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Asymptotic analysis of the linearized Boltzmann collision operator from angular cutoff to non-cutoff

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Abstract. We give quantitative estimates on the asymptotics of the linearized Boltzmann collision operator and its associated equation from angular cutoff to non-cutoff. On one hand, the results disclose the link between the hyperbolic property resulting from Grad's cutoff assumption and the smoothing property due to the long-range interaction. On the other hand, with the help of localization techniques in phase space, we observe some new phenomena in the asymptotic limit process.

Contents

1. Introduction

Let $\mathcal{L}^\varepsilon$ and $\mathcal{L}^0$ be linearized Boltzmann collision operators with and without angular cutoff respectively. Here, ε is the threshold of the angular cutoff $\theta \gtrsim \varepsilon$. The definitions of $\mathcal{L}^\varepsilon$ and $\mathcal{L}^0$ are given in (1.8) based on the Boltzmann collision operator Q^ε in (1.4). The present work aims to find quantitative estimates for the asymptotic behavior of the operator $\mathcal{L}^\varepsilon$ and its associated equation from angular cutoff to non-cutoff, which corresponds to the limit as $\varepsilon \to 0$. Our main motivation comes from the fact that the following properties of the collision operator are totally changed in the limit process:

(1) For fixed $\varepsilon > 0$, $\mathcal{L}^\varepsilon$ behaves like a damping term for the Boltzmann equation with angular cutoff, while $\mathcal{L}^0$ behaves like a fractional Laplace operator for the equation without cutoff.

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(2) For moderate soft potentials ($\gamma \in [-2s, 0)$), the operator $\mathcal{L}^\varepsilon$ has no spectral gap for fixed $\varepsilon > 0$ but the limiting point $\mathcal{L}^0$ of $\{\mathcal{L}^\varepsilon\}_{\varepsilon>0}$ does.

Another motivation arises from the approximation problem of the non-cutoff Boltzmann equation by the cutoff equations. It is of great importance to find some asymptotic formula to quantify the approximation accuracy.

1.1. Boltzmann operator and its linearized version

We first recall the Boltzmann collision operator and the linearized Boltzmann collision operator.

1.1.1. Boltzmann collision operator. The Boltzmann collision operator Q is a bilinear operator defined by

$$Q(g,h)(v) := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)(g'_*h' - g_*h)\,\mathrm{d}v_*\,\mathrm{d}\sigma.$$

Here we use the usual shorthand $h = h(v)$, $g_* = g(v_*)$, $h' = h(v')$, $g'_* = g(v'_*)$ where v', v'_* are given by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \quad \sigma \in \mathbb{S}^2.$$

The nonnegative function $B(v-v_*,\sigma)$ in the collision operator is called the Boltzmann collision kernel. It is always assumed to depend only on $|v-v_*|$ and $\frac{v-v_*}{|v-v_*|}\cdot\sigma$. It is convenient to introduce the angle variable θ through $\cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma$. Without loss of generality, we may assume that $B(v-v_*,\sigma)$ is supported in the set $0 \le \theta \le \frac{\pi}{2}$, i.e., $\cos\theta \ge 0$.

We now state some physically relevant assumptions on the collision kernel. The kernel $B(v-v_*,\sigma)$ satisfies

(A1) The cross-section $B(v-v_*,\sigma)$ takes the product form

$$B(v-v_*,\sigma) = |v-v_*|^\gamma b(\cos\theta),$$

where $-3 < \gamma \le 1$ and b is a nonnegative function satisfying

$$K^{-1}\sin^{-2-2s}\frac{\theta}{2} \le b(\cos\theta) \le K\sin^{-2-2s}\frac{\theta}{2} \quad \text{for any } 0 < \theta \le \frac{\pi}{2}, \tag{1.1}$$

where $0 < s < 1$, $K \ge 1$. The parameters γ and s verify $\gamma + 2s > -1$.

Assumption (A1) covers inverse power law interactions. For inverse repulsive potentials r^{-p}, $p > 1$, one has $\gamma = \frac{p-4}{p}$ and $s = \frac{1}{p}$. Usually, $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$ are called hard, Maxwellian and soft potentials respectively.

The Cauchy problem of the inhomogeneous Boltzmann equation without cutoff reads

$$\begin{cases} \partial_t F + v\cdot\nabla_x F = Q(F,F), & t > 0, x \in \mathbb{T}^3, v \in \mathbb{R}^3, \\ F|_{t=0} = F_0. \end{cases} \tag{1.2}$$

Here $F(t,x,v) \geq 0$ is the density function of collision particles which move with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$, position $x \in \mathbb{T}^3 := [-\pi,\pi]^3$.

Let us denote by $B^\varepsilon(v-v_*,\sigma)$ the cutoff Boltzmann collision kernel.

(A2) For $0 < \varepsilon \leq \frac{\sqrt{2}}{2}$, the angular cutoff kernel $B^\varepsilon(v-v_*,\sigma)$ is defined by

$$\begin{aligned} b^\varepsilon(\cos\theta) &:= b(\cos\theta)(1-\phi(\sin\tfrac{\theta}{2}/\varepsilon)), \\ B^\varepsilon(v-v_*,\sigma) &:= |v-v_*|^\gamma b^\varepsilon(\cos\theta), \end{aligned} \tag{1.3}$$

where ϕ is a smooth function defined in (1.19).

Note that ϕ has support in $[0,\frac{4}{3}]$ and equals 1 in $[0,\frac{3}{4}]$. As a result, $B^\varepsilon(v-v_*,\sigma)$ is supported in $\sin\frac{\theta}{2} \geq \frac{3}{4}\varepsilon$ and thus satisfies the famous Grad angular cutoff assumption.

Note that as $\varepsilon \to 0$, we have $b^\varepsilon \to b$ pointwise. For convenience, let $b^0 := b$, $B^0 := B$. For $\varepsilon \geq 0$, let Q^ε be the Boltzmann operator with kernel B^ε. That is,

$$Q^\varepsilon(g,h)(v) := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B^\varepsilon(v-v_*,\sigma)(g'_*h' - g_*h)\,\mathrm{d}v_*\,\mathrm{d}\sigma. \tag{1.4}$$

The Cauchy problem of the Boltzmann equation with the Boltzmann operator Q^ε is then given by

$$\begin{cases} \partial_t F + v\cdot\nabla_x F = Q^\varepsilon(F,F), & t>0, x\in\mathbb{T}^3, v\in\mathbb{R}^3, \\ F|_{t=0} = F_0. \end{cases} \tag{1.5}$$

Note that when $\varepsilon = 0$, $Q^0 = Q$ and equation (1.5) is the same as (1.2).

We remark that the solutions to (1.2) and (1.5) have the fundamental physical properties of conserving total mass, momentum and kinetic energy, that is, for all $t \geq 0$,

$$\begin{aligned} &\int_{\mathbb{T}^3\times\mathbb{R}^3} [1, v_1, v_2, v_3, |v|^2] F(t,x,v)\,\mathrm{d}x\,\mathrm{d}v \\ &\quad = \int_{\mathbb{T}^3\times\mathbb{R}^3} [1, v_1, v_2, v_3, |v|^2] F_0(x,v)\,\mathrm{d}x\,\mathrm{d}v. \end{aligned} \tag{1.6}$$

Without loss of generality, we assume $F_0(x,v)$ has the same mass, momentum and energy as the Maxwellian $\mu(v) := (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}$. By (1.6), one has for any $t \geq 0$,

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} [1, v_1, v_2, v_3, |v|^2](F(t,x,v)-\mu(v))\,\mathrm{d}x\,\mathrm{d}v = 0. \tag{1.7}$$

1.1.2. Linearized Boltzmann collision operator. For the cutoff case $\varepsilon > 0$ or the non-cutoff case $\varepsilon = 0$, the operators based on Q^ε are defined by

$$\begin{aligned} \Gamma^\varepsilon(g,h) &:= \mu^{-\frac{1}{2}} Q^\varepsilon(\mu^{\frac{1}{2}} g, \mu^{\frac{1}{2}} h), \\ \mathcal{L}_1^\varepsilon g := -\Gamma^\varepsilon(\mu^{\frac{1}{2}}, g), \quad &\mathcal{L}_2^\varepsilon g := -\Gamma^\varepsilon(g, \mu^{\frac{1}{2}}), \quad \mathcal{L}^\varepsilon g := \mathcal{L}_1^\varepsilon g + \mathcal{L}_2^\varepsilon g. \end{aligned} \tag{1.8}$$

We recall that the null space $\mathcal{N}(\mathcal{L}^\varepsilon)$ of $\mathcal{L}^\varepsilon$ reads

$$\mathcal{N}(\mathcal{L}^\varepsilon) = \mathcal{N} := \operatorname{span}\{\mu^{\frac{1}{2}}, \mu^{\frac{1}{2}} v_1, \mu^{\frac{1}{2}} v_2, \mu^{\frac{1}{2}} v_3, \mu^{\frac{1}{2}} |v|^2\}.$$

With the expansion $F = \mu + \mu^{\frac{1}{2}} f$, the two problems (1.5) and (1.2) reduce to

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathcal{L}^\varepsilon f = \Gamma^\varepsilon(f, f), & t > 0, x \in \mathbb{T}^3, v \in \mathbb{R}^3, \\ f|_{t=0} = f_0, \end{cases} \tag{1.9}$$

and

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathcal{L}^0 f = \Gamma^0(f, f), & t > 0, x \in \mathbb{T}^3, v \in \mathbb{R}^3, \\ f|_{t=0} = f_0, \end{cases} \tag{1.10}$$

where $f_0 = \mu^{-\frac{1}{2}}(F_0 - \mu)$ verifies

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} [1, v_1, v_2, v_3, |v|^2] \mu^{\frac{1}{2}}(v) f_0(x, v) \, \mathrm{d}x \, \mathrm{d}v = 0. \tag{1.11}$$

1.2. Problems and difficulties

The main purpose of the paper is to understand what happens to the linear operator $\mathcal{L}^\varepsilon$ and the nonlinear equation (1.9) in the limit as $\varepsilon \to 0$. More concretely, we are concerned with the following three problems.

Problem 1. What is the behavior change of the operator $\mathcal{L}^\varepsilon$ in the limit process?

We recall that $\mathcal{L}^\varepsilon$ behaves like a damping term for equation (1.9) while $\mathcal{L}^0$ behaves like a fractional Laplace operator for equation (1.10). The motivation of Problem 1 is to see clearly the kind of link between these two different properties in the limit process. Obviously, it is a fundamental but challenging problem.

To explain the main difficulty of the problem, we focus on the Maxwellian molecules ($\gamma = 0$), which is simpler than the other cases. Previous works [3,4,7,8,10] show that for $\gamma = 0$, there holds

$$\langle \mathcal{L}^0 f, f \rangle_v + |f|^2_{L^2} \sim |f|^2_{L^2_s} + |f|^2_{H^s} + |(-\Delta_{\mathbb{S}^2})^{\frac{s}{2}} f|^2_{L^2}. \tag{1.12}$$

Here $\langle f, g \rangle_v$ denotes the inner product for the v variable. On the right-hand side of the equivalence (1.12), there are three parts, which correspond to gain of weight $|f|^2_{L^2_s}$, gain of Sobolev regularity $|f|^2_{H^s}$ and gain of tangential derivative on the sphere $|(-\Delta_{\mathbb{S}^2})^{\frac{s}{2}} f|^2_{L^2}$ respectively. Observe that (1.12) can be rewritten as

$$\langle \mathcal{L}^0 f, f \rangle_v + |f|^2_{L^2} \sim |W_s f|^2_{L^2} + |W_s(D) f|^2_{L^2} + |W_s((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}) f|^2_{L^2}, \tag{1.13}$$

where $W_s(x) := (1 + |x|^2)^{\frac{s}{2}}$. Here, $W_s(D)$ is the pseudo-differential operator with symbol W_s and the operator $W_s((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})$ is defined in (1.23). As W_s serves as a common weight function in the three parts, we call W_s the *characteristic function* of $\mathcal{L}^0$.

Considering $\langle \mathcal{L}^\varepsilon f, f \rangle_v \to \langle \mathcal{L}^0 f, f \rangle_v$ as $\varepsilon \to 0$, we guess that $\langle \mathcal{L}^\varepsilon f, f \rangle_v$ has the same structure as the right-hand side of (1.13). If so, what is the characteristic function of $\mathcal{L}^\varepsilon$ when $\varepsilon > 0$? To find a good candidate, we go back to the original proof of the coercivity estimate for the collision operator in [1]. Following the computation used there, we can derive that

$$-\langle Q^{\varepsilon}(g,f),f\rangle_v+|f|^2_{L^2}\geq C_g|W^{\varepsilon}(D)f|^2_{L^2}, \tag{1.14}$$

where W^{ε} is defined by

$$W^{\varepsilon}(v)=W_s\phi(\varepsilon v)+\varepsilon^{-s}(1-\phi(\varepsilon v)). \tag{1.15}$$

Here, $\phi\in C_0^{\infty}(B_{\frac{4}{3}})$ is the smooth compactly supported function in (1.19). Note that as $\varepsilon\to 0$, we have $W^{\varepsilon}\to W_s$ at least pointwise. For convenience, let $W^0:=W_s$. We conjecture that W^{ε} is the characteristic function of $\mathcal{L}^{\varepsilon}$ in the following sense:

$$\langle\mathcal{L}^{\varepsilon}f,f\rangle_v+|f|^2_{L^2}\sim|W^{\varepsilon}f|^2_{L^2}+|W^{\varepsilon}(D)f|^2_{L^2}+|W^{\varepsilon}((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})f|^2_{L^2}. \tag{1.16}$$

The operator $W^{\varepsilon}((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})$ is defined in (1.23).

Let us give some comments on conjecture (1.16). Firstly, it is easy to see that when ε goes to zero, (1.16) will coincide with (1.13). This shows that the characteristic function W^{ε} connects the cutoff case and the non-cutoff case. Secondly, on the right-hand side of (1.16), gain of weight only happens in the region $|v|\lesssim\frac{1}{\varepsilon}$ in phase space, gain of Sobolev regularity only happens in the region $|\xi|\lesssim\frac{1}{\varepsilon}$ in frequency space and gain of tangential derivative only happens in the region that the eigenvalue λ of the operator $(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}}$ verifies $\lambda\lesssim\frac{1}{\varepsilon}$. These properties are consistent with the fact that the operator $\mathcal{L}^{\varepsilon}$ has a hyperbolic structure due to the angular cutoff, that is, $\theta\gtrsim\varepsilon$. Thirdly, because of the hyperbolic structure of $\mathcal{L}^{\varepsilon}$, it is unclear how to derive $|W^{\varepsilon}((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})f|_{L^2}$ and $|W^{\varepsilon}f|_{L^2}$ in (1.16) using the methods in the previous works [2–4, 7, 8, 10, 14, 17]. Therefore we need some new ideas to prove the conjecture.

Problem 2. What is the longtime behavior of $e^{-\mathcal{L}^{\varepsilon}t}f$ with $f\in\mathcal{N}^{\perp}$ for moderate soft potentials in the limit process as $\varepsilon\to 0$? Here, $e^{-\mathcal{L}^{\varepsilon}t}$ is the semigroup generated by $\mathcal{L}^{\varepsilon}$.

As we know, for $\gamma\in[-2s,0)$, the operator $\mathcal{L}^{\varepsilon}$ has no spectral gap for any fixed $\varepsilon>0$ but the limiting point $\mathcal{L}^0$ of $\{\mathcal{L}^{\varepsilon}\}_{\varepsilon>0}$ does. It seems that there is a jump. Rather than investigating the spectrum of the operator, which looks extremely difficult, we instead turn to consider the longtime behavior of $e^{-\mathcal{L}^{\varepsilon}t}f$ because the spectrum information of an operator has a strong connection with the corresponding semigroup.

Thanks to spectral gap of $\mathcal{L}^0$, it is easy to see that for any $f\in\mathcal{N}^{\perp}$,

$$\|e^{-\mathcal{L}^0t}f\|_{L^2}\leq e^{-ct}\|f\|_{L^2}.$$

As for the operator $\mathcal{L}^{\varepsilon}$, by imposing the additional assumption that $f\in L^2_l$, we can derive that $e^{-\mathcal{L}^{\varepsilon}t}f$ will decay to zero with polynomial rate. However, we have no idea about the explicit rate of this relaxation for $f\in\mathcal{N}^{\perp}$ if we only impose $f\in L^2$. By an approximation argument, we can only prove that

$$\lim_{t\to\infty}\|e^{-\mathcal{L}^{\varepsilon}t}f\|_{L^2}=0.$$

Therefore, from these two estimates, it is hard to find the link between these two different longtime behaviors. We emphasize that this difficulty matches the fact that $\mathcal{L}^{\varepsilon}$ does not have a spectral gap but $\mathcal{L}^0$ does.

Problem 3. Which kind of asymptotic formula describes the limit as $\varepsilon \to 0$ for the solutions of the nonlinear equations (1.9) and (1.10)?

Formally, when ε goes to zero, the solution f^ε to (1.9) will converge to the solution f^0 to (1.10). To answer Problem 3 is to justify the convergence and find an asymptotic formula.

To guess the relation between f^ε and f^0, we first take a look at the stationary case. By Taylor expansion, we can prove that for any smooth compactly supported functions f,

$$|Q^\varepsilon(f,f) - Q^0(f,f)| \sim O(\varepsilon^{2-2s}). \tag{1.17}$$

Thus, it is natural to conjecture

$$f^\varepsilon - f^0 = O(\varepsilon^{2-2s}). \tag{1.18}$$

Obviously, the main difficulty in establishing (1.18) lies in bringing the error order (1.17) from operator level to solution level. To this end, we need some uniform (with respect to ε) estimates of $\mathcal{L}^\varepsilon$ and Γ^ε, and also estimates of the differences $\mathcal{L}^\varepsilon - \mathcal{L}^0$ and $\Gamma^\varepsilon - \Gamma^0$.

1.3. Notation

We list the function spaces and notation that will be used throughout the paper.

