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Singular solutions for divergence-form elliptic equations involving regular variation theory: Existence and classification ☆

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Abstract

We generalise and sharpen several recent results in the literature regarding the existence and complete classification of the isolated singularities for a broad class of nonlinear elliptic equations of the form

$$-\text{div}\,(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) + b(x)\,h(u) = 0 \quad \text{in } B_1 \setminus \{0\}, \tag{0.1}$$

where B_r denotes the open ball with radius $r > 0$ centred at 0 in $\mathbb{R}^N$ $(N \geq 2)$. We assume that $\mathcal{A} \in C^1(0, 1]$, $b \in C(\overline{B_1} \setminus \{0\})$ and $h \in C[0, \infty)$ are positive functions associated with regularly varying functions of index ϑ, σ and q at 0, 0 and ∞ respectively, satisfying $q > p - 1 > 0$ and $\vartheta - \sigma < p < N + \vartheta$. We prove that the condition $b(x)\,h(\Phi) \notin L^1(B_{1/2})$ is sharp for the removability of all singularities at 0 for the positive solutions of (0.1), where Φ denotes the "fundamental solution" of $-\text{div}\,(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) = \delta_0$ (the Dirac mass at 0) in B_1, subject to $\Phi|_{\partial B_1} = 0$. If $b(x)\,h(\Phi) \in L^1(B_{1/2})$, we show that any non-removable singularity at 0 for a positive solution of (0.1) is either *weak* (i.e., $\lim_{|x|\to 0} u(x)/\Phi(|x|) \in (0, \infty)$) or *strong* ($\lim_{|x|\to 0} u(x)/\Phi(|x|) = \infty$). The main difficulty and novelty of this paper, for which we develop new techniques, come from the explicit asymptotic behaviour of the strong singularity solutions in the critical case, which had previously remained open even for $\mathcal{A} = 1$. We also study the existence and uniqueness of the positive solution of (0.1) with a prescribed admissible behaviour at 0 and a Dirichlet condition on ∂B_1.

Keywords: Divergence-form elliptic equations; Isolated singularities; Regular variation theory

1. Introduction and main results

In this paper, we aim to fully classify the isolated singularities for nonlinear elliptic equations of the form

$$\text{div}\,(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) = b(x)\,h(u) \tag{1.1}$$

in the punctured unit ball $B^* := B_1 \setminus \{0\}$ in $\mathbb{R}^N$ $(N \geq 2)$ under the following structural conditions:

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($\mathbf{A}_1$) The function $\mathcal{A} \in C^1(0,1]$ is positive such that $\mathcal{A}(t) = t^{\vartheta} L_{\mathcal{A}}(t)$ with $1 < p < N + \vartheta$ and $L_{\mathcal{A}}$ satisfies

$$\lim_{t \to 0^+} \frac{t L'_{\mathcal{A}}(t)}{L_{\mathcal{A}}(t)} = 0. \tag{1.2}$$

($\mathbf{A}_2$) The function h is continuous on $\mathbb{R}$ and positive on $(0,\infty)$ with $h(0) = 0$ and $h(t)/t^{p-1}$ bounded for small $t > 0$, whereas b is a positive continuous function on $\overline{B_1} \setminus \{0\}$.

($\mathbf{A}_3$) There exist $q, \sigma \in \mathbb{R}$ and functions L_h, L_b that are slowly varying at ∞ and at 0 respectively, such that

$$\lim_{t \to \infty} \frac{h(t)}{t^q L_h(t)} = 1 \text{ and } \lim_{|x| \to 0} \frac{b(x)}{|x|^{\sigma} L_b(|x|)} = 1 \text{ with } q + 1 > p > \vartheta - \sigma. \tag{1.3}$$

For the definition of a slowly varying function, see Appendix A.

We are interested in positive solutions of (1.1) since any non-negative solution of (1.1) is either identically zero or positive in B^* by the strong maximum principle (see [18, Theorem 2.5.1]). A function u is said to be a *solution* (*sub-solution, super-solution*) of (1.1) if $u(x) \in C^1(B^*)$ and

$$\int_{B_1} \mathcal{A}(|x|)\, |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx + \int_{B_1} b(x)\, h(u)\, \varphi \, dx = 0 \qquad (\leq 0, \ \geq 0) \tag{1.4}$$

for all functions (non-negative functions) $\varphi \in C^1_c(B^*)$, the space of all $C^1(B^*)$-functions with compact support in B^*. We say that a positive solution u of (1.1) can be extended as a positive continuous solution of (1.1) in B_1 if there exists $\lim_{|x| \to 0} u(x) \in (0,\infty)$, the function $\mathcal{A}(|x|)\, |\nabla u|^{p-1} \in L^1_{\text{loc}}(B_1)$ and (1.4) holds for every $\varphi \in C^1_c(B_1)$ (see Remark 3.1).

For $\mathcal{A} = b = 1$, the profile of all positive solutions of (1.1) with $h(t) = |t|^{q-1} t$ is known (see [13,25]), depending on the position of q relative to the *critical exponent* $q_* = \frac{N(p-1)}{N-p}$ (with $q_* = \infty$ for $p = N$):

(a) If $p - 1 < q < q_*$, then as $|x| \to 0$, exactly one of the following holds (see Friedman–Véron [13]):
  (i) u can be extended as a continuous solution of the same equation in B_1 (removable singularity);
  (ii) There exists a positive number λ such that $u(x)/\mu(x) \to \lambda$ (weak singularity) and, moreover,

$$-\text{div}\,(|\nabla u|^{p-2} \nabla u) + |u|^{q-1} u = \lambda^{p-1} \delta_0 \quad \text{in } \mathcal{D}'(B_1).$$

  Here, δ_0 is the Dirac mass at 0 and μ is the fundamental solution of $-\text{div}\,(|\nabla u|^{p-2} \nabla u) = \delta_0$ in $\mathcal{D}'(\mathbb{R}^N)$.
  (iii) $|x|^{p/(q-p+1)} u(x) \to \gamma_{N,p,q}$, where $\gamma_{N,p,q} := \left[\left(\frac{p}{q-p+1} \right)^{p-1} \left(\frac{pq}{q-p+1} - N \right) \right]^{1/(q-p+1)}$ (strong singularity).

(b) If, in turn, $q \geq q_*$ (for $1 < p < N$), then only (a)(i) occurs, (see Vázquez–Véron [25]).

If $g \in C^1(\partial B_1)$ is a non-negative function, then the singular Dirichlet problem $\text{div}\,(|\nabla u|^{p-2} \nabla u) = |u|^{q-1} u$ in B^*, with $\lim_{|x| \to 0} u(x)/\mu(x) = \lambda \in (0,\infty)$ and $u = g$ on ∂B_1 admits a unique non-negative solution if and only if $q < q_*$ (see [13]). The strong singularities at 0 are *all* obtained as limits of solutions with a weak singularity at 0. However, beyond the power nonlinearities, the understanding of strong singularities and their removability had hitherto remained elusive even for Laplacian-type equations. The following question formulated by Vázquez and Véron [26] remains open: *What is the weakest condition on a continuous non-decreasing function h such that any isolated singularity of a non-negative solution of* $\Delta u = h(u)$ *in* B^* *with* $N \geq 3$ *is removable?* This question, together with a complete classification of the isolated singularities, has recently been settled by Cîrstea [9] in the framework of regular variation theory for more general semilinear elliptic equations.

Under suitable assumptions where the growth of B is at most the growth of $\mathbf{A}$ (see (2) in [20]), Serrin established that if u is a non-negative continuous solution of $\text{div}\,\mathbf{A}(x,u,\nabla u) = B(x,u,\nabla u)$ in $\Omega \setminus \{0\}$ for a domain Ω in $\mathbb{R}^N$ with $0 \in \Omega$, then *either u has a removable singularity at* 0 *or* $c_1 \leq u(x)/\mu(|x|) \leq c_2$ *near* 0 *for positive constants* c_1 *and* c_2.

Our paper addresses the singularity problem for related divergence form elliptic equations when *the growth of B is bigger than that of* $\mathbf{A}$, which is a challenge formulated by Véron [29]. We set $C_{N,p} := (N\omega_N)^{-1/(p-1)}$ with ω_N denoting the volume of the unit ball in $\mathbb{R}^N$. Let Φ denote a "fundamental solution" of $-\text{div}\,(\mathcal{A}(|x|)\, |\nabla \Phi|^{p-2} \nabla \Phi) = \delta_0$ in $\mathcal{D}'(B_1)$ with $\Phi = 0$ on ∂B_1, namely

$$\Phi(r) := C_{N,p} \int_r^1 \left(\frac{t^{1-N-\vartheta}}{L_{\mathcal{A}}(t)} \right)^{\frac{1}{p-1}} dt \quad \text{for all } r \in (0,1]. \tag{1.5}$$

In Theorem 1.1(b), we show that $b(x)\, h(\Phi) \notin L^1(B_{1/2})$ is sharp for the removability of all singularities of (1.1), whereas in Theorem 1.1(a), we fully classify the singularities of (1.1) provided that $b(x)\, h(\Phi) \in L^1(B_{1/2})$. Note that $\lim_{r\to 0^+} \Phi(r) = \infty$ since $1 < p < N + \vartheta$. We now define q_*, henceforth referred to as a *critical exponent*, by

$$q_* := \frac{(N+\sigma)(p-1)}{N+\vartheta-p}. \tag{1.6}$$

By Remark B.1, we need to check $b(x)\, h(\Phi) \in L^1(B_{1/2})$ only for $q = q_*$. In such a critical case, assuming

$$\text{either } t \longmapsto L_h(e^t) \text{ is regularly varying at } \infty \text{ with index } \gamma \in \mathbb{R} \tag{1.7}$$

$$\text{or } t \longmapsto \left[L_{\mathcal{A}}(e^{-t})\right]^{-\frac{q}{p-1}} L_b(e^{-t}) \text{ is regularly varying at } \infty \text{ with index } j \in \mathbb{R}, \tag{1.8}$$

then $b(x)\, h(\Phi) \in L^1(B_{1/2})$ if and only if $F(r) < \infty$, where we define

$$F(r) := \int_0^r \xi^{-1} \left[L_{\mathcal{A}}(\xi)\right]^{-\frac{q_*}{p-1}} L_b(\xi)\, L_h(1/\xi)\, d\xi \quad \text{for } r > 0 \text{ small.} \tag{1.9}$$

From Assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$, it follows that m_0, m_1 and m_2 are all *positive*, where we define

$$m_0 := \frac{p+\sigma-\vartheta}{q-p+1}, \qquad m_1 := \frac{q-p+1}{p-1}, \qquad m_2 := \frac{N+\vartheta-p}{p-1}. \tag{1.10}$$

We now state our first main result, where F plays an important role for strong singularity solutions of (1.1).

Theorem 1.1 *(Classification of singularities and sharp removability results). Let Assumptions* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *hold.*

(a) *If $b(x)\, h(\Phi) \in L^1(B_{1/2})$, then for every positive solution u of (1.1), we find that:*
  (i) *Either there exists $\lambda \in [0,\infty)$ such that the following holds*

$$-\mathrm{div}\left(\mathcal{A}(|x|)\, |\nabla u|^{p-2} \nabla u\right) + b(x)\, h(u) = \lambda^{p-1} \delta_0 \quad \text{in } \mathcal{D}'(B_1). \tag{1.11}$$

  If $\lambda \in (0,\infty)$, then $\lim_{|x|\to 0} u(x)/\Phi(x) = \lambda$ (weak singularity), while if $\lambda = 0$, then u can be extended as a positive continuous solution of (1.1) in the whole ball B_1.
  (ii) *Or u has a strong singularity at 0. Moreover, $\lim_{|x|\to 0} u(x)/\tilde{u}(|x|) = 1$, where $\tilde{u}$ is given by*

$$\int_{\tilde{u}(r)}^{\infty} \frac{t^{-\frac{q+1}{p}}}{[L_h(t)]^{\frac{1}{p}}}\, dt = \int_0^r \left[M \frac{\xi^{\sigma-\vartheta} L_b(\xi)}{L_{\mathcal{A}}(\xi)} \right]^{\frac{1}{p}} d\xi \quad \text{with } \frac{1}{M} := q - \frac{N+\sigma}{m_0} \quad \text{if } q < q_*. \tag{1.12}$$

  On the other hand, in the critical case $q = q_*$, then $\lim_{|x|\to 0} u(x)/\tilde{u}(|x|) = 1$ for $\tilde{u}$ given by

$$\begin{cases} \tilde{u}(r) = \left[m_1 m_0^{\gamma+1-p} F(r) \right]^{-\frac{1}{q_*-p+1}} L_{\mathcal{A}}^{-\frac{1}{p-1}}(r)\, r^{-m_0} & \text{if (1.7) holds,} \\ \displaystyle\int_c^{\tilde{u}(r)} \left[F(1/t)\right]^{\frac{1}{q_*-p+1}} dt = \left(m_1 m_0^{-p-j}\right)^{-\frac{1}{q_*-p+1}} L_{\mathcal{A}}^{-\frac{1}{p-1}}(r)\, r^{-m_0} & \text{if (1.8) holds,} \end{cases} \tag{1.13}$$

  where F, m_0 and m_1 are prescribed by (1.9) and (1.10) respectively. In (1.13), $c > 0$ is a large constant.

(b) *If $b(x)\, h(\Phi) \notin L^1(B_{1/2})$, then $q \geq q_*$ and every positive solution of (1.1) can be extended as a positive continuous solution of (1.1) in the whole ball B_1.*

Remark 1.1.

(i) When $\mathcal{A} = b = 1$ and $h(t) = |t|^{q-1}t$, our Theorem 1.1(a) recovers [13, Theorem 2.1]. Moreover, Theorem 1.1(a) generalises and sharpens [10, Theorem 1.1] where $\mathcal{A} = 1$ and $q < q_*$. Our Theorem 1.1 is also established under the optimal condition for the existence of solutions with singularities at 0 for (1.1). Even for $\mathcal{A} = 1$, the behaviour of the strong singularity solutions in the critical case $q = q_*$ is new, being obtained via a perturbation technique we devise in this paper (see §2.1). While the understanding of strong singularities for Laplacian-type equations with power-like non-linearities in [3] relied on the work by Taliaferro [21], this is no longer possible in our general context of quasi-linear equations such as (1.1).

(ii) When $\mathcal{A} = b = 1$ and $h(t) = t^q$, Theorem 1.1(b) recovers the removability result of [4] for $p = 2$ and [25] for $1 < p < N$. By letting $\mathcal{A} = 1$ in Theorem 1.1(b), we also obtain a sharp version of [10, Theorem 1.3].

Next, in our second main result, under suitable conditions, we show that there exist positive solutions of (1.1) in any of the categories appearing in the complete classification of Theorem 1.1. Furthermore, we obtain a uniqueness result for (1.1) subject to a Dirichlet condition on ∂B_1 with a prescribed, admissible behaviour at zero.

