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ALMOST CONTINUOUS SOLUTIONS OF GEOMETRIC HAMILTON–JACOBI EQUATIONS ☆

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ABSTRACT. – We study the Hamilton–Jacobi equation

$$u_t + H(x, Du) = 0$$

in $\mathbb{R}^N \times]0, +\infty[$, where H is a continuous positively homogeneous Hamiltonian with constant sign and verifying suitable assumptions but no convexity properties. We look for discontinuous (viscosity) solutions verifying certain initial conditions with discontinuous data. Our aim is to give representation formulae as well as uniqueness and stability results.

We find that the condition

$$(u^\#)_\# = u_\# \quad \text{and} \quad (u_\#)^\# = u^\#$$

where $u^\#$ ($u_\#$) denotes the upper (lower) semicontinuous envelope of u, can be used as a uniqueness criterion and determines a class of solutions defined and continuous on certain dense subsets of $\mathbb{R}^N \times]0, +\infty[$ that we call almost continuous.

Such solutions can be represented by a formula which is a generalization of the Lax–Hopf one for the eikonal equation.

MSC: 35F20; 49L25; 35C99

Keywords: Hamilton–Jacobi equations; Discontinuous viscosity solutions; Representation formulae

RÉSUMÉ. – Nous étudions l'équation de Hamilton–Jacobi

$$u_t + H(x, Du) = 0$$

en $\mathbb{R}^N \times]0, +\infty[$, où H est un Hamiltomien continu et positivement homogène qui ne change pas de signe et qui ne vérifie aucune hypothèse de convexité. On cherche des solutions de viscosité discontinues qui vérifient certaines conditions initiales avec des données discontinues. Le but est de donner des formules de représentation aussi bien que des résultats d'unicité et de stabilité.

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Nous prouvons que la condition

$$(u^\#)_\# = u_\# \quad \text{et} \quad (u_\#)^\# = u^\#$$

oú $u^\#$ ($u_\#$) est l'enveloppe s.c.s. (s.c.i.) de u, peut être utilisée comme un critère d'unicité et détermine une classe de solutions définies et continues sur des ensembles denses de $\mathbb{R}^N \times]0, +\infty[$ que nous appellons presque continues.

Nous représentons ces solutions à l'aide d'une formule qui généralise celle de Lax–Hopf pour l'équation eiconale.

Introduction

We study the Hamilton–Jacobi time dependent equation

$$u_t + H(x, Du) = 0 \tag{I}$$

in $\mathbb{R}^N \times]0, +\infty[$, where H has constant sign and is homogeneous in the second argument but does not verify any convexity or uniform continuity assumptions with respect to the state variable.

We are interested in discontinuous viscosity solutions verifying initial conditions with discontinuous data. More precisely we look for locally bounded solutions u verifying

$$\limsup_{(x,t)\to(x_0,0)} u^\#(x,t) \leqslant u_0^\#(x_0), \qquad \liminf_{(x,t)\to(x_0,0)} u_\#(x,t) \geqslant u_{0\#}(x_0) \tag{II}$$

for any $x_0 \in \mathbb{R}^N$, where u_0 is a locally bounded initial datum and $u^\#$ ($u_\#$) denotes the upper (lower) semicontinuous envelope of u. Our aim is to give representation formulae as well as uniqueness (in a sense that will be specified later) and stability results. It is clear that the usual comparison principles between u.s.c. subsolutions and l.s.c. supersolutions of (I) are useless for such a purpose.

In the case where H is convex (concave) the definition of viscosity solution has been adapted in [6], see also [11], for l.s.c. (u.s.c.) function; these solutions are now called bilateral, see [1]. Then a comparison principle for the equation (I) coupled with the initial condition $\liminf_{(x,t)\to(x_0,0)} u(x,t) = u_0(x_0)$ ($\limsup_{(x,t)\to(x_0,0)} u(x,t) = u_0(x_0)$) has been established for l.s.c (u.s.c.) bilateral solutions and l.s.c. (u.s.c.) initial data . However the application of this theory is strictly confined to the convex case and cannot be generalized skipping such an assumption.

Several attempts have been made to give new definitions of solutions or to select some special (sub, super) viscosity solution in order to recover uniqueness and stability in a more general setting. In this line of research there are notions as envelope and minimax solutions (see Bardi's survey in [2]), and more recently of L-solutions given applying the level set method to the evolution of certain hypographs, see [12].

We follow the approach of [3] where suitable relations between l.s.c. and u.s.c. envelopes of solutions are required in order to obtain uniqueness results.

In [4] it has been investigated how in the convex case the definition of bilateral solution is related to the usual viscosity one for discontinuous functions clarifying that they are in a sense equivalent if a function u verifies

$$(u^{\#})_{\#} = u_{\#} \quad \text{and} \quad (u_{\#})^{\#} = u^{\#}. \tag{III}$$

If the convexity assumption is removed, however, it has been pointed out that essential nonuniqueness phenomena can appear in the sense that two solutions verifying the same initial condition and (III) may have different lower and upper semicontinuous envelopes.

The main achievement of this paper is to show that under our assumptions, the condition (III) instead can be used as a uniqueness criterion.

It is clear that this must be understood in a generalized sense since the very definition of viscosity solutions in the discontinuous case forces to identify functions with the same lower and upper semicontinuous envelopes.

We introduce as in [17] the equivalence relation which express this identification and consider the space of these equivalence classes. It turns out that for functions verifying (III) the structure of these classes is particularly simple and it is a consequence of Baire's theorem that two equivalent functions coincide on a residual set where they are continuous, see also [3] for related results. We show that these classes of functions can be characterized knowing the values attained on this residual set.

This justifies the definition of almost continuous functions as those which are defined and continuous on dense subsets and that cannot be extended continuously. We explain in Remark 4.1 the choice of this terminology.

The uniqueness results are based on some representation formulae of the maximal u.s.c. subsolution of (I) and (the first inequality of) (II) and the minimal l.s.c. supersolution of (I) and (the second inequality of) (II). They are obtained adapting similar ones given in [19] for the continuous case and can be viewed as a generalization of the Lax–Hopf formula for the eikonal equation. In the stability result we use a convergence naturally related to the class of almost continuous functions.

The paper is organized as follows:

In the first section the problem we deal with is stated with all the assumptions. The definition of viscosity solution is then rephrased, thanks to the geometric character of the Hamiltonian, using the perpendiculars to some level sets. Finally it is introduced a distance on $\mathbb{R}^N$ on which the representation formulae are based.

In Section 2 the Hamiltonian is assumed convex (concave) in p and it is proved the equivalence between bilateral solutions and solutions verifying (III) with different techniques with respect to [4]. The representation formulae are presented in Section 3. Section 4 is devoted to illustrate the definition and the basic properties of almost continuous functions. Finally the uniqueness and stability results are given in Section 5.

1. Statement of the problem and preliminary results

We study the time dependent Hamilton–Jacobi equation

$$u_t + H(x, Du) = 0 \tag{1.1}$$

in $\mathbb{R}^N \times \,]0, +\infty[$ coupled with initial conditions we will specify later and initial data which will not be taken continuous but only locally bounded.

We assume the following conditions on H:

$$H \text{ is continuous in } (x, p), \tag{1.2}$$

$$H(x, \lambda p) = \lambda H(x, p), \tag{1.3}$$

for any (x, p) and $\lambda \geqslant 0$.

$$H(x, p) > 0 \quad (< 0) \tag{1.4}$$

for any x and for $p \neq 0$

$$\bigl|H(x, p_1) - H(x, p_2)\bigr| \leqslant \bigl(a|x| + b\bigr)|p_1 - p_2| \tag{1.5}$$

for any x, p_1, p_2 and suitable positive constants a, b.

By the assumption (1.3), (1.1) belongs to the class of the so-called geometric equations, see [7].

We fix some notation and terminology. We recall that a general treatment of viscosity solutions and all the basic definitions we will use in the paper can be found in [1,5].

Given an element z, a subset K of an Euclidean space and a positive number r, we denote by $B(z, r)$ the Euclidean ball with radius r centered at z and by K^c the complementary set of K. We define the distance and the signed distance from K through the formulae:

$$d(z, K) = \inf_K |z - y|,$$

$$d^\#(z, K) = d(z, K) - d\bigl(z, K^c\bigr).$$

The expression φ is supertangent (subtangent) to f at a certain point z_0 for φ continuous and f u.s.c. (l.s.c.) will mean that z_0 is a local maximizer (minimizer) of $f - \varphi$.

Given a locally bounded function f, $f^\#$ ($f_\#$) will stand for its upper (lower) semicontinuous envelope.

In the whole paper we will call (sub, super) solution of (1.1) a locally bounded viscosity (sub, super) solution, see [13,1,7]. In the case where H is convex or concave in p we will consider l.s.c. or u.s.c. viscosity solutions in the sense of Barron and Jensen, see [6,11,4] we will refer to it as bilateral solutions.

We emphasize that all the (sub, super) solution of (1.1) we will consider are defined in $\mathbb{R}^N \times]0, +\infty[$, consequently the level sets of it are subsets of $\mathbb{R}^N \times]0, +\infty[$ and the boundaries and the closures of such level sets are with respect to the relative topology of $\mathbb{R}^N \times]0, +\infty[$. Moreover all the arguments of the type (x, t), (y, s) we will write in the remainder of the paper are understood to have the time component positive unless otherwise specified.

Thanks to the geometric character of (1.1) the previous notions of solutions can be restated using the notion of perpendicular (see [8]) to certain level sets.

DEFINITION 1.1. – *Given a closed set K of an Euclidean space and $z_0 \in K$, we say that a nonvanishing vector p is perpendicular to K ($p \perp K$) provided there exists $\varepsilon > 0$ such that*

$$\text{proj}_K(z_0 + \varepsilon p) \ni z_0$$

where for any z

$$\mathrm{proj}_K(z) = \{y \in K\colon |y - z| = d(z, K)\}.$$

The following characterizations hold, see [19]:

PROPOSITION 1.1. – *Let* f *be a locally bounded function is* $\mathbb{R}^N \times]0, +\infty[$*. The two following assertions are equivalent*:

(i) f *is a sub* (*super*) *solution of* (1.1).

