 108657 (2020)

(B) *A problem in geometric analysis*

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The last 40 years have seen enormous progress in the application of variational methods to problems in geometric analysis, which in general are characterized by the possibility of "bubbling" and topological degeneration of sequences of approximate solutions obtained either by regularization of the problem, or as "Palais–Smale sequences" for the energy functional involved. In critical point theory therefore it is vital to understand the possible interaction of the problem at hand with its "cousins" that characterize the "bubbling", in particular, when the sought-after critical points are of "mountain-pass" type.

As an example consider the (by now classical) "Nirenberg problem" of finding conformal metrics of prescribed Gauss curvature on the standard 2-sphere, which has given rise to sophisticated analytic approaches and deep insights into the interplay of analysis and geometry, but which still poses a challenge, even though many partial answers have been obtained.

Nirenberg's problem

After the work of Berger [1] and Kazdan–Warner [4] on conformal metrics of prescribed Gauss curvature on closed Riemann surfaces, the particular case, proposed by Nirenberg, of finding conformal metrics $g = e^{2u}g_0$ on the sphere S^2 with its standard round metric g_0 having a given function f as Gauss curvature $K_g = f$ has attracted the attention of geometric analysts.

In view of the equation

$$K_g = e^{-2u}(-\Delta_0 u + 1)$$

relating K_g and u, where Δ_0 is the Laplace–Beltrami operator in the metric g_0, for given $f\colon S^2 \to \mathbb{R}$, we need to solve the nonlinear partial differential equation

$$-\Delta_0 u + 1 = fe^{2u} \quad \text{on } S^2. \tag{1}$$

The problem is variational. Indeed, introducing the Liouville energy

$$S(u) = \fint_{S^2} (|\nabla u|^2 + 2u)\, d\mu_0,$$

where $d\mu_0$ is the area element in the metric g_0 and $\fint_{S^2} = \frac{1}{4\pi}\int_{S^2}$ denotes the average, and setting

$$E(u) = S(u) - \log\Big(\fint_{S^2} fe^{2u}\, d\mu_0\Big) \tag{2}$$

for $u \in H^1(S^2)$, the standard Sobolev space of L^2-functions on S^2 with square-integrable weak derivatives, solutions of (1) may be characterized as critical points of E.

Via the Möbius group M of conformal diffeomorphisms of the sphere, for any point $p \in S^2$ the functional E may be compared with the functional

$$E_p(u) = S(u) - \log\Big(\fint_{S^2} f(p)e^{2u}\, d\mu_0\Big),$$

where f is replaced by the constant $f(p)$. Indeed, given any $p \in S^2$, any $t \geq 1$, letting $\Phi_p\colon S^2 \setminus \{-p\} \to \mathbb{R}^2$ be the stereographic projection from the point $-p \in S^2$ and letting $\delta_t\colon \mathbb{R}^2 \ni z \to tz \in \mathbb{R}^2$ be the standard dilation, we obtain the Möbius map

$$\Phi_{p,t} = \Phi_p^{-1} \circ \delta_t \circ \Phi_p \in M.$$

Letting $u_{p,t} = u \circ \Phi_{p,t} + \log|\Phi'_{p,t}|$, where we write $|\Phi'| = \sqrt{\det d\Phi}$ for brevity, we then have

$$S(u_{p,t}) = S(u)$$

(see for instance [2, Proposition 2.1]) and thus

$$\begin{aligned} E(u_{p,t}) &= S(u_{p,t}) - \log\Big(\fint_{S^2} fe^{2u_{p,t}}\, d\mu_0\Big) \\ &= S(u) - \log\Big(\fint_{S^2} (f \circ \Phi_{p,t}^{-1})e^{2u}\, d\mu_0\Big) \to E_p(u) \quad \text{as } t \to \infty. \end{aligned}$$

For large $t > 1$, it was shown by Chang–Yang [2] that the first and second variation of E at $u_{p,t}$ may be related to $\nabla f(p)$ and $\nabla^2 f(p)$, respectively. From this observation, they deduce the following existence result.