Theorem 1.2 *(Existence and uniqueness). Let Assumptions* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *hold. Assume that h is a non-decreasing function on* $(0, \infty)$ *and* $g \in C^1(\partial B_1)$ *is an arbitrary non-negative function. We consider the following problem*

$$\begin{cases} \operatorname{div}(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) = b(x)\,h(u) & \text{in } B^* := B_1 \setminus \{0\}, \\ \lim\limits_{|x|\to 0} \dfrac{u(x)}{\Phi(x)} = \lambda, \quad u\big|_{\partial B_1} = g, \quad u > 0 & \text{in } B^*. \end{cases} \tag{1.14}$$

(i) *If* $\lambda = 0$ *and* $g \not\equiv 0$ *on* ∂B_1, *then* (1.14) *has a unique solution.*
(ii) *If* $\lambda \in (0, \infty]$, *then* (1.14) *admits solutions if and only if* $b(x)\,h(\Phi) \in L^1(B_{1/2})$.
(iii) *Assume that* $b(x)\,h(\Phi) \in L^1(B_{1/2})$ *and* $h(t)/t^{p-1}$ *is non-decreasing for* $t > 0$.
  (a) *For* $\lambda \in (0, \infty)$, *then* (1.14) *has a unique solution. The same conclusion holds for* $\lambda = \infty$ *and* $q < q_*$.
  (b) *For* $\lambda = \infty$ *and* $q = q_*$, *then* (1.14) *has a unique solution provided that either* (1.7) *or* (1.8) *holds.*

Remark 1.2. When $b = 1$ and $h(t) = |t|^{q-1}t$, our Theorem 1.2 recovers previous results such as [13, Theorems 1.2] (with $\mathcal{A} = 1$) and [3, Theorem 2] (with $p = 2$). Moreover, in Theorem 1.2, we generalise [10, Theorem 1.2] (where $\mathcal{A} = 1$) by sharpening the condition under which there exists a unique singular solution to (1.14).

Remark 1.3. In this paper, we focus on the case $p < N + \vartheta$ in Assumption $(\mathbf{A}_1)$. We mention that Theorem 1.1 and Theorem 1.2 remain valid also for $p = N + \vartheta$ provided that $\limsup_{r\to 0^+} L_{\mathcal{A}}(r) < \infty$ (which ensures that $\Phi(r) \to \infty$ as $r \to 0^+$). For $p = N + \vartheta$, we understand $q_* = \infty$ since $b(x)\,h(\Phi) \in L^1(B_{1/2})$ holds for any $q \in (p-1, \infty)$ and thus in Theorem 1.1 only the assertion of (a) is meaningful in which the strong singularity behaviour of (ii) is given by (1.12).

For relevant background on isolated singularities, see [27–29]. Recent contributions include boundary singularities [15,17], interior singularities for the fractional Laplacian [5,6], non-homogeneous divergence-form operators [16], nonlinear equations with singular potentials [9,12] or with nonlinearities depending on the gradient [1,7].

Structure of the paper. We shall always assume $(\mathbf{A}_1)$–$(\mathbf{A}_3)$. In Section 2, we prove Theorem 1.1(a) which fully classifies the nature of all possible singularities at 0 for the positive solutions of (1.1) when $b(x)\,h(\Phi) \in L^1(B_{1/2})$. We emphasise that this is an optimal condition under which, besides weak singularity solutions, there can arise strong singularity solutions of (1.1) (that is $\lim_{|x|\to 0} u(x)/\Phi(x) = \infty$) as stated by Theorem 1.2 to be proved in Section 5. The proof of Theorem 1.1(a), and in particular, the analysis of (radial) solutions with strong singularities at 0 in Theorem 2.1, represent the crux of this paper. Even in the case $\mathcal{A} = 1$, Theorem 1.1(a) is new with regard to the explicit derivation of the asymptotic behaviour near 0 of solutions with strong singularities in the critical case $q = q_*$. To establish Theorem 1.1(a), we invoke some auxiliary results such as *a priori* estimates, a spherical Harnack-type inequality and regularity results, whose proofs are deferred until Section 4. In Section 3, we prove Theorem 1.1(b), which establishes $b(x)\,h(\Phi) \notin L^1(B_{1/2})$ as a sharp condition such that all positive solutions of (1.1) have a removable singularity at zero. In Appendix A, we gather the necessary concepts and properties related to the regular variation theory as they play a prominent role in our analysis. The condition in (1.2) implies that $L_{\mathcal{A}}$ is slowly varying at 0 (see Definition A.1 and Remark A.2). A complete characterisation of slowly varying function at 0 is provided by Theorem A.2. *Without loss of generality, we assume that L_h and L_b satisfy the properties in* (A.2) (see Remark A.4). In Appendix B, we apply Theorem 1.1 on specific examples.

Notation. By $f_1(t) \sim f_2(t)$ as $t \to t_0$ for $t_0 \in \mathbb{R} \cup \{\infty\}$, we mean that $\lim_{t\to t_0} f_1(t)/f_2(t) = 1$.

2. Proof of Theorem 1.1(a): classification of singularities

Let $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ hold and $b(x)\,h(\Phi) \in L^1(B_{1/2})$. Let u be a positive solution of (1.1) and $\lambda := \limsup_{|x|\to 0} \frac{u(x)}{\Phi(|x|)}$.

(i) When $\lambda = 0$, the assertion of (i) in Theorem 1.1(a) follows from Lemma 3.1 whereas when $\lambda \in (0, \infty)$, one can show that u has a weak singularity at 0 and can verify (1.11) by using the same argument as in [10, Theorem 5.1] (see also [3, Proposition 6]). We thus omit the details.

(ii) When $\lambda = \infty$, then (4.16) yields that $\lim_{|x|\to 0} u(x)/\Phi(x) = \infty$. We show below how to reduce the proof of (ii) in Theorem 1.1(a) to the case of strong singularities for radial solutions of an approximate problem (2.1) treated in Theorem 2.1. We reason as in [9, Lemma 4.12], using Lemmas 4.1 and 4.3 to deduce that for every $\varepsilon \in (0, 1)$, there exists $r_\varepsilon \in (0, 1)$ and a function v_ε satisfying $(1-\varepsilon)\,u \le v_\varepsilon \le (1+\varepsilon)\,u$ in $B^*_{r_\varepsilon}$ with v_ε a positive solution of

$$-\mathrm{div}\,(\mathcal{A}(|x|)\,|\nabla v|^{p-2}\nabla v) + |x|^\sigma\, v^q\, L_b(|x|)\, L_h(v) = 0 \quad \text{in } B^*_{r_\varepsilon} := B_{r_\varepsilon} \setminus \{0\}. \tag{2.1}$$

Moreover, if v is any positive solution of (2.1), then as in [3, Lemma 4], we can obtain two positive *radial* solutions of (2.1) in $B^*_{r_\varepsilon/2}$, say v_* and v^*, such that for a sufficiently large constant $K > 1$, we have

$$K^{-1} v \le v_* \le v \le v^* \le K v \quad \text{in } B^*_{r_\varepsilon/2}. \tag{2.2}$$

We observe that any positive radial solution of (2.1) in B^* satisfies

$$\frac{d}{dr}\left(r^{N-1+\vartheta} L_{\mathcal{A}}(r)|v'(r)|^{p-2}v'(r)\right) = r^{N-1+\sigma} L_b(r)\, L_h(v(r))\, v^q(r) \quad \text{for } r = |x| \in (0, 1). \tag{2.3}$$

In view of (2.2), to conclude the assertion of (ii) in Theorem 1.1(a), it is enough to prove Theorem 2.1 below.

Theorem 2.1. *Let Assumptions* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *hold. Suppose that* $b(x)\,h(\Phi) \in L^1(B_{1/2})$. *Let v be any positive solution of* (2.3) *with a strong singularity at* 0.

(a) *If $q < q_*$, then $v(r) \sim \tilde{u}(r)$ as $r \to 0$, where $\tilde{u}$ is given by* (1.12).
(b) *If $q = q_*$, then assuming either* (1.7) *or* (1.8), *we have $v(r) \sim \tilde{u}(r)$ as $r \to 0$, where $\tilde{u}$ is given by* (1.13).

A major advance in this paper compared with Cîrstea and Du [10] (where $\mathcal{A} = 1$) is the analysis of the *critical case* and the derivation of the asymptotic behaviour of the strong singularities. Our contribution here is the development of a perturbation technique suitable for the *critical case* $q = q_*$. Unlike the subcritical case, where the power model corresponding to $\mathcal{A} = b = 1$ and $h(t) = |t|^{q-1}t$ was completely understood due to Friedman and Véron [13] (see also Remark 1.1), in the critical case we had no model in the literature to provide us with intuition on the asymptotics of strong singularity solutions. As we reveal in our paper, the critical case is important in the *non-power nonlinearity* case as it represents the threshold between having *a trichotomy* classification (as in Theorem 1.1(a)) or *no singularities at all* as in Theorem 1.1(b), all depending on whether or not $b(x)\,h(\Phi)$ belongs to $L^1(B_{1/2})$.

The proofs of Theorem 2.1(a) and (b) are intricate, each being composed of three main steps. First, we shall prove here the critical case $q = q_* < \infty$, while also pointing out the major differences between the subcritical and critical cases. Under the assumptions of Theorem 2.1, let v be any positive solution of (2.3) with a strong singularity at 0. A change of variable $y(s) = v(r)$ with $s = \Phi(r)$ moves the singularity from $r = 0$ to $s = \infty$ for the equation

$$(p-1)\left|y'(s)\right|^{p-2} y''(s) = C_{N,p}^{-p+1} r^{N-1+\sigma} L_b(r) L_h(y(s)) \left[y(s)\right]^q \left|\frac{dr}{ds}\right| \quad \text{for } s \in (0,\infty), \tag{2.4}$$

where, for simplicity, we denote $y'(s) = dy/ds$ and $y''(s) = d^2y/ds^2$.

Step 1. *Fix $\eta_0 > 0$ small. For every $\varepsilon \in (0,1)$ small, there exists $r_\varepsilon \in (0,1)$ such that $(1-\varepsilon)\, v_{-\eta}$ and $(1+\varepsilon)\, v_\eta$ are a sub-solution and super-solution of (2.3) for $0 < r < r_\varepsilon$, respectively, for every $\eta \in [0,\eta_0]$. Moreover, it holds that $\lim_{\eta\to 0^+} v_{\pm\eta}(r) = \tilde{u}(r)$ for every $r \in (0, r_\varepsilon]$, where $\tilde{u}$ is as in Theorem 2.1.*

The local one-parameter family $v_{\pm\eta}$ of sub- and super-solutions of (2.3) is constructed such that $v_{\pm\eta}(r)$ converges to $\tilde{u}(r)$ as η approaches 0^+. The function $\tilde{u}$ in Theorem 2.1 is regularly varying at 0 with index $-m_0$, where m_0 and m_2 are given by (1.10). The definition of $v_{\pm\eta}$ in the subcritical case is different from that of the critical case as follows.

In the subcritical case $q < q_*$, we define $v_{\pm\eta}$ in (2.26) as a regularly varying function at 0 with index $-(1\pm\eta)\, m_0$ (here $m_0 > m_2$). We shall check the assertion of Step 1 in §2.2.

In the critical case $q = q_* < \infty$, we have $m_0 = m_2$, that is, $\tilde{u}$ has the same index of regular variation at 0 as the fundamental solution Φ in (1.5), namely $-m_2$. In this case, $v_{\pm\eta}$ is defined by (2.19) as a regularly varying function at 0 with index $-m_2$. We shall verify Step 1 in §2.1 with the change of variable $y_{\pm\eta}(s) = v_{\pm\eta}(r)$ where $s = \Phi(r)$. Notice that when either (1.7) holds or (1.8) holds, by the definitions of $\tilde{u}$ in (1.13) and $v_{\pm\eta}$ in (2.19), we infer that

$$\lim_{r\to 0^+} \frac{\tilde{u}(r)}{v_\eta(r)} = 0 \ \text{ and } \ \lim_{r\to 0^+} \frac{\tilde{u}(r)}{v_{-\eta}(r)} = \infty \ \text{ for every } \eta \in [0,\eta_0]. \tag{2.5}$$

Step 2. *The functions v_η and $v_{-\eta}$ constructed in Step 1 satisfy the following property:*

$$\lim_{r\to 0^+} \frac{v(r)}{v_\eta(r)} = 0 \quad \textit{and} \quad \lim_{r\to 0^+} \frac{v(r)}{v_{-\eta}(r)} = \infty. \tag{2.6}$$

In both the subcritical and critical cases, since v has a strong singularity at 0, that is $v(r)/\Phi(r) \to \infty$ as $r \to 0^+$, then we have $y(s)/s \to \infty$ as $s \to \infty$. Using that $y''(s) \geq 0$, we find that $y'(s)$ is increasing so that $\lim_{s\to\infty} y'(s) = \infty$. As the function $s \longmapsto sy'(s) - y(s)$ is increasing on $(0,\infty)$ and $\lim_{s\to\infty} y(s) = \infty$, we see that

$$\liminf_{s\to\infty} \frac{sy'(s)}{y(s)} \geq 1. \tag{2.7}$$

In the *subcritical* case, we shall use (2.7) in Lemma 2.3(b) of §2.2 to improve the behaviour of the solution v of (2.3) from dominating near zero the fundamental solution Φ (of index $-m_2$) to dominating *any* function f regularly varying at zero with index $-\kappa$, where $m_2 < \kappa < m_0$. We deduce (2.6) by using Lemma 2.3 with $f = v_{\pm\eta}$ since the index of regular variation at 0 for the function v_η (respectively, $v_{-\eta}$) is smaller (respectively, bigger) than $-m_0$. We point out that Lemma 2.3 relies essentially on the assumption that $q < q_*$ and cannot be adapted to the critical case.

Hence, in the *critical* case, we need a new argument that takes into account that $v_{\pm\eta}$ varies regularly at 0 with the same index as $\tilde{u}$. We now prove Step 2 in the critical case.