(ii) *For any* $\alpha \in \mathbb{R}$, $(x_0, t_0) \in \{f^\# \geqslant \alpha\}$ $(\{f_\# \leqslant \alpha\})$, $(p, s) \perp \{f^\# \geqslant \alpha\}$ $(\{f_\# \leqslant \alpha\})$ *at* (x_0, t_0) *one has*

$$-s + H(x_0, -p) \leqslant 0 \quad \big(s + H(x_0, p) \geqslant 0\big).$$

PROPOSITION 1.2. – *Assume* H *to be convex* (*concave*) *in* p *and let* f *be a l.s.c.* (*u.s.c.*) *function defined in* $\mathbb{R}^N \times]0, +\infty[$*. The following two assertions are equivalent*:

(i) f *is a bilateral solution.*

(ii) *For any* $\alpha \in \mathbb{R}$, $(x_0, t_0) \in \{f \leqslant \alpha\}$ $(\{f \geqslant \alpha\})$, $(p, s) \perp \{f \leqslant \alpha\}$ $(\{f \geqslant \alpha\})$ *at* (x_0, t_0) *one has*

$$s + H(x_0, p) = 0 \quad \big(-s + H(x_0, -p) = 0\big).$$

The next result shows that a subsolution f can be also tested using the perpendiculars to the level sets $\mathrm{cl}\{f > \alpha\}$.

COROLLARY 1.1. – f *is a subsolution of* (1.1) *if and only if for any* $\alpha \in \mathbb{R}$, $(x_0, t_0) \in \mathrm{cl}\{f > \alpha\}$, $(p_0, s_0) \perp \mathrm{cl}\{f > \alpha\}$ *at* (x_0, t_0) *it results*

$$-s_0 + H(x_0, -p_0) \leqslant 0. \tag{1.6}$$

Proof. – Assume f to verify (1.6) and observe that by the very definition of u.s.c. envelope

$$\mathrm{cl}\{f > \alpha\} = \mathrm{cl}\{f^\# > \alpha\} \quad \text{for any } \alpha.$$

Fix $\alpha \in \mathbb{R}$, $(x_0, t_0) \in \{f^\# \geqslant \alpha\}$, $(p_0, s_0) \perp \{f^\# \geqslant \alpha\}$ at (x_0, t_0) and $\varepsilon > 0$ such that $(x_0 + \varepsilon p_0, t_0 + \varepsilon s_0)$ has (x_0, t_0) as unique projection on $\{f^\# \geqslant \alpha\}$.

Put

$$\beta = f^\#(x_0 + \varepsilon p_0, t_0 + \varepsilon s_0) < \alpha$$

and consider a sequence α_n contained in $]\beta, \alpha[$ and converging to α.

Note that

$$\{f^\# \geqslant \alpha\} \subseteq \mathrm{cl}\{f > \alpha_n\} \quad \text{for any } n \tag{1.7}$$

and any limit point (x, t) of sequences (x_n, t_n) with $(x_n, t_n) \in \mathrm{cl}\{f > \alpha_n\}$ for any n verifies by the upper semicontinuity of $f^\#$

$$(x, t) \in \{f^\# \geqslant \alpha\}. \tag{1.8}$$

Select

$$(x_n, t_n) \in \mathrm{proj}_{\mathrm{cl}\{f>\alpha_n\}}(x_0 + \varepsilon p_0, t_0 + \varepsilon s_0)$$

and exploit (1.7), (1.8) to find

$$(x_0, t_0) = \lim_n (x_n, t_n). \tag{1.9}$$

By (1.6)

$$t_n - t_0 - \varepsilon s_0 + H(x_n, x_n - x_0 - \varepsilon p_0) \leqslant 0$$

then pass to the limit for n going to infinity and take in account (1.9) and the continuity of H to get

$$-s_0 + H(x_0, -p_0) \leqslant 0$$

which shows that f is a subsolution of (1.1) by Proposition 1.1.

The converse implication can be obtained arguing similarly. Then the proof is complete. □

We now give some estimates on the distance function from certain level sets of a subsolution and a bilateral solution of (1.1) exploiting the assumption (1.3). For simplicity we assume H nonnegative in the first proposition and convex in the second. Similar results hold with suitable modifications for H nonpositive or concave.

PROPOSITION 1.3. – *Assume H nonnegative.*

Let f be a subsolution of (1.1)*, α a constant with $\{f > \alpha\} \neq \emptyset$ and K a bounded subset of* $\mathbb{R}^N \times]0, +\infty[$.

If $T > \sup\{t\colon (x,t) \in K\}$ there is $R > 0$ verifying

$$d\big((x_0, t), \mathrm{cl}\{f > \beta\}\big) \geqslant \frac{R}{(R+1)^{1/2}}(t - t_0)$$

for any $\beta \leqslant \alpha$, $(x_0, t_0) \in K \cap \mathrm{int}\{f \leqslant \beta\}$, $t \in [t_0, T]$.

Proof. – Let β, (x_0, t_0), T be as in the statement.

Set

$$h(t) = d\big((x_0, t), \mathrm{cl}\{f > \beta\}\big)$$

and consider $t_1 \in \mathrm{dom}\, h' \cap]t_0, T]$ with $(x_0, t_1) \in \mathrm{int}\{f \leqslant \beta\}$.

Assume

$$h(t_1) < t_1 = d\big((x_0, t_1), \{t = 0\}\big)$$

so that $\mathrm{proj}_{\mathrm{cl}\{f>\beta\}}(x_0, t_1)$ is nonempty and an element (y_0, s_0) can be selected in it.

The function

$$\sigma(t) = \big(|x_0 - y_0|^2 + |t - s_0|^2\big)^{1/2}$$

is supertangent to h at t_1 and consequently

$$h'(t_1) = \sigma'(t_1) = \frac{1}{h(t_1)}(t_1 - s_0). \tag{1.10}$$

Since $((x_0 - y_0), (t_1 - s_0)) \perp \mathrm{cl}\{f > \beta\}$ at (y_0, s_0) it results by Corollary 1.1

$$H(y_0, y_0 - x_0) \leqslant (t_1 - s_0), \tag{1.11}$$

and so

$$t_1 - s_0 > 0$$

and

$$h'(t_1) > 0$$

which shows that

$$(x_0, t) \in \mathrm{int}\{f \leqslant \beta\}$$

for any $t \in [t_0, T]$.

Set

$$\begin{aligned}
K_1 &= \{x\colon\ (x, s) \in K \text{ for some } s\} \times]0, T],\\
r &= \sup\{d\big((x, t), \mathrm{cl}\{f > \alpha\}\big)\colon\ (x, t) \in K_1\},\\
r_0 &= \sup\{|x|\colon\ (x, s) \in K \text{ for some } s\},\\
R &= \min\{H(y, p)\colon\ |y| \leqslant r + r_0,\ |p| = 1\},
\end{aligned}$$

and get from (1.11)

$$|y_0 - x_0| \leqslant \frac{t_1 - s_0}{R}.$$

Plug this inequality in (1.10) to discover

$$h'(t_1) \geqslant \frac{t_1 - s_0}{(1 + \frac{1}{R^2})^{1/2}(t_1 - s_0)} = \frac{R}{(R^2 + 1)^{1/2}}. \tag{1.12}$$

Fix $t \in [t_0, T]$ and put

$$\begin{aligned}
I &= \{s \in [t_0, t]\colon\ h(s) \geqslant s\},\\
t_1 &= \begin{cases} \max I & \text{if } I \neq \emptyset,\\ t_0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Finally use (1.12) to obtain for $t \in [t_0, T]$

$$h(t) = h(t_1) + \int_{t_1}^{t} h'\,\mathrm{d}s \geqslant t_1 + \frac{R}{(R^2 + 1)^{1/2}}(t - t_1) \geqslant \frac{R}{(R^2 + 1)^{1/2}}(t - t_0). \qquad \square$$

PROPOSITION 1.4. – *Assume H convex.*

Let f be a bilateral solution of (1.1)*, α a constant with $\{f \leqslant \alpha\} \neq \emptyset$ and K a bounded subset of $\mathbb{R}^N \times]0, +\infty[$ verifying*

$$d\big((x, t), \{f \leqslant \alpha\}\big) < t \quad \textit{for any } (x, t) \in K. \tag{1.13}$$

If $T > \sup\{t\colon (x,t) \in K\}$ *there is* $R > 0$ *such that*

$$d\big((x_0,t),\{f \leqslant \alpha\}\big) \leqslant \max\left(d\big((x_0,t_0),\{f \leqslant \alpha\}\big) - \frac{R}{(R+1)^{1/2}}(t-t_0), 0\right)$$

for any $(x_0,t_0) \in K$*,* $t \in [t_0,T]$.

Proof. – Given $(x_0,t_0) \in K$ define

$$h(t) = d\big((x_0,t),\{f \leqslant \alpha\}\big)$$

and consider $t_1 \in \operatorname{dom} h' \cap [t_0,T]$ with $h(t_1) > 0$. By (1.13) $h(t_1) < t_1$ and so an element (y_0,s_0) can be selected in $\operatorname{proj}_{\{f \leqslant \alpha\}}(x_0,t_1)$, hence

$$h'(t_1) = \frac{1}{h(t_1)}(t_1 - s_0).$$

Since $((x_0-y_0),(t_1-s_0)) \perp \{f \leqslant \alpha\}$ at (y_0,s_0) it results by Proposition 1.2

$$H(y_0, x_0-y_0) = s_0 - t_1$$

and so $h'(t_1) < 0$.

Therefore arguing as in Proposition 1.3 a positive constant R depending only on H and K can be determined so that

$$|x_0 - y_0| \leqslant \frac{s_0-t_1}{R} \quad \text{and} \quad h'(t_2) \leqslant -\frac{R}{(R+1)^{1/2}}.$$

The proof can be thus completed arguing as in Proposition 1.2. □

We proceed to define a distance between a point and a closed set of $\mathbb{R}^N$ related to the equation

$$\big|H(x,Du)\big| = 1 \tag{1.14}$$

see [18,19]. It will be used in the representation formulae of Section 3.

We start by introducing some terminology and a definition, see [9,10].

For any $T > 0$ we shall denote by B^T the space of measurable essentially bounded functions defined in $]0,T[$ with values in $\mathbb{R}^N$.

DEFINITION 1.2. – *Given* $T > 0$ *and two subset* B_1*,* B_2 *of* B^T *we call a nonanticipative strategy a mapping*

$$\gamma : B_1 \to B_2$$

such that if $t \in]0,T[$*,* $\eta_1,\eta_2 \in B_1$ *with*

$$\eta_1(s) = \eta_2(s) \text{ a.e. } s \in]0,t[$$

then

$$\gamma[\eta_1](s) = \gamma[\eta_2](s) \text{ a.e. } s \in]0,t[.$$

Given a closed set K of $\mathbb{R}^N$ and $x \in \mathbb{R}^N$ we set for any $T > 0$

$$B^T_{K,x} = \left\{ \zeta \in B^T \colon \left(x - \int_0^T \zeta \, \mathrm{d}t \right) \in K \right\}$$

and denote by Γ^T, $\Gamma^T_{K,x}$ the nonanticipative strategies from B^T to B^T and from B^T to $B^T_{K,x}$, respectively.