Theorem 6 (Chang–Yang [2], Theorem II′). *Suppose that $f > 0$ is a smooth function satisfying the non-degeneracy condition*

$$\Delta_0 f(p) \neq 0 \quad \text{at any } p \in S^2 \text{ with } \nabla f(p) = 0 \tag{3}$$

and the index count condition

$$\sum_{\nabla f(p) = 0,\, \Delta_0 f(p) < 0} (-1)^{\operatorname{ind}(p)} \neq 1. \tag{4}$$

Then there is a smooth solution u to (1).

Interpretation

Condition (4) in Theorem 6 may be interpreted in terms of the "last Morse inequality" related to the variational integral (2), that is, in terms of an equation identifying the "topological degree" $d = 1$ of the (contractible) set of admissible functions $H^1(S^2)$ with the sum of the topological degrees of all critical points of E, including the contributions of the degenerate variational problems related to the functionals E_p, $p \in S^2$. With what we remarked above, the latter contribution is given by the left-hand side of (4). Thus, if that term is different from 1, there has to be a further contribution to the total topological degree of all critical points, then necessarily coming from a solution u to (1).

Open problem

In [8] an example was given showing that condition (4) in Theorem 6 in general cannot be removed; thus, with the non-degeneracy condition (3), condition (4) is not only sufficient but also in general necessary for the existence of a solution to (1).

However, we are still lacking a precise characterization of *all* solutions of (1). In particular, we should be able to obtain the existence of multiple solutions in certain cases. A simple instance of such a case, where – hopefully – the problem is feasible, would be when the given function f is symmetric with respect to reflection in a plane and is a Morse function similar to the example studied in [8] but satisfying the Chang–Yang condition (4).

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Problem. *Let F be the set of functions $0 < f \in C^\infty(S^2)$ with*

$$f(x_1, x_2, x_3) = f(-x_1, x_2, x_3) \quad \text{for } x = (x_1, x_2, x_3) \in S^2,$$

having a saddle point at the north pole $x_3 = 1$, a minimum at the south pole $x_3 = -1$, and precisely two maxima as critical points, all of which are non-degenerate and satisfy (3)*, and such that condition* (4) *holds. Find conditions for $f \in F$ such that there is more than one solution of* (1)*, and characterize the set of all solutions of* (1) *in the sense of Morse theory.*

Of course, the question may easily be widened to a larger class F of functions.

Related challenges

Note that Chang–Yang [2] showed that when $f \not\equiv 1$ solutions of (1) never are relative minima of the energy E.

The Nirenberg problem thus can be seen in the larger context of finding critical points of "mountain-pass" type for variational problems characterized by conformal invariance and "bubbling". A classic instance of such problems is in 4-dimensional gauge theory, in particular, in the question concerning the existence of 1-equivariant, non-minimal Yang–Mills connections in the trivial $\mathrm{SU}(2)$-bundle over S^4, which remained open after Sibner–Sibner–Uhlenbeck [6] obtained m-equivariant, non-minimal Yang–Mills connections for any $m \geq 2$; see also Donaldson [3, pp. 309–310] for further details. Moreover, conformal invariance is responsible for many of the difficulties encountered by Rivière [5] in his recent work on "min-max" critical points for the Willmore energy related to sphere eversion.

Recall that Smale [7] famously showed that it is possible to "turn a sphere inside out" via a continuous path of C^2-immersions of S^2 into $\mathbb{R}^3$. Moreover, Bryant characterized all immersed Willmore spheres in $\mathbb{R}^3$ as being given by the images by inversions of simply connected, complete, non-compact minimal surfaces with planar ends, with Willmore energy given by $4\pi k$, where k is the number of ends, and index equal to $k - 3$. Finally, a topological result of Banchoff–Max shows that any path everting the sphere has to contain at least one immersion with a quadruple point and therefore, by a result of Li–Yau, with Willmore energy $\beta \geq 16\pi$. Combining these pieces of information, Rivière conjectured that the inversion of a simply connected, complete minimal surface with $k = 4$ planar ends, thus having index $m = 1$ and Willmore energy 16π, should give a "min-max Willmore sphere", achieving the least maximal Willmore energy along paths of immersions of S^2 into $\mathbb{R}^3$ that "turn the sphere inside out". But in the variational ansatz "bubbling" may occur, and many questions remain to be solved. See [5] for further details and references.