Proof of Step 2 for the critical case $q = q_*$. The main ingredient in the proof of (2.6) is given by the following

$$0 < \liminf_{r\to 0^+} \frac{v(r)}{\tilde{u}(r)} \leq \limsup_{r\to 0^+} \frac{v(r)}{\tilde{u}(r)} < \infty. \tag{2.8}$$

By combining (2.5) and (2.8), we conclude (2.6) in the critical case. We note that both $r \mapsto \Phi(r)$ and $r \mapsto -r\,\Phi'(r)$ are regularly varying at 0^+ of index $-m_2$, where m_2 is defined in (1.10). Under Assumption $(\mathbf{A}_1)$, using Karamata's Theorem (see Theorem A.3), we find that

$$\lim_{r\to 0^+} \frac{\ln \Phi(r)}{\ln(1/r)} = \lim_{r\to 0^+} \Upsilon(r) = m_2, \quad \text{where } \Upsilon(r) := \frac{r\left|\Phi'(r)\right|}{\Phi(r)} = \frac{C_{N,p} r^{-m_0} \left[L_{\mathcal{A}}(r)\right]^{-\frac{1}{p-1}}}{\Phi(r)}. \tag{2.9}$$

Proof of (2.8). Using (4.17) and (2.7), we infer that $\limsup_{s\to\infty} sy''(s)/y'(s) < \infty$. Indeed, by (2.4), we have

$$\frac{sy''(s)}{y'(s)} = \frac{1}{p-1}\left[\frac{y(s)}{sy'(s)}\right]^{p-1} [\Upsilon(r)]^{-p} \frac{L_b(r)}{L_{\mathcal{A}}(r)} r^{p+\sigma-\vartheta} L_h(y(s)) \left[y(s)\right]^{q_*-p+1}, \tag{2.10}$$

where Υ is given by (2.9). For $s_0 > 0$, there exists a large constant $C > 0$ so that $s \mapsto sy'(s) - Cy(s)$ is non-increasing for all $s > s_0$. It follows that $\ell = \limsup_{s\to\infty} sy'(s)/y(s) < \infty$. From (2.7), we can take $s_0 > 0$ large such that

$$\frac{1}{2} \le \frac{s\,y'(s)}{y(s)} \le 2\ell \quad \text{for all } s \ge s_0. \tag{2.11}$$

In view of (4.18), we find that $\ln y(s) \sim \ln s$ as $s \to \infty$. Consequently, as $s \to \infty$, we obtain that

$$\begin{cases} m_0^{\gamma} L_h(1/r) \sim L_h(s) \sim L_h(y(s)) & \text{if (1.7) holds;} \\ (m_0)^{-j} \left[L_{\mathcal{A}}(1/y(s))\right]^{-\frac{q_*}{p-1}} L_b(1/y(s)) \sim \left[L_{\mathcal{A}}(\Phi^{-1}(s))\right]^{-\frac{q_*}{p-1}} L_b(\Phi^{-1}(s)) & \text{if (1.8) holds.} \end{cases} \tag{2.12}$$

For all $s \ge s_0$, by using (2.11) and (2.12) in (2.4), we find positive constants c_1 and c_2 so that

$$\begin{cases} c_1 r^{N-1+\sigma} L_b(r)\, h(\Phi(r)) \le \left[y'(s)\right]^{-q_*+p-2} y''(s) \left|\frac{ds}{dr}\right| \le c_2 r^{N-1+\sigma} L_b(r)\, h(\Phi(r)) & \text{if (1.7) holds;} \\ c_1 \frac{d}{ds}[F(1/y(s))] \le \left[y'(s)\right]^{-q_*+p-2} y''(s) \le c_2 \frac{d}{ds}[F(1/y(s))] & \text{if (1.8) holds,} \end{cases} \tag{2.13}$$

where F is defined by (1.9).

Case 1. Assume that (1.7) holds.

Since $y'(s) \to \infty$ as $s \to \infty$, by integrating (2.13), we obtain that

$$c_3 F(\Phi^{-1}(s)) \le \left[y'(s)\right]^{-q_*+p-1} \le c_4 F(\Phi^{-1}(s)) \quad \text{for all } s > s_0, \tag{2.14}$$

where c_3 and c_4 are positive constants. Using (2.11) in (2.14), then reversing the change of variable $y(s) = v(r)$ with $s = \Phi(r)$, we infer that there exist positive constants c_5 and c_6 such that

$$c_5 \left[F(r)\right]^{-\frac{1}{q_*-p+1}} \Phi(r) \le v(r) \le c_6 \left[F(r)\right]^{-\frac{1}{q_*-p+1}} \Phi(r) \quad \text{for all } r \in (0, \Phi^{-1}(s_0)). \tag{2.15}$$

Hence, using (2.9) and the definition of $\tilde{u}$ in (1.13), we conclude Step 2 in Case 1.

Remark 2.1. Notice that when (1.7) holds, the existence of a solution v of (2.3) with a strong singularity at zero implies that $b(x)\, h(\Phi(|x|)) \in L^1(B_{1/2})$. Indeed, fixing $r_0 \in (0, \Phi^{-1}(s_0))$, then for every $\varepsilon \in (0, r_0)$, by integrating the first inequality in (2.13) with respect to r from ε to r_0, and letting $\varepsilon \to 0$, we conclude the claim (using Remark B.1). A more general statement is proven later in Lemma 3.2.

Case 2. Assume that (1.8) holds.

By twice integrating (2.13), we find positive constants c_3 and c_4 such that

$$c_3 \le \frac{d}{ds}\left(\int_{y(s_0)}^{y(s)} \left[F(1/t)\right]^{\frac{1}{q_*-p+1}} dt \right) \le c_4 \quad \text{for every } s > s_0.$$

We thus conclude that

$$0 < \liminf_{s\to\infty} \frac{\int_{y(s_0)}^{y(s)} \left[F(1/t)\right]^{\frac{1}{q_*-p+1}} dt}{s} \le \limsup_{s\to\infty} \frac{\int_{y(s_0)}^{y(s)} \left[F(1/t)\right]^{\frac{1}{q_*-p+1}} dt}{s} < \infty.$$

This, jointly with (2.9) and the definition of $\tilde{u}$ in (1.13), proves the assertion of Step 2 in Case 2. □

Step 3. *Proof of Theorem 2.1 concluded.*

Proof of Step 3. The reasoning is the same for the subcritical and critical case. It is based on the previous two steps and the following comparison principle to be used frequently in the paper.

Lemma 2.2 *(Comparison principle, see Theorem 2.4.1 in [18]). Let Ω be a bounded domain in $\mathbb{R}^N$ with $N \geq 2$. Let $u, v \in C^1(\Omega)$ satisfy (in the sense of distributions in $\mathcal{D}'(\Omega)$) the pair of differential inequalities*

$$-\operatorname{div} \boldsymbol{A}(x, \nabla u) + B(x, u) \leq 0 \quad \textit{and} \quad -\operatorname{div} \boldsymbol{A}(x, \nabla v) + B(x, v) \geq 0 \quad \textit{in } \Omega.$$

Suppose that $\boldsymbol{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is in $L^\infty_{\rm loc}(\Omega \times \mathbb{R}^N)$ and $B : \Omega \times \mathbb{R} \to \mathbb{R}$ is in $L^\infty_{\rm loc}(\Omega \times \mathbb{R})$ such that $B = B(x, z)$ is independent of $\boldsymbol{\xi}$ and non-decreasing in z, whereas $\boldsymbol{A} = \boldsymbol{A}(x, \boldsymbol{\xi})$ is independent of z and monotone in $\boldsymbol{\xi}$, that is

$$\langle \boldsymbol{A}(x, \boldsymbol{\xi}) - \boldsymbol{A}(x, \boldsymbol{\eta}), \boldsymbol{\xi} - \boldsymbol{\eta} \rangle > 0 \quad \textit{when } \boldsymbol{\xi} \neq \boldsymbol{\eta}.$$

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Let $\varepsilon \in (0, 1)$ be small and $r_\varepsilon \in (0, 1)$ be as in Step 1. Fix $\eta \in [0, \eta_0]$ arbitrarily. Then, $(1 + \varepsilon)\, v_\eta(r) + v(r_\varepsilon)$ and $v(r) + \tilde{u}(r_\varepsilon)$ are super-solutions of (2.3) for $r \in (0, r_\varepsilon)$. By (2.6) and Lemma 2.2, we have

$$v(r) \leq (1 + \varepsilon)\, v_\eta(r) + v(r_\varepsilon) \quad \text{and} \quad (1 - \varepsilon)\, v_{-\eta}(r) \leq v(r) + \tilde{u}(r_\varepsilon) \quad \text{for all } 0 < r \leq r_\varepsilon. \tag{2.16}$$

Since r_ε is independent of $\eta \in [0, \eta_0]$, by letting $\eta \to 0^+$ in (2.16), we find that

$$v(r) \leq (1 + \varepsilon)\, \tilde{u}(r) + v(r_\varepsilon) \quad \text{and} \quad (1 - \varepsilon)\, \tilde{u}(r) \leq v(r) + \tilde{u}(r_\varepsilon) \quad \text{for all } 0 < r \leq r_\varepsilon. \tag{2.17}$$

By letting $r \to 0^+$ in (2.17), we deduce that

$$1 - \varepsilon \leq \liminf_{r \to 0^+} \frac{v(r)}{\tilde{u}(r)} \leq \limsup_{r \to 0^+} \frac{v(r)}{\tilde{u}(r)} \leq 1 + \varepsilon. \tag{2.18}$$

Finally, by passing to the limit $\varepsilon \to 0^+$ in (2.18), we conclude that $v(r) \sim \tilde{u}(r)$ as $r \to 0^+$. □

2.1. *Proof of Step 1 in the critical case $q = q_*$ of Theorem 2.1*

In this subsection, it remains only for us to establish the claim of Step 1 as outlined in the proof of Theorem 2.1. We first give the construction of a local family of sub- and super-solutions of (2.3). Let F be given by (1.9) and $c > 0$ be a large constant. Fix $\eta_0 \in (0, 1)$ small. Then for any $\eta \in [0, \eta_0]$, we define $v_{\pm\eta}(r)$ for $r > 0$ small, as follows

$$\begin{cases} v_{\pm\eta}(r) := C_{N,p}^{-1} \left(\dfrac{m_1 m_0^{\gamma - q}}{1 \pm \eta} \right)^{-\frac{1}{q-p+1}} \displaystyle\int_c^{\Phi(r)} \left[F(\Phi^{-1}(t)) \right]^{-\frac{1 \pm \eta}{q-p+1}} dt & \text{if (1.7) holds,} \\ \displaystyle\int_c^{v_{\pm\eta}(r)} [F(1/t)]^{\frac{1 \pm \eta}{q_* - p+1}} \, dt = C_{N,p}^{-1} \left(\dfrac{m_1 m_0^{-q-1-j}}{1 \pm \eta} \right)^{-\frac{1}{q-p+1}} \Phi(r) & \text{if (1.8) holds.} \end{cases} \tag{2.19}$$

We set $y_{\pm\eta}(s) = v_{\pm\eta}(r)$ with $s = \Phi(r)$. Using $y'_{\pm\eta}(s)$ and $y''_{\pm\eta}(s)$ to denote $dy_{\pm\eta}/ds$ and $d^2 y_{\pm\eta}/ds^2$, respectively, then

$$(p - 1) \left(y'_{\pm\eta}(s) \right)^{p-2} y''_{\pm\eta}(s) = \frac{1}{m_1} \left(y'_{\pm\eta}(s) \right)^{q_*} \left| \frac{d}{ds} \left[\left(y'_{\pm\eta}(s) \right)^{-q_* + p - 1} \right] \right|. \tag{2.20}$$

Step 1. *For every $\varepsilon \in (0, 1)$ small, there exists $s_\varepsilon > 0$ large such that $(1 - \varepsilon)\, y_{-\eta}$ and $(1 + \varepsilon)\, y_\eta$ are a sub-solution and super-solution of (2.4) for $s > s_\varepsilon$, respectively, for every $\eta \in [0, \eta_0]$.*

From (2.19), we find that

$$y'_{\pm\eta}(s) = \begin{cases} C_{N,p}^{-1} \left(\dfrac{m_1 m_0^{\gamma - q_*}}{1 \pm \eta} \right)^{-\frac{1}{q_* - p + 1}} [F(r)]^{-\frac{1 \pm \eta}{q_* - p + 1}} & \text{if (1.7) holds,} \\ C_{N,p}^{-1} \left(\dfrac{m_1 m_0^{-q_* - 1 - j}}{1 \pm \eta} \right)^{-\frac{1}{q_* - p + 1}} \left[F(1/y_{\pm\eta}(s)) \right]^{-\frac{1 \pm \eta}{q_* - p + 1}} & \text{if (1.8) holds.} \end{cases} \tag{2.21}$$

Moreover, we obtain the following asymptotic equivalence (uniform with respect to η)

$$\ln y_{\pm\eta}(s) \sim \ln s \quad \text{and} \quad s y'_{\pm\eta}(s) \sim y_{\pm\eta}(s) \ \text{ as } s \to \infty. \tag{2.22}$$

From (2.22), we deduce the following asymptotic equivalence as $s \to \infty$ (uniform with respect to η)

$$\begin{cases} m_0^{\gamma} L_h(1/r) \sim L_h(s) \sim L_h(y_{\pm\eta}(s)) & \text{if (1.7) holds;} \\ (m_0)^{-j} \left[L_{\mathcal{A}}(1/y_{\pm\eta}(s)) \right]^{-\frac{q_*}{p-1}} L_b(1/y_{\pm\eta}(s)) \sim \left[L_{\mathcal{A}}(\Phi^{-1}(s)) \right]^{-\frac{q_*}{p-1}} L_b(\Phi^{-1}(s)) & \text{if (1.8) holds.} \end{cases} \tag{2.23}$$

We introduce the notation $\mathcal{K}_{\pm\eta}(s) := \dfrac{\Upsilon(r)}{m_0} \dfrac{s\, y'_{\pm\eta}(s)}{y_{\pm\eta}(s)}$, where Υ is given by (2.9). We also denote $R_{\pm\eta}(s)$ as follows

$$R_{\pm\eta}(s) = \begin{cases} \dfrac{m_0^{\gamma} L_h(1/r)}{L_h(y_{\pm\eta}(s))} [F(r)]^{\pm\eta} \left[\mathcal{K}_{\pm\eta}(s) \right]^{q_*} & \text{if (1.7) holds,} \\ m_0^{-j} \left[\dfrac{L_{\mathcal{A}}(1/y_{\pm\eta}(s))}{L_{\mathcal{A}}(\Phi^{-1}(s))} \right]^{-\frac{q_*}{p-1}} \dfrac{L_b(1/y_{\pm\eta}(s))}{L_b(\Phi^{-1}(s))} \left[F(1/y_{\pm\eta}(s)) \right]^{\pm\eta} \left[\mathcal{K}_{\pm\eta}(s) \right]^{q_* + 1} & \text{if (1.8) holds.} \end{cases} \tag{2.24}$$

Since $m_0 = m_2$ for $q = q_*$, using (2.9) and (2.22), we infer that $\lim_{s \to \infty} \mathcal{K}_{\pm\eta}(s) = 1$ uniformly with respect to η. Hence, using (2.23), we derive the following asymptotics as $s \to \infty$ (uniform with respect to η)

$$R_{\pm\eta}(s) \sim \begin{cases} [F(r)]^{\pm\eta} & \text{if (1.7) holds,} \\ \left[F(1/y_{\pm\eta}(s)) \right]^{\pm\eta} & \text{if (1.8) holds.} \end{cases} \tag{2.25}$$

The right-hand side of (2.20) equals the product between $R_{\pm\eta}(s)$ and the right-hand side of (2.4) for $y = y_{\pm\eta}$. By the definition of F in (1.9), we have $\lim_{r \to 0^+} F(r) = 0$. Since $q > p - 1$, using (2.25), we conclude Step 1.

2.2. Proof of Steps 1 and 2 in the subcritical case $q < q_*$ of Theorem 2.1

We need only to justify the first two steps in the outline of the proof of Theorem 2.1. We shall adapt the perturbation method initiated by Cîrstea and Du in [10]. We construct a local family of sub- and super-solutions of (2.3). Fix $\eta_0 \in (0, 1)$ such that $2\eta_0 (p - 1) M < 1$, where M is the positive constant given by (1.12). For every $\eta \in [0, \eta_0]$, we define the function $v_{\pm\eta}$ and the constant $C_{\pm\eta} > 0$ as

$$v_{\pm\eta}(r) = C_{\pm\eta} [\tilde{u}(r)]^{1 \pm \eta} \ \text{ for } r \in (0, 1) \quad \text{where } C_{\pm\eta}^{q - p + 1} := (1 \pm \eta)^{p - 1} \left[1 \pm \eta M (p - 1) \right]. \tag{2.26}$$

From this definition, we have that $\lim_{\eta \to 0^+} v_{\pm\eta}(r) = \tilde{u}(r)$ for every $r \in (0, 1)$ and $\lim_{\eta \to 0} C_{\pm\eta} = 1$.