If $\eta \in B^T$, $\gamma \in \Gamma^T$, $x \in \mathbb{R}^N$ we write $\overline{\xi}(\eta, \gamma, x, \cdot)$ for the integral curve of $\gamma[\eta]$ defined in $[0, T]$ which equals x at T.

For $\eta \in B^T$, $\gamma \in \Gamma^T$, $x \in \mathbb{R}^N$, K closed subset of $\mathbb{R}^N$ we define

$$\overline{\mathcal{I}}^T_x(\eta, \gamma) = \int_0^T \gamma[\eta]\eta - |\gamma[\eta]| d^{\#}\big(\eta, Z(\overline{\xi}(\eta, \gamma, x, \cdot))\big) \, \mathrm{d}t$$

and

$$S(K, x) = \inf_{\Gamma^1_{K,x}} \sup_{B^1} \overline{\mathcal{I}}^1_x(\eta, \gamma). \tag{1.15}$$

PROPOSITION 1.5. – *$S(K, \cdot)$ is a Lipschitz continuous solution of* (1.14) *in K^c.*

The following comparison principle holds for Eq. (1.14), see [19].

PROPOSITION 1.6. – *Let Ω be an open set of $\mathbb{R}^N$ and g, f an u.s.c. subsolution and a l.s.c. supersolution of* (1.14) *in Ω, respectively.*

Assume

$$\liminf_{\substack{x \to x_0 \\ x \in \Omega}} f(x) \geqslant \limsup_{\substack{x \to x_0 \\ x \in \Omega}} g(x) \quad \textit{for any } x_0 \in \partial\Omega.$$

If Ω is unbounded assume in addition

$$\lim_{|x| \to +\infty} f(x) = +\infty$$

then $f \geqslant g$ in Ω.

2. Bilateral solutions

Here we assume H convex, and so nonnegative, or concave and consequently nonpositive.

The aim of this section is to characterize the bilateral solutions of (1.1) as solutions whose u.s.c. and l.s.c. envelopes verify a certain relation. This result is already in [4], here we prove it in our setting without using any uniform continuity condition on H with respect to the state variable or Rademacher's theorem. Our proof is instead based on a reflection principle for normal cones (in the Clarke sense) to certain closed sets.

This is the statement of the main result of the section:

THEOREM 2.1. – *Assume H convex (concave) in p and f l.s.c. (u.s.c.) then the following two assertion are equivalent*:
(i) *f is a bilateral solution of* (1.1).
(ii) *f is a solution of* (1.1) *with* $(f^{\#})_{\#} = f$ $((f_{\#})^{\#} = f)$.

We will show it for H convex (and so nonnegative). The case where the Hamiltonian is concave can be treated similarly.

The proof will be divided in steps.

We recall that for a given closed subset K of an Euclidean space and $z \in K$ the (Clarke) normal cone $N_K(z)$ to K at z is given by the formula

$$\mathrm{co}\{p\colon\ p = \lim p_i \text{ with } p_i \perp K \text{ at } z_i \text{ and } z_i \to z\},$$

where co indicates the convex hull.

The tangent cone $T_K(z)$ is the polar set of $N_K(z)$, i.e.

$$T_K(z) = \{p\colon\ pq \leqslant 0 \text{ for any } q \in N_K(z)\}.$$

The following reflection principle for normal cones holds, see [16] for related results:

LEMMA 2.1. – *Let K be a closed set and $\widehat{K} = (\mathrm{int}\, K)^c$. Assume $z_0 \in \partial K$ and $\mathrm{int}\, T_K(z_0) \neq \emptyset$ then $N_{\widehat{K}}(z_0) \subseteq -N_K(z_0)$.*

Proof. – The proof is based on the following characterization, see [8]: $\overline{q} \in \mathrm{int}\, T_{\widehat{K}}(z_0)$ if and only if there exists $\varepsilon > 0$ such that $z + tq \in \widehat{K}$ for any $z \in B(z_0, \varepsilon) \cap \widehat{K}$, $q \in B(\overline{q}, \varepsilon)$, $t \in]0, \varepsilon[$.

Take

$$q_0 \in \bigl(\mathrm{int}\, T_{\widehat{K}}(z_0)\bigr)^c$$

then there are sequences z_n in $\widehat{K}$, q_n, $t_n > 0$ converging to z_0, q_0, 0 respectively, which satisfy

$$z_n + t_n q_n \notin \widehat{K} \quad \text{for any } n.$$

Observe that $\widehat{K} = \mathrm{cl}\,\mathrm{int}\, \widehat{K}$, then it is possible to select a positive sequence ε_n converging to 0 and a sequence of unit vectors a_n such that

$$z_n + \varepsilon_n a_n + t_n q_n =: y_n \in K$$

and

$$y_n - t_n q_n = z_n + \varepsilon_n a_n \notin K.$$

This implies

$$-q_0 \in \bigl(\mathrm{int}\, T_K(z_0)\bigr)^c.$$

Therefore

$$\mathrm{int}\, T_{\widehat{K}}(z_0) \supseteq -\mathrm{int}\, T_K(z_0) \neq \emptyset \quad \text{and} \quad T_{\widehat{K}}(z_0) \supseteq -T_K(z_0)$$

which gives the thesis. □

Remark 2.1. – The previous assertion does not hold if we skip the assumption $\operatorname{int} T_K(z_0) \neq \emptyset$. To see this through an example, define in $\mathbb{R}^2$ the functions

$$f_1(x_1, x_2) = x_1^2 + (x_2 + 1)^2, \qquad f_2(x_1, x_2) = x_1^2 + (x_2 - 1)^2$$

and the closed set

$$K = \{x_1 \geqslant 0\} \cup \{f_1 \leqslant 1\} \cup \{f_2 \leqslant 1\}.$$

One has

$$N_K(0,0) = \{x_1 = 0\}, \qquad N_{\widehat{K}}(0,0) = \{x_1 \geqslant 0\}.$$

Then $T_K(0,0) = \{x_2 = 0\}$ has empty interior and obviously

$$N_{\widehat{K}}(0,0) \nsubseteq -N_K(0,0).$$

PROPOSITION 2.1. – *If f is a bilateral solution of* (1.1) *then* $(f^\#)_\# = f$.

Proof. – Fix α and $(x_0, t_0) \in \partial\{f \leqslant \alpha\}$.

Let $\varepsilon_0 > 0$ be so small that

$$d\big((x,t), \{f \leqslant \alpha\}\big) < t \quad \text{for any } (x,t) \in B\big((x_0,t_0), \varepsilon_0\big)$$

then by Proposition 1.4 for any $\delta > 0$ there is $\varepsilon \in]0, \varepsilon_0[$ such that

$$\big((x, t+\delta) \in \{f \leqslant \alpha\}\big) \quad \text{for any } (x,t) \in B\big((x_0,t_0), \varepsilon\big),$$

hence

$$(x_0, t_0 + \delta) \in \operatorname{int}\{f \leqslant \alpha\}.$$

Therefore

$$\operatorname{cl}\operatorname{int}\{f \leqslant \alpha\} = \{f \leqslant \alpha\}. \tag{2.1}$$

Recall that

$$\operatorname{int}\{f \leqslant \alpha\} \subseteq \{f^\# \leqslant \alpha\} \tag{2.2}$$

and

$$\operatorname{cl}\{f^\# \leqslant \alpha\} \subseteq \big\{(f^\#)_\# \leqslant \alpha\big\} \tag{2.3}$$

for any α, and get from (2.1), (2.2) and (2.3)

$$\{f \leqslant \alpha\} \subseteq \big\{(f^\#)_\# \leqslant \alpha\big\} \quad \text{for any } \alpha \in \mathbb{R},$$

and so $f \geqslant (f^\#)_\#$ which gives the thesis being the converse inequality obvious. □

PROPOSITION 2.2. – *Any bilateral solution f of* (1.1) *is also a solution.*

Proof. – Fix $\alpha \in \mathbb{R}$. Exploit the convex character of H and the definition of normal cone to see that

$$s + H(x,p) \leqslant 0 \tag{2.4}$$

for any $(x,t) \in \partial\{f \leqslant \alpha\}$, $(p,s) \in N_{\{f\leqslant\alpha\}}(x,t)$.

Consequently since H is nonnegative, $N_{\{f\leqslant\alpha\}}(x,t)$ cannot contain any vector subspace and then

$$\operatorname{int} T_{\{f\leqslant\alpha\}}(x,t) \neq \emptyset \quad \text{for any } (x,t) \in \partial\{f \leqslant \alpha\}. \tag{2.5}$$

Lemma 2.1 can thus be applied to

$$K = \{f \leqslant \alpha\}, \qquad \widehat{K} = \big(\operatorname{int}\{f \leqslant \alpha\}\big)^c = \operatorname{cl}\{f > \alpha\}$$

yielding the relation

$$N_{\operatorname{cl}\{f>\alpha\}}(x,t) \subseteq -N_{\{f\leqslant\alpha\}}(x,t) \quad \text{for any } (x,t) \in \partial\{f \leqslant \alpha\}. \tag{2.6}$$

Use (2.4) and (2.6) to discover

$$-s + H(x,-p) \leqslant 0 \tag{2.7}$$

for any $(x,t) \in \operatorname{cl}\{f > \alpha\}$, $(p,s) \perp \operatorname{cl}\{f > \alpha\}$ at (x,t).

This implies in the light of Corollary 1.1 that f is a subsolution of (1.1) and so the thesis. □

PROPOSITION 2.3. – *Assume f to be a l.s.c. solution of* (1.1) *verifying* $(f^{\#})_{\#} = f$ *then f is a bilateral solution of* (1.1).

Proof. – Fix $\alpha \in \mathbb{R}$. Use Corollary 1.1 and the convex character of H to find the relation

$$-s + H(x,-p) \leqslant 0 \tag{2.8}$$

for any $(x,t) \in \partial \operatorname{cl}\{f > \alpha\}$, $(p,s) \in N_{\operatorname{cl}\{f>\alpha\}}(x,t)$. This implies since H is nonnegative

$$\operatorname{int} T_{\operatorname{cl}\{u>\alpha\}}(x,t) \neq \emptyset \quad \text{for any } (x,t) \in \partial \operatorname{cl}\{f > \alpha\}. \tag{2.9}$$

Now consider an element $(y_0,s_0) \in \partial\{f \leqslant \alpha\}$. Since $(f^{\#})_{\#} = f$, there is a sequence (y_n,s_n) converging to (y_0,s_0) such that

$$\lim_n f^{\#}(y_n,s_n) = f(y_0,s_0) \leqslant \alpha.$$

Fix $\bar{s} > s_0$ and $\beta_0 > \alpha$ with $\{f > \beta_0\} \neq \emptyset$.