1.3.1. Basic notation. We denote a multi-index by $\alpha = (\alpha^1, \alpha^2, \alpha^3) \in \mathbb{N}^3$ with $|\alpha| = \alpha^1 + \alpha^2 + \alpha^3$. We write $a \lesssim b$ to indicate that there is a universal constant C, which may be different on different lines, such that $a \le Cb$. We use the notation $a \sim b$ whenever $a \lesssim b$ and $b \lesssim a$. The notation $[a]$ denotes the maximum integer which does not exceed a. The bracket $\langle \cdot \rangle$ is defined by $\langle v \rangle := (1 + |v|^2)^{\frac{1}{2}}$. Then the weight function W_l is defined by $W_l(v) := \langle v \rangle^l$. We denote by $C(\lambda_1, \lambda_2, \dots, \lambda_n)$ or $C_{\lambda_1,\lambda_2,\dots,\lambda_n}$ a constant depending on parameters $\lambda_1, \lambda_2, \dots, \lambda_n$. The notation $\langle f, g \rangle_v := \int_{\mathbb{R}^3} f(v)g(v)\,\mathrm{d}v$, $\langle f, g \rangle_x := \int_{\mathbb{T}^3} f(x)g(x)\,\mathrm{d}x$ and $(f, g) := \int_{\mathbb{T}^3 \times \mathbb{R}^3} f(x,v)g(x,v)\,\mathrm{d}x\,\mathrm{d}v$ is used to denote the inner products in $L^2(\mathbb{R}^3)$, $L^2(\mathbb{T}^3)$ and $L^2(\mathbb{T}^3 \times \mathbb{R}^3)$ respectively. As usual, 1_A is the characteristic function of a set A. If A, B are two operators, then their commutator $[A, B] := AB - BA$. Recall that $|f|_{L \log L} := \int_{\mathbb{R}^3} |f(v)| \log(1 + |f(v)|)\,\mathrm{d}v$.

1.3.2. Function spaces. Several spaces are introduced, as follows.

- For $n, l \in \mathbb{R}$, we define the weighted Sobolev space on $\mathbb{R}^3$ by

$$H^n_l := \left\{ f(v) \,\middle|\, |f|^2_{H^n_l} := \int_{\mathbb{R}^3} |(W_n(D) W_l f)(v)|^2\,\mathrm{d}v < \infty \right\}.$$

  For any symbol $a: \mathbb{R}^3 \to \mathbb{R}$, recall that $a(D)$ is the pseudo-differential operator defined by

$$(a(D)f)(v) := \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{\mathrm{i}(v-y)\cdot\xi} a(\xi) f(y)\,\mathrm{d}y\,\mathrm{d}\xi.$$

- For $p \ge 1, l \in \mathbb{R}$, we introduce the L^p_l space on $\mathbb{R}^3$ as

$$L^p_l := \left\{ f(v) \,\middle|\, |f|^p_{L^p_l} := \int_{\mathbb{R}^3} |f(v)|^p \langle v \rangle^{lp}\,\mathrm{d}v < \infty \right\}.$$

- For $m \in \mathbb{N}$, we denote the Sobolev space on $\mathbb{T}^3$ by

$$H_x^m := \big\{ f(x) \,\big|\, |f|_{H_x^m}^2 := \textstyle\sum_{|\alpha|\le m} \int_{\mathbb{T}^3} |\partial_x^\alpha f(x)|^2 \,\mathrm{d}x < \infty \big\}.$$

- For a function $f(x,v)$, we define the following weighted Sobolev spaces with weight on velocity variable v. For $m, n \in \mathbb{N}$, $l \in \mathbb{R}$, the weighted (in v) Sobolev space on $\mathbb{T}^3 \times \mathbb{R}^3$ is defined by

$$H_x^m H_l^n := \big\{ f(x,v) \,\big|\, \|f\|_{H_x^m H_l^n}^2 := \textstyle\sum_{|\alpha|\le m, |\beta|\le n} \int_{\mathbb{T}^3} |\partial_x^\alpha \partial_v^\beta f(x,\cdot)|_{L_l^2}^2 \,\mathrm{d}x < \infty \big\}.$$

For simplicity, we write $\|f\|_{H_x^m L_l^2} := \|f\|_{H_x^m H_l^0}$ if $n = 0$ and $\|f\|_{L_l^2} := \|f\|_{H_x^0 H_l^0}$ if $m = n = 0$. We can define the homogeneous space $\dot{H}_x^m \dot{H}_l^n$ if we replace $|\alpha| \le m$, $|\beta| \le n$ with $|\alpha| = m$, $|\beta| = n$. Similarly, we can introduce the partial homogeneous spaces $\dot{H}_x^m H_l^n$ and $H_x^m \dot{H}_l^n$.

1.3.3. Dyadic decompositions. We will now recall dyadic decomposition. Let

$$B_{\frac{4}{3}} := \{v \in \mathbb{R}^3 \mid |v| \le \tfrac{4}{3}\}, \quad C := \{v \in \mathbb{R}^3 \mid \tfrac{3}{4} \le |v| \le \tfrac{8}{3}\}.$$

Then one may introduce two radial functions $\phi \in C_0^\infty(B_{\frac{4}{3}})$ and $\psi \in C_0^\infty(C)$ which satisfy

$$0 \le \phi, \psi \le 1 \quad \text{and} \quad \phi(v) + \sum_{j\ge 0} \psi(2^{-j} v) = 1 \ \text{ for all } v \in \mathbb{R}^3. \tag{1.19}$$

Since ϕ is a radial function, we can interchangeably use $\phi(v)$ and $\phi(|v|)$. Now define $\varphi_{-1}(v) := \phi(v)$ and $\varphi_j(v) := \psi(2^{-j} v)$ for any $v \in \mathbb{R}^3$ and $j \ge 0$. Let $(\mathcal{P}_j f)(v) := \varphi_j(v) f(v)$; then one has the following dyadic decomposition:

$$f = \sum_{j=-1}^{\infty} \mathcal{P}_j f = \sum_{j=-1}^{\infty} \varphi_j f$$

for any function f defined on $\mathbb{R}^3$. We will use the notation

$$f_\phi := \phi(\varepsilon D) f, \quad f^\phi := (1 - \phi(\varepsilon D)) f, \quad f^l := \phi(\varepsilon \cdot) f, \quad f^h := (1 - \phi(\varepsilon \cdot)) f. \tag{1.20}$$

1.3.4. Projection on the null space. Recalling that $\mathcal{N} = \mathrm{span}\{\mu^{\frac12}, \mu^{\frac12} v_1, \mu^{\frac12} v_2, \mu^{\frac12} v_3, \mu^{\frac12} |v|^2\}$, the projection operator $\mathbb{P}$ on $\mathcal{N}$ is defined by

$$\mathbb{P} f := (a + b \cdot v + c|v|^2) \mu^{\frac12}, \tag{1.21}$$

where for $1 \le i \le 3$,

$$\begin{aligned} a &= \int_{\mathbb{R}^3} \Big(\frac{5}{2} - \frac{|v|^2}{2} \Big) \mu^{\frac12} f \,\mathrm{d}v, \quad b_i = \int_{\mathbb{R}^3} v_i \mu^{\frac12} f \,\mathrm{d}v, \\ c &= \int_{\mathbb{R}^3} \Big(\frac{|v|^2}{6} - \frac{1}{2} \Big) \mu^{\frac12} f \,\mathrm{d}v. \end{aligned} \tag{1.22}$$

1.3.5. Anisotropic function spaces. Let Y_l^m with $l \in \mathbb{N}, -l \leq m \leq l$ be real spherical harmonics verifying that $(-\Delta_{\mathbb{S}^2})Y_l^m = l(l+1)Y_l^m$. Then the operator $W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})$ for $\varepsilon \geq 0$ is defined as follows: if $v = r\sigma$, then

$$\big(W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})f\big)(v) := \sum_{l=0}^{\infty}\sum_{m=-l}^{l} W^\varepsilon\big((l(l+1))^{\frac{1}{2}}\big)Y_l^m(\sigma)f_l^m(r), \tag{1.23}$$

where $f_l^m(r) = \int_{\mathbb{S}^2} Y_l^m(\sigma)f(r\sigma)\,d\sigma$. We recall that when $\varepsilon > 0$, W^ε is defined in (1.15). When $\varepsilon = 0$, $W^0(v) = \langle v\rangle^s$.

Now we introduce several anisotropic function spaces induced by $\mathcal{L}^\varepsilon$.

- *The space $L^2_{\varepsilon,l}$ with $l \in \mathbb{R}$.* For functions on $\mathbb{R}^3$, the space $L^2_{\varepsilon,l}$ is defined by

$$L^2_{\varepsilon,l} := \big\{f(v) \,\big|\, |f|^2_{\varepsilon,l} := |W^\varepsilon W_l f|^2_{L^2} + |W^\varepsilon(D)W_l f|^2_{L^2} + |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})W_l f|^2_{L^2} < \infty\big\}. \tag{1.24}$$

- *The space $H^m_x H^n_{\varepsilon,l}$ with $m,n \in \mathbb{N}$, $l \in \mathbb{R}$.* For functions on $\mathbb{T}^3 \times \mathbb{R}^3$, the space $H^m_x H^n_{\varepsilon,l}$ is defined by

$$H^m_x H^n_{\varepsilon,l} := \big\{f(x,v) \,\big|\, \|f\|^2_{H^m_x H^n_{\varepsilon,l}} := \textstyle\sum_{|\alpha|\leq m,|\beta|\leq n}\int_{\mathbb{T}^3} |\partial^\alpha_x \partial^\beta_v f(x,\cdot)|^2_{\varepsilon,l}\,dx < \infty\big\}.$$

  For simplicity, we set $\|f\|_{H^m_x L^2_{\varepsilon,l}} := \|f\|_{H^m_x H^0_{\varepsilon,l}}$ if $n = 0$ and $\|f\|_{L^2_{\varepsilon,l}} := \|f\|_{H^0_x H^0_{\varepsilon,l}}$ if $m = n = 0$. Similarly we can introduce the spaces $\dot{H}^m_x \dot{H}^n_{\varepsilon,l}$, $\dot{H}^m_x H^n_{\varepsilon,l}$ and $H^m_x \dot{H}^n_{\varepsilon,l}$.

- *Functionals related to $\mathcal{L}^\varepsilon$.* We introduce

$$\mathcal{R}^{\varepsilon,\gamma}_g(f) := \int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} b^\varepsilon(\cos\theta)|v-v_*|^\gamma g_*(f'-f)^2\,d\sigma\,dv_*\,dv, \tag{1.25}$$

$$\mathcal{R}^{\varepsilon,\gamma}_{*,g}(f) := \int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma g_*(f'-f)^2\,d\sigma\,dv_*\,dv, \tag{1.26}$$

$$\mathcal{M}^{\varepsilon,\gamma}(f) := \int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} b^\varepsilon(\cos\theta)|v-v_*|^\gamma f_*^2((\mu^{\frac{1}{2}})'-\mu^{\frac{1}{2}})^2\,d\sigma\,dv_*\,dv. \tag{1.27}$$

  As we will show in Section 2,

$$\langle\mathcal{L}^\varepsilon f, f\rangle_v + |f|^2_{L^2_{\gamma/2}} \gtrsim \mathcal{R}^{\varepsilon,\gamma}_\mu(f) + \mathcal{M}^{\varepsilon,\gamma}(f).$$

The quantities $\mathcal{R}^{\varepsilon,\gamma}_g(f)$ and $\mathcal{M}^{\varepsilon,\gamma}(f)$ correspond to gain of regularity and gain of weight respectively. In contrast to $\mathcal{R}^{\varepsilon,\gamma}_g(f)$, when $\gamma < 0$, $\mathcal{R}^{\varepsilon,\gamma}_{*,g}(f)$ contains no singularity in the relative velocity $v - v_*$ near the origin.

1.4. Main results

Now we are ready to state our main results. The first one is a uniform coercivity estimate for $\mathcal{L}^\varepsilon$, which fully solves Problem 1.

Theorem 1.1. *There exists a constant* $\varepsilon_0 > 0$ *such that for* $0 \le \varepsilon \le \varepsilon_0$ *and any suitable function* f,

$$\langle \mathcal{L}^\varepsilon f, f\rangle_v + |f|^2_{L^2_{\gamma/2}} \sim |f|^2_{\varepsilon,\gamma/2}. \tag{1.28}$$

Here the norm $|\cdot|_{\varepsilon,\gamma/2}$ *is defined in* (1.24).

Some remarks are in order.

Remark 1.1. Through the characteristic function W^ε, the coercivity estimate (1.28) discloses the link between the hyperbolic structure due to the cutoff assumption ($\varepsilon > 0$) and the smoothing property due to the long-range interaction ($\varepsilon = 0$).

Remark 1.2. Recall f^l and f^h in (1.20); then

$$|W^\varepsilon f|^2_{L^2_{\gamma/2}} \sim |f^l|^2_{L^2_{\gamma/2+s}} + \varepsilon^{-2s}|f^h|^2_{L^2_{\gamma/2}}.$$

Let us consider the moderate soft potentials, i.e., $\gamma \in [-2s, 0)$. In the region $|v| \lesssim \frac{1}{\varepsilon}$ of phase space, the operator $\mathcal{L}^\varepsilon$ produces some weight since $\gamma + 2s \ge 0$, while in the region $|v| \gtrsim \frac{1}{\varepsilon}$, the operator $\mathcal{L}^\varepsilon$ loses some weight since $\gamma < 0$. This observation is consistent with the fact that $\mathcal{L}^\varepsilon$ has no spectral gap for any fixed $\varepsilon > 0$ but $\mathcal{L}^0$ does.

Our second result is on the diversity of the longtime behavior of $e^{-\mathcal{L}^\varepsilon t} f_0$ with $f_0 \in \mathcal{N}^\perp$ for moderate soft potentials, which solves Problem 2.

Theorem 1.2. *Suppose* $0 < \varepsilon \le \varepsilon_0$, $-2s \le \gamma < 0$ *and* $f_0 \in \mathcal{N}^\perp$. *There is a universal constant* $c > 0$ *such that*

$$|e^{-\mathcal{L}^\varepsilon t} f_0|^2_{L^2} \lesssim e^{-ct}|f_0^l|^2_{L^2} + |f_0^h|^2_{L^2} + \varepsilon^{2s}|f_0|^2_{L^2}, \tag{1.29}$$

where f_0^l *and* f_0^h *are given by* (1.20). *Furthermore, the following statements are valid:*

(1) *Let* $q > 0$ *and* $-\gamma q/2 \ge 2$. *Suppose* $f_0 \in L^2_{-\gamma q/2}$. *Depending on the relation between* $|f_0|_{L^2}$ *and* $\varepsilon^{sq}|f_0|_{L^2_{-\gamma q/2}}$, *we have two estimates:*

(a) *If* $|f_0|_{L^2} > 2C_0^q\varepsilon^{sq}|f_0|_{L^2_{-\gamma q/2}}$, *then*

$$|e^{-\mathcal{L}^\varepsilon t} f_0|^2_{L^2} \le |f_0|^2_{L^2} e^{-\lambda_0 t} 1_{t<t_*} + \frac{|e^{-\mathcal{L}^\varepsilon t_*} f_0|^2_{L^2}}{(1 + \lambda_0 q^{-1}(t - t_*))^q} 1_{t \ge t_*}, \tag{1.30}$$

where t_* *is the time verifying*

$$|e^{-\mathcal{L}^\varepsilon t_*} f_0|_{L^2} = 2C_0^q \varepsilon^{sq}|f_0|_{L^2_{-\gamma q/2}}, \quad t_* \le 2\lambda_0^{-1} \ln \frac{|f_0|_{L^2}}{2C_0^q \varepsilon^{sq}|f_0|_{L^2_{-\gamma q/2}}}.$$

Here C_0, λ_0 *are the constants given in* (3.34).

(b) *If* $|f_0|_{L^2} \le 2C_0^q\varepsilon^{sq}|f_0|_{L^2_{-\gamma q/2}}$, *then*

$$|e^{-\mathcal{L}^\varepsilon t} f_0|^2_{L^2} \le \frac{|f_0|^2_{L^2}}{(1 + C(f_0)t)^q}, \tag{1.31}$$

where

$$C(f_0) = C_1 4^{-\frac{1}{q}} q^{-1} \Big(\frac{|f_0|_{L^2}}{|f_0|_{L^2_{-\gamma q/2}}} \Big)^{\frac{2}{q}} \varepsilon^{-2s}$$

and C_1 is the constant given in (3.34).

(2) *Let $0 < \eta < 1$. Let $j \in \mathbb{N}$ be large enough such that $2^j \geq \varepsilon^{-1}$. Let f_0 verify $|f_0|_{L^2} = 1$ and $|\mathcal{P}_j f_0|^2_{L^2} = 1 - \eta$. Then for $t \in [0, C^{-1}\eta 2^{-j\gamma}\varepsilon^{2s}]$, it holds that*

$$|e^{-\mathcal{L}^\varepsilon t} f_0|^2_{L^2} \geq |\mathcal{P}_j e^{-\mathcal{L}^\varepsilon t} f_0|^2_{L^2} \geq 1 - 2\eta - C\varepsilon^{2s}, \tag{1.32}$$

where C is a universal constant.

As a consequence, for any fixed sufficiently small $\varepsilon > 0$, the estimate

$$\lim_{t\to\infty} |e^{-\mathcal{L}^\varepsilon t} f_0|_{L^2} = 0$$

is sharp.

We have some remarks on Theorem 1.2.

Remark 1.3. We have three comments on estimate (1.29). Firstly, (1.29) shows that the longtime behavior of $e^{-\mathcal{L}^\varepsilon t} f_0$ depends heavily on the energy distribution of f_0. Secondly, the estimate is sharp for general data $f_0 \in \mathcal{N}^\perp$ thanks to estimates (1.30) and (1.32), which deal with the case that the energy of f_0 is concentrated in the ball $B_{1/\varepsilon}$ and the case that the energy of f_0 is concentrated far away from the ball $B_{1/\varepsilon}$. Thirdly, by passing to the limit $\varepsilon \to 0$, we recover from (1.29) that for all $t \geq 0$,

$$|e^{-\mathcal{L}^0 t} f_0|^2_{L^2} \lesssim e^{-ct} |f_0|^2_{L^2}. \tag{1.33}$$

This demonstrates that there is no jump for the fact that the operator $\mathcal{L}^\varepsilon$ does not have a spectral gap for fixed $\varepsilon > 0$ but $\mathcal{L}^0$ does.