Similarly, in gauge theory the desired 1-equivariant, non-minimal Yang–Mills connections in the trivial $\mathrm{SU}(2)$-bundle over S^4 should achieve the least maximal Yang–Mills energy along paths of connections beginning at a 1-equivariant Yang–Mills instanton and ending at a 1-equivariant anti-instanton. But again "bubbling" comes in the way.

References

[1] M. S. Berger, Riemannian structures of prescribed Gaussian curvature for compact 2-manifolds. *J. Differential Geom.* **5**, 325–332 (1971)

[2] S.-Y. A. Chang and P. C. Yang, Prescribing Gaussian curvature on S^2. *Acta Math.* **159**, 215–259 (1987)

[3] S. Donaldson, Karen Uhlenbeck and the calculus of variations. *Notices Amer. Math. Soc.* **66**, 303–313 (2019)

[4] J. L. Kazdan and F. W. Warner, Curvature functions for compact 2-manifolds. *Ann. of Math. (2)* **99**, 14–47 (1974)

[5] T. Rivière, Willmore minmax surfaces and the cost of the sphere eversion. *J. Eur. Math. Soc. (JEMS)* **23**, 349–423 (2021)

[6] L. M. Sibner, R. J. Sibner and K. Uhlenbeck, Solutions to Yang–Mills equations that are not self-dual. *Proc. Nat. Acad. Sci. U.S.A.* **86**, 8610–8613 (1989)

[7] S. Smale, A classification of immersions of the two-sphere. *Trans. Amer. Math. Soc.* **90**, 281–290 (1958)

[8] M. Struwe, A flow approach to Nirenberg's problem. *Duke Math. J.* **128**, 19–64 (2005)

III Solutions

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Prove that the space of unordered couples of distinct points of a circle is the (open) Möbius band. More formally, consider

$$(S^1 \times S^1) \setminus \{(x, x) \mid x \in S^1\}$$

and the equivalence relation on this space $(x, y) \equiv (y, x)$; prove that the quotient topological space is the (open) Möbius band.

Costante Bellettini (Department of Mathematics, University College London, UK)

Solution by the proposer

The space of ordered pairs of points of a circle is the cartesian product $S^1 \times S^1$, hence a torus. This is the same as the unit square $[0, 1] \times [0, 1]$ with the following identifications: $[0, 1] \times \{0\}$ is identified with $[0,1]\times\{1\}$, both with orientation from left to right; $\{0\}\times[0,1]$ is identified with $\{1\}\times[0,1]$, both with orientation from bottom to top. Note that the four vertices of the square are the same point. We now need to remove couples of the type (x,x) (same point of the circle), which implicitly removes $(1,0)$ and $(0,1)$ as well. Hence, we are now looking at the square with the diagonal from $(0,0)$ to $(1,1)$ removed, and with $(1,0)$ and $(0,1)$ removed, keeping the identification we had earlier. Next, we need to identify couples (x,y) and (y,x) (since we want to study unordered couples). This amounts to removing one of the two triangles that have been obtained after removing the diagonal of the square; without loss of generality we assume that we remove the top-left triangle. What is left is the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$, with the longer side removed (the one that was the diagonal of the square), with the point $(1,0)$ removed, and with the following identification: any point $(x,0)$ was identified (in the original torus) with the point $(x,1)$, which has then been identified with the point $(1,x)$. Hence, the triangle has the horizontal side oriented left-to-right identified with the vertical side oriented bottom-to-top. We can check that this is the (open) Möbius band as follows: the point $(1,0)$ is not in the triangle (nor is the longer side of the triangle), so we can stretch the point $(1,0)$ until the triangle becomes a rectangle, with the stretched point that has become the side opposite to the side that was the diagonal of the square. The identification of the remaining two sides gives the (open) Möbius band.

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In the Euclidean plane, let γ_1 and γ_2 be two concentric circles of radius respectively r_1 and r_2, with $r_1 < r_2$. Show that the locus γ of points P such that the polar line of P with respect to γ_2 is tangent to γ_1 is a circle of radius r_2^2/r_1.