There is no loss of generality in assuming

$$s_n < \bar{s} \quad \text{for any } n$$

moreover for any $\beta \in]\alpha,\beta_0[$ there is an index n_β such that

$$(y_n,s_n) \in \{f^{\#} < \beta\} \subseteq \operatorname{int}\{f \leqslant \beta\} \quad \text{for } n > n_\beta.$$

Proposition 1.3 can be applied with $\alpha = \beta_0$, $K = \{(y_n, s_n)\colon\ n \in \mathbb{N}\}$ yielding the inequality

$$d\big((y_n, \bar{s}), \mathrm{cl}\{f > \beta\}\big) \geqslant \frac{R}{(R^2+1)^{1/2}}(\bar{s} - s_n)$$

for $\beta \in]\alpha, \beta_0[$, $n > n_\beta$ and a suitable positive constant R.

Passing to the limit one gets

$$d\big((y_0, \bar{s}), \mathrm{cl}\{f > \beta\}\big) \geqslant \frac{R}{(R^2+1)^{1/2}}(\bar{s} - s_0)$$

for $\beta \in]\alpha, \beta_0[$.

Hence

$$(y_0, \bar{s}) \in \mathrm{int}\{f \leqslant \alpha\}$$

and so taking in account that $\bar{s} > s_0$ has been arbitrarily chosen

$$(y_0, s_0) \in \mathrm{cl}\,\mathrm{int}\{f \leqslant \alpha\}.$$

Therefore

$$\{f \leqslant \alpha\} = \mathrm{cl}\,\mathrm{int}\{f \leqslant \alpha\}$$

or equivalently

$$\{f > \alpha\} = \mathrm{int}\,\mathrm{cl}\{f > \alpha\}.$$

Thanks to (2.9) Lemma 2.1 can be applied to

$$K = \mathrm{cl}\{f > \alpha\}, \qquad \widehat{K} = \big(\mathrm{int}\,\mathrm{cl}\{f > \alpha\}\big)^c = \{f \leqslant \alpha\}$$

Then for $(x_0, t_0) \in \{f \leqslant \alpha\}$, $(p_0, s_0) \perp \{f \leqslant \alpha\}$, it results

$$-(p_0, s_0) \in N_{\mathrm{cl}\{f>\alpha\}}(x_0, t_0)$$

and by (2.8)

$$s_0 + H(x_0, p_0) \leqslant 0.$$

The converse inequality comes from the fact that f is a supersolution of (1.1) and so the proof is complete in view of Proposition 1.2. □

3. Representation formulae

In this section we remove the convexity assumptions on H and define a solution f and a subsolution g of (1.1) starting from a locally bounded initial datum f_0. This will be done adapting a formula given in [19] for continuous initial data.

The relevant fact is that f verifies the same condition ($(f^\#)_\# = f$ and $(f_\#)^\# = f$) we have introduced in the previous section to characterize bilateral solutions.

Such a condition will be used in the sequel as a criterion for uniqueness.

H will be taken nonnegative. At the end of the section it will be outlined the modifications needed to formulae and statements to fit the case where the Hamiltonian is nonpositive.

We first consider a locally bounded function f_0 defined in $\mathbb{R}^N$ and verifying

$$f_0 \text{ is constant outside a certain compact set} \tag{3.1}$$

or

$$\Big|\lim_{|x|\to+\infty} f_0(x)\Big| = +\infty. \tag{3.2}$$

We will get rid of these restrictions later. We adopt the convention

$$S(K,\cdot) \equiv +\infty \quad \text{if } K = \emptyset$$

and set

$$f(x,t) = \min\bigl\{\alpha\colon\ S(\{f_{0\#} \leqslant \alpha\}, x) \leqslant t\bigr\}, \tag{3.3}$$

$$g(x,t) = \inf\bigl\{\alpha\colon\ S(\mathrm{cl}\{f_0^{\#} \leqslant \alpha\}, x) < t\bigr\}. \tag{3.4}$$

PROPOSITION 3.1. –

(i) *The minimum in formula* (3.3) *is achieved and* f *is l.s.c.*

(ii) g *is u.s.c.*

Proof. – The first assertion can be proved arguing as in [19], Lemmata 3.1, 3.2 where the assumptions (3.1), (3.2) are essentially exploited.

To show (ii) consider (x_n, t_n) converging to a point (x_0, t_0) and put

$$\alpha_n = g(x_n, t_n), \qquad \alpha = g(x_0, t_0).$$

Assume by contradiction $\alpha < \limsup_n \alpha_n$ and fix $\beta \in]\alpha, \limsup_n \alpha_n[$, then

$$S\bigl(\mathrm{cl}\{f_0^{\#} \leqslant \beta\}, x_n\bigr) \geqslant t_n$$

up to a subsequence and so

$$S\bigl(\mathrm{cl}\{f_0^{\#} \leqslant \beta\}, x_0\bigr) \geqslant t_0.$$

This contradicts the equality

$$\alpha = g(x_0, t_0)$$

taking into account the definition of g. □

PROPOSITION 3.2. – $(f^{\#})_{\#} = f$.

Proof. – If the inequality $f(x_0, t_0) \leqslant \alpha$ holds for a certain (x_0, t_0) and $\alpha \in \mathbb{R}$, then by the definition of f and the continuity of $S(\{f_{0\#} \leqslant \alpha\}, \cdot)$

$$(x_0, t_0 + \varepsilon) \in \mathrm{int}\{f \leqslant \alpha\} \quad \text{for any } \varepsilon > 0.$$

This implies

$$\operatorname{cl}\operatorname{int}\{f \leqslant \alpha\} = \{f \leqslant \alpha\} \quad \text{for any } \alpha \in \mathbb{R} \tag{3.5}$$

and one can conclude as in the proof of Proposition 2.1. □

THEOREM 3.1. – *f is a solution of* (1.1) *verifying*

$$\liminf_{(x,t)\to(x_0,0)} f(x,t) = f_{0\#}(x_0) \quad \textit{for any } x_0 \in \mathbb{R}^N. \tag{3.6}$$

Proof. – The first step is to show the relation

$$\operatorname{graph} S\big(\{f_{0\#} \leqslant \alpha\}, \cdot\big) \cap \{t > 0\} \subset \{f \leqslant \alpha\} \cap \partial \operatorname{cl}\{f > \alpha\} \tag{3.7}$$

for any $\alpha \in \mathbb{R}$.

Let (x_0, t_0) be an element of graph $S(\{f_{0\#} \leqslant \alpha\}, \cdot)$ then by the definition of f

$$f(x_0, t_0) \leqslant \alpha.$$

One has

$$(x_0, t_0 + \varepsilon) \in \operatorname{int}\{f \leqslant \alpha\}, \tag{3.8}$$

$$f(x_0, t_0 - \varepsilon) > \alpha, \tag{3.9}$$

for any $\varepsilon > 0$.

(3.8), (3.9) show

$$(x_0, t_0) \in \partial \operatorname{cl}\{f > \alpha\}.$$

At this point the arguments of Theorem 3.1 of [19] can be adapted to get that f is a solution of (1.1). We sketch it for reader's convenience.

Assume the relation

$$(p_0, s_0) \perp \{f \leqslant \alpha\} \quad \text{at } (x_0, t_0) \tag{3.10}$$

for certain (p_0, s_0), (x_0, t_0) and α, denote by ε a positive constant verifying

$$\operatorname{proj}_{\{f \leqslant \alpha\}}(x_0 + \varepsilon p_0, t_0 + \varepsilon s_0) = \big\{(x_0, t_0)\big\}.$$

The notation $h(\cdot) = S(\{f_{0\#} \leqslant \alpha\}, \cdot)$ will be adopted for simplicity in the remainder of the proof.

By the definition of f

$$h(x_0) \leqslant t_0$$

and if the previous inequality was strict then $(x_0, t_0) \in \operatorname{int}\{f \leqslant \alpha\}$ which cannot be. So it results

$$h(x_0) = t_0 \tag{3.11}$$

then (3.7) and (3.11) yield the inequality

$$|x - x_0 - \varepsilon p_0|^2 + \big(h(x) - h(x_0) - \varepsilon s_0\big)^2 \geqslant \varepsilon^2\big(|p_0|^2 + s_0^2\big) \tag{3.12}$$

for x close to x_0.

To show that s_0 is nonvanishing, write (3.12) with $s_0 = 0$, $x = x_0 + r\varepsilon p_0$ for r small and exploit the Lipschitz character of h to obtain a contradiction.

Therefore s_0 is negative.

Set

$$\varphi(x) = -|x - x_0 - \varepsilon p_0|^2 + \varepsilon^2\left(|p_0|^2 + s_0^2\right)$$

and deduce from (3.12) that $\varphi^{1/2}$ is subtangent to h at x_0.

Therefore

$$H\left(x_0, D\varphi^{1/2}(x_0)\right) \geqslant 1$$

where

$$D\varphi^{1/2}(x_0) = \frac{1}{2}\frac{1}{\varphi^{1/2}(x_0)} D\varphi(x_0) = -\frac{p_0}{s_0},$$

and so

$$H(x_0, p_0) \geqslant -s_0 H\left(x_0, \frac{p_0}{-s_0}\right) = -s_0$$

which shows that f is a supersolution of (1.1).

To prove that it is also a subsolution assume that

$$(p_0, s_0) \perp \mathrm{cl}\{f > \alpha\} \quad \text{at } (x_0, t_0) \tag{3.13}$$

and show the relations (3.11), (3.12) and that $s_0 \neq 0$ as in the first part.

Therefore s_0 is positive and $-\varphi^{1/2}$ is supertangent to h at x_0. This implies the inequality

$$H\left(x_0, -D\varphi^{1/2}(x_0)\right) \leqslant 1$$

and

$$H(x_0, -p_0) \leqslant s_0$$

which completes the proof in view of Corollary 1.1.

To prove (3.6) consider an element x_0 of $\mathbb{R}^N$ and (x_n, t_n) a sequence converging to $(x_0, 0)$.