Step 1. *For every $\varepsilon \in (0, 1)$ small, there exists $r_\varepsilon \in (0, 1)$ such that $(1 - \varepsilon)\, v_{-\eta}$ and $(1 + \varepsilon)\, v_{\eta}$ are a sub-solution and super-solution of (2.3) for $0 < r < r_\varepsilon$, respectively, for every $\eta \in [0, \eta_0]$.*

Claim. *We see that $\tilde{u}$ satisfies (2.3) asymptotically as $r \to 0^+$.*

Proof of Claim. Let $r_0 \in (0,1)$ be small so that $\tilde{u}(r_0) > t_0$, where t_0 is as in Remark A.4. For all $r \in (0, r_0)$, we set

$$\begin{cases} Q_{\pm\eta}(r) := r^{N-1+\vartheta} L_{\mathcal{A}}(r) \left| v'_{\pm\eta}(r) \right|^{p-2} v'_{\pm\eta}(r), \\ P(r) := M \left[q + 1 + \dfrac{\tilde{u}(r)\, L'_h(\tilde{u}(r))}{L_h(\tilde{u}(r))} - \dfrac{\tilde{u}(r)\, \tilde{u}''(r)}{\left[\tilde{u}'(r)\right]^2} + \left(N - 1 + \sigma + \dfrac{r L'_b(r)}{L_b(r)} \right) \dfrac{\tilde{u}(r)}{r\, \tilde{u}'(r)} \right]. \end{cases} \tag{2.27}$$

One can verify that $\lim_{r\to 0^+} P(r) = 1$ using the definition of M in (1.12). By differentiating (1.12), we find that

$$Q_0(r) = M r^{N-1+\sigma} L_b(r) \frac{\left[\tilde{u}(r)\right]^{q+1}}{\tilde{u}'(r)} L_h(\tilde{u}(r)) \quad \text{for all } r \in (0, r_0). \tag{2.28}$$

The claim follows since $Q'_0(r)$ equals the product between $P(r)$ in (2.27) and the right-hand side of (2.3) for $v = \tilde{u}$. □

By twice differentiating (2.26), we obtain that

$$\begin{cases} Q_{\pm\eta}(r) = \left[C_{\pm\eta}(1 \pm \eta)\right]^{p-1} [\tilde{u}(r)]^{\pm\eta(p-1)}\, Q_0(r), \\ \dfrac{dQ_{\pm\eta}}{dr} = \left[C_{\pm\eta}(1 \pm \eta)\right]^{p-1} [\tilde{u}(r)]^{\pm\eta(p-1)} \left\{ \pm\eta\, (p-1) M \left[\tilde{u}(r)\right]^q L_h(\tilde{u}(r))\, L_b(r)\, r^{N-1+\sigma} + \dfrac{dQ_0}{dr} \right\}. \end{cases} \tag{2.29}$$

Hence, using (2.26) and the above claim, we find the following asymptotics (uniform with respect to η)

$$\frac{dQ_{\pm\eta}}{dr} \sim C^q_{\pm\eta} r^{N-1+\sigma} L_b(r)\, L_h(\tilde{u}(r))\, [\tilde{u}(r)]^{q \pm \eta(p-1)} \quad \text{as } r \to 0^+. \tag{2.30}$$

From Remark A.4 in Appendix A, the function $t \longmapsto t^{q-p+1} L_h(t)$ is increasing on $(0, \infty)$ so that

$$L_h(\tilde{u}^{1-\eta})\, [\tilde{u}(r)]^{-\eta\,(q-p+1)} \leq L_h(\tilde{u}(r)) \leq L_h(\tilde{u}^{1+\eta})\, [\tilde{u}(r)]^{\eta\,(q-p+1)}$$

for every $r \in (0, r_0)$ and all $\eta \in [0, \eta_0]$. This, together with (2.26), implies that for every $r \in (0, r_0)$ and all $\eta \in [0, \eta_0]$

$$\pm C^q_{\pm\eta}\, L_h(\tilde{u}(r)) \left[\tilde{u}(r)\right]^{q \pm \eta(p-1)} \leq \pm L_h(v_{\pm\eta}(r)/C_{\pm\eta}) \left[v_{\pm\eta}(r)\right]^q. \tag{2.31}$$

Since $q > p - 1$, from (2.30), (2.31) and Proposition A.1 in Appendix A, we conclude the proof of Step 1.

Step 2. *Any positive solution v of (2.3) with a strong singularity at 0 satisfies (2.6).*

Since $v_{\pm\eta}$ is regularly varying at 0 with index $-(1 \pm \eta)\, m_0$, we conclude Step 2 based on Lemma 2.3 with $f = v_{\pm\eta}$.

Lemma 2.3. *Let* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *hold and $q < q_*$. Suppose that v is a positive solution of (2.3) with a strong singularity at zero. Let f be a regularly varying function at zero with real index $-\kappa$. With m_0 given by (1.10), the following hold:*

(a) *If $\kappa > m_0$, then* $\lim_{r\to 0^+} v(r)/f(r) = 0$.
(b) *If $\kappa < m_0$, then* $\lim_{r\to 0^+} v(r)/f(r) = \infty$.

Proof. We adapt ideas from Cîrstea and Du [10, Theorem 1.4].

(a) The *a priori* estimates in (4.1) (see Lemma 4.1 for a proof) show that v is bounded from above near zero by a regularly varying function at 0 with index $-m_0$. The assertion now follows easily since every regularly varying function at 0 with positive (respectively, negative) index must converge to 0 (respectively, ∞).

(b) Since $\kappa < m_0$, we can choose $q_1 \in (q, q_*)$ sufficiently close to q such that $\kappa < (p + \sigma - \vartheta)/(q_1 - p + 1)$. Then, $\lim_{t\to\infty} t^{q-q_1} L_h(t) = 0$ (see Remark A.1 in Appendix A) and using (2.7), we can let $s_0 > 0$ large and find that

$$L_h(y(s))\, [y(s)]^q \leq [y(s)/2]^{q_1} \leq s^{q_1} [y'(s)]^{q_1} \quad \text{for all } s \geq s_0. \tag{2.32}$$

We set $f_{q_1}(r) := r^{N-1+\sigma} L_b(r) [\Phi(r)]^{q_1}$ for $r \in (0,1)$. Since Φ is regularly varying at 0 with index $-m_2$ (see (2.9)), we find that f_{q_1} is regularly varying at 0 with index $N + \sigma - q_1 m_2 - 1$, which is greater than -1. This gives that

$\int_{0^+} f_{q_1}(\xi)\, d\xi < \infty$. Moreover, the function $F_{q_1}(r) = \int_r^{\Phi^{-1}(s_0)} \left[\int_0^\tau f_{q_1}(\xi)\, d\xi\right]^{-\frac{1}{q_1-p+1}} |\Phi'(\tau)|\, d\tau$ is regularly varying at zero with index $-(p+\sigma-\vartheta)(q_1-p+1)$, which is less than $-\kappa$ from our choice of q_1. We thus have $\lim_{r\to 0^+} F_{q_1}(r)/f(r) = \infty$.

We conclude that $\lim_{r\to 0^+} v(r)/f(r) = \infty$ by showing that $\liminf_{r\to 0^+} v(r)/F_{q_1}(r) > 0$. Indeed, we see that

$$\liminf_{r\to 0^+} \frac{v(r)}{F_{q_1}(r)} = \liminf_{s\to\infty} \frac{y(s)}{\int_{s_0}^s \left[\int_0^{\Phi^{-1}(t)} f_{q_1}(\xi)\, d\xi\right]^{-\frac{1}{q_1-p+1}} dt}. \tag{2.33}$$

From (2.4) and (2.32), we deduce that

$$\left[y'(s)\right]^{p-2-q_1} y''(s) \le -\frac{C_{N,p}^{-p+1}}{p-1} f_{q_1}(\Phi^{-1}(s)) \frac{d(\Phi^{-1}(s))}{ds} \quad \text{for all } s > s_0. \tag{2.34}$$

Recall that $\lim_{s\to\infty} y'(s) = \infty$ since v has a strong singularity at 0. Thus, by integrating (2.34), we obtain that

$$y'(s) \ge \left[\frac{(q_1-p+1)\, C_{N,p}^{-p+1}}{p-1} \int_0^{\Phi^{-1}(s)} f_{q_1}(\xi)\, d\xi\right]^{-\frac{1}{q_1-p+1}} \quad \text{for all } s > s_0,$$

which shows that the right-hand side of (2.33) is positive. This concludes the assertion of Lemma 2.3(b). □

3. Proof of Theorem 1.1(b): removability of singularities

Throughout this section, we let Assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ hold. The proof of Theorem 1.1(b) relies on two main ingredients, whose verification is postponed to the end of this section.

Lemma 3.1. *If u is a positive solution of (1.1) such that $\lim_{|x|\to 0} u(x)/\Phi(x) = 0$, then there exists $\lim_{|x|\to 0} u(x) \in (0,\infty)$ and $\lim_{|x|\to 0} |x||\nabla u(x)| = 0$. Moreover, u can be extended as a continuous positive solution of (1.1) in B_1.*

This result, which was also invoked in the proof of Theorem 1.1(a)(i), generalises [10, Lemma 3.2(ii)] (where $\mathcal{A} = 1$) and [3, Proposition 3] (where $p = 2$, $b = 1$ and $h(u) = u^q$).

Remark 3.1. If u is a positive solution of (1.1) with $\limsup_{|x|\to 0} u(x) < \infty$, then both integrals in (1.4) are well-defined for every $\varphi \in C_c^1(B_1)$. Indeed, $b \in L^1_{\mathrm{loc}}(B_1)$ since $\sigma > -N$ (from $(\mathbf{A}_1)$ and $(\mathbf{A}_3)$), whereas Lemma 3.1 gives that $\mathcal{A}(|x|)\, |\nabla u|^{p-1} \in L^1_{\mathrm{loc}}(B_1)$ since $t^{1-p}\mathcal{A}(t)$ is regularly varying at 0 with index $\vartheta - p + 1$ (greater than $-N$).

Lemma 3.2. *If (2.3) has a positive solution with either a weak or a strong singularity at 0, then $b(x)\, h(\Phi) \in L^1(B_{1/2})$.*

We show how to use Lemma 3.1 and Lemma 3.2 to finish the proof of Theorem 1.1(b). We thus assume that $b(x)\, h(\Phi) \notin L^1(B_{1/2})$ and prove that any positive solution of (1.1) can be extended as a positive solution of (1.1) in B_1. By Remark B.1, we have $q \ge q_*$, with q_* as in (1.6). Our argument is twofold:

Case 1. $q > q_*$.

Since $m_0 < m_2$, the claim follows from Lemma 3.1 and the *a priori* estimates in (4.17). Indeed, we have $\limsup_{|x|\to 0} u(x)/T(|x|) < \infty$ for a function T regularly varying at 0 with index $-m_0$. Using that $\Phi \in RV_{-m_2}(0+)$, by Remark A.1 and Definition A.1 in Appendix A, we find that $\lim_{r\to 0^+} T(r)/\Phi(r) = 0$ so that $\lim_{|x|\to 0} u(x)/\Phi(x) = 0$ for any positive solution u of (1.1). Then, by Lemma 3.1, we conclude the proof of Theorem 1.1(b).

Case 2. $q = q_*$.

The previous argument no longer applies since T and Φ are now regularly varying at 0 with the same index $-m_0$. Hence, T/Φ is slowly varying at 0, whose behaviour at 0 is, in general, undetermined as illustrated by Example 1 in Appendix A. In view of Lemma 3.1, we conclude the proof by showing that $\lim_{|x|\to 0} u(x)/\Phi(x) = 0$.

Assuming the contrary and using (4.16), we deduce $\lim_{|x|\to 0} u(x) = \infty$. Then there exists $k \in (0, 1/2)$ and a positive solution v_* of (2.3) for $0 < r < k$ such that $C_1 u \le v_* \le C_2$ in B_k^*, where C_1 and C_2 are positive constants. Thus, by Lemma 3.2, we cannot have $\limsup_{|x|\to 0} u(x)/\Phi(x) \in (0, \infty]$. This completes the proof of Theorem 1.1(b).

Proof of Lemma 3.1. Let u be a positive solution of (1.1) such that $\lim_{|x|\to 0} u(x)/\Phi(x) = 0$. For convenience, we define

$$\theta := \limsup_{|x|\to 0} u(x).$$

By the comparison principle (Lemma 2.2), we find as in [10, Lemma 3.2] that $\theta < \infty$. Due to our general assumption $(\mathbf{A}_1)$, we cannot invoke [20, Theorem 1] to conclude the proof, unlike the case $\mathcal{A} = 1$ treated in [10].

We show below that $\theta > 0$. In the special case $p = 2$ and $h(t) = t^q$ of [3], the claim follows by a reduction to radial solutions, coupled with a change of variable and [21, Theorem 1.1]. For our general divergence-form equation, we require different ideas that are inspired by [9, Lemma 5.2].