Acknowledgement. I want to thank the professors who guided me in the first part of my career for giving me the ideas for these problems.

Paola Bonacini (Mathematics and Computer Science Department, University of Catania, Italy)

Solution by the proposer

Let P be any point such that the polar line r of P with respect to γ_2 is tangent to γ_1. Then clearly P is external to γ_2. Let C be the centre of the two circles, $\{Q\} = \gamma_1 \cap r$ and $\{T_1, T_2\} = \gamma_2 \cap r$. Since r is tangent to γ_1 in Q, we can assert that the line CQ is orthogonal to r and, since r is the polar of P with respect to γ_2, the line CP is orthogonal to the line r. So, the points P, Q and C are collinear and the angles $\widehat{T_1QP}$, $\widehat{T_2QP}$, $\widehat{T_1QC}$ and $\widehat{T_2QC}$ are right. We also know that $\widehat{T_1CQ} = \widehat{T_2CQ} = \alpha$, where $0 < \alpha < \frac{\pi}{2}$. So we can assert that

$$\overline{QT_1} = \overline{QT_2} = \sqrt{r_2^2 - r_1^2},$$

and by looking at the triangle CQT_1, we see that

$$r_1 = r_2 \cos\alpha \implies \cos\alpha = \frac{r_1}{r_2}.$$

Clearly, $\widehat{T_1PQ} = \frac{\pi}{2} - \alpha$, and consequently,

$$\overline{QP} = \overline{QT_1}\cot\Big(\frac{\pi}{2} - \alpha\Big) = \frac{r_2^2 - r_1^2}{r_1}.$$

This implies that

$$\overline{PC} = \overline{QP} + \overline{QC} = \frac{r_2^2}{r_1},$$

which clearly shows that γ is a circle of centre C and radius r_2^2/r_1.

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Let $A \subseteq \mathbb{R}^3$ be a connected open subset of Euclidean space, and suppose that the following conditions hold:

(1) Every smooth irrotational vector field on A admits a potential (i.e., it is the gradient of a smooth function).
(2) The closure $\overline{A}$ of A is a smooth compact submanifold of $\mathbb{R}^3$ (of course, with non-empty boundary).

Show that A is simply connected. Does this conclusion hold even if we drop condition (2) on A?

Roberto Frigerio (Dipartimento di Matematica, Università di Pisa, Italy)

Solution by the proposer

The usual scalar product on $\mathbb{R}^3$ induces an identification between smooth vector fields and differential 1-forms, which identifies irrotational vector fields with closed forms, and fields admitting a potential with exact forms. Therefore, condition (1) may be restated as follows: every smooth 1-form on A is exact, i.e., the first de Rham cohomology group of A vanishes. By the de Rham Theorem, this is in turn equivalent to the fact that the *singular* homology module $H^1(A,\mathbb{R})$ vanishes.

Since any compact manifold with boundary is homotopy equivalent to its interior, we may thus assume that $H^1(\overline{A},\mathbb{R}) = 0$. A well-known consequence of the Poincaré Duality Theorem is that, for any compact orientable 3-manifold with boundary M, the dimension of $H^1(\partial M,\mathbb{R})$ is twice the dimension of $H^1(M,\mathbb{R})$. Since every codimension-0 submanifold of $\mathbb{R}^3$ is obviously orientable, we thus have $H^1(\partial\overline{A},\mathbb{R}) = 0$. Let $S_1,\dots,S_k$ be the components of the boundary ∂A. Since

$$H^1(\partial\overline{A},\mathbb{R}) = \bigoplus_{i=1}^{k} H_1(S_i,\mathbb{R})$$

and the 2-sphere is the only compact orientable 2-manifold without boundary with vanishing first cohomology group, we can conclude that S_i is diffeomorphic to the 2-sphere for every $i = 1,\dots,k$.