Put

$$K_n = \left\{f_{0\#} \leqslant f(x_n, t_n)\right\}$$

and select for any n

$$y_n \in \mathrm{proj}_{K_n}(x_n).$$

Since the relation

$$d(x_n, K_n) \leqslant r S(K_n, x_n) \leqslant r t_n$$

holds for a certain $r > 0$, see [18], it results

$$\lim_n y_n = x_0.$$

Therefore

$$\liminf_n f(x_n, t_n) \geqslant \liminf_n f_{0\#}(y_n) \geqslant f_{0\#}(x_0). \tag{3.14}$$

On the other side setting $x_n = x_0$ for any n it comes

$$f(x_n, t_n) = f(x_0, t_n) \leqslant f_{0\#}(x_0) \tag{3.15}$$

and so (3.6) is proved. □

THEOREM 3.2. – *g is a subsolution of* (1.1) *verifying*

$$\limsup_{(x,t)\to(x_0,0)} g(x,t) \leqslant f_0^{\#}(x_0) \quad \textit{for any } x_0 \in \mathbb{R}^N. \tag{3.16}$$

Proof. – Take $(\bar{x}, \bar{t}) \in \operatorname{graph} S(\operatorname{cl}\{f_0^{\#} \leqslant \alpha\}, \cdot)$ then by the very definition of g

$$g(\bar{x}, \bar{t}) \geqslant \alpha$$

which shows

$$\operatorname{graph} S\big(\{f_0^{\#} \leqslant \alpha\}, \cdot\big) \cap \{t > 0\} \subset \{g \geqslant \alpha\}. \tag{3.17}$$

Now assume the relation

$$(p_0, s_0) \perp \{g \geqslant \alpha\} \quad \text{at } (x_0, t_0)$$

for certain (p_0, s_0), (x_0, t_0) and α, denote by $\varepsilon > 0$ a constant verifying

$$\operatorname{proj}_{\{g \geqslant \alpha\}}(x_0 + \varepsilon p_0, t_0 + \varepsilon s_0) = \big\{(x_0, t_0)\big\}.$$

By the definition of g

$$S\big(\{f_0^{\#} \leqslant \alpha\}, x_0\big) \geqslant t_0$$

and if the previous inequality was strict then $(x_0, t_0) \in \operatorname{int}\{g \geqslant \alpha\}$ which cannot be. So it results

$$S\big(\{f_0^{\#} \leqslant \alpha\}, x_0\big) = t_0.$$

From this point one can continue as in the Theorem 3.1 to show that g is subsolution of (1.1).

It results

$$g(x,t) \leqslant f_0^{\#}(x) \quad \text{for any } (x,t).$$

Then if (x_n, t_n) converges to $(x_0, 0)$

$$\limsup_n g(x_n, t_n) \leqslant \limsup_n f_0^{\#}(x_n) \leqslant f_0^{\#}(x)$$

which completes the proof. □

We proceed to prove an extremality property for the functions f, g:

THEOREM 3.3. – *Assume* f_0 *to satisfy* (3.1) *or* (3.2) *then* g *is the maximal u.s.c. subsolution of* (1.1), (3.16).

THEOREM 3.4. – *Assume* f_0 *to satisfy* (3.1) *or* (3.2) *then* f *is the minimal l.s.c. supersolution of* (1.1) *verifying*

$$\liminf_{(x,t)\to(x_0,0)} f(x,t) \geqslant f_{0\#}(x_0) \quad \text{for any } x_0 \in \mathbb{R}^N. \tag{3.18}$$

To prove Theorems 3.3 and 3.4 we follow the same ideas of [19], Section 4.

We just recall the main steps of the construction given there. We emphasize that assumptions (3.1), (3.2) are needed because some compactness of the level sets of f_0 is required.

Let $\bar{g}$ be an u.s.c. subsolution of (1.1) verifying (3.16). We set

$$\mathcal{A} = \{\alpha\colon \{\bar{g} < \alpha\} \text{ is a nonempty proper subset of } \mathbb{R}^N \times]0, +\infty[\}$$

and

$$\Omega_\alpha = \{x\colon \text{ there exists } t \text{ with } (x,t) \in \operatorname{int}\{\bar{g} \geqslant \alpha\}\} \tag{3.19}$$

for any $\alpha \in \mathcal{A}$.

PROPOSITION 3.3. – *Let* $\alpha \in \mathcal{A}$.

(i) *If* $x_0 \in \operatorname{cl}\{f_0^\# < \alpha\}$ *then* $\bar{g}(x_0, t_0) < \alpha$ *for any* $t_0 > 0$ *and consequently* $\Omega_\alpha \subset \operatorname{int}\{f_0^\# \geqslant \alpha\}$.

(ii) *If* $(x_0, t_0) \in \operatorname{cl}\{\bar{g} < \alpha\}$ *then* $\bar{g}(x_0, t) < \alpha$ *for any* $t > t_0$.

Proof. – We prove (i). The other assertion can be obtained similarly.

Take $x_0 \in \operatorname{cl}\{f_0^\# < \alpha\}$ and fix t_0. By (3.16) $\bar{g}(x_n, t_n) < \alpha$ for a suitable sequence converging to $(x_0, 0)$ with $t_n < t_0$ for any n.

It can be assumed

$$\{\bar{g} < \alpha\} \subsetneq \mathbb{R}^N \times]0, +\infty[.$$

Then arguing as in Proposition 1.3 and Lemma 4.1 of [19] it can be proved the existence of $R > 0$ such that

$$d\big((x_n, t_0), \partial\{\bar{g} < \alpha\}\big) \geqslant \frac{R}{(R^2+1)^{1/2}}(t_n - t) \quad \text{for any } n.$$

Therefore $\bar{g}(x_0, t_0) < \alpha$. □

PROPOSITION 3.4. – *Let* α *be an element of* $\mathcal{A}$. *For any* $x \in \Omega_\alpha$ *there exists one and only one* t *such that*

$$(x,t) \in \partial\{\bar{g} < \alpha\}.$$

As a consequence of the previous proposition a function h_α can be defined in $\mathbb{R}^N$ putting

$$h_\alpha(x) = \{t\colon (x,t) \in \partial\{\bar{g} < \alpha\}\} \quad \text{for any } x \in \Omega_\alpha. \tag{3.20}$$

PROPOSITION 3.5. – *For any* $\alpha \in \mathcal{A}$ h_α *is continuous in* Ω_α, *verifies*

$$\lim_{\substack{x \to x_0 \\ x \in \Omega_\alpha}} h_\alpha(x) = 0 \quad \text{for any } x_0 \in \partial\Omega_\alpha$$

and it is a subsolution of (1.14) *in* Ω_α.

Taking in account Propositions 3.3, 3.5 and using Proposition 1.6 with $\Omega = \Omega_\alpha$, we can prove:

PROPOSITION 3.6. – *For any* $\alpha \in \mathcal{A}$, $\varepsilon > 0$, $x \in \Omega_\alpha$

$$h_\alpha(x) \leqslant S\big(\mathrm{cl}\{f_0^\# \leqslant \alpha - \varepsilon\}, x\big).$$

We now consider a l.s.c. supersolution $\bar{f}$ of (1.1) verifying (3.18).

We set

$$\mathcal{A}' = \{\alpha\colon \{\bar{f} > \alpha\} \text{ is a nonempty proper subset of } \mathbb{R}^N \times]0, +\infty[\}$$

and define

$$l_\alpha(x) = \inf\{t\colon (x,t) \in \partial\{\bar{f} > \alpha\}\}$$

for $\alpha \in \mathcal{A}'$, $x \in \{f_{0\#} > \alpha\}$. Note that l_α takes values in $[0, +\infty[$.

PROPOSITION 3.7. – *For any* $\alpha \in \mathcal{A}'$ l_α *is l.s.c. and positive in* $\{f_{0\#} > \alpha\}$. *Moreover it is a supersolution of* (1.14) *in* $\{f_{0\#} > \alpha\}$ *and if this set is unbounded then it verifies*

$$\lim_{|x| \to +\infty} l_\alpha(x) = +\infty.$$

Using Propositions 3.7 and 1.6 with $\Omega = \{f_{0\#} > \alpha\}$ we finally get

PROPOSITION 3.8. – *Assume* $\alpha \in \mathcal{A}'$. *Then*

$$l_\alpha(x) \geqslant S\big(\{f_{0\#} \leqslant \alpha\}, x\big) \quad \text{for any } x \in \{f_{0\#} > \alpha\}.$$

Proof of Theorem 3.3. – Assume

$$\alpha =: g(x_0, t_0) < \bar{g}(x_0, t_0) := \beta \tag{3.21}$$

for a certain (x_0, t_0).

Observe that

$$]\alpha, \beta[\subset \mathcal{A}.$$

In fact if this is not the case there exists $\alpha_0 \in]\alpha, \beta[$ such that

$$\{\bar{g} < \alpha_0\} = \emptyset$$

then by (3.16)

$$\{f_0^\# < \alpha_0\} = \emptyset$$

and by the definition of g

$$\{g < \alpha_0\} = \emptyset$$

which contradicts (3.21).

Note that if $\alpha_0 \in]\alpha, \beta[$ then by Proposition 3.3 $x_0 \in \Omega_{\alpha_0}$ and so thanks to (3.21), Proposition 3.6 and the definition of h_α

$$t_0 \leqslant h_{\alpha_0}(x_0) \leqslant S\big(\mathrm{cl}\{f_0^\# \leqslant \alpha_0 - \varepsilon\}, x_0\big)$$

for any $\varepsilon > 0$.

Then the definition of g gives

$$g(x_0, t_0) \geqslant \alpha_0 - \varepsilon$$

which contradicts (3.21) for ε sufficiently small. □

Proof of Theorem 3.4. – Assume by contradiction that there is (x_0, t_0) such that

$$\alpha =: \bar{f}(x_0, t_0) < f(x_0, t_0) := \beta.$$

It results

$$]\alpha, \beta[\subset \mathcal{A}'$$

in fact if this relation was false there should be $\alpha_0 \in]\alpha, \beta[$ with

$$\{\bar{f} > \alpha_0\} = \emptyset,$$

then by (3.18)

$$\{f_{0\#} > \alpha_0\} = \emptyset$$

and consequently

$$\{f > \alpha_0\} = \emptyset$$

which is impossible.

By Proposition 3.8 and the definition of l_{α_0} it comes

$$t_0 \geqslant l_{\alpha_0}(x_0) \geqslant S\big(\{f_{0\#} \leqslant \alpha_0\}, x_0\big)$$

for $\alpha_0 \in]\alpha, \beta[$.

Consequently $f(x_0, t_0) \leqslant \alpha_0$ which cannot be. □

To remove the restrictions on the initial datum, we first establish two propositions exploiting the finite propagation speed property for Eq. (1.1) due to the assumption (1.5).

We define a distance L on $\mathbb{R}^N$ setting for any x, y

$$L(x, y) = \inf\left\{ \int_0^1 \frac{1}{a|\xi| + b} |\dot{\xi}| \,\mathrm{d}t \colon \xi \in B_{x,y} \right\}$$

where $B_{x,y}$ the set of Lipschitz continuous curves joining x and y, and a, b the constants appearing in (1.5).

PROPOSITION 3.9. – *Assume* $\tilde{f}_0$ *and* $\bar{f}_0$ *to be Lipschitz-continuous functions verifying* (3.1)*, denote by* $\tilde{f}$, $\bar{f}$ *the solutions of* (1.1) *equaling* $\tilde{f}_0$ *and* $\bar{f}_0$ *respectively, at* $t = 0$*, and fix* (x_0, t_0).