Since Assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ hold and $\theta < \infty$, there exists a positive constant C such that

$$b(x)\, h(u) \le C|x|^\sigma L_b(|x|)\, u^{p-1} \quad \text{for all } 0 < |x| \le 1/2.$$

Similar to Step 2 of [9, Lemma 5.2], we construct a positive radial solution v_∞ of

$$-\mathrm{div}\,(\mathcal{A}(|x|)\, |\nabla v|^{p-2}\nabla v) + C|x|^\sigma L_b(|x|)\, v^{p-1} = 0 \quad \text{for } 0 < |x| < 1/2 \tag{3.1}$$

such that $v_\infty(|x|) \le u(x)$ for $0 < |x| \le 1/2$. By a contradiction argument and Lemma 2.2, we find that the radial solution v_∞ of (3.1) has a non-negative limit at 0. To conclude that $\theta > 0$, it suffices to show that $\lim_{r\to 0^+} v_\infty(r) > 0$. By assuming that $\lim_{r\to 0^+} v_\infty(r) = 0$, we arrive at a contradiction as follows. We use the change of variable $z(s) = v_\infty(r)$ with $s = \Phi(r)$. Then, we have $\lim_{s\to\infty} z(s) = 0$. Moreover, z is a positive solution of the ordinary differential equation

$$\left|\frac{dz}{ds}\right|^{p-2} \frac{d^2 z}{ds^2} = C_1 r^{N-1+\sigma} L_b(r)\, [z(s)]^{p-1} \left|\frac{dr}{ds}\right| \quad \text{for } s \in (\Phi(1/2), \infty), \tag{3.2}$$

where C_1 denotes a positive constant. Since $z''(s) > 0$, then $z'(s)$ is increasing on $(\Phi(1/2), \infty)$ with $\lim_{s\to\infty} z'(s) = 0$. Therefore, using (3.2), we find that

$$z(s) = C_2 \int_s^\infty \left(\int_0^{\Phi^{-1}(t)} \xi^{N-1+\sigma} L_b(\xi)\, [z(\Phi(\xi))]^{p-1}\, \mathrm{d}\xi \right)^{\frac{1}{p-1}} dt \quad \text{for } s > \Phi(1/2),$$

where C_2 is a positive constant. Since z is decreasing, we infer that

$$1/C_2 \le \int_s^\infty \left(\int_0^{\Phi^{-1}(t)} \xi^{N-1+\sigma} L_b(\xi)\, \mathrm{d}\xi \right)^{\frac{1}{p-1}} dt \quad \text{for every } s > \Phi(1/2). \tag{3.3}$$

Let $V(s)$ denote the right-hand side of (3.3). We claim that $V(s)$ is well-defined and $V(s) \to 0$ as $s \to \infty$. Indeed, we have $\Phi \in RV_{-m_2}(0+)$ and thus $\Phi^{-1} \in RV_{-1/m_2}(\infty)$. Note that $r \longmapsto \int_0^r \xi^{N-1+\sigma} L_b(\xi)\, d\xi$ is regularly varying at 0^+ with positive index given by $\sigma + N$. Consequently, V is regularly varying at ∞ with *negative* index $(p + \sigma - \vartheta)/(p - N - \vartheta)$ so that the claim follows. Then, (3.3) leads to a contradiction, which proves that $\lim_{r\to 0^+} v_\infty(r) > 0$ and, hence, $\theta > 0$.

To obtain that $\lim_{|x|\to 0} u(x) = \theta$, $\lim_{|x|\to 0} |x||\nabla u(x)| = 0$ and (1.4) holds for all $\varphi \in C_c^1(B_1)$, we proceed as in the special case of [3, Proposition 3]. Since the ideas are very similar, we skip the details. □

Proof of Lemma 3.2. We show that $b(x)\,h(\Phi) \in L^1(B_{1/2})$ is a necessary condition for the existence of a positive solution of (2.3) with a weak or strong singularity at 0. Let v be a positive solution of (2.3) with $\lim_{r\to 0^+} v(r)/\Phi(r) = \lambda \neq 0$.

First, we consider the case $\lambda \in (0,\infty)$. Let $\Phi^{-1}(t)$ denote the inverse of Φ, which exists for any $t > 0$. By the change of variable $y(s) = v(r)$ with $s = \Phi(r)$, we find (2.4). Since $v(r) \sim \lambda\Phi(r)$ as $r \to 0^+$, we have $y(s) \sim \lambda s$ as $s \to \infty$. Using that $d^2y/ds^2 \geq 0$, we get that dy/ds is increasing on $(0,\infty)$ so that $\lim_{s\to\infty} dy/ds = \lambda$. We define Λ by

$$\Lambda(s) := \frac{C_{N,p}^{-p+1}}{p-1}[\Phi^{-1}(s)]^{N-1+\sigma} L_b(\Phi^{-1}(s))\, L_h(s)\, s^{p-2} \left|\frac{dr}{ds}\right| \quad \text{for } s > 0 \text{ large.} \tag{3.4}$$

Since $L_h \in RV_0(\infty)$ and $y(s) \sim \lambda s$ as $s \to \infty$, we have $L_h(y(s)) \sim L_h(s)$ as $s \to \infty$. We apply (2.22) to (2.4) to get that

$$\begin{cases} \dfrac{d^2y}{ds^2} \sim \Lambda(s)[y(s)]^{q-p+2} & \text{as } s \to \infty, \\ y'(s) \to \lambda & \text{as } s \to \infty. \end{cases} \tag{3.5}$$

By Taliaferro [21, p. 96], we get that $\int^\infty t^{q-p+2}\Lambda(t)\,dt < \infty$. Then applying a change of variable $r = \Phi^{-1}(t)$ and using Remark B.1, we obtain that $b(x)\,h(\Phi) \in L^1(B_{1/2})$.

Secondly, let $\lambda = \infty$. We adapt ideas from the proof of [9, Lemma 5.8]. Choose $m \in (p-1, q_*)$ and for $t > 0$, set $\chi(t) = t^{q_*-m}L_h(t)$. By the property in (A.2) in the Appendix A, we have $\lim_{t\to\infty} t\chi'(t)/\chi(t) = q_* - m > 0$ and, hence, $\chi(t)$ is increasing for $t > 0$ sufficiently large. Since $\lim_{r\to 0^+} v_*(r)/\Phi(r) = \infty$, there exists a constant $a_0 > 0$ such that $v_*(r) \geq a_0\Phi(r)$ for all $0 < r \leq 1/2$. Then there exists a constant $c > 0$ such that

$$L_h(v_*)\, v_*^{q_*} \geq c\,\chi(\Phi(r))\, v_*^m \quad \text{for all } r \in (0, 1/2]. \tag{3.6}$$

Define a function $\tilde{b}(r) := c\, r^\sigma L_b(r)\, \chi(\Phi(r))$ for $r \in (0,1/2]$. We construct a positive radial solution v_∞ of

$$-\text{div}\,(\mathcal{A}(|x|)\,|\nabla v|^{p-2}\nabla v) + \tilde{b}(|x|)\, v^m = 0 \quad \text{in } B_{1/2}^* \tag{3.7}$$

such that $v_* \leq v_\infty$ in $B_{1/2}^*$. Then, v_∞ has a strong singularity at 0^+. Since $\chi \in RV_{q_*-m}(\infty)$, we find that $\tilde{b} \in RV_{\tilde{\sigma}}(0+)$ with $\tilde{\sigma}$ given by $m(N+\sigma)/q_* - N$, which is greater than $\vartheta - p$ from our choice of m. We note that (3.7) corresponds to (2.3) in the *critical* case with $r^\sigma L_b(r) = \tilde{b}(r)$, $L_h \equiv 1$ and $q = m$, where (1.7) holds. Using Remark 2.1 on (3.7), and the definition of $\tilde{b}$, we conclude that $b(x)\,h(\Phi) \in L^1(B_{1/2})$. This completes the proof of Lemma 3.2. □

4. Basic tools

Throughout this section, let Assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ hold. Our aim is to prove the basic tools used in this paper: *a priori* estimates (Lemma 4.1), a spherical Harnack-type inequality (Lemma 4.2) and a regularity result (Lemma 4.3).

Lemma 4.1 *(A priori estimates). For any $r_0 \in (0,1/2)$, there exists a positive constant C, depending on r_0, such that for every positive (sub-)solution of (1.1), we have*

$$\frac{|x|^p b(x)}{\mathcal{A}(|x|)} \frac{h(u(x))}{[u(x)]^{p-1}} \leq C \quad \textit{for every } 0 < |x| \leq r_0. \tag{4.1}$$

Proof. Fix $x_0 \in \mathbb{R}^N$ with $0 < |x_0| \leq r_0$. We denote $\rho := |x_0|/2$ and $p' := p/(p-1)$. Let

$$\zeta(r) := r^{\frac{\sigma-\vartheta+p}{p}} \left[\frac{L_b(r)}{L_{\mathcal{A}}(r)}\right]^{\frac{1}{p}} \quad \text{for } r \in (0, r_0] \;\text{ and }\; f(t) := \frac{t^{1-\frac{q+1}{p}}[L_h(t)]^{-\frac{1}{p}}}{\int_t^\infty \xi^{-\frac{q+1}{p}}[L_h(\xi)]^{-\frac{1}{p}}\,d\xi} \quad \text{for } t > 0 \text{ large.} \tag{4.2}$$

Let $c > 0$ be a positive constant. We define $S = S_{x_0} : B_\rho(x_0) \to \mathbb{R}$ by

$$\int_{S(x)}^{\infty} t^{-\frac{q+1}{p}}[L_h(t)]^{-\frac{1}{p}}\,dt = c\zeta(|x_0|)\left[1 - \left(\frac{|x-x_0|}{\rho}\right)^{p'}\right] \quad \text{for every } x \in B_\rho(x_0). \tag{4.3}$$

Claim. *There exists a small positive constant c depending on r_0, but independent of x_0 such that the function S defined by (4.3) is a super-solution of (1.1) in $B_\rho(x_0)$, namely for h_1 as in Remark A.4, it holds*

$$\mathrm{div}\,(\mathcal{A}(|x|)\,|\nabla S|^{p-2}\nabla S) \le b(x)\,h_1(S) \quad \text{in } B_\rho(x_0). \tag{4.4}$$

Suppose the claim holds. Since $S(x) \to \infty$ as $|x-x_0| \to \rho$, by the comparison principle of Lemma 2.2, we find that $u \le S$ in $B_\rho(x_0)$. In particular, we have $u(x_0) \le S(x_0)$. Since ζ is regularly varying at 0^+ with positive index $(p+\sigma-\vartheta)/p$, we have $\lim_{r\to 0^+}\zeta(r) = 0$ so that $\sup_{0<r\le r_0}\zeta(r) < \infty$. Since the right-hand side of (4.3) is bounded from above by $c\sup_{0<r\le r_0}\zeta(r)$, for every $M > 0$ there exists a small positive constant c (depending on M and r_0) such that $S \ge M$ in $B_\rho(x_0)$ for every $0 < |x_0| \le r_0$. Using (4.2) and (4.3), we find that

$$[S(x_0)]^{q-p+1}\,L_h(S(x_0)) = \left[c\zeta(|x_0|)\,f(S(x_0))\right]^{-p}. \tag{4.5}$$

We fix $M > 0$ as large as needed. Let h_1 and h_2 be as in Remark A.4 of Appendix A. We can thus assume that $h_2(t) \le 2t^q L_h(t)$ for all $t \ge M$. By Karamata's Theorem in Appendix A, we have $\lim_{t\to\infty} f(t) = (q-p+1)/p > 0$. Since $u(x_0) \le S(x_0)$, using (4.5) and (A.1), we can find a positive constant $C_1 = C_1(r_0)$ independent of x_0 such that

$$\frac{|x_0|^p b(x_0)}{\mathcal{A}(|x_0|)}\,\frac{h(u(x_0))}{[u(x_0)]^{p-1}} \le \frac{|x_0|^p b(x_0)}{\mathcal{A}(|x_0|)}\,\frac{h_2(S(x_0))}{[S(x_0)]^{p-1}} \le \frac{2}{[cf(S(x_0))]^p}\,\frac{b(x_0)}{|x_0|^\sigma L_b(|x_0|)} \le C_1. \tag{4.6}$$

Since (4.6) holds for every $0 < |x_0| \le r_0$, we conclude the assertion of Lemma 4.1.

Proof of Claim. By (4.3), we find that

$$|\nabla S(x)|^{p-2}\nabla S(x) = \left(cp'\right)^{p-1}\rho^{-p}\,[\zeta(|x_0|)]^{p-1}\left[S^{q+1}(x)\,L_h(S(x))\right]^{\frac{1}{p'}}(x-x_0) \quad \text{in } B_\rho(x_0). \tag{4.7}$$

Using f given by (4.2), we denote by $T_{x_0}(x)$ the following quantity

$$\left(\frac{|x-x_0|}{\rho}\right)^{p'}\left(q+1+\frac{S(x)L_h'(S(x))}{L_h(S(x))}\right) + f(S(x))\left[1-\left(\frac{|x-x_0|}{\rho}\right)^{p'}\right]\left(N+\frac{|x|\mathcal{A}'(|x|)}{\mathcal{A}(|x|)}\,\frac{(x-x_0)\cdot x}{|x|^2}\right). \tag{4.8}$$

With $T_{x_0}(x)$ given by (4.8), we derive that

$$\mathrm{div}\,(\mathcal{A}(|x|)\,|\nabla S|^{p-2}\nabla S) = \left(p'\right)^{p-1}(2c)^p\left(\frac{|x|}{|x_0|}\right)^{\vartheta}\frac{L_{\mathcal{A}}(|x|)}{L_{\mathcal{A}}(|x_0|)}\,|x_0|^\sigma L_b(|x_0|)\,S^q L_h(S)\,T_{x_0}(x). \tag{4.9}$$

By Assumption $(\mathbf{A}_1)$ and Remark A.4 in Appendix A, we have $\lim_{r\to 0^+} r\mathcal{A}'(r)/\mathcal{A}(r) = \vartheta$ and $\lim_{t\to\infty} tL_h'(t)/L_h(t) = 0$. Recall that $\lim_{t\to\infty} f(t) = (q-p+1)/p$. Moreover, by Proposition A.1 in Appendix A, there exist positive constants c_i $(0 \le i \le 3)$ depending on r_0, but independent of x_0 such that

$$c_0\,L_{\mathcal{A}}(|x_0|) \le L_{\mathcal{A}}(|x|) \le c_1\,L_{\mathcal{A}}(|x_0|) \ \text{ and } \ c_2\,L_b(|x|) \le L_b(|x_0|) \le c_3\,L_b(|x|)$$

for every x, x_0 such that $0 < |x_0| \le r_0$ and $|x|/|x_0| \in [1/2, 3/2]$. Thus, using (1.3) and (4.9), we conclude (4.4) by taking in (4.3) a small constant $c > 0$ depending on r_0, but independent of x_0. This completes the proof of Lemma 4.1. □

Lemma 4.2 *(Harnack-type inequality). Fix $r_0 \in (0, 1/2)$. There exists a positive constant K (depending on p, N and r_0) such that for every positive solution u of (1.1), we have*

$$\max_{|x|=r} u(x) \le K \min_{|x|=r} u(x) \quad \textit{for all } 0 < r \le r_0/2. \tag{4.10}$$

Proof. We first observe that (1.1) is equivalent to

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) + \frac{\mathcal{A}'(|x|)}{\mathcal{A}(|x|)}\,|\nabla u|^{p-2}\frac{\nabla u\cdot x}{|x|} + \frac{b(x)\,h(u)}{\mathcal{A}(|x|)\,u^{p-1}}\,u^{p-1} = 0 \quad \text{in } B^*. \tag{4.11}$$

Let b_1 and b_2 denote two non-negative functions as follows

$$b_1(x) := \frac{|\mathcal{A}'(|x|)|}{\mathcal{A}(|x|)} \quad \text{and} \quad [b_2(x)]^p := \frac{b(x)\, h(u)}{\mathcal{A}(|x|)\, u^{p-1}} \quad \text{for } 0 < |x| \le r_0. \tag{4.12}$$

By (1.2) and Lemma 4.1, there exists a positive constant C_1, depending on r_0, such that

$$|x|\, b_1(x) \le C_1 \quad \text{and} \quad |x|\, b_2(x) \le C_1 \quad \text{for all } 0 < |x| \le r_0. \tag{4.13}$$

Fix $x_0 \in \mathbb{R}^N$ such that $0 < |x_0| \le r_0/2$ and set $\rho := |x_0|/2$. We use μ to denote

$$\mu = \mu_{x_0} := \max\{\|b_1\|_{L^\infty(B_\rho(x_0))}, \|b_2\|_{L^\infty(B_\rho(x_0))}\}.$$