It is well known that every smooth sphere in $\mathbb{R}^3$ bounds a smooth closed disc (this is no longer true for non-smooth spheres; see below); hence, for every $i = 1, \ldots, k$, we have $S_i = \partial B_i$, where $B_i \subseteq \mathbb{R}^3$ is a smooth disc. Since $\overline{A}$ is connected, it readily follows that there exists one of these closed discs, say B_1, such that

$$\overline{A} = B_1 \setminus (\operatorname{int}(B_2) \cup \cdots \cup \operatorname{int}(B_k)).$$

In other words, $\overline{A}$ is a closed disc with some open discs removed, and in particular it is simply connected.

In order to prove that A is simply connected, the condition that A be the interior of a compact smooth manifold with boundary is essential. Indeed, let $S \subseteq \mathbb{R}^3 \subseteq S^3$ be the well-known Alexander horned sphere. Then S separates S^3 into two connected components: one of them, say A_1, is homeomorphic to an open ball; the other one, say A_2, is not simply connected. However, Alexander duality implies that

$$H^1(A_1, \mathbb{R}) \oplus H^1(A_2, \mathbb{R}) = H^1(S^3 \setminus S) \cong H_1(S, \mathbb{R}) = 0.$$

We thus have $H^1(A_2, \mathbb{R}) = 0$ while $\pi_1(A_2) \neq \{1\}$. By setting $A = A_2$, we thus get a non-simply connected open connected subset A of $\mathbb{R}^3$ such that every smooth irrotational vector field on A admits a potential.

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A *regulus* is a surface in $\mathbb{R}^3$ that is formed as follows: We consider pairwise skew lines $\ell_1, \ell_2, \ell_3 \subset \mathbb{R}^3$ and take the union of all lines that intersect each of ℓ_1, ℓ_2, and ℓ_3. Prove that, for every regulus U, there exists an irreducible polynomial $f \in \mathbb{R}[x, y, z]$ of degree two that vanishes on U.

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Proof

Let $\mathcal{P}$ be a set of 9 points that is obtained by arbitrarily choosing three points from each of ℓ_1, ℓ_2, and ℓ_3. We write

$$f(x, y, z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz + a_7x + a_8y + a_9z + a_{10}.$$

Asking f to vanish at a specific point is equivalent to a linear equation in the variables $a_1, \ldots, a_{10}$. Thus, asking f to vanish at all points of $\mathcal{P}$ yields a system of 9 linear equations with 10 variables. Since the number of variables is larger, this system admits a nontrivial solution. Thus, there exists a nonzero polynomial $f \in \mathbb{R}[x, y, z]$ of degree at most two that vanishes on $\mathcal{P}$. Let $W \subset \mathbb{R}^3$ be the set of points at which f vanishes.

Let $f_1 \in \mathbb{R}[s]$ be the restriction of f to the line ℓ_1. Since f vanishes on at least three points of ℓ_1, the polynomial f_1 has at least three roots. Since $\deg f_1 \leq 2$ but this polynomial has more than two roots, we have that $f_1(s) = 0$. In other words, $\ell_1 \subset W$. By repeating the above argument, we get that $\ell_1, \ell_2, \ell_3 \subset W$. By definition, no plane contains a pair of skew lines, so W cannot contain a plane. This implies that f is irreducible of degree two.

Consider a line ℓ' that intersects ℓ_1, ℓ_2 and ℓ_3. Since these three lines are pairwise skew, the three intersection points are distinct, so $|\ell' \cap U| \geq 3$. By restricting f to ℓ' as above, we get that $\ell' \subset W$. Since U is the union of all such lines ℓ', we get that $U \subseteq W$. This proof is by Larry Guth, although it may have also existed earlier.

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(Enumerative Geometry). How many lines pass through 4 generic lines in a 3-dimensional complex projective space $\mathbb{CP}^3$?

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Introductory remarks

This is a problem in an over a century-old area of mathematics called enumerative geometry. Enumerative geometry is concerned with finding or counting geometric objects (mainly curves, i.e., 1-dimensional objects over the ground field) satisfying certain geometric conditions (e.g., passing through a specified set of objects or having a particular degree, genus, and types of singularities). Enumerative geometry was revolutionized in the mid-1990s by the novel predictions of mirror symmetry that led to the creation of Gromov–Witten theory and extensive study of such questions in complex algebraic geometry, symplectic geometry, and string-theoretic physics.