If $\tilde{f}_0 = \bar{f}_0$ *in a compact subset* K *with*

$$L(y, x_0) > t_0 \quad \textit{for } y \in (\operatorname{int} K)^c, \tag{3.22}$$

then $\tilde{f} = \bar{f}$ *in a neighborhood of* (x_0, t_0).

Proof. – $\tilde{f}$ and $\bar{f}$ are locally Lipschitz-continuous (see [19]) and so verify for a.e. (x, t) the relation

$$0 = \tilde{f}_t + H(x, D\tilde{f}) - \bar{f}_t - H(x, D\bar{f}) \geqslant (\tilde{f}_t - \bar{f}_t) - \bigl(a|x| + b\bigr)|D\tilde{f} - D\bar{f}|$$

where the last inequality has been obtained using (1.5).

Consequently the representation formula and the comparison results for the equation $u_t - (a|x| + b)|Du| = 0$ (see [19]) give

$$\tilde{f}(x, t) - \bar{f}(x, t) \leqslant \max\bigl\{\tilde{f}_0(y) - \bar{f}_0(y) \colon L(y, x) \leqslant t\bigr\} \quad \text{for any } (x, t).$$

Then thanks to (3.22)

$$\tilde{f} \leqslant \bar{f} \text{ in a suitable neighborhood of } (x_0, t_0).$$

The thesis is obtained exchanging the roles of $\tilde{f}$ and $\bar{f}$. □

We generalize the previous proposition to more general initial data:

PROPOSITION 3.10. – *Assume* $\tilde{f}_0$ *and* $\bar{f}_0$ *to be bounded initial data verifying* (3.1)*, denote by* $\tilde{f}$, $\bar{f}$ *the functions given by the formula* (3.3) ((3.4)) *with* f_0 *replaced by* $\tilde{f}_0$ *and* $\bar{f}_0$*, respectively and fix* (x_0, t_0).

If $\tilde{f}_0 = \bar{f}_0$ *in a compact subset* K *verifying* (3.22) *then* $\tilde{f} = \bar{f}$ *in a neighborhood of* (x_0, t_0).

Proof. – The proof will be given for $\tilde{f}$ and $\bar{f}$ defined by (3.3).

Consider for any $x, \varepsilon > 0$ the inf-convolutions

$$\tilde{f}_{0\varepsilon}(x) = \inf_y \left\{ \tilde{f}_0(y) + \frac{1}{2\varepsilon}|x - y|^2 \right\},$$

$$\bar{f}_{0\varepsilon}(x) = \inf_y \left\{ \bar{f}_0(y) + \frac{1}{2\varepsilon}|x - y|^2 \right\},$$

and denote by $\tilde{f}_\varepsilon$, $\bar{f}_\varepsilon$ the solutions of (1.1) equaling $\tilde{f}_{0\varepsilon}$, $\bar{f}_{0\varepsilon}$ at $t = 0$, respectively.

The equality

$$\tilde{f}_{0\varepsilon} = \bar{f}_{0\varepsilon}$$

holds in

$$K_\varepsilon := \{y \in K:\ d(y, \partial K) \geqslant 2\sqrt{\varepsilon R}\}, \quad \text{where } R = \max\{\sup_{\mathbb{R}^N} \tilde{f}_0, \sup_{\mathbb{R}^N} \bar{f}_0\}$$

and

$$L(y, x_0) > t_0 \quad \text{for } y \in K_\varepsilon^c$$

if ε is sufficiently small.

Then by Proposition 3.9

$$\tilde{f}_\varepsilon = \bar{f}_\varepsilon \text{ in a neighborhood of } (x_0, t_0) \tag{3.23}$$

for ε small.

Claim. –

$$\tilde{\tilde{f}} := \liminf{}_\# \tilde{f}_\varepsilon = \tilde{f}, \qquad \bar{\bar{f}} := \liminf{}_\# \bar{f}_\varepsilon = \bar{f}. \tag{3.24}$$

Since

$$\liminf_{(y,t)\to(x,0)} \tilde{f}(y, t) \geqslant \tilde{f}_{0\#}(x) \geqslant \tilde{f}_{0\varepsilon}(x) \quad \text{for any } \varepsilon, x$$

it comes by the comparison results of [19]

$$\tilde{f} \geqslant \tilde{f}_\varepsilon \quad \text{for any } \varepsilon$$

and so

$$\tilde{f} \geqslant \tilde{\tilde{f}}.$$

Conversely set

$$\tilde{\tilde{f}}_0(x) = \liminf_{(y,t)\to(x,0)} \tilde{\tilde{f}}(y, t) \quad \text{for any } x$$

and assume for purposes of contradiction

$$\tilde{\tilde{f}}_0(y_0) < \tilde{f}_{0\#}(y_0) \quad \text{for a certain } y_0. \tag{3.25}$$

Exploiting the convergence of $\tilde{f}_{0\varepsilon}$ to $\tilde{f}_{0\#}$ and (3.25) one can select r and η positive, a sequence $(x_\varepsilon, t_\varepsilon)$ converging to $(y_0, 0)$ and $\varepsilon_0 > 0$ verifying

$$\tilde{f}_{0\varepsilon}(x) > \tilde{f}_{0\#}(y_0) - \eta/2, \tag{3.26}$$

$$\tilde{f}_\varepsilon(x_\varepsilon, t_\varepsilon) < \tilde{f}_{0\#}(y_0) - \eta, \tag{3.27}$$

for $x \in B(y_0, r)$, $< 0 < \varepsilon < \varepsilon_0$.

Thanks to (3.26) and to the local equiboundedness of $\tilde{f}_\varepsilon$, two positive constants M_0 and k_0 can be fixed so that the function

$$\psi_M(x, t) := \tilde{f}_{0\#}(y_0) - Mt - k_0|x - y_0|^2$$

verifies for $0 < \varepsilon < \varepsilon_0$, $M \geqslant M_0$

$$\tilde{f}_{0\varepsilon} - \psi_M(\cdot, 0) > -\eta/2 \quad \text{in } \operatorname{cl} B(y_0, r), \tag{3.28}$$

$$\tilde{f}_\varepsilon - \psi_M > 0 \quad \text{in } \operatorname{cl} B(y_0, r) \times \{1\} \cup \partial \operatorname{cl} B(y_0, r) \times [0, 1]. \tag{3.29}$$

For any $M \geqslant M_0$ it can be found $0 < \varepsilon_M < \varepsilon_0$ for which

$$M t_{\varepsilon_M} + k_0 |x_{\varepsilon_M} - y_0|^2 < \eta/4$$

and

$$(x_{\varepsilon_M}, t_{\varepsilon_M}) \in B(y_0, r) \times]0, 1[,$$

therefore by (3.27)

$$\tilde{f}_{\varepsilon_M}(x_{\varepsilon_M}, t_{\varepsilon_M}) - \psi_M(x_{\varepsilon_M}, t_{\varepsilon_M}) < -\frac{3}{4}\eta. \tag{3.30}$$

By (3.28), (3.29), (3.30), ψ_M is subtangent to $\tilde{f}_{\varepsilon_M}$ for any $M > M_0$ at a certain point of $B(x_0, r) \times]0, 1[$ (depending on M).

This is impossible taking in account the definition of ψ_M and that $\tilde{f}_{\varepsilon_M}$ is supersolution of (1.1).Then an y_0 verifying (3.25) cannot exist, consequently

$$\tilde{\tilde{f}}_0 \geqslant \tilde{f}_{0\#} \quad \text{and} \quad \tilde{\tilde{f}} \geqslant \tilde{f}$$

by the minimality of $\tilde{f}$, see Theorem 3.4.

The proof of the claim is thus complete.

It results by (3.24) that for any fixed (x, t)

$$\tilde{f}(x, t) = \lim_n \tilde{f}_{\varepsilon_n}(x_{\varepsilon_n}, t_{\varepsilon_n})$$

with $x_{\varepsilon_n}, t_{\varepsilon_n}$ sequences suitably chosen.

Then in force of (3.23), (3.24)

$$\tilde{f}(x, t) = \lim_n \bar{f}_{\varepsilon_n}(x_{\varepsilon_n}, t_{\varepsilon_n}) \geqslant \bar{f}(x, t)$$

for (x, t) in a suitable neighborhood of (x_0, t_0).

The thesis is finally obtained exchanging the roles of $\tilde{f}$ and $\bar{f}$.

□

From now on f_0 will denote a general locally bounded initial datum and f (g) the function given by (3.3) ((3.4)).

THEOREM 3.5. – *f is a l.s.c. solution of* (1.1) *verifying* (3.6) *and* $(f^\#)_\# = f$, *g an u.s.c. subsolution of* (1.1) *verifying* (3.16).

Proof. – The proof will be given for f.

Fix (x_0, t_0) and a compact set K satisfying (3.22).

Define two l.s.c. functions $\tilde{f}_0$, $\bar{f}_0$ on $\mathbb{R}^N$ through the formulae

$$\tilde{f}_0 = f_{0\#} \quad \text{in } K, \tag{3.31}$$

$$\bar{f}_0 = f_{0\#} \quad \text{in int} K, \tag{3.32}$$

$$\tilde{f}_0 = (\sup_K f_0) + 1 \quad \text{in } K^c, \tag{3.33}$$

$$\bar{f}_0 = (\inf_K f_0) - 1 \quad \text{in } (\text{int} K)^c. \tag{3.34}$$

By (3.22), (3.33), (3.34) and Proposition 3.10

$$\tilde{f} = \bar{f} \quad \text{in a neighborhood of } (x_0, t_0) \tag{3.35}$$

where $\tilde{f}$, $\bar{f}$ are given by (3.3) with $\tilde{f}_0$, $\bar{f}_0$ at the place of f, respectively.

By (3.31), (3.32), (3.33), (3.34) it results

$$\begin{aligned}\bar{f}(x,t) &= \min\bigl\{\alpha \in \text{cl} f_0(K):\ S\bigl(\{\bar{f}_0 \leqslant \alpha\}, x\bigr) \leqslant t\bigr\} \\ &\leqslant \inf\bigl\{\alpha \in \text{cl} f_0(K):\ S\bigl(\{f_{0\#} \leqslant \alpha\}, x\bigr) \leqslant t\bigr\} \\ &\leqslant \min\bigl\{\alpha \in \text{cl} f_0(K):\ S\bigl(\{\tilde{f}_0 \leqslant \alpha\}, x\bigr) \leqslant t\bigr\} = \tilde{f}(x,t)\end{aligned}$$

for (x,t) close to (x_0,t_0).

Then using (3.35) one obtains

$$\bar{f} = f = \tilde{f}$$

in a neighborhood of (x_0,t_0) and so the thesis. □

Remark 3.1. – Taking in account the proof of the previous theorem, we see that Proposition 3.10 holds true for general locally bounded initial data $\tilde{f}_0$, $\bar{f}_0$.