Since $\rho \le |x|$ for every $x \in B_\rho(x_0)$, from (4.13) it follows that

$$\rho\mu \le C_1 \quad \text{for every } x \in B_\rho(x_0). \tag{4.14}$$

We apply the Harnack inequality of [23, Theorem 1.1] for (4.11) on $B_{|x_0|/2}(x_0)$ where the structure conditions in (1.2) and (1.3) of [23] are satisfied with $a_0 = 1$ and $a_i = b_0 = b_3 = 0$ for $i \in \{1, 2, 3, 4\}$. Hence, there exists a positive constant k, depending only on p, N and $\rho\mu$, such that

$$\sup_{x \in B_{\rho/3}(x_0)} u(x) \le k \inf_{x \in B_{\rho/3}(x_0)} u(x). \tag{4.15}$$

By the covering argument in [13], any two points x_1 and x_2 in $\mathbb{R}^N$ such that $0 < |x_1| = |x_2| \le r_0/2$ can be joined by ten overlapping balls of radius $|x_1|/6$ with centres positioned on $\partial B_{|x_1|}(0)$. Thus, by (4.14) and (4.15), we obtain (4.10) with $K = k^{10}$, where K is a positive constant depending on p, N and r_0. □

Remark 4.1. Using (4.10) and the same argument as in [3, Corollary 4] and [9, Corollary 4.5], we can show that

$$\begin{cases} \text{If } \limsup_{|x|\to 0} \dfrac{u(x)}{[\Phi(x)]^j} = \infty, \text{ then } \lim_{|x|\to 0} \dfrac{u(x)}{[\Phi(x)]^j} = \infty \quad \text{for } j \in \{0, 1\}. \\ \text{If } \liminf_{|x|\to 0} \dfrac{u(x)}{\Phi(x)} = 0, \text{ then } \lim_{|x|\to 0} \dfrac{u(x)}{\Phi(x)} = 0. \end{cases} \tag{4.16}$$

Consequently, we either have $\limsup_{|x|\to 0} u(x) < \infty$ or $\lim_{|x|\to 0} u(x) = \infty$. In the latter case, the *a priori* estimate in (4.1), together with Assumptions $(\mathbf{A}_1)$ and $(\mathbf{A}_3)$, gives that

$$\limsup_{|x|\to 0} \frac{L_b(|x|)}{L_{\mathcal{A}}(|x|)} |x|^{p+\sigma-\vartheta} [u(x)]^{q-p+1} L_h(u(x)) < \infty. \tag{4.17}$$

In particular, (4.17) yields that $\limsup_{|x|\to 0} u(x)/T(|x|) < \infty$ for some function T regularly varying at 0 with index $-m_0$. Since $\lim_{r\to 0^+} \ln T(r)/\ln(1/r) = m_0$, we find that $\limsup_{|x|\to 0} \ln u(x)/\ln(1/|x|) \le m_0$. Furthermore, if $q = q_*$, then $m_0 = m_2$ and any positive solution u of (1.1) with a strong singularity at zero satisfies

$$\lim_{|x|\to 0} \frac{\ln u(x)}{\ln(1/|x|)} = m_0. \tag{4.18}$$

Lemma 4.3 *(A regularity result). Fix $r_0 \in (0, 1/4)$ and $\delta \ge 0$. Let $g \in C(0, 1)$ be a positive function such that g is regularly varying at 0 with index $-\delta$. Suppose that u is a positive solution of (1.1) and $C_0 > 0$ is a constant such that*

$$0 < u(x) \le C_0\, g(|x|) \quad \text{for } 0 < |x| < 2r_0. \tag{4.19}$$

Then there exist positive constants $C > 0$ and $\alpha \in (0, 1)$ such that

$$|\nabla u(x)| \le C \frac{g(|x|)}{|x|} \quad \text{and} \quad |\nabla u(x) - \nabla u(x')| \le C \frac{g(|x|)}{|x|^{1+\alpha}} |x - x'|^\alpha \tag{4.20}$$

for any x, x' in $\mathbb{R}^N$ satisfying $0 < |x| \le |x'| < r_0$.

Proof. We use an argument close to [10, Lemma 4.1], which is similar to [13, Lemma 1.1] (see also [3, Lemma 3]). There is, however, one essential difference with respect to the derivation of the first inequality in (4.20). We show below the main modifications compared with [10, Lemma 4.1].

Using (4.11) and defining Ψ_β as in (4.5) of [10], that is $\Psi_\beta(\xi) := u(\beta\xi)/g(\beta)$ for $\xi \in \bar{\Gamma}$, where $\beta \in (0, r_0/6)$ is fixed, we see that Ψ_β satisfies an equation of the form (4.3) of [10], namely

$$-\mathrm{div}\,(|\nabla\Psi_\beta|^{p-2}\nabla\Psi_\beta) + B_\beta = 0 \quad \text{in } \Gamma, \quad \text{where } \Gamma := \{y \in \mathbb{R}^N : 1 < |y| < 7\}. \tag{4.21}$$

However, instead of (4.7) in [10], the expression of B_β is more complicated here, involving a gradient term, namely

$$B_\beta(\xi) := \frac{\beta^p}{[g(\beta)]^{p-1}}\, b(\beta\xi)\frac{h(u(\beta\xi))}{\mathcal{A}(\beta|\xi|)} - \frac{\beta\mathcal{A}'(\beta|\xi|)}{\mathcal{A}(\beta|\xi|)}\,|\nabla\Psi_\beta|^{p-2}\,\frac{\nabla\Psi_\beta(\xi)\cdot\xi}{|\xi|} \quad \text{for } \xi \in \Gamma. \tag{4.22}$$

Claim. *The functions Ψ_β and B_β are in $L^\infty(\Gamma)$ with their L^∞-norms bounded above by a positive constant independent of $\beta \in (0, r_0/6)$.*

Proof of Claim. For Ψ_β, we can proceed exactly as in [10]. We thus need to prove the claim only for B_β. Using Lemma 4.1 and (4.19), jointly with (4.10) in [10], we find that the $L^\infty(\Gamma)$-norm of the *first term* in the right-hand side of (4.22) is bounded above by a constant independent of β.

Assume for now that the first inequality in (4.20) is proved. Then we can infer that $|\nabla\Psi_\beta(\xi)| \le Cg(\beta|\xi|)/g(\beta)$ for every $\xi \in \Gamma$. Hence, using (4.13), as well as (4.10) in [10], we could conclude the claim for B_β given by (4.22). □

Since $B \in L^\infty(\Gamma)$ and $\Psi \in L^\infty(\Gamma) \cap W^{1,p}(\Gamma)$ is a weak solution of (4.21), from the $C^{1,\alpha}$-regularity result of Tolksdorf [22], we conclude that there exist constants $\alpha = \alpha(N, p) \in (0, 1)$ and $\tilde{C} = \tilde{C}\left(N, p, \|\Psi\|_{L^\infty(\Gamma)}, \|B\|_{L^\infty(\Gamma)}\right) > 0$ such that

$$\|\nabla\Psi\|_{C^{0,\alpha}(\Gamma^*)} \le \tilde{C}, \quad \text{where } \Gamma^* := \{y \in \mathbb{R}^N : 2 < |y| < 6\}. \tag{4.23}$$

This fact is then used to derive the second inequality in (4.20) (see [10] for details).

Proof of the first inequality in (4.20). Our proof here is different from both [10, Lemma 4.1] and [3, Lemma 3]. We require a new argument to that of [10] as we used the first inequality in (4.20) to derive (4.23). The ideas in [3] work for the special case $p = 2$. In our general situation, we apply Theorem 1 in Tolksdorf [22] for the function v in (4.24). More precisely, let $x_0 \in \mathbb{R}^N$ be fixed such that $0 < |x_0| \le r_0$ and set $\rho := |x_0|/2$. We define $v = v_{x_0} : B_1 \to (0, \infty)$ by

$$v(y) := \frac{u(x_0 + \rho y)}{g(|x_0|)} \quad \text{for every } y \in B_1. \tag{4.24}$$

Since u satisfies (4.11), by using the formula for ∇v derived from (4.24), that is

$$\nabla v(y) = \frac{\rho}{g(|x_0|)}(\nabla u)(x_0 + \rho y) \quad \text{for } y \in B_1, \tag{4.25}$$

we obtain that v is a positive solution of the following equation

$$-\mathrm{div}\,(|\nabla v|^{p-2}\nabla v) + \tilde{B}(y, v, \nabla v) = 0 \quad \text{in } B_1,$$

where we define $\tilde{B}(y, v, \nabla v)$ to be

$$\tilde{B}(y, v, \nabla v) = -\frac{\rho\,\mathcal{A}'(|x_0 + \rho y|)}{\mathcal{A}(|x_0 + \rho y|)}|\nabla v|^{p-2}\frac{\nabla v(y)\cdot(x_0 + \rho y)}{|x_0 + \rho y|} + \rho^p\frac{b(x_0 + \rho y)\,h(v)}{\mathcal{A}(|x_0 + \rho y|)\,v^{p-1}}\,v^{p-1}.$$

Since $|x_0 + \rho y| \in [\rho, 3\rho]$ for all $y \in B_1$, in view of (1.2) and (4.13), we find that

$$|\tilde{B}(y, v, \nabla v)| \le A_1|\nabla v|^{p-1} + A_2\,v^{p-1} \tag{4.26}$$

for some positive constants A_1 and A_2, which depend on r_0, but are independent of x_0. Using the assumptions on g, namely g is regularly varying at 0, we obtain (similar to (4.10) in [10]) that

$$\underline{c}\,g(|x_0|) \le g(|x_0 + \rho y|) \le \overline{c}\,g(|x_0|) \quad \text{for all } y \in B_1,$$

where $\underline{c}$ and $\overline{c}$ are positive constants, which depend on r_0, but are independent of x_0 satisfying $0 < |x_0| < r_0$. Moreover, from (4.19) and (4.24), we deduce that

$$v(y) \le \overline{c}\, C_0 \quad \text{for every } y \in B_1.$$

Thus, in view of (4.26), we can find a positive constant $A_3 = A_3(r_0)$, which is independent of x_0 such that

$$|\tilde{B}(y, v, \eta)| \le A_3(1 + |\eta|)^p \quad \text{for all } y \in B_1 \text{ and } \eta \in \mathbb{R}^N.$$

Hence, we can apply Theorem 1 in Tolksdorf [22] to obtain a constant A_4, which depends on N, p and A_3, but is independent of x_0, such that $|\nabla v(0)| \le A_4$. This, jointly with (4.25), proves that

$$|\nabla u(x_0)| \le 2A_4 \frac{g(|x_0|)}{|x_0|} \quad \text{for every } 0 < |x_0| < r.$$

This completes the proof of Lemma 4.3. □

5. Proof of Theorem 1.2: existence and uniqueness

Let Assumptions $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ hold. Let h be non-decreasing on $[0, \infty)$ and $g \in C^1(\partial B_1)$ be a non-negative function. We study the existence of solutions for the following problem

$$\begin{cases} \operatorname{div}(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) = b(x)\,h(u) \quad \text{in } B^* := B_1 \setminus \{0\}, \\ \lim\limits_{|x|\to 0} \dfrac{u(x)}{\Phi(x)} = \lambda, \quad u\big|_{\partial B_1} = g, \quad u > 0 \quad \text{in } B^*. \end{cases} \tag{5.1}$$

We treat separately the following cases: $\lambda = 0$, $\lambda \in (0, \infty)$ and $\lambda = \infty$. For the construction of a solution of (5.1), we adapt ideas from [10, Theorem 1.2] (where $\mathcal{A} = 1$), see also [3, Proposition 5], where $p = 2$, $b = 1$ and $h(t) = t^q$. We denote $C_0 := \max_{|x|=1} g(x)$. For every $n \ge 2$ and $0 \le \lambda < \infty$, we consider the auxiliary problem

$$\begin{cases} \operatorname{div}(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) = b(x)\,h(u) \quad \text{in } D_n := B_1 \setminus \overline{B_{1/n}}, \\ u(x) = \lambda\,\Phi(|x|) + C_0 \quad \text{for } |x| = 1/n, \\ u\,\big|_{\partial B_1} = g. \end{cases} \tag{5.2}$$

For $\lambda = 0$, we further assume that $g \not\equiv 0$ on ∂B_1. By the method of sub-super-solutions and Lemma 2.2, the problem (5.2) admits a unique non-negative solution $u_{n,\lambda,g}$, which is continuous on $\overline{D_n}$ (see, for example, [14]). For simplicity, whenever λ and g are fixed, we simply write u_n instead of $u_{n,\lambda,g}$. By the strong maximum principle (see Theorem 2.5.1 of [18]), we see that u_n is positive in D_n. Moreover, by Lemma 2.2, we infer that

$$0 < u_{n+1} \le u_n \le \lambda\Phi(|x|) + C_0 \quad \text{in } D_n. \tag{5.3}$$

By Lemma 4.3, we have that, up to a subsequence, $u_n \to u_{\lambda,g}$ in $C^1_{\text{loc}}(B^*)$ and, moreover, for some $\alpha \in (0, 1)$, we find that $u_{\lambda,g}$ is a non-negative $C^{1,\alpha}_{\text{loc}}(B^*) \cap C(\overline{B_1} \setminus \{0\})$-solution of the problem

$$\begin{cases} \operatorname{div}(\mathcal{A}(|x|)\,|\nabla u|^{p-2}\nabla u) = b(x)\,h(u) \quad \text{in } B^* := B_1 \setminus \{0\}, \\ u\big|_{\partial B_1} = g. \end{cases} \tag{5.4}$$

By the strong maximum principle, $u_{\lambda,g}$ is positive in B^* (using here that $g \not\equiv 0$ on ∂B_1 when $\lambda = 0$). From (5.3), we find that $\limsup_{|x|\to 0} u_{\lambda,g}(x)/\Phi(|x|) \le \lambda$. In particular, the problem (5.1) with $\lambda = 0$ admits $u_{\lambda,g}$ as a solution.

Proof of Theorem 1.2(i). It remains to show the uniqueness of the solution of (5.1) with $\lambda = 0$. Let u_1 and u_2 be two solutions of (5.1) with $\lambda = 0$. To show that $u_1 = u_2$ in B^*, we proceed as in Proposition 4 in [3] with modifications appearing here due to our more general setting. By Lemma 3.1, u_1 and u_2 can be extended by continuity at 0. Since $u_1, u_2 \in C^1(B^*) \cap C(\overline{B_1})$ with $u_1 = u_2 = g$ on ∂B_1, then $u_1 = u_2$ in B_1 would be a consequence of the following claim.

Claim. *We have* $\nabla(u_1 - u_2)(x) = 0$ *for all* $x \in B^*$.