The most straightforward example in this area is the number of lines passing through two points, where the answer is 1. Here, one can interpret the word "line" as a real line in the real Euclidean space $\mathbb{R}^n$, a complex line in the complex Euclidean space $\mathbb{C}^n$, or a complex projective line (i.e., $\mathbb{CP}^1 \cong S^2$) in the complex projective space $\mathbb{CP}^n$. The answer is the same regardless of the context. The same is not true in most other questions. Gromov–Witten theory is mostly about counting complex curves in complex projective varieties or closed symplectic manifolds. The benefits of studying complex curves in compact complex/almost complex manifolds is two-fold. First, the compactness of the spaces involved results in finite counts. Second, working over complex numbers ensures that count of such objects does not depend on the choices involved. Recall that a degree-d polynomial over $\mathbb{C}$ has always d roots (when counted with multiplicities), but a degree-d polynomial over $\mathbb{R}$ has at most d roots.

Solution by the proposer

Before finding the answer, let us indeed argue that the expected answer is a finite number. As in linear algebra, this is done by computing degrees of freedom and the number of equations imposed by the constraints. As we mentioned above, there is exactly one line passing through two distinct points in $\mathbb{CP}^3$; the dimension of the space of pairs of such points is $3+3=6$. However, for each line, there is a $(1+1=2)$-dimensional family of pairs of points that yield that particular line. Therefore, assuming that the set of lines in $\mathbb{CP}^3$ is a nice geometric space, its dimension should be $6-2=4$. The reduction in the dimension caused by the condition of intersecting any of the given lines is 1. It follows that the reduction in the dimension caused by the condition of intersecting all given four lines is 4. Since $4-4=0$, the solution set should be discrete. Since we are working with compact spaces, it will indeed be finite. Bellow, using Schubert calculus on Grassmannians, we will compute this number. We challenge the reader to think about the following real affine version of the question using elementary techniques: How many lines pass through 4 generic lines in $\mathbb{R}^3$?

The n-dimensional complex projective space $\mathbb{CP}^n$ is the projectivization of $\mathbb{C}^{n+1}$ in the sense that each point in the former corresponds to a line in the latter. In one dimension higher, every projective line in $\mathbb{CP}^n$ is the projectivization of a plane in $\mathbb{C}^{n+1}$. Therefore, the space of lines in $\mathbb{CP}^3$ is the same as the space of planes in $\mathbb{C}^4$, which is known as the (complex) Grassmannian manifold $\mathrm{Gr}(2,4)$. More generally, the Grassmannian $\mathrm{Gr}(r,n)$ is a compact complex $(r\times(n-r))$-dimensional manifold that parametrizes the r-dimensional subspaces of $\mathbb{C}^n$. Let $\ell\subset\mathbb{CP}^3$ be a line that is the projectivization of a two-dimensional subspace $V\subset\mathbb{C}^4$. The subspace of lines in $\mathbb{CP}^3$ that intersect ℓ is a submanifold X_ℓ of $\mathrm{Gr}(2,4)$ with $\dim_{\mathbb{C}} X_\ell = 4-1=3$. The points of X_ℓ correspond to two-dimensional subspaces $V'\subset\mathbb{C}^4$ such that $\dim_{\mathbb{C}}(V\cap V')\geq 1$. Even though X_ℓ depends on ℓ, the homology class $A\in H_6(\mathrm{Gr}(2,4),\mathbb{Z})\cong\mathbb{Z}$ of X_ℓ does not depend on ℓ. The homology groups of the Grassmannian are generated by a specific class of complex submanifolds known as Schubert cycles. All the odd degree homology groups are trivial.