Consequently, given (x_0,t_0), any locally bounded datum f_0 can be perturbed outside a certain compact set so that the modified function $\tilde{f}_0$ verifies (3.1) or (3.2) and

$$\tilde{f}_0 \leqslant f_0 \ (\tilde{f}_0 \geqslant f_0) \quad \text{in } \mathbb{R}^N,$$

$$f(x,t) = \tilde{f}(x,t) \ \bigl(g(x,t) = \tilde{g}(x,t)\bigr) \quad \text{in a neighborhood of } (x_0,t_0),$$

where $\tilde{f}$ ($\tilde{g}$) is given by (3.3) ((3.4)) with $\tilde{f}_0$ replacing f_0.

From the previous remark we immediately derive:

THEOREM 3.6. – *For any locally bounded initial datum f_0, f is the minimal l.s.c. supersolution of* (1.1) *verifying* (3.18) *and g the maximal u.s.c. subsolution of* (1.1) *verifying* (3.16).

Note that the previous theorem says that g is an L-solution of (1.1), (3.16) in the sense of Giga and Sato, see [12].

Remark 3.2. – If H is nonpositive we define

$$f(x,t) = \max\bigl\{\alpha:\ S\bigl(\{f_0^{\#} \geqslant \alpha\}, x\bigr) \leqslant t\bigr\},$$

$$g(x,t) = \sup\bigl\{\alpha:\ S\bigl(\text{cl}\{f_{0^{\#}} \geqslant \alpha\}, x\bigr) < t\bigr\}.$$

It results that f is an u.s.c. solution of (1.1) verifying

$$\limsup_{(x,t)\to(x_0,0)} f(x,t) = f_0^{\#}(x_0) \quad \text{for any } x_0 \in \mathbb{R}^N.$$

Moreover it is the maximal u.s.c. subsolution verifying

$$\limsup_{(x,t)\to(x_0,0)} f(x,t) \leqslant f_0^{\#}(x_0) \quad \text{for any } x_0 \in \mathbb{R}^N.$$

g is minimal l.s.c. supersolution verifying

$$\liminf_{(x,t)\to(x_0,0)} g(x,t) \geqslant f_{0\#}(x_0) \quad \text{for any } x_0 \in \mathbb{R}^N.$$

4. Almost continuous functions

We introduce on the space of locally bounded functions defined on the Euclidean space $\mathbb{R}^M$, the following equivalence relation, see [3,17]:

$$f \sim g \quad \text{if } f^{\#} = g^{\#} \text{ and } f_{\#} = g_{\#}. \tag{4.1}$$

This is motivated by the very definition of solution. In fact it comes directly from (4.1) that if a function f is a solution of a certain equation then every function equivalent to f has the same property. Moreover it is easy to check that the weak limits of sequences of locally bounded functions depend only on the equivalence classes of such functions. More precisely:

LEMMA 4.1. – *Assume f_n to be a sequence of locally equibounded functions defined on $\mathbb{R}^M$ and $g_n \sim f_n$ for any n.*

Then g_n is locally equibounded and

$$\limsup{}^{\#} f_n = \limsup{}^{\#} g_n,$$
$$\liminf{}_{\#} f_n = \liminf{}_{\#} g_n.$$

We will denote by $LB(\mathbb{R}^M)$ the set of equivalence classes with respect to the relation (4.1).

If $u \in LB(\mathbb{R}^M)$ $u^{\#}$, $u_{\#}$ will denote the u.s.c. an the l.s.c. envelope of any of its representatives.

Similarly thanks to Lemma 4.1 we can talk of locally equibounded sequences u_n in $LB(\mathbb{R}^M)$ and use without ambiguity the expressions

$$\limsup{}^{\#} u_n, \qquad \liminf{}_{\#} u_n.$$

We want to distinguish the equivalence classes having an u.s.c. representative or a l.s.c. representative, we will indicate such subsets of $LB(\mathbb{R}^M)$ by $ALC^+(\mathbb{R}^M)$ and $ALC^-(\mathbb{R}^M)$, respectively.

We define

$$ALC(\mathbb{R}^M) = ALC^+(\mathbb{R}^M) \cap ALC^-(\mathbb{R}^M).$$

We will call almost continuous the elements of $ALC(\mathbb{R}^M)$, this terminology will be justified below.

The next lemma summarizes some simple characterization of sets of (equivalence class of) functions that we have introduced:

LEMMA 4.2. –
(i) *$u \in ALC^+(\mathbb{R}^M)$ if and only if $(u^\#)_\# = u_\#$.*
(ii) *$u \in ALC^-(\mathbb{R}^M)$ if and only if $(u_\#)^\# = u^\#$.*
(iii) *$u \in ALC(\mathbb{R}^M)$ if and only if $(u^\#)_\# = u_\#$ and $(u_\#)^\# = u^\#$.*
(iv) *$u \in ALC(\mathbb{R}^M)$ if and only if $u^\# \sim u_\#$.*

For any continuous function defined on $\mathbb{R}^M$ the equivalence class of (4.1) is a singleton, if $C(\mathbb{R}^M)$ denotes the space of such functions we have the inclusion

$$C(\mathbb{R}^M) \subset ALC(\mathbb{R}^M).$$

For any couple f, g of locally bounded functions defined in $\mathbb{R}^M$ with

$$f(z) \leqslant g(z) \quad \text{for any } z \in \mathbb{R}^M \tag{4.2}$$

we set

$$[f, g] = \{h\colon\ f(z) \leqslant h(z) \leqslant g(z) \text{ for any } z \in \mathbb{R}^M\}.$$

LEMMA 4.3. – *Let u be in $ALC(\mathbb{R}^M)$, f is a representative of u if and only if $f \in [u_\#, u^\#]$.*

Proof. – Since u is almost continuous then $u^\#$ and $u_\#$ are representatives of u and $u^\#(u_\#)$ is the unique u.s.c. (l.s.c.) function in $[u_\#, u^\#]$.

From this the thesis follows. □

The next proposition is based on Baire's theorem. The proof goes as in [1], Corollary V.4.30.

PROPOSITION 4.1. – *Assume u to be an element of $ALC^-(\mathbb{R}^M) \cup ALC^+(\Omega)$ then $\{u^\# = u_\#\}$ is residual in $\mathbb{R}^M$ and it results $f(z) = g(z)$ for any couple f, g of representatives of u and any $z \in \{u^\# = u_\#\}$. Moreover $f|_{\{u^\#=u_\#\}} = g|_{\{u^\#=u_\#\}}$ is continuous.*

Proof. – The proof will be given for $u \in ALC^-(\mathbb{R}^M)$. The first step is to show that the set

$$\{u^\# - u_\# < 1/n\}$$

is dense in $\mathbb{R}^M$ for any $n \in \mathbb{N}$.

In fact if this is not the case there exists $z_0 \in \mathbb{R}^M$, $r > 0$, n_0 such that

$$B(z_0, r) \cap \{u^\# - u_\# < 1/n_0\} = \emptyset.$$

This implies

$$(u_\#)^\#(z_0) \leqslant u^\#(z_0) - 1/n_0$$

which contradicts the characterization of the elements of $ALC^-(\mathbb{R}^M)$ given in Lemma 4.2.

Consequently by Baire's theorem the set

$$K := \{u_\# = u^\#\} = \bigcap_n \{u^\# - u_\# < 1/n\}$$

being the intersection of a countable family of open dense sets, is residual in $\mathbb{R}^M$.

Since any representative of u belongs to $[u_\#, u^\#]$ the equality

$$f(z) = g(z)$$

holds for any couple f, g of representatives of u and $z \in K$.

If a sequence z_n of elements of K converges to $z \in K$ then

$$f(z) = f_\#(z) \leqslant \liminf f(z_n) \leqslant \limsup f(z_n) \leqslant f^\#(z) = f(z)$$

which proves that $f|_K = g|_K$ is continuous. □

If in addition u is almost continuous the previous result can be strengthened in the sense that the whole equivalence class can be recovered from the values of any representative on $\{u^\# = u_\#\}$.

PROPOSITION 4.2. – *Let u be almost continuous and f one of its representative. Set $K_u = \{u^\# = u_\#\}$.*

Then

$$u^\#(z_0) = \limsup_{\substack{z \to z_0 \\ z \in K_u}} f(z), \qquad u_\#(z_0) = \liminf_{\substack{z \to z_0 \\ z \in K_u}} f(z)$$

for any $z_0 \in \mathbb{R}^M$, and consequently $f|_{K_u}$ cannot be extended outside K_u keeping its continuity.

Proof. – Set

$$\overline{u}(z_0) = \limsup_{\substack{z \to z_0 \\ z \in K_u}} f(z), \tag{4.3}$$

$$\underline{u}(z_0) = \liminf_{\substack{z \to z_0 \\ z \in K_u}} f(z) \tag{4.4}$$

for any $z_0 \in \mathbb{R}^M$.

Fix z_0 and consider z_n converging to it. By (4.3), (4.4) for any n there are $z_n', z_n'' \in K_u$ such that

$$|z_n - z_n'| < 1/n, \qquad |z_n - z_n''| < 1/n,$$

$$\left|\overline{u}(z_n) - f(z_n')\right| < 1/n, \qquad \left|\underline{u}(z_n) - f(z_n'')\right| < 1/n.$$

Consequently

$$\limsup_n \overline{u}(z_n) = \limsup_n f(z_n') \leqslant \overline{u}(z),$$

$$\liminf_n \underline{u}(z_n) = \liminf_n f(z_n'') \geqslant \underline{u}(z).$$

These relations show that $\overline{u}$ and $\underline{u}$ are u.s.c. and l.s.c., respectively.

Moreover by (4.3), (4.4)

$$\overline{u}, \underline{u} \in [u_\#, u^\#]$$

and this implies the equalities

$$\overline{u} = u^{\#}, \qquad \underline{u} = u_{\#}$$

since u is almost continuous.

Finally if $z_0 \notin K_u$ then

$$u^{\#}(z_0) = \limsup_{\substack{z \to z_0 \\ z \in K_u}} f(z) > \liminf_{\substack{z \to z_0 \\ z \in K_u}} f(z) = u_{\#}(z_0)$$

and this strict inequality proves the last part of the assertion. □

Conversely it is easily seen:

PROPOSITION 4.3. – *Let f be a function defined and continuous in a dense subset K of $\mathbb{R}^M$ which does not admit any proper continuous extension.*

Set

$$f^{\#}(z_0) = \limsup_{\substack{z \to z_0 \\ z \in K}} f(z), \qquad f_{\#}(z) = \limsup_{\substack{z \to z_0 \\ z \in K}} f(z).$$

Then $u = [f_{\#}, f^{\#}] \in ALC(\mathbb{R}^M)$ and $K = \{f_{\#} = f^{\#}\}$.