Remark 1.4. Estimates (1.30) and (1.33) show that up to a critical time $t_* = O(|\ln \varepsilon|)$, in terms of decay pattern, there is no difference between $e^{-\mathcal{L}^\varepsilon t} f_0$ and $e^{-\mathcal{L}^0 t} f_0$. The difference appears only after the critical time t_*. In fact, after t_* the hyperbolic structure will take over the behavior of the semigroup $e^{-\mathcal{L}^\varepsilon t}$, which corresponds to the polynomial decay in (1.30). To the best of our knowledge, this phenomenon is being observed for the first time.

Remark 1.5. We have two remarks on (1.32). Firstly, by taking η sufficiently small and j sufficiently large, the total energy of f_0 can be almost conserved in $e^{-\mathcal{L}^\varepsilon t} f_0$ in any given time interval. Such a datum prevents the formation of a spectral gap for $\mathcal{L}^\varepsilon$, no matter how small $\varepsilon > 0$ is. Secondly, we want to show there are extensive data f_0 verifying all the assumptions. Take an arbitrary function $f \in L^2$ with $|f|_{L^2} = 1$ and the support of f belonging to the ring $\{v \in \mathbb{R}^3 \mid \frac{4}{3} \times 2^j \leq |v| \leq \frac{3}{2} \times 2^j\}$. Let $f_0 = f - \mathbb{P} f$. Then f_0 verifies $f_0 \in \mathcal{N}^\perp$, $|\mathcal{P}_j f_0|_{L^2} \geq 1 - O(e^{-\frac{1}{8}\times 2^{2j}})$. Then $f_0/|f_0|_{L^2}$ fulfills all the assumptions.

Remark 1.6. The sharpness of the estimate $\lim_{t\to\infty}|e^{-\mathcal{L}^{\varepsilon}t}f_0|_{L^2}=0$ directly follows from (1.30)–(1.32). On one hand, the estimate can be derived thanks to (1.30) and (1.31). On the other hand, due to (1.32), it is impossible to get an explicit and uniform decay rate for the above relaxation. These two facts reveal the diversity of the longtime behavior of $e^{-\mathcal{L}^{\varepsilon}t}f_0$. Our results are comparable to the results for the homogeneous Boltzmann equation with moderate soft potentials. As is shown in [5], the rate of convergence to equilibrium can be very slow if we only assume that a solution conserves mass, momentum and energy.

Remark 1.7. Let us comment on the connection between the constant λ_0 in (1.30) and the spectral gap λ of the operator $\mathcal{L}^0$. Obviously, $\lambda_0\le\lambda$. It is interesting but challenging to see the dependence of λ on λ_0. By Lemma 4.3, there holds $|(\mathcal{L}^{\varepsilon}-\mathcal{L}^0)f|_{L^2}\lesssim\varepsilon^{2-2s}|f|_{H^2_{\gamma+2}}$. Therefore if $f_0\in H^2_{\gamma+2}$, besides (1.30), it also holds that

$$|e^{-\mathcal{L}^{\varepsilon}t}f_0|^2_{L^2}\lesssim e^{-\lambda t}|f_0|^2_{L^2}+\varepsilon^{2-2s}|f_0|^2_{H^2_{\gamma+2}}.$$

Our third result is on global well-posedness, propagation of regularity and global dynamics of equation (1.9). Based on propagation of regularity, we derive an asymptotic formula for solutions to (1.9) and (1.10), which solves Problem 3.

Let $\gamma<0$. For $N\in\mathbb{N}$ with $N\ge 2$, we introduce a sequence of weight functions $\{W_{l_j}\}_{0\le j\le N}$ with $l_j\in\mathbb{R}$ verifying

$$l_N\ge 2,\quad l_j\ge l_{j+1}-\gamma. \tag{1.34}$$

We remark that l_j is the weight order for the v derivative of order j. Note that (1.34) means that the weight order increases by $-\gamma$ ($\gamma<0$) if the v derivative order decreases by 1. This type of weight sequence is designed to control the term $v\cdot\nabla_x f$ and is used in [9] in the angular cutoff case.

Let $\partial^{\alpha}_{\beta}:=\partial^{\alpha}_{x}\partial^{\beta}_{v}$. For $0\le k\le N-1$, $0\le J\le N$, we define energy and dissipation functionals

$$\begin{aligned}\dot{\mathcal{E}}^{m,j}(f)&:=\sum_{|\alpha|=m,|\beta|=j}\|W_{l_j}\partial^{\alpha}_{\beta}f\|^2_{L^2},\\ \dot{\mathcal{D}}^{m,j}(f)&:=\sum_{|\alpha|=m,|\beta|=j}\|W_{l_j}\partial^{\alpha}_{\beta}f\|^2_{L^2_{\varepsilon,\gamma/2}},\end{aligned}\tag{1.35}$$

$$\dot{\mathcal{E}}^{k}(f):=\sum_{j=0}^{k}\dot{\mathcal{E}}^{k-j,j}(f),\quad \dot{\mathcal{D}}^{k}(f):=\sum_{j=0}^{k}\dot{\mathcal{D}}^{k-j,j}(f),\tag{1.36}$$

$$\begin{aligned}\mathcal{E}^{N,J}(f)&:=\sum_{k=0}^{N-1}\dot{\mathcal{E}}^{k}(f)+\sum_{j=0}^{J}\dot{\mathcal{E}}^{N-j,j}(f),\\ \mathcal{D}^{N,J}(f)&:=\sum_{k=0}^{N-1}\dot{\mathcal{D}}^{k}(f)+\sum_{j=0}^{J}\dot{\mathcal{D}}^{N-j,j}(f).\end{aligned}\tag{1.37}$$

Here $\dot{\mathcal{E}}^{m,j}$ contains all x derivatives of order m and v derivatives of order j. The functional $\dot{\mathcal{E}}^k$ contains all mixed x and v derivatives of total order k, i.e., $|\alpha| + |\beta| = k$. The functional $\mathcal{E}^{N,J}$ contains derivatives ∂^α_β with either $|\alpha| + |\beta| \leq N - 1$ or $|\alpha| + |\beta| = N$, $|\beta| \leq J$. The functionals $\dot{\mathcal{D}}^{m,j}$, $\dot{\mathcal{D}}^k$ and $\mathcal{D}^{N,J}$ are the corresponding dissipations. If $J = N$, we simplify the notation to $\mathcal{E}^N(f) := \mathcal{E}^{N,N}(f)$, $\mathcal{D}^N(f) := \mathcal{D}^{N,N}(f)$. The energy functional $\mathcal{E}^{N,J}$ is introduced to prove the propagation of full regularity of the solution.

We are ready to present our last main result.

Theorem 1.3. *Let $0 \leq \varepsilon \leq \varepsilon_0$, $\gamma \in (-\frac{3}{2}, 0) \cap [-2s, 0)$. There is a constant $\delta_0 > 0$ independent of ε such that the following statements are valid. Let f_0 verify (1.11) and $\|f_0\|_{H^2_x L^2} \leq \delta_0$.*

(1) (Global well-posedness and propagation of regularity) *The Cauchy problem (1.9) (which is problem (1.10) if $\varepsilon = 0$) admits a unique and global solution f^ε verifying $\sup_{t\geq 0} \|f^\varepsilon(t)\|_{H^2_x L^2} \lesssim \|f_0\|_{H^2_x L^2}$.*

(i) *If, additionally, $f_0 \in H^N_x L^2_l$ with $N, l \geq 2$, then*

$$\sup_{t\geq 0} \|f^\varepsilon(t)\|^2_{H^N_x L^2_l} + \int_0^\infty \|f^\varepsilon(\tau)\|^2_{H^N_x L^2_{\varepsilon,l+\gamma/2}} \mathrm{d}\tau \leq C(\|f_0\|^2_{H^N_x L^2_l}). \quad (1.38)$$

(ii) *If, additionally, $\mathcal{E}^{N,J}(f_0) < \infty$ with $N \geq 2$, $0 \leq J \leq N$, then*

$$\sup_{t\geq 0} \mathcal{E}^{N,J}(f^\varepsilon(t)) + \int_0^\infty \mathcal{D}^{N,J}(f^\varepsilon(\tau)) \,\mathrm{d}\tau \leq C(\mathcal{E}^{N,J}(f_0)). \quad (1.39)$$

Here $C(\cdot)$ is a continuous increasing function verifying $C(0) = 0$.

(2) (Global dynamics) *There are two results.*

(i) *If $f_0 \in H^2_x L^2_{-q\gamma/2}$ with $q > 0$, $-q\gamma/2 \geq 2$, then, depending on the relation between $\mathcal{E}_{2,M}(f_0)$ and $M\varepsilon^{2sq}\|f_0\|^2_{H^2_x L^2_{-\gamma q/2}}$, we have two estimates. Here, the functional $\mathcal{E}_{2,M}(\cdot)$ is defined in (4.3) and verifies*

$$\frac{1}{2} M\|\cdot\|^2_{H^2_x L^2} \leq \mathcal{E}_{2,M}(\cdot) \leq 2M\|\cdot\|^2_{H^2_x L^2}$$

for some universal constant M.

(a) *If $\mathcal{E}_{2,M}(f_0) > MC_q\varepsilon^{2sq}\|f_0\|^2_{H^2_x L^2_{-\gamma q/2}}$, then*

$$\mathcal{E}_{2,M}(f^\varepsilon(t)) \leq \mathcal{E}_{2,M}(f_0)e^{-\lambda_0 t}1_{t<t_*} + \frac{\mathcal{E}_{2,M}(f^\varepsilon(t_*))}{(1 + \lambda_0 q^{-1}(t - t_*))^q} 1_{t\geq t_*}, \quad (1.40)$$

where t_* is the time such that $\mathcal{E}_{2,M}(f^\varepsilon(t_*)) = MC_q\varepsilon^{2sq}\|f_0\|^2_{H^2_x L^2_{-\gamma q/2}}$ and verifies

$$t_* \leq \lambda_0^{-1} \ln \frac{\mathcal{E}_{2,M}(f_0)}{MC_q\varepsilon^{2sq}\|f_0\|^2_{H^2_x L^2_{-\gamma q/2}}}.$$

Here C_q, λ_0 *are given in* (4.24).

(b) *If* $\mathcal{E}_{2,M}(f_0) \le M C_q \varepsilon^{2sq} \|f_0\|^2_{H^2_x L^2_{-\gamma q/2}}$, *then*

$$\mathcal{E}_{2,M}(f^\varepsilon(t)) \le \frac{\mathcal{E}_{2,M}(f_0)}{(1 + C(f_0)t)^q}, \tag{1.41}$$

where $C(f_0)$ *is given in* (4.24) *and verifies*

$$C(f_0) \sim \Big(\frac{\|f_0\|_{H^2_x L^2}}{\|f_0\|_{H^2_x L^2_{-\gamma q/2}}}\Big)^{\frac{2}{q}} \varepsilon^{-2s}.$$

(ii) *Let* $0 < \eta < 1$. *Let* $j \in \mathbb{N}$ *be large enough such that* $2^j \ge \varepsilon^{-1}$. *Then for* $t \in [0, C^{-1}\eta 2^{-j\gamma}\varepsilon^{2s}]$, *it holds that*

$$\|\mathcal{P}_j f^\varepsilon(t)\|^2_{L^2} \ge \|\mathcal{P}_j f_0\|^2_{L^2} - \eta\delta_0 - C\varepsilon^{2s}\delta_0, \tag{1.42}$$

where C *is a universal constant.*

(3) (Global asymptotic formula) *If* $\mathcal{E}^{N+2,2}(f_0) < \infty$ *with* $N \ge 2$, *then*

$$\sup_{t\ge 0} \|f^\varepsilon(t) - f^0(t)\|^2_{H^N_x L^2} \le C(\mathcal{E}^{N+2,2}(f_0))\varepsilon^{4-4s}. \tag{1.43}$$

Some comments are in order.

Remark 1.8. Theorem 1.3 gives global well-posedness of (1.9) for all $0 \le \varepsilon \le \varepsilon_0$. The non-cutoff case ($\varepsilon = 0$) is established in [4] and [7], and the cutoff case (somehow can be considered as $\varepsilon = \varepsilon_0$) is proved in [9]. We get global well-posedness of (1.9) uniformly in the whole range $0 \le \varepsilon \le \varepsilon_0$.

Remark 1.9. The asymptotic formula (1.43) is global in time. A local-in-time result is proved in [12] for solutions to (1.5) and (1.2).

Remark 1.10. Estimates (1.40)–(1.42) show that the diversity of semigroup $e^{-\mathcal{L}^\varepsilon t}$ in Theorem 1.2 can also be observed at the nonlinear level. In other words, even in the perturbation framework, the solution F of the original problem (1.5) converges to the equilibrium without any explicit rate. That is, we can only derive

$$\lim_{t\to\infty} \|\mu^{-\frac{1}{2}}(F(t) - \mu)\|_{L^2} = 0.$$

However, by the energy–entropy method introduced in [11], it holds that

$$\lim_{t\to\infty} \|F(t) - \mu\|_{L^2} = O(t^{-\infty}).$$

Remark 1.11. To the best of our knowledge, the results in Theorem 1.3 are new for moderate soft potentials. To keep the paper to a reasonable size, we refrain from generalizing the results to other potentials, but this can be done by noticing that all the estimates involving $\mathcal{L}^\varepsilon$ and Γ^ε in this article are valid for $\gamma > -3$. Using the estimates in this article, very soft potentials $-3 < \gamma < -2s$ are considered in [13].

1.5. Ideas and novelties

Let us illustrate the ideas and novelties of the proofs of our main results.

1.5.1. Proof of Theorem 1.1. We illustrate our strategy in the Maxwellian molecules case $\gamma = 0$. It is not difficult to see (in the proof of Theorem 2.1) that the coercivity estimate of $\langle \mathcal{L}^\varepsilon f, f\rangle_v$ can be reduced to the control of quantities $\mathcal{M}^{\varepsilon,0}(f)$ and $\mathcal{R}^{\varepsilon,0}_\mu(f)$, which correspond to gain of weight and gain of regularity respectively.

- Instead of using the Carleman representation of the collision operator, in Lemma 2.1 we introduce a new coordinate system that enables us to make full use of the cancellation and the law of sines to estimate $\mathcal{M}^{\varepsilon,0}(f)$. The method is elementary but effective in catching the hyperbolic structure of $\mathcal{L}^\varepsilon$ uniformly in ε.
- To give a precise description of $\mathcal{R}^{\varepsilon,0}_\mu(f)$, we develop some new techniques. The first new idea is to apply the geometric decomposition to $\mathcal{R}^{\varepsilon,0}_\mu(f)$ in frequency space rather than phase space. More precisely, by Bobylev's equality, we have

$$\begin{aligned}\mathcal{R}^{\varepsilon,0}_\mu(f) = {} & \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3\times\mathbb{S}^2} b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)\big(\hat\mu(0)|\hat f(\xi)-\hat f(\xi^+)|^2 \\ & \qquad + 2\Re((\hat\mu(0)-\hat\mu(\xi^-))\hat f(\xi^+)\bar{\hat f}(\xi))\big)\,\mathrm{d}\xi\,\mathrm{d}\sigma \\ := {} & \frac{\hat\mu(0)}{(2\pi)^3}\mathcal{I}_1 + \frac{2}{(2\pi)^3}\mathcal{I}_2,\end{aligned}$$

where $\xi^+ = \frac{\xi+|\xi|\sigma}{2}$ and $\xi^- = \frac{\xi-|\xi|\sigma}{2}$. It is not difficult to prove that

$$|\mathcal{I}_2| \lesssim |W^\varepsilon(D)f|^2_{L^2} \lesssim \langle \mathcal{L}^\varepsilon f, f\rangle_v + |f|^2_{L^2},$$

where the latter $\lesssim$ is given by (1.14). Therefore, we only need to consider the estimate of $\mathcal{I}_1$. By the geometric decomposition introduced in [10],

$$\hat f(\xi) - \hat f(\xi^+) = \hat f(\xi) - \hat f(|\xi|\tfrac{\xi^+}{|\xi^+|}) + \hat f(|\xi|\tfrac{\xi^+}{|\xi^+|}) - \hat f(\xi^+),$$

we have

$$\begin{aligned}\mathcal{I}_1 = {} & \int_{\mathbb{R}^3\times\mathbb{S}^2} b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)|\hat f(\xi)-\hat f(\xi^+)|^2\,\mathrm{d}\xi\,\mathrm{d}\sigma \\ \geq {} & \frac12\int_{\mathbb{R}^3\times\mathbb{S}^2} b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)|\hat f(\xi)-\hat f(|\xi|\tfrac{\xi^+}{|\xi^+|})|^2\,\mathrm{d}\xi\,\mathrm{d}\sigma \\ & -\int_{\mathbb{R}^3\times\mathbb{S}^2} b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)|\hat f(|\xi|\tfrac{\xi^+}{|\xi^+|})-\hat f(\xi^+)|^2\,\mathrm{d}\xi\,\mathrm{d}\sigma \\ := {} & \frac12\mathcal{I}_{1,1} - \mathcal{I}_{1,2}.\end{aligned}$$

Thanks to the fact that the Fourier transform is commutative with $W^\varepsilon((-\triangle_{\mathbb{S}^2})^{\frac12})$, we obtain the anisotropic regularity from $\mathcal{I}_{1,1}$ (see Proposition 2.3 and Lemma 2.3

for details). Now it remains to estimate $\mathcal{I}_{1,2}$ from above. The key observation is that $\hat{f}(\frac{\xi^+}{|\xi^+|})$ and $\hat{f}(\xi^+)$ can be localized in the same region in both frequency space and phase space, which enables us to derive that $\mathcal{I}_{1,2}$ can be bounded by $|W^\varepsilon(D)f|_{L^2}^2 + |W^\varepsilon f|_{L^2}^2$.

To get the $\lesssim$ direction of (1.28), we have to give some upper bounds for $\langle Q^\varepsilon(g,h), f\rangle_v$ and $\langle \Gamma^\varepsilon(g,h), f\rangle_v$. To this end, our new idea is to separate the integration domain into two regions $|v-v_*| \leq 1$ and $|v-v_*| \geq 1$ to manifest the hyperbolic structure and the smoothing property of the operator.

- In the region $|v-v_*| \leq 1$, the hyperbolic structure prevails over the anisotropic structure, which can be checked from the proof of the sharp bounds for the operator in weighted Sobolev spaces (see [10] for details). It suggests that Sobolev regularity is enough to bound the inner product. See Proposition 2.5 for more details.
- In the region $|v-v_*| \geq 1$, the operator is dominated by the anisotropic structure. We resort to a geometric decomposition in phase space. In particular, we make full use of the symmetric property of the structure inside the operator and also the dissipation $\mathcal{R}^{\varepsilon,\gamma}_{*,g}(f)$ obtained from the lower bound of the operator. See Proposition 2.6 for more details.