Digression on Schubert calculus

Let $\lambda \overset{\text{def}}{=} (\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_r)$ be a sequence of non-negative integers between 0 and $n-r$, and define $|\lambda|=\sum\lambda_i$. Given a sequence of vector spaces $W\overset{\text{def}}{=}(0\subsetneq W_1\subsetneq\cdots\subsetneq W_n=\mathbb{C}^n)$, the Schubert cycle $\sigma_\lambda=\sigma_\lambda(W)$, with Poincaré dual $\mathrm{PD}(\sigma_\lambda)\in H^{2|\lambda|}(\mathrm{Gr}(r,n),\mathbb{Z})$, is defined to be

$$\sigma_\lambda(W)=\{V\in\mathrm{Gr}(r,n) : \dim(V\cap W_{n-r+i-\lambda_i})\geq i\}. \tag{1}$$

The homology class of $\sigma_\lambda(W)$ does not depend on W. There is a geometric way of describing a non-decreasing sequence λ which helps with understanding the computations involving Schubert cycles. A Young diagram is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths weakly decreasing (each row has the same or shorter length than its predecessor). Listing the number of boxes in each row gives a sequence λ of non-negative integers, such that $|\lambda|$ is the total number of boxes of the diagram. Figure 1 shows the Young diagram of $\lambda=(5,4,1)$.

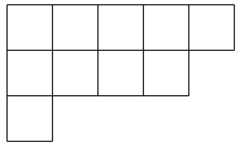

Figure 1. Young diagram of $\lambda=(5,4,1)$

A special case of the so-called Pieri formula states that

$$\sigma_{(1,0,\ldots,0)}\cdot\sigma_\lambda=\sum\sigma_\nu,$$

where the left-hand side is the intersection of two cycles and the sum on the right-hand side is over all partitions ν which can be obtained by adding one box to the Young diagram of λ.

Going back to the counting of the proposed problem, it follows from (1) that $A=\sigma_{(1,0)}$. Therefore,

$$[X_{\ell_1}]\cdot[X_{\ell_2}]\cdot[X_{\ell_3}]\cdot[X_{\ell_4}]=\sigma_{(1,0)}^4\in H_0(\mathrm{Gr}(2,4),\mathbb{Z})\cong\mathbb{Z}.$$

By Pieri's formula, we have

$$\begin{aligned}\sigma_{(1,0)}\cdot\sigma_{(1,0)}=\sigma_{(2,0)}+\sigma_{(1,1)} &\implies \sigma_{(1,0)}^3=\sigma_{(2,1)}+\sigma_{(2,1)}\\ &\implies \sigma_{(1,0)}^4=2\sigma_{(2,2)}=2.\end{aligned}$$

Note that the Schubert cycle $\sigma_{(2,2)}(W)$ is the point $W_2\in\mathrm{Gr}(2,4)$ (i.e., it generates $H_0(\mathrm{Gr}(2,4),\mathbb{Z})$). It is straightforward (but crucial) to show that for generic 4 lines $\ell_1,\ell_2,\ell_3,\ell_4$, the intersection $X_{\ell_1}\cap X_{\ell_2}\cap X_{\ell_3}\cap X_{\ell_4}$ is transverse. We conclude that the answer to the proposed problem is 2.

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I learned about the following problem from Shmuel Weinberger. It can be viewed as a topological analogue of Arrow's Impossibility Theorem.

(a) A group of n friends have decided to spend their summer cottaging together on an undeveloped island, which happens to be a perfect copy of the closed 2-disk D^2. Their first task is to decide where on this island to build their cabin. Being democratically-minded, the friends decide to vote on the question. Each friend chooses his or her favourite point on D^2. The friends want a function that will take as input their n votes, and give as output a suitable point on D^2 to build. They believe, to be reasonable and fair, their "choice" function should have the following properties:

- (*Continuity*) It should be continuous as a function $(D^2)^n\to D^2$. This means, if one friend changes their vote by a small amount, the output will change only a small amount.
- (*Symmetry*) The n friends should be indistinguishable from each other. If two friends swap votes, the final choice should be unaffected.
- (*Unanimity*) If all n friends chose the same point x, then x should be the final choice.

For which values of n does such a choice function exist?

(b) The friends' second task is to decide where along the shoreline of the island they will build their dock. The shoreline happens to be a perfect copy of the circle S^1. Again, they decide to take the problem to a vote. For which values of n does a continuous, symmetric, and unanimous choice function $(S^1)^n \to S^1$ exist?