Propositions 4.2 and 4.3 justify that from now on we identify the equivalence classes belonging to $ALC(\mathbb{R}^M)$ and the functions defined and continuous on a dense subset of $\mathbb{R}^M$ which do not admit any proper continuous extension.

For a function u of this type, K_u will denote the set $\{u^{\#} = u_{\#}\}$.

Remark 4.1. – The domain of an almost continuous function is not only dense but residual. This is clear from Proposition 4.1 and conversely it is a consequence of the fact that the set of point of discontinuity of any function defined on the whole space is the countable union of closed sets.

The term almost continuous comes from the property that for the functions of this class the inverse image of any open set is almost open, i.e. the symmetric difference of an open and a meager set, see [15,14].

Given a sequence u_n and $u \in LB(\mathbb{R}^M)$, we write

$$u = \lim{}^{\#} u_n \tag{4.5}$$

to mean

$$u^{\#} = \limsup{}^{\#} u_n, \qquad u_{\#} = \liminf{}_{\#} u_n.$$

If u and u_n are almost continuous then (4.5) implies

$$\lim_n u_n(z_n) = u(z_0)$$

for any $z_0 \in K_u$ and z_n converging to z_0 with $z_n \in K_{u_n}$.

5. Almost continuous solutions

Here we use the terminology introduced in the previous section to give uniqueness and stability results for almost continuous solutions u of (1.1) verifying

$$\limsup_{\substack{(x,t)\to(x_0,0)\\(x,t)\in K_u}} u(x,t) \leqslant u_0^{\#}(x_0), \tag{5.1}$$

$$\liminf_{\substack{(x,t)\to(x_0,0)\\(x,t)\in K_u}} u(x,t) \geqslant u_{0\#}(x_0), \tag{5.2}$$

or

$$\liminf_{\substack{(x,t)\to(x_0,0)\\(x,t)\in K_u}} u(x,t) = u_{0\#}(x_0), \tag{5.3}$$

$$\limsup_{\substack{(x,t)\to(x_0,0)\\(x,t)\in K_u}} u(x,t) = u_0^{\#}(x_0), \tag{5.4}$$

for any $x_0 \in \mathbb{R}^N$, where the initial value u_0 is taken in $LB(\mathbb{R}^N)$.

We treat the case $H \geqslant 0$, the modifications for H nonnegative can be easily derived.

We consider the functions f and g defined as in (3.3), (3.4) with $u_{0\#}$, $u_0^{\#}$ replacing $f_{0\#}$ and $f_0^{\#}$, respectively and denote by w the almost continuous function having f as representative.

We observe that in general $f \nsim g$ as it can be seen taking for initial datum the function of $LB(\mathbb{R}^N)$ with $\chi_{\mathbb{R}^N\setminus\{0\}}$ as representative and setting $H(x,p) = |p|$.

In this case $u_0^{\#} \equiv 1$ and so $g \equiv 1$ while

$$f(x,t) = \begin{cases} 0 & \text{if } |x| < t, \\ 1 & \text{if } |x| \geqslant t. \end{cases}$$

This fact will have some consequences in the formulation of the uniqueness results we are going to present.

One condition for $f \sim g$ is indicated in the next result.

PROPOSITION 5.1. – *Assume $u_0 \in ALC^+(R^N)$ then $f \sim g$.*

Proof. – The argument of Theorem 3.3 will be adapted for the proof.

Let $\bar g$ an u.s.c. subsolution of (1.1) verifying (3.16) such that

$$\alpha =: f^{\#}(x_0,t_0) < \bar g(x_0,t_0) := \beta$$

for a certain (x_0,t_0).

In view of Remark 3.1 u_0 (any representative of u_0) can be assumed without loss of generality to verify (3.1) or (3.2).

Take $\alpha_0 \in]\alpha,\beta[$ and assume that

$$\{\bar g < \alpha_0\} = \emptyset$$

then by (3.16)

$$\{u_0^{\#} < \alpha_0\} = \emptyset$$

and since $u_0 \in ALC^+(\mathbb{R}^N)$

$$\{u_{0\#} < \alpha_0\} = \emptyset.$$

This implies by the definition of f

$$\{f < \alpha_0\} = \emptyset$$

and

$$\{f^\# < \alpha_0\} = \emptyset$$

which cannot be.

Therefore $]\alpha, \beta[\subset \mathcal{A}$, and if $\alpha_0 \in]\alpha, \beta[$ then $x_0 \in \Omega_{\alpha_0}$, see (3.19), by Proposition 3.3 and so

$$t_0 \leqslant h_{\alpha_0}(x_0) \leqslant S\big(\mathrm{cl}\{u_0^\# \leqslant \alpha_0 - \varepsilon\}, x_0\big)$$

for any $\varepsilon > 0$, see (3.20) for the definition of h_{α_0}.

Since $u_0 \in ALC^+(R^N)$ it results

$$\mathrm{cl}\{u_0^\# \leqslant \alpha_0 - \varepsilon\} \supset \{u_{0\#} \leqslant \alpha_0 - 2\varepsilon\} \quad \text{for any } \varepsilon > 0$$

then

$$t_0 \leqslant h_{\alpha_0}(x_0) \leqslant S\big(\{u_{0\#} \leqslant \alpha_0 - 2\varepsilon\}, x_0\big)$$

and the definition of f gives

$$f(x_0, t_0 - \delta) > \alpha_0 - 2\varepsilon \quad \text{for any } \delta > 0$$

and

$$f^\#(x_0, t_0) \geqslant \alpha_0 - 2\varepsilon$$

which is impossible for ε sufficiently small.

Consequently $f^\#$ is the maximal u.s.c. subsolution of (1.1) verifying (3.16) with $f_0^\#$ replaced by $u_0^\#$ which gives the thesis in view of Theorem 3.3. □

The first uniqueness result is the following:

THEOREM 5.1. – *Let $u_0 \in LB(\mathbb{R}^N)$. Then w is the unique solution of* (1.1), (5.3) *in* $ALC(\mathbb{R}^N \times]0, +\infty[)$.

Proof. – Let $u \in ALC(\mathbb{R}^N \times]0, +\infty[)$ be a solution of (1.1), (5.3). The crucial point is to prove the inequality

$$u^\# \leqslant w^\#. \tag{5.5}$$

As in the previous proposition in view of Remark 3.1 u_0 can be assumed to verify (3.1) or (3.2).

Define $\mathcal{A}$, Ω_α and h_α as in (3.19), (3.20) replacing $\bar{g}$ by $u^\#$.

Take $\alpha \in \mathcal{A}$, $x \in \mathrm{cl}\{u_{0\#} < \alpha\}$ and fix t. By (5.3) $u(x_n, t_n) < \alpha$ for a sequence (x_n, t_n) of elements of K_u converging to (x, t) with $t_n < t$ for any n, therefore $u^\#(x_n, t_n) < \alpha$ and arguing as in Proposition 3.3 it comes

$$u^\#(x, t) < \alpha$$

this shows the relation

$$\Omega_\alpha \subset \mathrm{int}\{u_{0\#} \geqslant \alpha\}$$

and so using Propositions 3.5, 1.6 the inequality

$$h_\alpha(x) \leqslant S\big(\{u_{0\#} \leqslant \alpha - \varepsilon\}, x\big)$$

can be obtained for any $\alpha \in \mathcal{A}$, ε and $x \in \Omega_\alpha$.

From this arguing as in Proposition 5.1 and exploiting (5.3) and the almost continuity of u it can be derived (5.5) and so

$$u_\# \leqslant w_\#$$

which completes the proof in view of Theorem 3.4. □

Strengthening the assumptions on the initial datum we obtain that the solution w is complete in the terminology of [1], Definition V.4.28.

THEOREM 5.2. – *Assume $u_0 \in ALC^+(\mathbb{R}^N)$. Then w is the unique solution in $LB(\mathbb{R}^N \times]0, +\infty[)$ of* (1.1), (5.1), (5.2).

Proof. – Let $u \in LB(\mathbb{R}^N \times]0, +\infty[)$ a solution of (1.1), (5.1), (5.2) then by Theorem 3.6 and Proposition 5.1

$$w_\# \leqslant u_\# \leqslant u^\# \leqslant w^\#$$

which gives the thesis in view of the almost continuity of w. □

Note that if u_0 is almost continuous the conditions (5.1), (5.2) are equivalent to (5.3), (5.4) and it results

$$\lim_{\substack{(x,t)\to(x_0,0)\\ (x,t)\in K_w}} w(x, t) = u_0(x_0) \quad \text{for any } x_0 \in K_{u_0}.$$

This relation implies that the function which equals w in K_w and u_0 in $K_{u_0} \times \{0\}$ is almost continuous in $\mathbb{R}^N \times [0, +\infty]$.

Finally we give a stability result which will be proved without using explicitly the representation formulae.

We consider a sequence of almost continuous initial data u_{0n} with $\lim^\# u_{0n} = u_0$ and denote by w_n the solutions of (1.1), (5.1), (5.2) with u_0 replaced by u_{0n}.

THEOREM 5.3. – $\lim^\# w_n = w$.

Proof. – Recall that w_n is almost continuous for any n in the light of Theorem 5.2.

By assumption (1.5) $w_{n\#}$ is a l.s.c. supersolution of

$$u_t + \big(a|x| + b\big)|Du| = 0$$

verifying

$$\liminf_{(x,t)\to(x_0,0)} w_{n\#}(x,t) = u_{0n\#}(x_0)$$

for any n, $x_0 \in \mathbb{R}^N$ and $w_n^\#$ is an u.s.c. subsolution of

$$u_t - \big(a|x| + b\big)|Du| = 0$$

verifying

$$\limsup_{(x,t)\to(x_0,0)} w_n^\#(x,t) = u_{0n}^\#(x_0).$$

Consequently the sequence w_n is locally equibounded since u_{0n} is so.

Set

$$\bar{g} = \limsup{}^\# w_n,$$
$$\bar{f} = \liminf{}_\# w_n,$$

and argue as in the proof of claim (3.24) in the Proposition 3.10 to show

$$\limsup_{(y,t)\to(x,0)} \bar{g}(y,t) \leqslant u_0^\#(x) \quad \text{for any } x \in \mathbb{R}^N,$$
$$\liminf_{(y,t)\to(x,0)} \bar{f}(y,t) \geqslant u_{0\#}(x) \quad \text{for any } x \in \mathbb{R}^N.$$

Then Theorem 3.6 yields

$$w_\# \leqslant \bar{f} \leqslant \bar{g} \leqslant w^\#$$

and so the thesis. □

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