1.5.2. Proof of Theorem 1.2. We have two novelties in the proof.

- The first one lies in the localization techniques in phase space which are totally new and important considering that the Boltzmann equation is a nonlocal equation. It shows that the linear or even nonlinear Boltzmann equations can be almost localized thanks to the commutator estimates (in Lemma 3.1) between $\mathcal{L}^\varepsilon$ and the localization function. This fact enables us to consider the evolution of the local energy which is the key to proving diversity of longtime behavior of $e^{-\mathcal{L}^\varepsilon t} f$.
- We reduce longtime behavior of $e^{-\mathcal{L}^\varepsilon t} f$ to some special ODE system. Based on a technical argument, we obtain a sharp estimate (in Proposition 3.1) for the ODE system, which in turn gives the precise behavior of the semigroup. The result shows that there exists a critical time t_* such that the decay rate is totally different before and after t_* which matches the complex property of $\mathcal{L}^\varepsilon$.

1.5.3. Proof of Theorem 1.3. The proof has some new features.

- Since we only impose the smallness assumption on $\|f\|_{H^2_x L^2}$, we have to find a new way to prove propagation of full regularity ((1.38) and (1.39)). To this end, we first close energy estimates for pure spatial regularity. Thanks to the well-designed weight functions in (1.34), propagation of $\dot{\mathcal{E}}^{N-j-1,j+1}(f)$ can be obtained after propagation of $\dot{\mathcal{E}}^{N-j,j}(f)$.
- To prove the global error estimate (1.43), the key idea is to regard the error equation as a linear equation since we already have high-order energy estimates (1.39) of the solutions to (1.9) and (1.10).

1.6. Plan of the article

In Section 2 we endeavor to prove Theorem 1.1 and some upper bound estimates for the nonlinear term Γ^ε. Theorems 1.2 and 1.3 are proved in Sections 3 and 4 respectively. In the appendix we give some necessary results for the sake of completeness.

2. Bounds of the linearized Boltzmann operator and the nonlinear term

In this section we will prove Theorem 1.1. To this end, we separate the proof into two parts: the lower bound of $\langle \mathcal{L}^\varepsilon f, f\rangle_v$ and the upper bound of $\langle \Gamma^\varepsilon(g,h), f\rangle_v$. Moreover, we give an estimate of the commutator between the collision operator $\Gamma^\varepsilon(g,\cdot)$ and the weight function W_l. The commutator estimate will be used in the proofs of Theorems 1.2 and 1.3.

Throughout the article we assume $0 \le \varepsilon \le \varepsilon_0$ with $\varepsilon_0 > 0$ sufficiently small. Recall that $-3 < \gamma \le 1$ unless otherwise specified. Since many variables are used frequently, we will sometimes omit their range in integrals. Usually, $\sigma, \tau, \varsigma \in \mathbb{S}^2$, $v, v_*, u, \xi \in \mathbb{R}^3$, $\kappa \in [0,1]$, $r \in \mathbb{R}_+$. For instance,

$$\int (\cdots)\, \mathrm{d}\sigma := \int_{\mathbb{S}^2} (\cdots)\, \mathrm{d}\sigma, \quad \int (\cdots)\, \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v := \int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} (\cdots)\, \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v.$$

2.1. Lower bound of the linearized operator

Our strategy for the proof can be summarized as follows. We first give the estimates of $\mathcal{R}_g^{\varepsilon,\gamma}(f)$ and $\mathcal{M}^{\varepsilon,\gamma}(f)$. Then the lower bound of $\mathcal{L}^\varepsilon$ is obtained by proving $\langle \mathcal{L}_1^\varepsilon f, f\rangle_v + |f|^2_{L^2_{\gamma/2}} \gtrsim \mathcal{R}_\mu^{\varepsilon,\gamma}(f) + \mathcal{M}^{\varepsilon,\gamma}(f)$ and the fact that $\langle \mathcal{L}_2^\varepsilon f, f\rangle_v$ is a lower-order term.

2.1.1. Estimate of $\mathcal{M}^{\varepsilon,\gamma}(f)$. Recall (1.27) for the definition of $\mathcal{M}^{\varepsilon,\gamma}(f)$. We derive the weight W^ε in phase space from the functional $\mathcal{M}^{\varepsilon,\gamma}(f)$ in the following result.

Proposition 2.1. *There exists $\varepsilon_0 > 0$ such that for any $0 \le \varepsilon \le \varepsilon_0$,*

$$\mathcal{M}^{\varepsilon,\gamma}(f) + |f|^2_{L^2_{\gamma/2}} \sim |W^\varepsilon f|^2_{L^2_{\gamma/2}}.$$

Proof. We only consider the case $\varepsilon > 0$. Note that with slight modification, our method also works for the case $\varepsilon = 0$. We divide the proof into two steps.

Step 1: Lower bound of $\mathcal{M}^{\varepsilon,\gamma}(f)$. Note that $\nabla\mu^{\frac12} = -\frac{\mu^{\frac12}}{2}v$ and $\nabla^2\mu^{\frac12} = \frac{\mu^{\frac12}}{4}(-2I_3 + v\otimes v)$, where I_3 is the 3×3 identity matrix. By Taylor expansion, we have

$$\begin{aligned}\mu^{\frac12}(v') - \mu^{\frac12}(v) = &-\frac{\mu^{\frac12}(v)}{2}v\cdot(v'-v)\\ &+ \int_0^1 (1-\kappa)(\nabla^2\mu^{\frac12})(v(\kappa)) : (v'-v)\otimes(v'-v)\,\mathrm{d}\kappa,\end{aligned}$$

where $v(\kappa) = v + \kappa(v' - v)$. Using the inequality $(a-b)^2 \geq \frac{a^2}{2} - b^2$, we have

$$(\mu^{\frac{1}{2}}(v') - \mu^{\frac{1}{2}}(v))^2 \geq \frac{\mu(v)}{8}|v \cdot (v'-v)|^2 - \int_0^1 |(\nabla^2\mu^{\frac{1}{2}})(v(\kappa))|^2|v'-v|^4 \,\mathrm{d}\kappa.$$

Step 1.1*:* $|v_*| \leq \frac{\eta}{\varepsilon}$. Here $0 < \eta < 1$ is a constant to be determined later. Set $r = 4\sqrt{2}$ and $A(\varepsilon, \eta, r) = \{(v_*, v, \sigma) \mid 2r \leq |v_*| \leq \frac{\eta}{\varepsilon},\ |v| \leq r,\ \frac{2\eta}{|v-v_*|} \leq \sin\frac{\theta}{2} \leq \frac{4\eta}{|v-v_*|}\}$. Recall $\mathcal{M}^{\varepsilon,\gamma}(f)$ defined in (1.27). For simplicity, let $B^{\varepsilon,\gamma} := b^{\varepsilon}(\cos\theta)|v-v_*|^{\gamma}$; then

$$\begin{aligned}
\mathcal{M}^{\varepsilon,\gamma}(f) &\geq \int B^{\varepsilon,\gamma} 1_{A(\varepsilon,\eta,r)} f_*^2((\mu^{\frac{1}{2}})' - \mu^{\frac{1}{2}})^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&\geq \frac{1}{8}\int B^{\varepsilon,\gamma} 1_{A(\varepsilon,\eta,r)} \mu(v)|v \cdot (v'-v)|^2 f_*^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&\quad - \int B^{\varepsilon,\gamma} 1_{A(\varepsilon,\eta,r)} |(\nabla^2\mu^{\frac{1}{2}})(v(\kappa))|^2|v'-v|^4 f_*^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\,\mathrm{d}\kappa \\
&:= \frac{1}{8}\mathcal{M}_1^{\varepsilon,\gamma}(\eta) - \mathcal{M}_2^{\varepsilon,\gamma}(\eta).
\end{aligned} \tag{2.1}$$

Estimate of $\mathcal{M}_1^{\varepsilon,\gamma}(\eta)$. For fixed v, v_*, we introduce an orthonormal basis $(h^1_{v,v_*}, h^2_{v,v_*}, \frac{v-v_*}{|v-v_*|})$ such that $\mathrm{d}\sigma = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$. Then one has

$$\begin{aligned}
\tfrac{v'-v}{|v'-v|} &= \cos\tfrac{\theta}{2}\cos\varphi h^1_{v,v_*} + \cos\tfrac{\theta}{2}\sin\varphi h^2_{v,v_*} - \sin\tfrac{\theta}{2}\tfrac{v-v_*}{|v-v_*|}, \\
\tfrac{v}{|v|} &= c_1 h^1_{v,v_*} + c_2 h^2_{v,v_*} + c_3\tfrac{v-v_*}{|v-v_*|},
\end{aligned}$$

where $c_3 = \frac{v}{|v|} \cdot \frac{v-v_*}{|v-v_*|}$ and c_1, c_2 are constants independent of θ and φ. Then we have

$$\begin{aligned}
|\tfrac{v}{|v|} \cdot \tfrac{v'-v}{|v'-v|}|^2 &= |c_1\cos\tfrac{\theta}{2}\cos\varphi + c_2\cos\tfrac{\theta}{2}\sin\varphi - c_3\sin\tfrac{\theta}{2}|^2 \\
&= c_1^2\cos^2\tfrac{\theta}{2}\cos^2\varphi + c_2^2\cos^2\tfrac{\theta}{2}\sin^2\varphi + c_3^2\sin^2\tfrac{\theta}{2} \\
&\quad + 2c_1c_2\cos^2\tfrac{\theta}{2}\cos\varphi\sin\varphi - 2c_3\cos\tfrac{\theta}{2}\sin\tfrac{\theta}{2}(c_1\cos\varphi + c_2\sin\varphi).
\end{aligned}$$

Integrating with respect to σ we have

$$\begin{aligned}
&\int b^{\varepsilon}(\cos\theta)1_{A(\varepsilon,\eta,r)}|v \cdot (v'-v)|^2 \,\mathrm{d}\sigma \\
&\quad = \int_0^{\pi/2}\int_0^{2\pi} b^{\varepsilon}(\cos\theta)\sin\theta 1_{A(\varepsilon,\eta,r)}|v \cdot (v'-v)|^2 \,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&\quad \geq \pi(c_1^2 + c_2^2)|v|^2|v-v_*|^2 \int_0^{\pi/2} b^{\varepsilon}(\cos\theta)\sin\theta\cos^2\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2} 1_{A(\varepsilon,\eta,r)} \,\mathrm{d}\theta.
\end{aligned} \tag{2.2}$$

If $(v_*, v, \sigma) \in A(\varepsilon, \eta, r)$, then $|v - v_*| \geq |v_*| - |v| \geq r$ and thus $4\eta|v-v_*|^{-1} \leq 4r^{-1} \leq \sqrt{2}/2$. Suppose $\varepsilon \leq \eta/2r$; then $|v-v_*| \leq |v| + |v_*| \leq r + \eta/\varepsilon \leq 3\eta/2\varepsilon$ and thus $2\eta|v-v_*|^{-1} \geq 4\varepsilon/3$. Recall ϕ in (1.19) and b^{ε} in (1.3) to see

$$b^{\varepsilon}(\cos\theta)1_{4\varepsilon/3 \leq \sin\frac{\theta}{2} \leq \sqrt{2}/2} = b(\cos\theta)1_{4\varepsilon/3 \leq \sin\frac{\theta}{2} \leq \sqrt{2}/2}, \tag{2.3}$$

which gives

$$\begin{aligned}
&\int_0^{\pi/2} b^\varepsilon(\cos\theta)\sin\theta\cos^2\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2}1_{A(\varepsilon,\eta,r)}\,\mathrm{d}\theta \\
&\quad = \int_0^{\pi/2} b(\cos\theta)\sin\theta\cos^2\tfrac{\theta}{2}\sin^2\tfrac{\theta}{2}1_{A(\varepsilon,\eta,r)}\,\mathrm{d}\theta \\
&\quad \gtrsim 1_{B(\varepsilon,\eta,r)}\int_{2\eta|v-v_*|^{-1}}^{4\eta|v-v_*|^{-1}} t^{1-2s}\,\mathrm{d}t \gtrsim \eta^{2-2s}|v-v_*|^{2s-2}1_{B(\varepsilon,\eta,r)},
\end{aligned} \tag{2.4}$$

where we use (1.1) and the change of variable $t=\sin\frac{\theta}{2}$. Here $B(\varepsilon,\eta,r)=\{(v_*,v)\mid 2r\le|v_*|\le\frac{\eta}{\varepsilon},\ |v|\le r\}$. Plugging (2.2) and (2.4) into the definition of $\mathcal{M}_1^{\varepsilon,\gamma}(\eta)$ we have

$$\begin{aligned}
\mathcal{M}_1^{\varepsilon,\gamma}(\eta) &\gtrsim \eta^{2-2s}\int (c_1^2+c_2^2)|v-v_*|^{\gamma+2s}|v|^2 1_{B(\varepsilon,\eta,r)}\mu(v)f_*^2\,\mathrm{d}v\,\mathrm{d}v_* \\
&= \eta^{2-2s}\int \big(1-(\tfrac{v}{|v|}\cdot\tfrac{v_*}{|v_*|})^2\big)|v_*|^2|v-v_*|^{\gamma+2s-2}|v|^2 1_{B(\varepsilon,\eta,r)}\mu(v)f_*^2\,\mathrm{d}v\,\mathrm{d}v_*,
\end{aligned}$$

where in the last line we use the fact that $c_1^2+c_2^2+c_3^2=1$ and the law of sines

$$\big(1-(\tfrac{v}{|v|}\cdot\tfrac{v_*}{|v_*|})^2\big)^{-1}|v-v_*|^2=(1-c_3^2)^{-1}|v_*|^2.$$

Note that in the region $B(\varepsilon,\eta,r)$, one has $|v-v_*|\sim|v_*|$ and thus $|v-v_*|^{\gamma+2s-2}\sim|v_*|^{\gamma+2s-2}$. Recalling $r=4\sqrt{2}$, then $\int(1-(\frac{v}{|v|}\cdot\frac{v_*}{|v_*|})^2)|v|^2\mu(v)1_{|v|\le4\sqrt{2}}\,\mathrm{d}v\gtrsim1$ and the value of the integral is independent of v_*. Therefore, we have

$$\begin{aligned}
\mathcal{M}_1^{\varepsilon,\gamma}(\eta) &\gtrsim \eta^{2-2s}\int |v_*|^{\gamma+2s}1_{2r\le|v_*|\le\eta/\varepsilon}f_*^2\,\mathrm{d}v_* \\
&\gtrsim \eta^{2-2s}\Big(\int\langle v_*\rangle^{\gamma+2s}1_{|v_*|\le\eta/\varepsilon}f_*^2\,\mathrm{d}v_*-(1+(8\sqrt{2})^2)^s|f|^2_{L^2_{\gamma/2}}\Big).
\end{aligned} \tag{2.5}$$

Estimate of $\mathcal{M}_2^{\varepsilon,\gamma}(\eta)$. By the change of variable $v\to v(\kappa)$, let $\cos\theta(\kappa)=\frac{v(\kappa)-v_*}{|v(\kappa)-v_*|}\cdot\sigma$; then $\frac{\theta}{2}\le\theta(\kappa)\le\theta$ and it is not difficult to check

$$\begin{aligned}
\mathcal{M}_2^{\varepsilon,\gamma}(\eta) &\lesssim \int\Big(\int_{\eta|v-v_*|^{-1}}^{8\eta|v-v_*|^{-1}} t^{3-2s}\,\mathrm{d}t\Big)|v-v_*|^{\gamma+4}1_{|v_*|\le\eta/\varepsilon}f_*^2\mu^{\frac12}\,\mathrm{d}v\,\mathrm{d}v_* \\
&\lesssim \eta^{4-2s}\int|v-v_*|^{\gamma+2s}1_{|v_*|\le\eta/\varepsilon}f_*^2\mu^{\frac12}\,\mathrm{d}v\,\mathrm{d}v_* \\
&\lesssim \eta^{4-2s}\int\langle v_*\rangle^{\gamma+2s}1_{|v_*|\le\eta/\varepsilon}f_*^2\,\mathrm{d}v_*,
\end{aligned} \tag{2.6}$$

where we use the following estimate (see [4]): for $a>-3$, $b>0$, there holds

$$\int|v-v_*|^a\mu^b(v)\,\mathrm{d}v\le C_{a,b}\langle v_*\rangle^a. \tag{2.7}$$

Plugging (2.5) and (2.6) into (2.1), for some universal constant $C \geq 1$, we have

$$\mathcal{M}^{\varepsilon,\gamma}(f) \gtrsim \eta^{2-2s}(1 - C\eta^2) \int \langle v_* \rangle^{\gamma+2s} 1_{|v_*| \leq \eta/\varepsilon} f_*^2 \, \mathrm{d}v_* - C\eta^{2-2s} |f|^2_{L^2_{\gamma/2}}.$$

Choosing η such that $C\eta^2 = \frac{1}{2}$, we have

$$\mathcal{M}^{\varepsilon,\gamma}(f) + |f|^2_{L^2_{\gamma/2}} \gtrsim \int \langle v_* \rangle^{\gamma+2s} 1_{|v_*| \leq \eta/\varepsilon} f_*^2 \, \mathrm{d}v_*. \tag{2.8}$$

Step 1.2*:* $|v_*| \geq R/\varepsilon$. Here $R \geq 1$. By direct computation we have

$$\begin{aligned} \mathcal{M}^{\varepsilon,\gamma}(f) = & \int B^{\varepsilon,\gamma} f_*^2 ((\mu^{\frac{1}{2}})' - \mu^{\frac{1}{2}})^2 \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \\ \geq & \int B^{\varepsilon,\gamma} 1_{|v_*| \geq R/\varepsilon} f_*^2 \mu \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \\ & - 2 \int B^{\varepsilon,\gamma} 1_{|v_*| \geq R/\varepsilon} f_*^2 (\mu^{\frac{1}{2}})' \mu^{\frac{1}{2}} \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \\ := & \mathcal{M}_1^{\varepsilon,\gamma,R} - \mathcal{M}_2^{\varepsilon,\gamma,R}. \end{aligned}$$