These are special cases of the following general problem in topological social choice theory: given a topological space X, for what values of n does X admit a social choice function that is continuous, symmetric, and unanimous? In other words, when is there a function $A\colon X^n \to X$ satisfying

- A is continuous,
- $A(x_1, \dots, x_n)$ is independent of the ordering of $x_1, \dots, x_n$, and
- $A(x, x, x, \dots, x) = x$ for all $x \in X$?

Jenny Wilson (Department of Mathematics,
University of Michigan, USA)

Solution by the proposer

Consider the general problem described in the last paragraph. The statement that $A(x_1, \dots, x_n)$ is independent of the ordering of $x_1, \dots, x_n$ is the statement that the function A factors through the *symmetric product*, the quotient of X^n by the action of the symmetric group Σ_n, endowed with the quotient topology. Elements of X^n/Σ_n are multisets of n (not necessarily distinct) points in X. The statement that $A(x, x, \dots, x) = x$ is the statement that the composition

$$X \xrightarrow{\Delta} X^n \to X^n/\Sigma_n \to X,$$
$$x \mapsto (x, x, \dots, x) \mapsto \{x, x, \dots, x\} \mapsto A(x, x, \dots, x) = x$$

is the identity function. Thus the problem is equivalent to the following: does there exist a retraction from the symmetric product X^n/Σ_n onto the image of the diagonal?

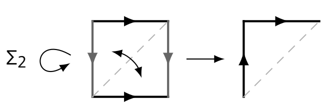
(a) An appropriate choice function exists for any n. Identify the island (up to homeomorphism) with the closed unit disk in $\mathbb{R}^2$. Since the disk is convex, we can (for example) let

$$A(x_1, x_2, \dots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

be the average value of the n points.

(b) Such a choice function only exists for $n = 1$. We first consider the case $n = 2$, since this case reduces to a problem that will be familiar to many algebraic topology students.

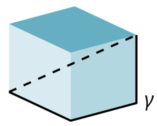
The symmetric product $(S^1 \times S^1)/\Sigma_2$ is the Möbius band and the image of the diagonal is its boundary, as pictured.



However, the boundary is not a retract of the Möbius band: the inclusion of the boundary induces the map $2\mathbb{Z} \hookrightarrow \mathbb{Z}$ on fundamental groups, which does not have a left inverse.

This argument generalizes for any $n \geq 2$. We can realize the n-torus $(S^1)^n$ as the unit cube in $\mathbb{R}^n$ with opposite faces identified. The 2^n corners are identified to a single point x, which we choose as basepoint.

Let γ be a path from the origin to the point $(1, 1, \dots, 1) \in \mathbb{R}^n$ along n mutually orthogonal edges of the cube, pictured here for $n = 3$. Construct (say, by straight-line homotopy) a based homotopy from the diagonal to γ.



The n orthogonal edges are identified in the symmetric product to a single loop. Thus, in $\pi_1((S^1)^n/\Sigma_n, \Sigma_n \cdot x)$ the image of the diagonal (which equals the image of γ) has an nth root. The image of the diagonal in $(S^1)^n/\Sigma_n$ cannot be a retract.

For the general problem, Eckmann–Ganea–Hilton and later (independently) Weinberger proved the following results; see Eckmann's survey [1].

Theorem. *Suppose that X is homotopy equivalent to a finite simplicial complex. If X is contractible, the function A exists for any n. If X is not contractible, it exists only for $n = 1$.*

Theorem. *Suppose that X is homotopy equivalent to a connected CW complex. Then the map A exists for all n if and only if X is a product of rational Eilenberg–MacLane spaces.*

Weinberger [2] notes that there exist other infinite CW complexes for which a choice function exists for (some) arbitrarily large values of n. For example, the infinite-dimensional real projective space $\mathbb{RP}^\infty$ admits a social choice function A for any odd value of n, but not for any even value.

References

[1] B. Eckmann, Social choice and topology: A case of pure and applied mathematics. *Expo. Math.* **22**, 385–393 (2004)

[2] S. Weinberger, On the topological social choice model. *J. Econom. Theory* **115**, 377–384 (2004)

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions to Michael Th. Rassias by email to mthrassias@yahoo.com.

We also solicit your new problems with their solutions for the next "Solved and unsolved problems" column, which will be devoted to Number Theory.