Proof of Claim. Assume by contradiction that there exists $x_0 \in B^*$ such that $|\nabla(u_1 - u_2)(x_0)| > 0$. We fix r_0 small such that $0 < r_0 < \min\{1 - |x_0|, |x_0|\}$, which ensures that $\overline{B_{r_0}(x_0)} \subset B^*$. Since $u_1 - u_2 \in C^1(B^*)$, by making r_0 smaller if necessary, we can assume that $|\nabla(u_1 - u_2)(x)| > 0$ on $\overline{B_{r_0}(x_0)}$ and thus $|\nabla u_1(x)| + |\nabla u_2(x)| > 0$ on $\overline{B_{r_0}(x_0)}$. Hence, there exists a positive constant c_0 such that

$$(|\nabla u_1(x)| + |\nabla u_2(x)|)^{p-2}\, |\nabla(u_1 - u_2)(x)|^2 \geq c_0 \quad \text{for all } x \in \overline{B_{r_0}(x_0)}. \tag{5.5}$$

By Proposition 17.3 in [8, p. 235], we know that there exists a positive constant c_p such that

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq c_p\, (|\xi| + |\eta|)^{p-2}\, |\xi - \eta|^2 \quad \text{for every } \xi, \eta \in \mathbb{R}^N. \tag{5.6}$$

Thus using (5.5) and (5.6), we find for all $x \in \overline{B_{r_0}(x_0)}$ that

$$\mathcal{H}(x) := (|\nabla u_1(x)|^{p-2}\nabla u_1(x) - |\nabla u_2(x)|^{p-2}\nabla u_2(x)) \cdot \nabla(u_1 - u_2)(x) \geq c_p c_0. \tag{5.7}$$

For any $\varepsilon \in (0, 1/2)$, we denote $D_\varepsilon := B_1 \setminus \overline{B_\varepsilon}$. Let w_ε be a non-decreasing and smooth function on $(0, \infty)$ such that

$$\begin{cases} w_\varepsilon(r) \in (0,1) & \text{if } \varepsilon < r < 2\varepsilon, \\ w_\varepsilon(r) = 1 & \text{if } r \geq 2\varepsilon, \\ w_\varepsilon(r) = 0 & \text{if } 0 < r \leq \varepsilon. \end{cases} \tag{5.8}$$

We choose $\varepsilon > 0$ small such that $2\varepsilon < |x_0| - r_0$, which yields that $\overline{B_{r_0}(x_0)} \subseteq D_{2\varepsilon} \subset D_\varepsilon$. Since $w_\varepsilon(|x|) = 1$ for all $x \in D_{2\varepsilon}$, by using (5.7), we arrive at

$$\int_{D_\varepsilon} w_\varepsilon(|x|)\, \mathcal{A}(|x|)\, \mathcal{H}(x)\, dx \geq \int_{B_{r_0}(x_0)} \mathcal{A}(|x|)\, \mathcal{H}(x)\, dx \geq c_p\, c_0\, \omega_N r_0^N \min_{x \in \overline{B_{r_0}(x_0)}} \mathcal{A}(|x|) := c_{p,\mathcal{A}}. \tag{5.9}$$

Since $\mathcal{A} \in C(0, 1]$ is a positive function and $\overline{B_{r_0}(x_0)} \subset B^*$, we then obtain that $c_{p,\mathcal{A}}$ is a positive constant.

Observe that u_1, u_2 and w_ε belong to $W^{1,p}(D_\varepsilon) \cap L^\infty(D_\varepsilon)$. We define $\varphi_\varepsilon(x) := (u_1 - u_2)(x)\, w_\varepsilon(|x|)$ for all $x \in B^*$. Since $\varphi_\varepsilon|_{\partial D_\varepsilon} = 0$, it follows by the product rule that $\varphi_\varepsilon \in W_0^{1,p}(D_\varepsilon)$. Using the density of $C_c^1(D_\varepsilon)$ in $W_0^{1,p}(D_\varepsilon)$, we have

$$\int_{D_\varepsilon} \mathcal{A}(|x|)\, |\nabla u_j|^{p-2}\nabla u_j \cdot \nabla\varphi_\varepsilon\, dx + \int_{D_\varepsilon} b(x)\, h(u_j)\, \varphi_\varepsilon\, dx = 0 \text{ with } j = 1, 2. \tag{5.10}$$

In particular, by subtracting the relation in (5.10) with $j = 2$ from the one corresponding to $j = 1$, we obtain that

$$\int_{D_\varepsilon} w_\varepsilon(|x|)\, \mathcal{A}(|x|)\, \mathcal{H}(x)\, dx + \int_{D_\varepsilon} b(x)\, (h(u_1) - h(u_2))\, (u_1 - u_2)\, w_\varepsilon(|x|)\, dx = -K_\varepsilon, \tag{5.11}$$

where $\mathcal{H}$ is given by (5.7) and K_ε is defined by

$$K_\varepsilon = \int_{\varepsilon < |x| < 2\varepsilon} |x|^\vartheta\, L_{\mathcal{A}}(|x|)\, w_\varepsilon'(|x|)\, (u_1 - u_2) \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right) \cdot \frac{x}{|x|}\, dx. \tag{5.12}$$

Since $w_\varepsilon(2\varepsilon) = 1$ and $w_\varepsilon(\varepsilon) = 0$ (see (5.8)), we observe that

$$\begin{aligned} \mathcal{L}_\varepsilon &:= \int_{\varepsilon < |x| < 2\varepsilon} |x|^{\vartheta - p + 1} L_{\mathcal{A}}(|x|)\, w_\varepsilon'(|x|)\, dx \\ &= |\partial B_1| \int_\varepsilon^{2\varepsilon} r^{\vartheta + N - p} L_{\mathcal{A}}(r)\, w_\varepsilon'(r)\, dr \leq |\partial B_1| \max_{r \in [\varepsilon, 2\varepsilon]} \{r^{\vartheta + N - p}\, L_{\mathcal{A}}(r)\}. \end{aligned}$$

Using that $\vartheta + N - p > 0$ and $L_{\mathcal{A}}$ is slowly varying at zero, we get that $\lim_{r \to 0^+} r^{\vartheta + N - p} L_{\mathcal{A}}(r) = 0$. (In relation to Remark 1.3, we note that if $\vartheta + N - p = 0$ and $\limsup_{r \to 0^+} L_{\mathcal{A}}(r) < \infty$, then we get that $\limsup_{\varepsilon \to 0^+} \mathcal{L}_\varepsilon \in (0, \infty)$.) Thus, using (5.12), jointly with $|x||\nabla u_j| \to 0$ as $|x| \to 0$ for $j = 1, 2$ (see Lemma 3.1), we find that

$$|K_\varepsilon| \le \left(\|u_1\|_{L^\infty(B_1)} + \|u_2\|_{L^\infty(B_1)}\right) \mathcal{L}_\varepsilon \max_{\varepsilon\le|x|\le 2\varepsilon} |x|^{p-1} \left(|\nabla u_1(x)|^{p-1} + |\nabla u_2(x)|^{p-1}\right) \to 0 \text{ as } \varepsilon \to 0^+.$$

Hence, we can fix $\varepsilon > 0$ small enough to ensure that $|K_\varepsilon| < c_{p,\mathcal{A}}$, where $c_{p,\mathcal{A}}$ is the positive constant appearing in (5.9). Since the second term in the left-hand side of (5.11) is non-negative, from (5.9) and (5.11), we get a contradiction. This proves the claim, which concludes the proof of the uniqueness of the solution of (5.1) with $\lambda = 0$. □

Proof of Theorem 1.2(ii). If (5.1) has a solution for $\lambda \in (0, \infty]$, then $b(x)\, h(\Phi) \in L^1(B_{1/2})$ from Theorem 1.1(b).

Claim 1. *If $b(x)\, h(\Phi) \in L^1(B_{1/2})$, then $u_{\lambda,g}$ constructed above for $\lambda \in (0, \infty)$ is a solution of* (5.1).

Proof of Claim 1. We need only show that $\liminf_{|x|\to 0} u_{\lambda,g}(x)/\Phi(|x|) \ge \lambda$. We note that (B.2) is equivalent to $\int^\infty t^{q-p+2}\Lambda(t)\, dt < \infty$, where Λ is defined by (3.4). Then, by [21, Theorem 2.4], if $R > 0$ is large, there exists a positive proper solution of the following problem

$$\begin{cases} \dfrac{d^2 y}{ds^2} = \Lambda(s)[y(s)]^{q-p+2} \ \text{ for } s \in (R, \infty), \\ y'(s) \to \lambda \ \text{ as } s \to \infty \ \text{ and } y(R) \in (0, \infty). \end{cases} \tag{5.13}$$

Using the transformation $w(r) = y(s)$ with $r = \Phi^{-1}(s)$ and Remark A.4, we obtain that

$$\begin{cases} \operatorname{div}(\mathcal{A}(|x|)\, |\nabla w|^{p-2}\nabla w) \sim b(x)\, h_2(w(|x|)) \ \text{ as } |x| \to 0^+, \\ w(r) \sim \lambda\Phi(r) \ \text{ as } r \to 0^+. \end{cases} \tag{5.14}$$

Hence, for every $\varepsilon \in (0, 1)$, there exists $r_\varepsilon \in (0, \Phi^{-1}(R))$ such that $(1-\varepsilon)\, w$ is a sub-solution of

$$\operatorname{div}(\mathcal{A}(|x|)\, |\nabla v|^{p-2}\nabla v) = b(x)\, h_2(v) \quad \text{in } B^*_{r_\varepsilon} := B_{r_\varepsilon} \setminus \{0\}. \tag{5.15}$$

Recall that $u_{n,\lambda,g}$, in short u_n, represents the unique non-negative solution of (5.2). Since $w(r) \sim \lambda\Phi(r)$ as $r \to 0^+$ (see (5.14)), there exists $n_\varepsilon \ge 1$ large such that

$$(1-\varepsilon)\, w(1/n) \le \lambda\, \Phi(1/n) \le u_n(x) \quad \text{for every } |x| = 1/n \text{ and all } n \ge n_\varepsilon.$$

Let $C_\varepsilon := \max_{r=r_\varepsilon} w(r)$. Since u_n is a positive super-solution of (5.15) due to our choice of h_2, by Lemma 2.2, we have

$$(1-\varepsilon)\, w \le u_n + C_\varepsilon \quad \text{for } 1/n < |x| < r_\varepsilon \ \text{ and all } n \ge n_\varepsilon.$$

By letting $n \to \infty$, we find that $(1-\varepsilon)\, w \le u_{\lambda,g} + C_\varepsilon$ in $B^*_{r_\varepsilon}$. Hence, we conclude that $\liminf_{|x|\to 0} u_{\lambda,g}(x)/\Phi(|x|) \ge (1-\varepsilon)\lambda$. Since $\varepsilon \in (0, 1)$ is arbitrary, we obtain that $\liminf_{|x|\to 0} u_{\lambda,g}(x)/\Phi(|x|) \ge \lambda$. Since $\limsup_{|x|\to 0} u_{\lambda,g}(x)/\Phi(|x|) \le \lambda$, it follows that $u_{\lambda,g}$ is a solution of (5.1) for $\lambda \in (0, \infty)$. □

Claim 2. *If $b(x)\, h(\Phi) \in L^1(B_{1/2})$, then there exists a solution of* (5.1) *with $\lambda = \infty$.*

Proof of Claim 2. Let k be any positive integer and denote by $u_{k,g}$ the solution we constructed earlier for (5.1) with λ replaced by k. Then, by the comparison principle (Lemma 2.2), we find that $0 < u_{k,g} \le u_{k+1,g}$ in B^*. We show that for every fixed $x \in B_1 \setminus \{0\}$, there exists $\lim_{k\to\infty} u_{k,g}(x) \in (0, \infty)$. Indeed, since $|x| > 0$, we can fix $\rho = \rho_x$ such that $0 < \rho < \min\{|x|, 1/4\}$. Hence, by Lemma 4.1, there exists $C_\rho > 0$ such that $u_{k,g}(y) \le C_\rho$ for all $|y| = \rho$ and every $k \ge 1$. By Lemma 2.2, it follows that $u_{k,g}(y) \le \max\{C_0, C_\rho\}$ for all $\rho \le |y| \le 1$ and all $k \ge 1$, where $C_0 = \max_{|x|=1} g(x)$. Hence, for all $x \in \overline{B_1} \setminus \{0\}$, we can define $u_{\infty,g}(x) := \lim_{k\to\infty} u_{k,g}(x)$. Moreover, by Lemma 4.3, we have that, up to a subsequence, $u_{k,g} \to u_{\infty,g}$ in $C^1_{\text{loc}}(B^*)$ and $u_{\infty,g}$ is a solution of (5.1) with $\lambda = \infty$. This concludes Claim 2 and the proof of Theorem 1.2(ii). □

Proof of Theorem 1.2(iii). Assume that $b(x)\, h(\Phi) \in L^1(B_{1/2})$ and $h(t)/t^{p-1}$ is non-decreasing for $t > 0$. We show the uniqueness of the solution of (5.1) in any of the following situations:

(A) $\lambda \in (0, \infty)$;
(B) $\lambda = \infty$ and $q < q_*$;
(C) $\lambda = \infty$ and $q = q_*$, assuming also that either (1.7) or (1.8) holds.

Indeed, if u_1 and u_2 are arbitrary solutions of (5.1) corresponding to the same λ and g, then $\lim_{|x|\to 0} u_1(x)/u_2(x) = 1$. This is evident in Case (A), while for the Cases (B) and (C), we use Theorem 1.1(a) to obtain the same asymptotic behaviour near zero for any positive solution of (1.1) with a strong singularity at 0. The uniqueness claim follows from Lemma 2.2 as in the proof of [10, Theorem 1.2]. This completes the proof of Theorem 1.2. □

Conflict of interest statement

There is no conflict of interest.

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Appendix A. Regular variation theory

The regular variation theory initiated by Karamata in the 1930's has been very fruitful in statistics in connection with extreme value theory (statistical estimation of tails, rates of convergence). It also plays a crucial role in probability theory (weak limit theorems such as central limit theorem and the weak law of large numbers; branching processes; stability and domains of attraction; fluctuation theory; renewal theory). The applications are much broader, including areas such as analytic number theory, financial engineering and complex analysis (see [2] for a comprehensive treatment of regular variation theory and its applications).

We recall below the concepts and properties of regularly varying functions needed in this paper, see [2,19,24].

Definition A.1 *(Regularly varying functions).*

(a) A positive measurable function L defined on a neighbourhood of ∞ is called *slowly varying* at ∞ if

$$\lim_{t\to\infty} \frac{L(\xi t)}{L(t)} = 1 \quad \text{for every } \xi > 0.$$

(b) The function $r \longmapsto L(r)$ is *slowly varying at (the right of) zero* if $t \longmapsto L(1/t)$ is slowly varying at ∞.

(c) A function f is *regularly varying at* ∞ (respectively, 0) with real index m, in short $f \in RV_m(\infty)$ (respectively, $f \in RV_m(0+)$) if $f(t)/t^m$ is slowly varying at ∞ (respectively, 0).

Example 1. Any positive constant function is trivially slowly varying at ∞. Other non-trivial examples of slowly varying functions at ∞ are given by:

(a) The logarithm $\ln t$, its iterates $\ln_n t$ (defined as $\ln \ln_{n-1} t$) and powers of $\ln_n t$ for any integer $n \ge 1$.

(b) $\exp\left(\frac{\ln t}{\ln\ln t}\right)$.

(c) $\exp((\ln t)^\nu)$ with $\nu \in (0, 1)$.