Recalling (2.3) and (1.1), using the change of variable $t = \sin \frac{\theta}{2}$, we have

$$\begin{aligned} \mathcal{M}_1^{\varepsilon,\gamma,R} & \gtrsim \left(\int_{4\varepsilon/3}^{\sqrt{2}/2} t^{-1-2s} \, \mathrm{d}t \right) \int |v - v_*|^\gamma 1_{|v_*| \geq R/\varepsilon} f_*^2 \mu \, \mathrm{d}v \, \mathrm{d}v_* \\ & \gtrsim \varepsilon^{-2s} \int \langle v_* \rangle^\gamma 1_{|v_*| \geq R/\varepsilon} f_*^2 \, \mathrm{d}v_*, \end{aligned}$$

where we use $\int |v - v_*|^\gamma \mu \, \mathrm{d}v \gtrsim \langle v_* \rangle^\gamma$ and $\int_{4\varepsilon/3}^{\sqrt{2}/2} t^{-1-2s} \, \mathrm{d}t \gtrsim \varepsilon^{-2s}$ when $0 < \varepsilon \leq \frac{1}{10}$. Recalling that the support of b^ε belongs to $\sin \frac{\theta}{2} \geq \frac{3}{4}\varepsilon$, there holds $|v'| + |v| \geq |v' - v| = \sin \frac{\theta}{2} |v - v_*| \geq \frac{3}{4}\varepsilon |v - v_*| \geq \frac{3}{4}\varepsilon(|v_*| - |v|)$ and thus $|v'| + (1 + \frac{3}{4}\varepsilon)|v| \geq \frac{3}{4}\varepsilon|v_*| \geq \frac{3}{4}R$. Then $R^2/2 \leq 4(|v'| + |v|)^2 \leq 8(|v'|^2 + |v|^2)$ and so

$$(\mu^{\frac{1}{2}})' \mu^{\frac{1}{2}} = (2\pi)^{-3/2} e^{-\frac{|v'|^2 + |v|^2}{4}} \lesssim e^{-\frac{|v|^2}{8}} e^{-\frac{R^2}{2^7}}.$$

From this, together with (2.7), we have

$$\mathcal{M}_2^{\varepsilon,\gamma,R} \lesssim e^{-\frac{R^2}{2^7}} \varepsilon^{-2s} \int \langle v_* \rangle^\gamma 1_{|v_*| \geq R/\varepsilon} f_*^2 \, \mathrm{d}v_*.$$

Patching together the above estimates of $\mathcal{M}_1^{\varepsilon,\gamma,R}$ and $\mathcal{M}_2^{\varepsilon,\gamma,R}$, we arrive at

$$\mathcal{M}^{\varepsilon,\gamma}(f) \geq (C_1 - C_2 e^{-\frac{R^2}{2^7}}) \varepsilon^{-2s} \int \langle v_* \rangle^\gamma 1_{|v_*| \geq R/\varepsilon} f_*^2 \, \mathrm{d}v_* \tag{2.9}$$

for some universal constants C_1 and C_2.

Step 1.3*:* $|v_*| \geq \eta/\varepsilon$*.* Here η is the fixed constant in Step 1.1. Note that estimate (2.9) is valid for any $R \geq 1$ and $\varepsilon \leq \frac{1}{10}$. We choose $R = N\eta$, where $N \geq \eta^{-1}$ is large enough such that $C_1 - C_2 e^{-\frac{(N\eta)^2}{2^7}} \geq \frac{C_1}{2}$. Then by (2.9), when $\varepsilon \leq \frac{1}{10N}$, we have

$$\begin{aligned}\mathcal{M}^{\varepsilon,\gamma}(f) \geq \mathcal{M}^{N\varepsilon,\gamma}(f) &\geq \big(C_1 - C_2 e^{-\frac{(N\eta)^2}{2^7}}\big)(N\varepsilon)^{-2s} \int \langle v_*\rangle^\gamma 1_{|v_*|\geq \eta/\varepsilon} f_*^2 \, dv_* \\ &\geq \frac{C_1}{2} N^{-2s} \varepsilon^{-2s} \int \langle v_*\rangle^\gamma 1_{|v_*|\geq \eta/\varepsilon} f_*^2 \, dv_*.\end{aligned}$$

From this, together with (2.8), taking $\varepsilon_0 := \min\{\frac{\eta}{8\sqrt{2}}, \frac{1}{10N}\}$, when $\varepsilon \leq \varepsilon_0$, we arrive at

$$\begin{aligned}\mathcal{M}^{\varepsilon,\gamma}(f) + |f|^2_{L^2_{\gamma/2}} &\gtrsim \int \langle v_*\rangle^{\gamma+2s} 1_{|v_*|\leq \eta/\varepsilon} f_*^2 \, dv_* \\ &\quad + \varepsilon^{-2s} \int \langle v_*\rangle^\gamma 1_{|v_*|\geq \eta/\varepsilon} f_*^2 \, dv_* \gtrsim |W^\varepsilon f|^2_{L^2_{\gamma/2}}.\end{aligned}$$

Step 2*: Upper bound of* $\mathcal{M}^{\varepsilon,\gamma}(f)$*.* Since $((\mu^{\frac{1}{2}})' - \mu^{\frac{1}{2}})^2 \leq 2((\mu^{\frac{1}{4}})' - \mu^{\frac{1}{4}})^2((\mu^{\frac{1}{2}})' + \mu^{\frac{1}{2}})$, we have

$$\begin{aligned}\mathcal{M}^{\varepsilon,\gamma}(f) &\lesssim \int B^{\varepsilon,\gamma} f_*^2 ((\mu^{\frac{1}{4}})' - \mu^{\frac{1}{4}})^2 (\mu^{\frac{1}{2}})' \, d\sigma \, dv_* \, dv \\ &\quad + \int B^{\varepsilon,\gamma} f_*^2 ((\mu^{\frac{1}{4}})' - \mu^{\frac{1}{4}})^2 \mu^{\frac{1}{2}} \, d\sigma \, dv_* \, dv := \mathcal{M}_1^{\varepsilon,\gamma}(f) + \mathcal{M}_2^{\varepsilon,\gamma}(f).\end{aligned}$$

By Taylor expansion, one has

$$((\mu^{\frac{1}{4}})' - \mu^{\frac{1}{4}})^2 \lesssim \min\{1, |v - v_*|^2 \sin^2 \tfrac{\theta}{2}\} \sim \min\{1, |v' - v_*|^2 \sin^2 \tfrac{\theta}{2}\}.$$

By Proposition A.1, we have

$$\int b^\varepsilon(\cos\theta) \min\{1, |v - v_*|^2 \sin^2 \tfrac{\theta}{2}\} \, d\sigma \lesssim (W^\varepsilon)^2 (v - v_*).$$

After checking

$$(W^\varepsilon)^2(v - v_*) \lesssim (W^\varepsilon)^2(v)(W^\varepsilon)^2(v_*), \tag{2.10}$$

we have

$$\int b^\varepsilon(\cos\theta) \min\{1, |v - v_*|^2 \sin^2 \tfrac{\theta}{2}\} \, d\sigma \lesssim (W^\varepsilon)^2(v)(W^\varepsilon)^2(v_*).$$

Thus we have

$$\mathcal{M}_2^{\varepsilon,\gamma}(f) \lesssim \int f_*^2 |v - v_*|^\gamma (W^\varepsilon)^2(v)(W^\varepsilon)^2(v_*) \mu^{\frac{1}{2}} \, dv \, dv_* \lesssim |W^\varepsilon f|^2_{L^2_{\gamma/2}},$$

where (2.7) is used.

The term $\mathcal{M}_1^{\varepsilon,\gamma}(f)$ can be similarly estimated by the change of variable $v \to v'$. Indeed, one has $\mathcal{M}_1^{\varepsilon,\gamma}(f) \lesssim \int b^\varepsilon(\cos(2\theta'))|v'-v_*|^\gamma f_*^2((\mu^{\frac{1}{4}})' - \mu^{\frac{1}{4}})^2(\mu^{\frac{1}{2}})' \,\mathrm{d}\sigma\,\mathrm{d}v'\,\mathrm{d}v_*$, where θ' is the angle between $v'-v_*$ and σ. With the fact that $\theta' = \frac{\theta}{2}$, we also have

$$\int b^\varepsilon(\cos(2\theta')) \min\{1, |v'-v_*|^2 \sin^2 \tfrac{\theta}{2}\} \,\mathrm{d}\sigma \lesssim (W^\varepsilon)^2(v')(W^\varepsilon)^2(v_*).$$

Thus, by exactly the same argument as that for $\mathcal{M}_2^{\varepsilon,\gamma}(f)$, we have $\mathcal{M}_1^{\varepsilon,\gamma}(f) \lesssim |W^\varepsilon f|^2_{L^2_{\gamma/2}}$. The proof is complete. ∎

2.1.2. Estimate of $\mathcal{R}_\mu^{\varepsilon,\gamma}(f)$. We recall from [1] that for $g \geq 0$ with $|g|_{L^1} \geq \delta > 0$ and $|g|_{L^1_1 \cap L\log L} \leq \lambda$,

$$\int b(\cos\theta) g_*(f'-f)^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v + |f|^2_{L^2} \geq C(\delta,\lambda)|a(D)f|^2_{L^2},$$

where $a(\xi) := \int b(\frac{\xi}{|\xi|}\cdot\sigma)\min\{|\xi|^2\sin^2\frac{\theta}{2}, 1\}\,\mathrm{d}\sigma + 1$. As an application, recalling (1.25) and using Proposition A.1, we get the following proposition:

Proposition 2.2. *It holds that*

$$\mathcal{R}_\mu^{\varepsilon,0}(f) + |f|^2_{L^2} \gtrsim |W^\varepsilon(D)f|^2_{L^2}.$$

Proposition 2.2 provides Sobolev regularity. We also need to derive the anisotropic regularity $|W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})W_{\gamma/2}f|^2_{L^2}$ from the lower bound of $\mathcal{R}_\mu^{\varepsilon,\gamma}(f)$. To this end, we first give three technical lemmas.

Lemma 2.1. *It holds that*

$$\mathcal{A} := \int_{\mathbb{R}^3}\int_\varepsilon^{\pi/4} \theta^{-1-2s}|f(v) - f(v/\cos\theta)|^2 \,\mathrm{d}v\,\mathrm{d}\theta \lesssim |W^\varepsilon(D)f|^2_{L^2} + |W^\varepsilon f|^2_{L^2}.$$

Proof. Applying dyadic decomposition in phase space, since $\frac{\sqrt{2}}{2} \leq \cos\theta \leq 1$ for $\theta \in [0, \frac{\pi}{4}]$, we have

$$\begin{aligned}\mathcal{A} &= \int_{\mathbb{R}^3}\int_\varepsilon^{\pi/4} \theta^{-1-2s}\left|\sum_{k=-1}^\infty (\varphi_k f)(v) - \sum_{k=-1}^\infty (\varphi_k f)(v/\cos\theta)\right|^2 \mathrm{d}v\,\mathrm{d}\theta \\ &\lesssim \sum_{k=-1}^\infty \int_{\mathbb{R}^3}\int_\varepsilon^{\pi/4} \theta^{-1-2s}|(\varphi_k f)(v) - (\varphi_k f)(v/\cos\theta)|^2 \,\mathrm{d}v\,\mathrm{d}\theta := \sum_{k=-1}^\infty \mathcal{A}_k.\end{aligned}$$

It is easy to check $\sum_{2^k \geq 1/\varepsilon}\mathcal{A}_k \lesssim |W^\varepsilon f|^2_{L^2}$ since $\int_\varepsilon^{\pi/4}\theta^{-1-2s}\,\mathrm{d}\theta \lesssim \varepsilon^{-2s}$. For the case $2^k \leq 1/\varepsilon$, by Plancherel's theorem and dyadic decomposition in frequency space, we have

$$\begin{aligned}\mathcal{A}_k &= \int_{\mathbb{R}^3}\int_\varepsilon^{\pi/4} \theta^{-1-2s}|\widehat{\varphi_k f}(\xi) - \cos^3\theta\,\widehat{\varphi_k f}(\xi\cos\theta)|^2 \,\mathrm{d}\xi\,\mathrm{d}\theta \\ &\lesssim \int_{\mathbb{R}^3}\int_\varepsilon^{\pi/4} \theta^{-1-2s}|\widehat{\varphi_k f}(\xi) - \widehat{\varphi_k f}(\xi\cos\theta)|^2 \,\mathrm{d}\xi\,\mathrm{d}\theta + |\varphi_k f|^2_{L^2}\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3}\int_{\varepsilon}^{\pi/4} \theta^{-1-2s}\left|\sum_{l=-1}^{\infty}(\varphi_l\widehat{\varphi_k f})(\xi) - \sum_{l=-1}^{\infty}(\varphi_l\widehat{\varphi_k f})(\xi\cos\theta)\right|^2 \mathrm{d}\xi\,\mathrm{d}\theta + |\varphi_k f|^2_{L^2}\\
&\lesssim \sum_{l=-1}^{\infty}\int_{\mathbb{R}^3}\int_{\varepsilon}^{\pi/4}\theta^{-1-2s}|(\varphi_l\widehat{\varphi_k f})(\xi) - (\varphi_l\widehat{\varphi_k f})(\xi\cos\theta)|^2\,\mathrm{d}\xi\,\mathrm{d}\theta + |\varphi_k f|^2_{L^2}\\
&:= \sum_{l=-1}^{\infty}\mathcal{A}_{k,l} + |\varphi_k f|^2_{L^2}.
\end{aligned}
$$

Note that $\sum_{2^l\geq 1/\varepsilon}\mathcal{A}_{k,l}\lesssim |W^\varepsilon(D)\varphi_k f|^2_{L^2}$, thus $\mathcal{A}_k \lesssim \sum_{2^l\leq 1/\varepsilon}\mathcal{A}_{k,l} + |W^\varepsilon(D)\varphi_k f|^2_{L^2} + |\varphi_k f|^2_{L^2}$. Using $\sum_{k\geq -1}^{\infty}|\varphi_k f|^2_{L^2}\lesssim |f|^2_{L^2}$ and (A.1), we have

$$
\mathcal{A}\lesssim \sum_{\substack{2^k\leq 1/\varepsilon,\\ 2^l\leq 1/\varepsilon}}\mathcal{A}_{k,l} + |W^\varepsilon(D)f|^2_{L^2} + |W^\varepsilon f|^2_{L^2}.
$$

For each k and l such that $2^k\leq 1/\varepsilon$, $2^l\leq 1/\varepsilon$, using $\int_{2^{-k/2-l/2}}^{\pi/4}\theta^{-1-2s}\,\mathrm{d}\theta\lesssim 2^{s(l+k)}$, we have

$$
\begin{aligned}
\mathcal{A}_{k,l} &= \int_{\mathbb{R}^3}\int_{\varepsilon}^{2^{-k/2-l/2}}\theta^{-1-2s}|(\varphi_l\widehat{\varphi_k f})(\xi) - (\varphi_l\widehat{\varphi_k f})(\xi\cos\theta)|^2\,\mathrm{d}\xi\,\mathrm{d}\theta\\
&\quad + \int_{\mathbb{R}^3}\int_{2^{-k/2-l/2}}^{\pi/4}\theta^{-1-2s}|(\varphi_l\widehat{\varphi_k f})(\xi) - (\varphi_l\widehat{\varphi_k f})(\xi\cos\theta)|^2\,\mathrm{d}\xi\,\mathrm{d}\theta\\
&\lesssim \int_{\mathbb{R}^3}\int_{0}^{2^{-k/2-l/2}}\theta^{-1-2s}|(\varphi_l\widehat{\varphi_k f})(\xi) - (\varphi_l\widehat{\varphi_k f})(\xi\cos\theta)|^2\,\mathrm{d}\xi\,\mathrm{d}\theta + 2^{s(l+k)}|\varphi_l\widehat{\varphi_k f}|^2_{L^2}\\
&:= \mathcal{B}_{k,l} + 2^{s(l+k)}|\varphi_l\widehat{\varphi_k f}|^2_{L^2}.
\end{aligned}
\tag{2.11}
$$

By Taylor expansion,

$$
(\varphi_l\widehat{\varphi_k f})(\xi) - (\varphi_l\widehat{\varphi_k f})(\xi\cos\theta) = (1-\cos\theta)\int_0^1(\nabla\varphi_l\widehat{\varphi_k f})(\xi(\kappa))\cdot\xi\,\mathrm{d}\kappa,
$$

where $\xi(\kappa) = (1-\kappa)\xi\cos\theta + \kappa\xi$. Thus we obtain

$$
\mathcal{B}_{k,l}\lesssim \int_0^1\int_{\mathbb{R}^3}\int_0^{2^{-k/2-l/2}}\theta^{3-2s}|\xi|^2|(\nabla\varphi_l\widehat{\varphi_k f})(\xi(\kappa))|^2\,\mathrm{d}\kappa\,\mathrm{d}\xi\,\mathrm{d}\theta.
$$

By the change of variable $\xi\to\eta=\xi(\kappa)$, we have

$$
\begin{aligned}
\mathcal{B}_{k,l} &= \int_0^1\int_{\mathbb{R}^3}\int_0^{2^{-k/2-l/2}}\theta^{3-2s}\frac{|\eta|^2}{((1-\kappa)\cos\theta+\kappa)^5}|(\nabla\varphi_l\widehat{\varphi_k f})(\eta)|^2\,\mathrm{d}\kappa\,\mathrm{d}\eta\,\mathrm{d}\theta\\
&\lesssim \int_{\mathbb{R}^3}\int_0^{2^{-k/2-l/2}}\theta^{3-2s}|\eta|^2|(\nabla\varphi_l\widehat{\varphi_k f})(\eta)|^2\,\mathrm{d}\eta\,\mathrm{d}\theta\\
&\lesssim 2^{-(2-s)(l+k)}\int_{\mathbb{R}^3}|\eta|^2|(\nabla\varphi_l\widehat{\varphi_k f})(\eta)|^2\,\mathrm{d}\eta.
\end{aligned}
\tag{2.12}
$$

Note that

$$\begin{aligned}(\nabla\widehat{\varphi_l\varphi_k f})(\eta) &= (\nabla\varphi_l)(\eta)\widehat{\varphi_k f}(\eta) + \varphi_l(\eta)(\nabla\widehat{\varphi_k f})(\eta)\\ &= 2^{-l}(\nabla\varphi)\Big(\frac{\eta}{2^l}\Big)\widehat{\varphi_k f}(\eta) - \mathrm{i}(\varphi_l\widehat{v\varphi_k f})(\eta),\end{aligned}$$

which gives

$$|\eta|^2|(\nabla\widehat{\varphi_l\varphi_k f})(\eta)|^2 \lesssim |\nabla\varphi|^2_{L^\infty}|(\widetilde{\varphi}_l\widehat{\varphi_k f})(\eta)|^2 + 2^{2l}|(\varphi_l\widehat{v\varphi_k f})(\eta)|^2, \tag{2.13}$$

where $\widetilde{\varphi}_l := \sum_{k\geq -1, |k-l|\leq 4}\varphi_k$. Plugging (2.13) into (2.12), we have

$$\mathcal{B}_{k,l} \lesssim 2^{-(2-s)(l+k)}|\widetilde{\varphi}_l\widehat{\varphi_k f}|^2_{L^2} + 2^{s(l+k)-2k}|\varphi_l\widehat{v\varphi_k f}|^2_{L^2}. \tag{2.14}$$

Recalling (2.11), we arrive at

$$\begin{aligned}\mathcal{A}_{k,l} &\lesssim 2^{-(2-s)(l+k)}|\widetilde{\varphi}_l\widehat{\varphi_k f}|^2_{L^2} + 2^{s(l+k)-2k}|\varphi_l\widehat{v\varphi_k f}|^2_{L^2} + 2^{s(l+k)}|\varphi_l\widehat{\varphi_k f}|^2_{L^2}\\ &:= \mathcal{A}_{k,l,1} + \mathcal{A}_{k,l,2} + \mathcal{A}_{k,l,3}.\end{aligned}$$

The first term is estimated by $\sum_{2^k\leq 1/\varepsilon, 2^l\leq 1/\varepsilon}\mathcal{A}_{k,l,1} \lesssim |f|^2_{L^2}$. For the second term $\mathcal{A}_{k,l,2}$, we have

$$\begin{aligned}\sum_{\substack{2^k\leq 1/\varepsilon,\\ 2^l\leq 1/\varepsilon}}\mathcal{A}_{k,l,2} &\lesssim \sum_{j=1}^{3}\sum_{\substack{2^k\leq 1/\varepsilon,\\ 2^l\leq 1/\varepsilon}} 2^{2sl}2^{-2k}|\varphi_l\widehat{v_j\varphi_k f}|^2_{L^2} + \sum_{j=1}^{3}\sum_{\substack{2^k\leq 1/\varepsilon,\\ 2^l\leq 1/\varepsilon}} 2^{2sk}2^{-2k}|\varphi_l\widehat{v_j\varphi_k f}|^2_{L^2}\\ &\lesssim \sum_{j=1}^{3}\sum_{2^k\leq 1/\varepsilon} 2^{-2k}|W^\varepsilon\widehat{v_j\varphi_k f}|^2_{L^2} + \sum_{j=1}^{3}\sum_{2^k\leq 1/\varepsilon} 2^{2sk}2^{-2k}|v_j\varphi_k f|^2_{L^2}\\ &\lesssim \sum_{j=1}^{3}\sum_{2^k\leq 1/\varepsilon} 2^{-2k}|W^\varepsilon(D)v_j\varphi_k f|^2_{L^2} + |W^\varepsilon f|^2_{L^2}\\ &\lesssim |W^\varepsilon(D)f|^2_{L^2} + |W^\varepsilon f|^2_{L^2}.\end{aligned}$$

In the last inequality we apply Lemma A.1 to get

$$|W^\varepsilon(D)v_j\varphi_k f|^2_{L^2} \lesssim |v_j\varphi_k W^\varepsilon(D)f|^2_{L^2} + |f|^2_{H^{s-1}},$$

thanks to $W^\varepsilon \in S^s_{1,0}$, $v_j\varphi_k \in S^1_{1,0}$ (see Definition A.1 for $S^m_{1,0}$). As for the last term $\mathcal{A}_{k,l,3}$, we have

$$\begin{aligned}\sum_{\substack{2^k\leq 1/\varepsilon,\\ 2^l\leq 1/\varepsilon}}\mathcal{A}_{k,l,3} &\lesssim \sum_{\substack{2^k\leq 1/\varepsilon,\\ 2^l\leq 1/\varepsilon}} 2^{2sl}|\varphi_l\widehat{\varphi_k f}|^2_{L^2} + \sum_{\substack{2^k\leq 1/\varepsilon,\\ 2^l\leq 1/\varepsilon}} 2^{2sk}|\varphi_l\widehat{\varphi_k f}|^2_{L^2}\\ &\lesssim \sum_{2^k\leq 1/\varepsilon}|W^\varepsilon(D)\varphi_k f|^2_{L^2} + \sum_{2^k\leq 1/\varepsilon} 2^{2sk}|\widehat{\varphi_k f}|^2_{L^2} \lesssim |W^\varepsilon(D)f|^2_{L^2} + |W^\varepsilon f|^2_{L^2}.\end{aligned}$$

Patching together the above estimates, we finish the proof. ∎

Remark 2.1. If we change the integral range $\int_{\varepsilon}^{\pi/4}$ in Lemma 2.1 to $\int_{3\varepsilon/4}^{\pi/4}$, the estimate still holds true.

Lemma 2.2. *Let*

$$\mathcal{Z}^{\varepsilon,\gamma}(f) := \int b^{\varepsilon}(\tfrac{u}{|u|}\cdot\sigma)\langle u\rangle^{\gamma}|f(|u|\tfrac{u^+}{|u^+|}) - f(u^+)|^2 \,\mathrm{d}\sigma\,\mathrm{d}u$$

with $u^+ = \frac{u+|u|\sigma}{2}$. *Then*

$$\mathcal{Z}^{\varepsilon,\gamma}(f) \lesssim |W^{\varepsilon}(D)W_{\gamma/2}f|_{L^2}^2 + |W^{\varepsilon}W_{\gamma/2}f|_{L^2}^2.$$

Proof. We divide the proof into two steps.

Step 1*:* $\gamma = 0$. By the change of variable $(u,\sigma) \to (r,\tau,\varsigma)$ with $u = r\tau$ and $\varsigma = \frac{\sigma+\tau}{|\sigma+\tau|}$, we have

$$\mathcal{Z}^{\varepsilon,0}(f) = 4\int_{\mathbb{R}_+\times\mathbb{S}^2\times\mathbb{S}^2} b^{\varepsilon}(2(\tau\cdot\varsigma)^2-1)|f(r\varsigma) - f((\tau\cdot\varsigma)r\varsigma)|^2(\tau\cdot\varsigma)r^2\,\mathrm{d}r\,\mathrm{d}\tau\,\mathrm{d}\varsigma.$$

Let $\eta = r\varsigma$ and θ be the angle between τ and ς. Recalling assumption (1.1) and b^{ε} in (1.3), we have $b^{\varepsilon}(2(\tau\cdot\varsigma)^2-1) = b^{\varepsilon}(\cos 2\theta) \lesssim \theta^{-2-2s}1_{3\varepsilon/4\le\theta\le\pi/4}$. Observing that $r^2\,\mathrm{d}r\,\mathrm{d}\tau\,\mathrm{d}\varsigma = \sin\theta\,\mathrm{d}\eta\,\mathrm{d}\theta\,\mathrm{d}\varphi$, we have

$$\begin{aligned}\mathcal{Z}^{\varepsilon,0}(f) &\lesssim \int_{\mathbb{R}^3}\int_{3\varepsilon/4}^{\pi/4}\theta^{-1-2s}|f(\eta) - f(\eta\cos\theta)|^2\,\mathrm{d}\eta\,\mathrm{d}\theta\\ &\lesssim \int_{\mathbb{R}^3}\int_{3\varepsilon/4}^{\pi/4}\theta^{-1-2s}|f(\eta) - f(\eta/\cos\theta)|^2\,\mathrm{d}\eta\,\mathrm{d}\theta\\ &\lesssim |W^{\varepsilon}(D)f|_{L^2}^2 + |W^{\varepsilon}f|_{L^2}^2,\end{aligned} \tag{2.15}$$

where the last inequality is given by Lemma 2.1 and Remark 2.1.

Step 2*:* $\gamma \neq 0$. We reduce the general case $\gamma \neq 0$ to the special case $\gamma = 0$. For simplicity, denote $w = |u|\frac{u^+}{|u^+|}$; then $W_{\gamma}(u) = W_{\gamma}(w)$. Then we have

$$\begin{aligned}&\langle u\rangle^{\gamma}|f(w) - f(u^+)|^2\\ &\quad= |((W_{\gamma/2}f)(w) - (W_{\gamma/2}f)(u^+)) + (W_{\gamma/2}f)(u^+)(1 - W_{\gamma/2}(w)W_{-\gamma/2}(u^+))|^2\\ &\quad\le 2|(W_{\gamma/2}f)(w) - (W_{\gamma/2}f)(u^+)|^2 + 2|(W_{\gamma/2}f)(u^+)|^2|1 - W_{\gamma/2}(w)W_{-\gamma/2}(u^+)|^2.\end{aligned}$$

Thus we have

$$\begin{aligned}&\mathcal{Z}^{\varepsilon,\gamma}(f) \lesssim \mathcal{Z}^{\varepsilon,0}(W_{\gamma/2}f) + \mathcal{B},\\ &\mathcal{B} := \int b^{\varepsilon}(\tfrac{u}{|u|}\cdot\sigma)|(W_{\gamma/2}f)(u^+)|^2|1 - W_{\gamma/2}(w)W_{-\gamma/2}(u^+)|^2\,\mathrm{d}u\,\mathrm{d}\sigma.\end{aligned}$$

By noticing that $|W_{\gamma/2}(w)W_{-\gamma/2}(u^+) - 1| \lesssim \sin^2\frac{\theta}{2}$, and using the change of variable $u \to u^+$, we have $|\mathcal{B}| \lesssim |W_{\gamma/2}f|_{L^2}^2$. The desired result follows by utilizing (2.15) for $\mathcal{Z}^{\varepsilon,0}(W_{\gamma/2}f)$. ∎

Next we want to show the following lemma.

Lemma 2.3. *Let $\varepsilon \geq 0$ be small enough. For a suitable function f defined on $\mathbb{S}^2$, there holds*

$$\int \frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}} 1_{|\sigma-\tau|\geq\varepsilon}\, d\sigma\, d\tau + |f|^2_{L^2(\mathbb{S}^2)} \sim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})f|^2_{L^2(\mathbb{S}^2)} + |f|^2_{L^2(\mathbb{S}^2)}.$$

As a direct result, for a suitable function f defined on $\mathbb{R}^3$, there holds

$$\int_{\mathbb{S}^2\times\mathbb{S}^2\times\mathbb{R}_+} \frac{|f(r\sigma)-f(r\tau)|^2}{|\sigma-\tau|^{2+2s}} 1_{|\sigma-\tau|\geq\varepsilon} r^2\, d\sigma\, d\tau\, dr + |f|^2_{L^2}$$
$$\sim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})f|^2_{L^2} + |f|^2_{L^2}. \tag{2.16}$$

Proof. We only prove the case $\varepsilon > 0$ since the case $\varepsilon = 0$ is already proved in [10]. By [10, Lemma 5.4], we have

$$\int \frac{|f(\sigma)-f(\tau)|^2}{|\sigma-\tau|^{2+2s}} 1_{|\sigma-\tau|\geq\varepsilon}\, d\sigma\, d\tau = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} (f_l^m)^2 \int \frac{|Y_l^m(\sigma)-Y_l^m(\tau)|^2}{|\sigma-\tau|^{2+2s}} 1_{|\sigma-\tau|\geq\varepsilon}\, d\sigma\, d\tau,$$

where $f_l^m = \int_{\mathbb{S}^2} f Y_l^m\, d\sigma$. For simplicity, let $\mathcal{A}_l^\varepsilon = \int \frac{|Y_l^m(\sigma)-Y_l^m(\tau)|^2}{|\sigma-\tau|^{2+2s}} 1_{|\sigma-\tau|\geq\varepsilon}\, d\sigma\, d\tau$. Now we will analyze $\mathcal{A}_l^\varepsilon$.

Case 1*:* $\varepsilon^2 l(l+1) \leq \eta$. We have

$$\mathcal{A}_l^\varepsilon = \int \frac{|Y_l^m(\sigma)-Y_l^m(\tau)|^2}{|\sigma-\tau|^{2+2s}}\, d\sigma\, d\tau - \int \frac{|Y_l^m(\sigma)-Y_l^m(\tau)|^2}{|\sigma-\tau|^{2+2s}} 1_{|\sigma-\tau|\leq\varepsilon}\, d\sigma\, d\tau.$$

From [10, Lemma 5.5] we get

$$|(-\Delta_{\mathbb{S}^2})^{\frac{s}{2}} Y_l^m|^2_{L^2(\mathbb{S}^2)} - |Y_l^m|^2_{L^2(\mathbb{S}^2)} - \varepsilon^{2-2s}|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} Y_l^m|^2_{L^2(\mathbb{S}^2)}$$
$$\lesssim \mathcal{A}_l^\varepsilon \lesssim |(-\Delta_{\mathbb{S}^2})^{\frac{s}{2}} Y_l^m|^2_{L^2(\mathbb{S}^2)} + |Y_l^m|^2_{L^2(\mathbb{S}^2)} + \varepsilon^{2-2s}|(-\Delta_{\mathbb{S}^2})^{\frac{1}{2}} Y_l^m|^2_{L^2(\mathbb{S}^2)}.$$

For $l \geq 1$ and $\varepsilon^2 l(l+1) \leq \eta$, we have

$$\begin{aligned}[l(l+1)]^s(1-2^{-s}-\eta^{1-s}) &\leq [l(l+1)]^s(1-[l(l+1)]^{-s}-[\varepsilon^2 l(l+1)]^{1-s}) \\ &= [l(l+1)]^s - 1 - \varepsilon^{2-2s} l(l+1) \\ &\leq \mathcal{A}_l^\varepsilon \leq (2+\eta^{1-s})[l(l+1)]^s.\end{aligned}$$

By taking η small enough, we have $\mathcal{A}_l^\varepsilon \sim [l(l+1)]^s$.

Case 2*:* $\varepsilon^2 l(l+1) \geq R^2$. Let ζ be a smooth function with compact support verifying that $0 \leq \zeta \leq 1$, $\zeta(x) = 1$ if $|x| \geq 2$ and $\zeta(x) = 0$ if $|x| \leq 1$. We have

$$\begin{aligned}\mathcal{A}_l^\varepsilon &\geq \int \frac{|Y_l^m(\sigma)|^2+|Y_l^m(\tau)|^2-2Y_l^m(\sigma)Y_l^m(\tau)}{|\sigma-\tau|^{2+2s}} \zeta(\varepsilon^{-1}|\sigma-\tau|)\, d\sigma\, d\tau \\ &\gtrsim \varepsilon^{-2s} - \int \frac{Y_l^m(\sigma)Y_l^m(\tau)}{|\sigma-\tau|^{2+2s}} \zeta(\varepsilon^{-1}|\sigma-\tau|)\, d\sigma\, d\tau := \varepsilon^{-2s} - \mathcal{B}_l^\varepsilon.\end{aligned}$$

Since $(-\Delta_{\mathbb{S}^2})Y_l^m = l(l+1)Y_l^m$, we have

$$\begin{aligned}\mathcal{B}_l^\varepsilon &= [l(l+1)]^{-1}\int \frac{(-\Delta_{\mathbb{S}^2})Y_l^m(\sigma)Y_l^m(\tau)}{|\sigma-\tau|^{2+2s}}\zeta(\varepsilon^{-1}|\sigma-\tau|)\,\mathrm{d}\sigma\,\mathrm{d}\tau\\ &\le C[l(l+1)]^{-1}|(-\Delta_{\mathbb{S}^2})^{\frac12}Y_l^m|_{L^2(\mathbb{S}^2)}|Y_l^m|_{L^2(\mathbb{S}^2)}\int|\sigma-\tau|^{-3-2s}1_{|\sigma-\tau|\ge\varepsilon}\,\mathrm{d}\sigma\\ &\le C\varepsilon^{-2s}[\varepsilon^2 l(l+1)]^{-\frac12}.\end{aligned}$$

Thus we have $\mathcal{A}_l^\varepsilon \gtrsim \varepsilon^{-2s}(1-C[\varepsilon^2 l(l+1)]^{-\frac12}) \ge \varepsilon^{-2s}(1-C/R)$. Since $\mathcal{A}_l^\varepsilon \le 4\varepsilon^{-2s}$, we obtain that $\mathcal{A}_l^\varepsilon \sim \varepsilon^{-2s}$ as long as R is large enough.

Case 3*:* $\varepsilon^2 l(l+1)\ge\eta$. Here η is the fixed constant in Case 1. Note that $(N\varepsilon)^2 l(l+1)\ge N^2\eta$. Applying the lower bound estimate in Case 2 with $\varepsilon := N\varepsilon$, $R := N\sqrt{\eta}$, we obtain that

$$\mathcal{A}_l^{N\varepsilon} \gtrsim N^{-2s}\varepsilon^{-2s}\Big(1-\frac{C}{N\sqrt{\eta}}\Big).$$

Choosing N large enough, for $\varepsilon^2 l(l+1)\ge\eta$, we have $\mathcal{A}_l^\varepsilon \ge \mathcal{A}_l^{N\varepsilon}\gtrsim\varepsilon^{-2s}$. Notice that there still holds $\mathcal{A}_l^\varepsilon\lesssim\varepsilon^{-2s}$ in this case. Thus we get $\mathcal{A}_l^\varepsilon\sim\varepsilon^{-2s}$.

Combining Cases 1 and 3, we finally obtain the desired result. ■

Remark 2.2. If the truncation $|\sigma-\tau|\ge\varepsilon$ in Lemma 2.3 is replaced by $|\sigma-\tau|\ge a\varepsilon$ for some $\frac13\le a\le 3$, the results still hold true.

Now we are in a position to derive the anisotropic regularity from $\mathcal{R}_\mu^{\varepsilon,\gamma}(f)$ defined in (1.25). Our key strategy is to apply geometric decomposition in frequency space.