(d) $\exp\{(\ln t)^{1/3} \cos((\ln t)^{1/3})\}$.

Remark A.1. Note that $\lim_{t\to\infty} f(t) = \infty$ (respectively, 0) for any function $f \in RV_m(\infty)$ with $m > 0$ (respectively, $m < 0$). However, the limit at ∞ of a slowly varying function L at ∞ cannot be determined in general, and it may not even exist (see example (d) above for which $\liminf_{t\to\infty} L(t) = 0$ and $\limsup_{t\to\infty} L(t) = \infty$).

Proposition A.1 *(Uniform Convergence Theorem). If L is a slowly varying function at zero, then $L(\xi t)/L(t) \to 1$ as $t \to 0$, uniformly on each compact ξ-set in $(0, \infty)$.*

Theorem A.2 *(Representation Theorem). The function L is slowly varying at 0 if and only if we have*

$$L(t) = \eta(t) \exp\left(\int_t^c \frac{\varepsilon(r)}{r} dr \right), \qquad 0 < t \le c$$

for some $c > 0$*, where* η *is a measurable function on* $(0, c]$ *satisfying* $\lim_{t\to 0^+} \eta(t) = \eta \in (0, \infty)$ *and* ε *is a continuous function on* $(0, c]$ *such that* $\lim_{t\to 0^+} \varepsilon(t) = 0$.

Remark A.2. If $\eta(t)$ is replaced by a positive constant η, then the new function η is referred to as a *normalised slowly varying function*. In this case, $\varepsilon(t) = -tL'(t)/L(t)$ for $0 < t \le c$. Conversely, any function $\tilde{L} \in C^1(0, c]$, which is positive and satisfies $\lim_{t\to 0^+} t\tilde{L}'(t)/\tilde{L}(t) = 0$, is a normalised slowly varying function.

Remark A.3. Any slowly varying function at zero is asymptotically equivalent to a normalised slowly varying one.

Theorem A.3 *(Karamata's Theorem at* 0*). Let* f *vary regularly at zero with index* ρ *and be locally bounded on* $(0, c]$*. The following assertions hold:*

(a) *For any* $j \le -(\rho + 1)$*, we have*

$$\lim_{t\to 0^+} \frac{t^{j+1} f(t)}{\int_t^c r^j f(r)\, dr} = -(j + \rho + 1);$$

(b) *For any* $j > -(\rho + 1)$ *(and for* $j = -(\rho + 1)$ *if* $\int_{0^+} r^{-\rho-1} f(r)\, dr < +\infty$*), we have*

$$\lim_{t\to 0^+} \frac{t^{j+1} f(t)}{\int_0^t r^j f(r)\, dr} = j + \rho + 1.$$

Proposition A.4 *(Karamata's Theorem at* ∞*). If* $f \in RV_\rho(\infty)$ *is locally bounded in* $[A, \infty)$*, then*

(a) *For any* $j \ge -(\rho + 1)$*, we have*

$$\lim_{t\to\infty} \frac{t^{j+1} f(t)}{\int_A^t \xi^j f(\xi)\, d\xi} = j + \rho + 1.$$

(b) *For any* $j < -(\rho + 1)$ *(and for* $j = -(\rho + 1)$ *if* $\int^\infty \xi^{-(\rho+1)} f(\xi)\, d\xi < \infty$*), we have*

$$\lim_{t\to\infty} \frac{t^{j+1} f(t)}{\int_t^\infty \xi^j f(\xi)\, d\xi} = -(j + \rho + 1).$$

As in [19], we denote by $f^{\leftarrow}$ the (left continuous) inverse of a non-decreasing function f on $\mathbb{R}$, namely

$$f^{\leftarrow}(t) = \inf\{s : \ f(s) \ge t\}.$$

Proposition A.5 *(see Proposition 0.8 in [19]). We have*

1. *If* $f \in RV_\rho(\infty)$*, then* $\lim_{t\to\infty} \ln f(t)/\ln t = \rho$.
2. *If* $f_1 \in RV_{\rho_1}(\infty)$ *and* $f_2 \in RV_{\rho_2}(\infty)$ *with* $\lim_{t\to\infty} f_2(t) = \infty$*, then*
$$f_1 \circ f_2 \in RV_{\rho_1\rho_2}.$$
3. *Suppose* f *is non-decreasing,* $f(\infty) = \infty$*, and* $f \in RV_\rho(\infty)$ *with* $0 < \rho < \infty$*. Then*
$$f^{\leftarrow} \in RV_{1/\rho}(\infty).$$

Remark A.4. If $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ hold, then by [10, Lemma A.7], there exist continuous functions h_1 and h_2 on $[0, \infty)$, positive on $(0, \infty)$ with $h_1(0) = h_2(0) = 0$ such that

$$\begin{cases} h_1(t) \le h(t) \le h_2(t) \quad \text{for } t \in [0, \infty), \\ h_1(t)/t^{p-1} \text{ and } h_2(t)/t^{p-1} \text{ are both increasing for } t \in (0, \infty), \\ h_1(t) \sim h_2(t) \sim h(t) \ \text{ as } t \to \infty. \end{cases} \tag{A.1}$$

Therefore, without loss of generality, we can assume that $t \longmapsto t^{q-p+1} L_h(t)$ is increasing on $(0,\infty)$ so that $t^q L_h(t)$ is non-decreasing on $(0,\infty)$. Moreover, as in [9, Section 1.2.4], we can take $L_h \in C^2[t_0,\infty)$ and $L_b \in C^2(0,r_0]$ for some large constant $t_0 > 0$ and $r_0 \in (0,1)$ such that

$$\lim_{t\to\infty} \frac{tL_h'(t)}{L_h(t)} = \lim_{t\to\infty} \frac{t^2 L_h''(t)}{L_h(t)} = 0, \quad \lim_{r\to 0^+} \frac{rL_b'(r)}{L_b(r)} = \lim_{r\to 0^+} \frac{r^2 L_b''(r)}{L_b(r)} = 0. \tag{A.2}$$

Appendix B. Applications

Our first application illustrates how *weighted* divergence-form equations such as (1.1) arise naturally in the study of p-Laplacian type equations in *exterior* domains.

Corollary B.1. *Assuming $2 \le N \le p < a$ and $q > p-1$, we consider the problem*

$$\operatorname{div}\left(|\nabla v(\tilde{x})|^{p-2}\nabla v(\tilde{x})\right) = |\tilde{x}|^{-a}[v(\tilde{x})]^q \quad \text{in } \mathbb{R}^N \setminus \overline{B_1}. \tag{B.1}$$

By a modified Kelvin transform where $u(x) = v(\tilde{x})$ with $x = \tilde{x}/|\tilde{x}|^2$ (see [11, Appendix A]), the behaviour near ∞ of the positive solutions of (B.1) can be obtained from the behaviour near 0 of the positive solutions of (1.1) with $\mathcal{A}(x) = |x|^{2(p-N)}$, $b(x) = |x|^{a-2N}$ and $h(u) = [u(x)]^q$. Hence, by applying our Theorem 1.1, we find that:

(1) *If $p > N$, then the following classification holds for the positive solutions $v(\tilde{x})$ of (B.1):*
  (a) *If $q < \frac{(a-N)(p-1)}{p-N}$, then as $|\tilde{x}| \to \infty$, exactly one of the following holds*
    (i) *$v(\tilde{x})$ converges to a positive number;*
    (ii) *$|\tilde{x}|^{-\frac{p-N}{p-1}} v(\tilde{x})$ converges to a positive number;*
    (iii) $|\tilde{x}|^{-(a-p)/(q-p+1)} v(\tilde{x}) \to \left[\left(\frac{a-p}{q-p+1}\right)^{p-1}\left(\frac{-pq+ap-a}{q-p+1} - N\right)\right]^{1/(q-p+1)}$.
  (b) *If, in turn, $q \ge \frac{(a-N)(p-1)}{p-N}$, then for every positive solution of (B.1), only (i) holds.*

(2) *If $p = N$, then for all $q > p-1$, only (1)(a) holds in which (ii) should read as $\lim_{|\tilde{x}|\to\infty} v(\tilde{x})/\ln(|\tilde{x}|) \in (0,\infty)$.*

For readers with specific examples of $\mathcal{A}$, b and h in mind, we supply below the sharp condition (B.2) for which Theorem 1.1 can be applied. From a practical point of view, $\tilde{u}$ in (1.12) can be rewritten asymptotically as in (B.4), provided that (1.7) holds. In Example 1, we choose the prototype model satisfying (1.7) introduced in Remark B.2 and give in Example 2 a special example where (1.8) is satisfied.

Remark B.1. Assuming $(\mathbf{A}_1)$–$(\mathbf{A}_3)$, we note that $b(x)\,h(\Phi) \in L^1(B_{1/2})$ is equivalent to

$$\int_{0^+} r^{N-1+\sigma} L_b(r)\,h(\Phi(r))\,\mathrm{d}r < \infty. \tag{B.2}$$

The integrand in (B.2) varies regularly at 0 with index $N-1+\sigma-m_2 q$. Hence, if $q \ne q_*$ then (B.2) holds if and only if $q < q_*$, where q_* is given by (1.6). If $q = q_*$, then (B.2) may hold in some cases and fail in others. For example, if $L_{\mathcal{A}} = L_b = 1$ and $h(t) = t^{q_*}(\ln t)^\alpha$ for $t > 0$ large, then (B.2) holds if and only if $\alpha < -1$.

Remark B.2. A prototype model for (1.7) is $L_h(t) \sim (\ln t)^\gamma$ as $t \to \infty$, where $\gamma \in \mathbb{R}$. More generally, (1.7) holds if $L_h(T) \sim \mathcal{L}(T)$ as $T \to \infty$ and $\mathcal{L}(T) = \Pi_{i=1}^k (\ln_{m_i} T)^{\beta_i}$ for $T > 0$ large, where k and m_i are positive integers and $\beta_i \in \mathbb{R}$ for every $1 \le i \le k$. We use the notation $\ln_{m_i}$ for the m_i-iterated natural logarithm. Without loss of generality, we can take $1 \le m_1 < m_2 < \ldots < m_k$. Then $t \longmapsto L_h(e^t)$ is regularly varying at ∞ with index equal to β_1 (respectively, 0) if $m_1 = 1$ (respectively, $m_1 > 1$). Similarly, (1.8) is verified if $\left[L_{\mathcal{A}}(1/T)\right]^{-\frac{q}{p-1}} L_b(1/T) \sim \mathcal{L}(T)$ as $T \to \infty$.

In Theorem 1.1 for $q < q_*$, the function $\tilde{u}$ in (1.12) is well-defined, regularly varying at 0 with index $-m_0$ and

$$\tilde{u}(r)\left[L_h(\tilde{u}(r))\right]^{\frac{1}{q-p+1}} \sim \left[\frac{m_0^p}{M}\frac{L_{\mathcal{A}}(r)}{L_b(r)}\right]^{\frac{1}{q-p+1}} r^{-m_0} \quad \text{as } r \to 0^+. \tag{B.3}$$

Indeed, the integral in the left-hand side of (1.12) is well-defined since the integrand is regularly varying at ∞ with index $-(q+1)/p < -1$ from the assumption $q > p-1$. The right-hand side of (1.12) also exists since the integrand is regularly varying at 0^+ with index $(\sigma - \vartheta)/p > -1$ by virtue of $\sigma > \vartheta - p$. By Karamata's Theorem in Appendix A, (1.12) implies (B.3). Furthermore, if (1.7) holds, then $L_h(\tilde{u}(r)) \sim m_0^\gamma L_h(1/r)$ as $r \to 0^+$ so that (B.3) is refined by

$$\tilde{u}(r) \sim \left[\frac{m_0^{p-\gamma}}{M} \frac{L_{\mathcal{A}}(r)}{L_h(1/r)\, L_b(r)} \right]^{\frac{1}{q-p+1}} r^{-m_0} \quad \text{as } r \to 0^+. \tag{B.4}$$

Corollary B.2. *Let Assumptions* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *hold. Let* $\alpha, \beta, \gamma \in \mathbb{R}$ *and* $\nu \in (0, 1/2)$ *be arbitrary.*

Example	$L_{\mathcal{A}}(r)$ *as* $r \to 0^+$	$L_b(r)$ *as* $r \to 0^+$	$L_h(t)$ *as* $t \to \infty$
1	$\left(\ln \frac{1}{r}\right)^\alpha$	$\left(\ln \frac{1}{r}\right)^\beta$	$(\ln t)^\gamma$
2	$\left(\ln \frac{1}{r}\right)^\alpha \exp\left\{-\frac{p-1}{q}\sqrt{\ln \frac{1}{r}}\right\}$	$\exp\left\{-\sqrt{\ln \frac{1}{r}}\right\}$	$\exp\left\{-(\ln t)^\nu\right\}$

(A) *If* $q < q_*$*, then for any solution* $u > 0$ *of* (1.1)*, either* Theorem 1.1*(a)(i) holds or* u *satisfies near zero*

$$u(x) \sim \begin{cases} |x|^{-m_0} \left[\dfrac{m_0^{p-\gamma}}{M} \left(\ln \dfrac{1}{|x|} \right)^{\alpha-\beta-\gamma} \right]^{\frac{1}{q-p+1}} & \text{in Example 1,} \\ |x|^{-m_0} \left[\dfrac{m_0^{p}}{M} \left(\ln \dfrac{1}{|x|} \right)^{\alpha} \right]^{\frac{1}{q-p+1}} \exp\left\{ \dfrac{1}{q} \left(\ln \dfrac{1}{|x|} \right)^{\frac{1}{2}} + \dfrac{1}{q-p+1} \left(m_0 \ln \dfrac{1}{|x|} \right)^{\nu} \right\} & \text{in Example 2.} \end{cases}$$

(B) *If* $q = q_*$ *(and, in addition,* $\alpha q_*/(p-1) > \beta + \gamma + 1$ *for Example 1), either* Theorem 1.1*(a)(i) holds or near zero*

$$u(x) \sim \begin{cases} |x|^{-m_0} \left[\dfrac{m_0^{p-1-\gamma} \left(\frac{\alpha q_*}{p-1} - \beta - \gamma - 1 \right)}{m_1} \left(\ln \dfrac{1}{|x|} \right)^{\alpha-\beta-\gamma-1} \right]^{\frac{1}{q-p+1}} & \text{in Example 1,} \\ |x|^{-m_0} \left[\dfrac{\nu\, m_0^{p-1+\nu}}{m_1} \left(\ln \dfrac{1}{|x|} \right)^{\alpha+\nu-1} \right]^{\frac{1}{q-p+1}} \exp\left\{ \dfrac{1}{q} \left(\ln \dfrac{1}{|x|} \right)^{\frac{1}{2}} + \dfrac{1}{q-p+1} \left(m_0 \ln \dfrac{1}{|x|} \right)^{\nu} \right\} & \text{in Example 2.} \end{cases}$$

(C) *If* $q > q_*$*, then any positive solution* u *of* (1.1) *can be extended as a positive continuous solution of* (1.1) *in* B_1*. For Example 1, this conclusion also holds for* $q = q_*$ *and* $\alpha q_*/(p-1) \le \beta + \gamma + 1$.

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