Proposition 2.3. *The following two estimates are valid:*

$$\mathcal{R}_\mu^{\varepsilon,0}(f)+|W^\varepsilon f|_{L^2}^2 \sim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12})f|_{L^2}^2+|W^\varepsilon(D)f|_{L^2}^2+|W^\varepsilon f|_{L^2}^2, \quad (2.17)$$

$$\mathcal{R}_\mu^{\varepsilon,\gamma}(f)+|W^\varepsilon f|_{L^2_{\gamma/2}}^2 \gtrsim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12})W_{\gamma/2}f|_{L^2}^2+|W^\varepsilon(D)W_{\gamma/2}f|_{L^2}^2. \quad (2.18)$$

Proof. The proof is split into two steps.

Step 1*:* (2.17) *and* (2.18) *with* $\gamma=0$. By Bobylev's formula, we have

$$\begin{aligned}\mathcal{R}_\mu^{\varepsilon,0}(f) = &\frac{1}{(2\pi)^3}\int b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)(\hat\mu(0)|\hat f(\xi)-\hat f(\xi^+)|^2\\ &\qquad+2\Re((\hat\mu(0)-\hat\mu(\xi^-))\hat f(\xi^+)\bar{\hat f}(\xi))\,\mathrm{d}\xi\,\mathrm{d}\sigma\\ :=&\frac{\hat\mu(0)}{(2\pi)^3}\mathcal{I}_1+\frac{2}{(2\pi)^3}\mathcal{I}_2,\end{aligned}$$

where $\xi^+=\frac{\xi+|\xi|\sigma}{2}$ and $\xi^-=\frac{\xi-|\xi|\sigma}{2}$. Thanks to the fact that

$$\hat\mu(0)-\hat\mu(\xi^-)=\int(1-\cos(v\cdot\xi^-))\mu(v)\,\mathrm{d}v,$$

we have

$$
\begin{aligned}
|\mathcal{I}_2| &= \left| \int b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)(1-\cos(v\cdot\xi^-))\mu(v)\Re(\hat f(\xi^+)\bar{\hat f}(\xi))\,\mathrm{d}\sigma\,\mathrm{d}\xi\,\mathrm{d}v \right| \\
&\lesssim \left(\int b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)(1-\cos(v\cdot\xi^-))\mu(v)|\hat f(\xi^+)|^2\,\mathrm{d}\sigma\,\mathrm{d}\xi\,\mathrm{d}v \right)^{\frac12} \\
&\quad \times \left(\int b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)(1-\cos(v\cdot\xi^-))\mu(v)|\hat f(\xi)|^2\,\mathrm{d}\sigma\,\mathrm{d}\xi\,\mathrm{d}v \right)^{\frac12}.
\end{aligned}
$$

Observe that

$$
1-\cos(v\cdot\xi^-) \lesssim |v|^2|\xi^-|^2 = \tfrac14|v|^2|\xi|^2|\tfrac{\xi}{|\xi|}-\sigma|^2 \sim |v|^2|\xi^+|^2|\tfrac{\xi^+}{|\xi^+|}-\sigma|^2,
$$

thus

$$
1-\cos(v\cdot\xi^-) \lesssim \min\{|v|^2|\xi|^2|\tfrac{\xi}{|\xi|}-\sigma|^2, 1\} \sim \min\{|v|^2|\xi^+|^2|\tfrac{\xi^+}{|\xi^+|}-\sigma|^2, 1\}.
$$

Noting that $\frac{\xi}{|\xi|}\cdot\sigma = 2(\frac{\xi^+}{|\xi^+|}\cdot\sigma)^2-1$, by the change of variable $\xi\to\xi^+$, Proposition A.1 and the fact that $W^\varepsilon(|v||\xi|) \lesssim W^\varepsilon(|v|)W^\varepsilon(|\xi|)$, we have

$$
\begin{aligned}
|\mathcal{I}_2| &\lesssim \int (W^\varepsilon)^2(|v|\,|\xi|)|\hat f(\xi)|^2\mu(v)\,\mathrm{d}\xi\,\mathrm{d}v \\
&\lesssim |W^\varepsilon\mu^{\frac12}|^2_{L^2}|W^\varepsilon(D)f|^2_{L^2} \lesssim |W^\varepsilon(D)f|^2_{L^2}.
\end{aligned}
$$

Now we will estimate $\mathcal{I}_1$. By the geometric decomposition

$$
\hat f(\xi)-\hat f(\xi^+) = \hat f(\xi)-\hat f(|\xi|\tfrac{\xi^+}{|\xi^+|}) + \hat f(|\xi|\tfrac{\xi^+}{|\xi^+|})-\hat f(\xi^+),
$$

we have

$$
\begin{aligned}
\mathcal{I}_1 &= \int b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)|\hat f(\xi)-\hat f(\xi^+)|^2\,\mathrm{d}\xi\,\mathrm{d}\sigma \\
&\geq \frac12\int b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)|\hat f(\xi)-\hat f(|\xi|\tfrac{\xi^+}{|\xi^+|})|^2\,\mathrm{d}\xi\,\mathrm{d}\sigma \\
&\quad -\int b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)|\hat f(|\xi|\tfrac{\xi^+}{|\xi^+|})-\hat f(\xi^+)|^2\,\mathrm{d}\xi\,\mathrm{d}\sigma \\
&:= \frac12\mathcal{I}_{1,1}-\mathcal{I}_{1,2}.
\end{aligned}
$$

Let $\xi = r\tau$ and $\varsigma = \frac{\tau+\sigma}{|\tau+\sigma|}$; then $\frac{\xi}{|\xi|}\cdot\sigma = 2(\tau\cdot\varsigma)^2-1$ and $|\xi|\frac{\xi^+}{|\xi^+|} = r\varsigma$. For the change of variable $(\xi,\sigma)\to(r,\tau,\varsigma)$, one has $\mathrm{d}\xi\,\mathrm{d}\sigma = 4(\tau\cdot\varsigma)r^2\,\mathrm{d}r\,\mathrm{d}\tau\,\mathrm{d}\varsigma$. Let θ be the angle between τ and σ; then $2\sin\frac{\theta}{2} = |\tau-\sigma|$, $|\tau-\varsigma| = 2(1-\cos\frac{\theta}{2})$ and thus $\sin\frac{\theta}{2} = \frac12|\tau-\sigma| \leq |\tau-\varsigma| \leq |\tau-\sigma| = 2\sin\frac{\theta}{2}$. Therefore,

$$
|\tau-\varsigma|^{-2-2s}1_{|\tau-\varsigma|\geq\frac83\varepsilon} \lesssim b^\varepsilon(\cos\theta) \lesssim |\tau-\varsigma|^{-2-2s}1_{|\tau-\varsigma|\geq\frac34\varepsilon}. \tag{2.19}
$$

By (2.16) in Lemma 2.3 and Remark 2.2, we have

$$\begin{aligned}\mathcal{I}_{1,1} + |f|_{L^2}^2 &= 4\int b^\varepsilon(2(\tau\cdot\varsigma)^2-1)|\hat f(r\tau)-\hat f(r\varsigma)|^2(\tau\cdot\varsigma)r^2\,\mathrm{d}r\,\mathrm{d}\tau\,\mathrm{d}\varsigma + |f|_{L^2}^2\\ &\gtrsim \int \frac{|\hat f(r\tau)-\hat f(r\varsigma)|^2}{|\tau-\varsigma|^{2+2s}} 1_{|\tau-\varsigma|\geq 8\varepsilon/3} r^2\,\mathrm{d}r\,\mathrm{d}\tau\,\mathrm{d}\varsigma + |f|_{L^2}^2\\ &\sim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12})\hat f|_{L^2}^2 + |\hat f|_{L^2}^2 \sim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12}) f|_{L^2}^2 + |f|_{L^2}^2. \quad (2.20)\end{aligned}$$

Here we use Lemma A.3 and Plancherel's theorem in the last line. Similarly, by (2.19), (2.16) in Lemma 2.3 and Remark 2.2, we have

$$\mathcal{I}_{1,1} + |f|_{L^2}^2 \lesssim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12}) f|_{L^2}^2 + |f|_{L^2}^2.$$

By Lemma 2.2, there holds

$$\mathcal{I}_{1,2} \lesssim |W^\varepsilon(D)f|_{L^2}^2 + |W^\varepsilon f|_{L^2}^2. \quad (2.21)$$

Patching together (2.20), (2.21) and Proposition 2.2, we get the $\gtrsim$ direction of (2.17), i.e.,

$$\mathcal{R}_\mu^{\varepsilon,0}(f) + |W^\varepsilon f|_{L^2}^2 \gtrsim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac12}) f|_{L^2}^2 + |W^\varepsilon(D)f|_{L^2}^2 + |W^\varepsilon f|_{L^2}^2. \quad (2.22)$$

The $\lesssim$ direction of (2.17) follows easily from $\mathcal{R}_\mu^{\varepsilon,0}(f) \lesssim \mathcal{I}_{1,1} + \mathcal{I}_{1,2} + |\mathcal{I}_2|$.

Step 2*:* (2.18) *with* $\gamma \neq 0$. Thanks to [10, Lemma 3.4] which reads

$$\mathcal{R}_\mu^{\varepsilon,\gamma}(f) + |f|_{L^2_{\gamma/2}}^2 \gtrsim \mathcal{R}_\mu^{\varepsilon,0}(W_{\gamma/2}f),$$

we obtain the desired result by using (2.22). The proof is complete. ■

2.1.3. Lower bound of $\langle \mathcal{L}^\varepsilon f, f\rangle_v$. We are ready to derive the following lower bound estimate for $\langle \mathcal{L}^\varepsilon f, f\rangle_v$.

Theorem 2.1. *It holds that*

$$\langle \mathcal{L}^\varepsilon f, f\rangle_v + |f|_{L^2_{\gamma/2}}^2 \gtrsim |f|_{\varepsilon,\gamma/2}^2.$$

Proof. We proceed in the spirit of [4]. Recalling (1.8) and $(a+b)^2 \geq a^2/2 - b^2$, there holds

$$\begin{aligned}2\langle \mathcal{L}_1^\varepsilon f, f\rangle_v &= \int B^\varepsilon(\mu_*^{\frac12} f - (\mu^{\frac12})'_* f')^2\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\\ &= \int B^\varepsilon(\mu_*^{\frac12}(f-f') + (\mu_*^{\frac12} - (\mu^{\frac12})'_*)f')^2\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\\ &\geq \frac12 \mathcal{R}_\mu^{\varepsilon,\gamma}(f) - \mathcal{M}^{\varepsilon,\gamma}(f). \quad (2.23)\end{aligned}$$

Note that

$$2\langle \mathcal{L}_1^\varepsilon f, f\rangle_v = \mathcal{R}_\mu^{\varepsilon,\gamma}(f) + \mathcal{M}^{\varepsilon,\gamma}(f) + 2\int B^\varepsilon(\mu_*^{\frac12} - (\mu^{\frac12})'_*)\mu_*^{\frac12}(f-f')f'\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma.$$

From this, together with the fact that $2(a-b)b = a^2 - b^2 - (a-b)^2$, one gets

$$\begin{aligned}
&2(\mu_*^{\frac12} - (\mu^{\frac12})'_*)\mu_*^{\frac12}(f-f')f' \\
&= \frac12(f^2 - f'^2 - (f-f')^2)(\mu_* - \mu'_* + (\mu_*^{\frac12} - (\mu^{\frac12})'_*)^2) \\
&= \frac12(f^2 - f'^2)(\mu_* - \mu'_*) - \frac12(f-f')^2(\mu_*^{\frac12} - (\mu^{\frac12})'_*)^2 \\
&\quad + \frac12(f^2 - f'^2)(\mu_*^{\frac12} - (\mu^{\frac12})'_*)^2 - \frac12(f-f')^2(\mu_* - \mu'_*) \\
&:= A_1 + A_2 + A_3 + A_4.
\end{aligned}$$

By the change of variable $(v, v_*) \to (v', v'_*)$, the cancellation lemma in [1] and (2.7), it holds that

$$\begin{aligned}
&\left|\int B^\varepsilon A_1 \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\right| = \left|\int B^\varepsilon \mu_*(f^2 - f'^2)\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma\right| \le C|f|^2_{L^2_{\gamma/2}}, \\
&\int B^\varepsilon A_3 \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma = \int B^\varepsilon A_4 \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma = 0.
\end{aligned}$$

Note that

$$\begin{aligned}
\int B^\varepsilon A_2 \,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma = &-\int B^\varepsilon \mu_*(f-f')^2\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma \\
&+ \int B^\varepsilon \mu_*^{\frac12}(\mu^{\frac12})'_*(f-f')^2\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\sigma \ge -\mathcal{R}^{\varepsilon,\gamma}_\mu(f).
\end{aligned}$$

Patching together the above estimates, we infer that $2\langle \mathcal{L}_1^\varepsilon f, f\rangle_v \ge \mathcal{M}^{\varepsilon,\gamma}(f) - C|f|^2_{L^2_{\gamma/2}}$, from which, together with (2.23), we have

$$5\langle \mathcal{L}_1^\varepsilon f, f\rangle_v \ge \tfrac12 \mathcal{R}^{\varepsilon,\gamma}_\mu(f) + \tfrac12 \mathcal{M}^{\varepsilon,\gamma}(f) - \tfrac32 C|f|^2_{L^2_{\gamma/2}} \gtrsim |f|^2_{\varepsilon,\gamma/2} - C|f|^2_{L^2_{\gamma/2}},$$

where in the last inequality we use Proposition 2.1 and (2.18). By a similar proof to that of [4, Lemma 2.15], it holds that

$$|\langle \mathcal{L}_2^\varepsilon g, h\rangle_v| \lesssim |\mu^{1/10^3} g|_{L^2} |\mu^{1/10^3} h|_{L^2} \lesssim |g|_{L^2_{\gamma/2}} |h|_{L^2_{\gamma/2}}. \tag{2.24}$$

Recalling that $\mathcal{L}^\varepsilon = \mathcal{L}_1^\varepsilon + \mathcal{L}_2^\varepsilon$, we finish the proof. ∎

We give the coercivity estimate of $\mathcal{L}^\varepsilon$ on the perpendicular space $\mathcal{N}^\perp$ in the following proposition.

Proposition 2.4. *It holds that*

$$\langle \mathcal{L}^\varepsilon f, f\rangle_v \gtrsim |(\mathbb{I} - \mathbb{P})f|^2_{\varepsilon,\gamma/2}.$$

Here $\mathbb{I}$ *stands for the identity operator.*

Proof. By [15, 16], there holds $\langle \mathcal{L}^\varepsilon f, f\rangle_v \gtrsim |(\mathbb{I}-\mathbb{P})f|^2_{L^2_{\gamma/2}}$. By the definition of $\mathbb{P}$ and Theorem 2.1, we have

$$\langle \mathcal{L}^\varepsilon f, f\rangle_v = \langle \mathcal{L}^\varepsilon(\mathbb{I}-\mathbb{P})f, (\mathbb{I}-\mathbb{P})f\rangle_v \gtrsim |(\mathbb{I}-\mathbb{P})f|^2_{\varepsilon,\gamma/2} - |(\mathbb{I}-\mathbb{P})f|^2_{L^2_{\gamma/2}}.$$

Making a suitable combination of the two estimates, we get the desired result. ■

2.2. Upper bound of the nonlinear term

In this subsection we will estimate the following inner product:

$$\langle \Gamma^\varepsilon(g,h), f\rangle_v = \langle Q^\varepsilon(\mu^{\frac{1}{2}}g,h), f\rangle_v + \mathcal{I}(g,h,f), \tag{2.25}$$

$$\mathcal{I}(g,h,f) := \int b^\varepsilon(\cos\theta)|v-v_*|^\gamma((\mu^{\frac{1}{2}})'_* - \mu^{\frac{1}{2}}_*)g_* h f' \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v. \tag{2.26}$$

We will estimate Q^ε in Section 2.2.1 and $\mathcal{I}(g,h,f)$ in Section 2.2.2. Using relation (2.25), the upper bounds of $\langle \Gamma^\varepsilon(g,h), f\rangle_v$ will be summarized in Theorem 2.3 in Section 2.2.3. At the end of Section 2.2, we will finish the proof of Theorem 1.1.

2.2.1. Upper bounds for the collision operator Q^ε. We perform the decomposition

$$\langle Q^\varepsilon(g,h), f\rangle_v = \langle Q^\varepsilon_{-1}(g,h), f\rangle_v + \langle Q^\varepsilon_{\geq 0}(g,h), f\rangle_v, \tag{2.27}$$

where

$$\langle Q^\varepsilon_{-1}(g,h), f\rangle_v := \int b^\varepsilon(\cos\theta)|v-v_*|^\gamma \phi(v-v_*) g_* h(f'-f)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v,$$

$$\langle Q^\varepsilon_{\geq 0}(g,h), f\rangle_v := \int b^\varepsilon(\cos\theta)|v-v_*|^\gamma (1-\phi(v-v_*)) g_* h(f'-f)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v.$$

Here ϕ is given in (1.19).

To give an estimate for Q^ε_{-1}, we begin with two lemmas.

Lemma 2.4. *Let* $A := \int |v-v_*|^\gamma \phi(v-v_*) g_* h f \,\mathrm{d}v_*\,\mathrm{d}v$, $B := \varepsilon^{2s}\int b^\varepsilon(\cos\theta)|v - v_*|^\gamma \phi(v-v_*) g_* h f' \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v$. *The following statements are valid:*

- *If* $\gamma > -\frac{3}{2}$, *then* $|A| + |B| \lesssim |g|_{L^2}|h|_{L^2}|f|_{L^2}$.
- *If* $\gamma = -\frac{3}{2}$, *for any* $\eta > 0$, *there exists a constant* C_η *such that*

$$|A|+|B| \leq \begin{cases} C_\eta |g|_{H^\eta}|h|_{L^2}|f|_{L^2}, \\ C_\eta(|g|_{L^1}+|g|_{L^2})|h|_{H^\eta}|f|_{L^2}. \end{cases}$$

- *If* $-3 < \gamma < -\frac{3}{2}$, *for any* $\eta > 0$, *there exists a constant* C_η *such that*

$$|A|+|B| \lesssim \begin{cases} C_\eta |g|_{H^{\eta-\frac{3}{2}-\gamma}}|h|_{L^2}|f|_{L^2}, \\ |g|_{H^{s_1}}|h|_{H^{s_2}}|f|_{H^{s_3}}. \end{cases}$$

Here the constants $s_1, s_2, s_3 \geq 0$ *verify* $s_1+s_2+s_3 = -\gamma-\frac{3}{2}$, $s_2+s_3>0$.

Proof. We first handle the term A. If $\gamma > -\frac{3}{2}$, the desired result comes from the inequality

$$\int |v - v_*|^\gamma \phi(v - v_*)|g_*| \, \mathrm{d}v_* \lesssim |g|_{L^2}.$$

If $\gamma = -\frac{3}{2}$, the first result follows from Hardy's inequality,

$$\int |v - v_*|^{-\frac{3}{2}} \phi(v - v_*)|g_*| \, \mathrm{d}v_* \leq C_\eta \left(\int |v - v_*|^{-2\eta} |g_*|^2 \, \mathrm{d}v_* \right)^{\frac{1}{2}} \lesssim C_\eta |g|_{H^\eta}.$$

The second result follows from the Hardy–Littlewood–Sobolev inequality, the Sobolev embedding theorem and the interpolation inequality. Indeed, we have

$$|A| \lesssim |g|_{L^{p_1}} |h|_{L^{p_2}} |f|_{L^2} \lesssim C_\eta (|g|_{L^1} + |g|_{L^2}) |h|_{H^\eta} |f|_{L^2},$$

where $\frac{1}{p_1} + \frac{1}{p_2} = 1$, $\frac{\eta}{3} = \frac{1}{2} - \frac{1}{p_2}$ with $p_2 > 2$, $1 < p_1 < 2$.

If $-3 < \gamma < -\frac{3}{2}$, the first result follows from Hardy's inequality,

$$\int |v - v_*|^\gamma \phi(v - v_*)|g_*| \, \mathrm{d}v_* \leq C_\eta \left(\int |v - v_*|^{2\gamma + 3 - 2\eta} |g_*|^2 \, \mathrm{d}v_* \right)^{\frac{1}{2}} \lesssim C_\eta |g|_{H^{\eta - \frac{3}{2} - \gamma}}.$$

The second result follows from the Hardy–Littlewood–Sobolev inequality and the Sobolev embedding theorem,

$$|A| \lesssim |g|_{L^{p_1}} |h|_{L^{p_2}} |f|_{L^{p_3}} \lesssim |g|_{H^{s_1}} |h|_{H^{s_2}} |f|_{H^{s_3}},$$

where $\frac{-\gamma}{3} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 2$, $p_i \geq 2$, $\frac{1}{p_2} + \frac{1}{p_3} < 1$, $\frac{s_i}{3} = \frac{1}{2} - \frac{1}{p_i}$ and thus $s_1 + s_2 + s_3 = -\frac{3}{2} - \gamma$, $s_2 + s_3 > 0$.

Now we point out how to derive the same estimates for B. From the above proof for A, thanks to the change of variable $v \to v'$ and the estimate $\varepsilon^{2s} \int b^\varepsilon(\cos\theta) \, \mathrm{d}\sigma \lesssim 1$, we only need to prove that the Hardy–Littlewood–Sobolev inequality is still valid for B. To this end, we observe that for $\frac{-\gamma}{3} + \frac{1}{p_1} + \frac{1}{r} = 2$ and $\frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}$,

$$\begin{aligned} |B| &\lesssim \left(\varepsilon^{2s} \int b^\varepsilon(\cos\theta)|v - v_*|^\gamma \phi(v - v_*)|g_*| \, |h|^{\frac{p_2}{r}} \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \right)^{\frac{r}{p_2}} \\ &\quad \times \left(\varepsilon^{2s} \int b^\varepsilon(\cos\theta)|v - v_*|^\gamma \phi(v - v_*)|g_*| \, |f'|^{\frac{p_3}{r}} \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \right)^{\frac{r}{p_3}} \\ &\lesssim |g|_{L^{p_1}} |h|_{L^{p_2}} |f|_{L^{p_3}}. \end{aligned}$$

Then we conclude the results for B by copying the same argument used for A. ∎

Lemma 2.5. *Set*

$$A := \int |v - v_*|^\gamma g_* h f \, \mathrm{d}v_* \, \mathrm{d}v, \quad B := \varepsilon^{2s} \int b^\varepsilon(\cos\theta)|v - v_*|^\gamma g_* h f' \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v.$$

The following statements are valid:

- *If* $\gamma \geq 0$, *then* $|A| + |B| \lesssim |g|_{L^1_\gamma} |h|_{L^2_{\gamma/2}} |f|_{L^2_{\gamma/2}}$.
- *If* $-\frac{3}{2} < \gamma < 0$, *then* $|A| + |B| \lesssim (|g|_{L^1_{|\gamma|}} + |g|_{L^2_{|\gamma|}}) |h|_{L^2_{\gamma/2}} |f|_{L^2_{\gamma/2}}$.
- *If* $\gamma = -\frac{3}{2}$, *for any* $\eta > 0$, *there exists a constant* C_η *such that*

$$|A| + |B| \leq \begin{cases} C_\eta (|g|_{L^1_{|\gamma|}} + |g|_{H^\eta_{|\gamma|}}) |h|_{L^2_{\gamma/2}} |f|_{L^2_{\gamma/2}}, \\ C_\eta (|g|_{L^1_{|\gamma|}} + |g|_{L^2_{|\gamma|}}) |h|_{H^\eta_{\gamma/2}} |f|_{L^2_{\gamma/2}}. \end{cases}$$

- *If* $-3 < \gamma < -\frac{3}{2}$, *for any* $\eta > 0$, *there exists a constant* C_η *such that*

$$|A| + |B| \lesssim \begin{cases} C_\eta (|g|_{L^1_{|\gamma|}} + |g|_{H^{\eta - \frac{3}{2} - \gamma}_{|\gamma|}}) |h|_{L^2_{\gamma/2}} |f|_{L^2_{\gamma/2}}, \\ (|g|_{L^1_{|\gamma|}} + |g|_{H^{s_1}_{|\gamma|}}) |h|_{H^{s_2}_{\gamma/2}} |f|_{H^{s_3}_{\gamma/2}}. \end{cases}$$

Here $s_1, s_2, s_3 \geq 0$ *are constants verifying* $s_1 + s_2 + s_3 = -\gamma - \frac{3}{2}$, $s_2 + s_3 > 0$.

Proof. Let $G = g W_{|\gamma|}$, $H = h W_{\gamma/2}$, $F = f W_{\gamma/2}$. Then we have

$$\begin{aligned} |A| &= \left| \int |v - v_*|^\gamma \langle v_* \rangle^{-|\gamma|} \langle v \rangle^{-\gamma} G_* H F \, \mathrm{d}v_* \, \mathrm{d}v \right| \\ &\lesssim \int (1 + 1_{\gamma < 0} |v - v_*|^\gamma \phi(v - v_*)) |G_* H F| \, \mathrm{d}v_* \, \mathrm{d}v. \end{aligned}$$

Then the estimates for A follow from Lemma 2.4. Since $|v - v_*| \sim |v' - v_*|$, by a similar argument, we can conclude the results for B. ∎

Now we are ready to give the following upper bounds for Q^ε_{-1}.

Proposition 2.5. *For any* $\eta > 0$, *the following estimates are valid:*

- *If* $\gamma > -\frac{3}{2}$, *then* $|\langle Q^\varepsilon_{-1}(g, h), f \rangle_v| \lesssim |g|_{L^2_{|\gamma|}} |W^\varepsilon(D) h|_{L^2_{\gamma/2}} |W^\varepsilon(D) f|_{L^2_{\gamma/2}}$.
- *If* $\gamma = -\frac{3}{2}$, *then* $|\langle Q^\varepsilon_{-1}(g, h), f \rangle_v| \lesssim (|g|_{L^1_{|\gamma|}} + |g|_{H^{s_1}_{|\gamma|}}) |W^\varepsilon(D) h|_{H^{s_2}_{\gamma/2}} |W^\varepsilon(D) f|_{L^2_{\gamma/2}}$. *Here,* $(s_1, s_2) = (0, \eta)$ *or* $(\eta, 0)$.
- *If* $-3 < \gamma < -\frac{3}{2}$, *then* $|\langle Q^\varepsilon_{-1}(g, h), f \rangle_v| \lesssim |g|_{H^{s_1}_{|\gamma|}} |W^\varepsilon(D) h|_{H^{s_2}_{\gamma/2}} |W^\varepsilon(D) f|_{H^{s_3}_{\gamma/2}}$. *Here,* $s_1, s_2, s_3 \geq 0$ *are constants verifying either* $s_1 + s_2 + s_3 = -\gamma - \frac{3}{2}$, $s_2 + s_3 > 0$ *or* $s_1 = -\gamma - \frac{3}{2} + \eta$, $s_2 = s_3 = 0$.

If $\gamma = -\frac{3}{2}$ *or* $-3 < \gamma < -\frac{3}{2}$, *the* $\lesssim$ *could produce a constant depending on* η *on the right-hand sides.*

Proof. We divide the proof into two steps.

Step 1*: Estimates without weight.* Following [10, proof of Theorem 1.1], we conclude that

$$|Q^\varepsilon_{-1}(g, h), f \rangle_v| \lesssim |g|_{L^2} |h|_{H^a} |f|_{H^b}, \tag{2.28}$$

where $a+b=2s$ with $a,b\in[0,2s]$. Recalling (1.20), we have the decomposition

$$\begin{aligned}\langle Q_{-1}^{\varepsilon}(g,h),f\rangle_v = \langle Q_{-1}^{\varepsilon}(g,h_\phi),f_\phi\rangle_v + \langle Q_{-1}^{\varepsilon}(g,h_\phi),f^\phi\rangle_v \\ + \langle Q_{-1}^{\varepsilon}(g,h^\phi),f_\phi\rangle_v + \langle Q_{-1}^{\varepsilon}(g,h^\phi),f^\phi\rangle_v.\end{aligned}$$

Using (2.28), we have

$$\begin{aligned}|\langle Q_{-1}^{\varepsilon}(g,h_\phi),f_\phi\rangle_v| &\lesssim |g|_{L^2}|h_\phi|_{H^s}|f_\phi|_{H^s},\\ |\langle Q_{-1}^{\varepsilon}(g,h_\phi),f^\phi\rangle_v| &\lesssim |g|_{L^2}|h_\phi|_{H^{2s}}|f^\phi|_{L^2},\\ |\langle Q_{-1}^{\varepsilon}(g,h^\phi),f_\phi\rangle_v| &\lesssim |g|_{L^2}|h^\phi|_{L^2}|f_\phi|_{H^{2s}}.\end{aligned}$$

Thanks to $|h_\phi|_{H^{2s}}\lesssim \varepsilon^{-s}|h_\phi|_{H^s}$ and $|f_\phi|_{H^{2s}}\lesssim \varepsilon^{-s}|f_\phi|_{H^s}$, we have

$$\begin{aligned}&|\langle Q_{-1}^{\varepsilon}(g,h_\phi),f_\phi\rangle_v| + |\langle Q_{-1}^{\varepsilon}(g,h_\phi),f^\phi\rangle_v| + |\langle Q_{-1}^{\varepsilon}(g,h^\phi),f_\phi\rangle_v|\\ &\quad\lesssim |g|_{L^2}|W^\varepsilon(D)h|_{L^2}|W^\varepsilon(D)f|_{L^2}.\end{aligned}$$

From this, together with Lemma 2.4 to deal with $\langle Q_{-1}^{\varepsilon}(g,h^\phi),f^\phi\rangle_v$, we conclude that

$$\begin{aligned}\gamma>-\tfrac32:&\quad |\langle Q_{-1}^{\varepsilon}(g,h),f\rangle_v| \lesssim |g|_{L^2}|W^\varepsilon(D)h|_{L^2}|W^\varepsilon(D)f|_{L^2},\\ \gamma=-\tfrac32:&\quad |\langle Q_{-1}^{\varepsilon}(g,h),f\rangle_v| \lesssim (|g|_{L^1}+|g|_{H^{s_1}})|W^\varepsilon(D)h|_{H^{s_2}}|W^\varepsilon(D)f|_{L^2},\\ -3<\gamma<-\tfrac32:&\quad |\langle Q_{-1}^{\varepsilon}(g,h),f\rangle_v| \lesssim |g|_{H^{s_1}}|W^\varepsilon(D)h|_{H^{s_2}}|W^\varepsilon(D)f|_{H^{s_3}}.\end{aligned}$$

That is, the results in the proposition are valid if we take $\gamma=0$ on the right-hand sides. In the next step, we recover the weights by using some commutator estimates.

Step 2: Estimates with weight. We recall that

$$\begin{aligned}\langle Q_{-1}^{\varepsilon}(g,h),f\rangle_v &= \sum_{j\ge3}\langle Q_{-1}^{\varepsilon}(\varphi_j g,\tilde\varphi_j h),\tilde\varphi_j f\rangle_v + \sum_{j\le2}\langle Q_{-1}^{\varepsilon}(\varphi_j g,\mathcal{U}_3 h),\mathcal{U}_3 f\rangle_v\\ &:= \mathcal{A}_1+\mathcal{A}_2,\end{aligned}$$

where $\tilde\varphi_j := \sum_{k\ge-1,|k-j|\le4}\varphi_k$ and $\mathcal{U}_3 := \sum_{-1\le k\le3}\varphi_k$. We only consider the most difficult case $-3<\gamma<-\frac32$. In this case, by Step 1 we have

$$|\mathcal{A}_1| \lesssim \sum_{j\ge3}|\langle D\rangle^{s_1}\varphi_j g|_{L^2}|\langle D\rangle^{s_2}W^\varepsilon(D)\tilde\varphi_j h|_{L^2}|\langle D\rangle^{s_3}W^\varepsilon(D)\tilde\varphi_j f|_{L^2} := \sum_{j\ge3}\mathcal{A}_{1,j}.$$

For simplicity we write $\mathcal{A}_{1,j}=\mathcal{B}_j\mathcal{C}_j\mathcal{D}_j$, where

$$\begin{aligned}\mathcal{B}_j &:= 2^{-j}|\langle D\rangle^{s_1}2^{(-\gamma+1)j}\langle\cdot\rangle^{\gamma}\varphi_j\langle\cdot\rangle^{-\gamma}g|_{L^2},\\ \mathcal{C}_j &:= 2^{-j}|\langle D\rangle^{s_2}W^\varepsilon(D)2^{(\gamma/2+1)j}\langle\cdot\rangle^{-\gamma/2}\tilde\varphi_j\langle\cdot\rangle^{\gamma/2}h|_{L^2},\\ \mathcal{D}_j &:= 2^{-j}|\langle D\rangle^{s_3}W^\varepsilon(D)2^{(\gamma/2+1)j}\langle\cdot\rangle^{-\gamma/2}\tilde\varphi_j\langle\cdot\rangle^{\gamma/2}f|_{L^2}.\end{aligned}$$

Thanks to $2^{(-\gamma+1)j}\langle\cdot\rangle^{\gamma}\varphi_j\in S^1_{1,0}$ and $\langle\cdot\rangle^{s_1}\in S^{s_1}_{1,0}$, Lemma A.1 yields

$$\mathcal{B}_j \lesssim |\varphi_j\langle D\rangle^{s_1}\langle\cdot\rangle^{-\gamma}g|_{L^2} + 2^{-j}|\langle\cdot\rangle^{-\gamma}g|_{H^{s_1-1}}.$$

Similarly, Lemma A.1 yields

$$\mathcal{C}_j \lesssim |\widetilde{\varphi}_j \langle D\rangle^{s_2} W^\varepsilon(D)\langle\cdot\rangle^{\gamma/2} h|_{L^2} + 2^{-j}|\langle\cdot\rangle^{\gamma/2} h|_{H^{s_2+s-1}},$$
$$\mathcal{D}_j \lesssim |\widetilde{\varphi}_j \langle D\rangle^{s_3} W^\varepsilon(D)\langle\cdot\rangle^{\gamma/2} f|_{L^2} + 2^{-j}|\langle\cdot\rangle^{\gamma/2} f|_{H^{s_3+s-1}}.$$

Thus, it is not difficult to conclude that

$$|\mathcal{A}_1| \lesssim |\langle D\rangle^{s_1}\langle\cdot\rangle^{-\gamma} g|_{L^2}|\langle D\rangle^{s_2} W^\varepsilon(D)\langle\cdot\rangle^{\gamma/2} h|_{L^2}|\langle D\rangle^{s_3} W^\varepsilon(D)\langle\cdot\rangle^{\gamma/2} f|_{L^2}.$$

The term $\mathcal{A}_2$ is much easier since it has only finite terms. Finally, we have

$$|\langle Q^\varepsilon_{-1}(g,h), f\rangle_v| \lesssim |\langle D\rangle^{s_1}\langle\cdot\rangle^{-\gamma} g|_{L^2}|\langle D\rangle^{s_2} W^\varepsilon(D)\langle\cdot\rangle^{\gamma/2} h|_{L^2}|\langle D\rangle^{s_3} W^\varepsilon(D)\langle\cdot\rangle^{\gamma/2} f|_{L^2}.$$

For the case $\gamma \geq -\frac{3}{2}$, we can repeat the above procedure to get the desired results. We finish the proof with the help of Lemma A.2. ∎

To give the upper bound for $Q^\varepsilon_{\geq 0}$, we need the next two lemmas.

Lemma 2.6. *Let* $\mathcal{Y}^{\varepsilon,\gamma}(h,f) := \int b^\varepsilon(\frac{u}{|u|}\cdot\sigma)\langle u\rangle^\gamma h(u)(f(u^+) - f(|u|\frac{u^+}{|u^+|}))\,\mathrm{d}u\,\mathrm{d}\sigma$*; then*

$$|\mathcal{Y}^{\varepsilon,\gamma}(h,f)| \lesssim (|W^\varepsilon W_{\gamma/2} h|_{L^2} + |W^\varepsilon(D) W_{\gamma/2} h|_{L^2})$$
$$\times (|W^\varepsilon W_{\gamma/2} f|_{L^2} + |W^\varepsilon(D) W_{\gamma/2} f|_{L^2}).$$

Proof. We divide the proof into two steps.

Step 1*:* $\gamma = 0$*.* The proof is similar to that of Lemma 2.1. First, applying dyadic decomposition in phase space we have

$$\mathcal{Y}^{\varepsilon,0}(h,f) = \sum_{k=-1}^{\infty}\int b^\varepsilon(\tfrac{u}{|u|}\cdot\sigma)(\widetilde{\varphi}_k h)(u)((\varphi_k f)(u^+) - (\varphi_k f)(|u|\tfrac{u^+}{|u^+|}))\,\mathrm{d}u\,\mathrm{d}\sigma$$
$$:= \sum_{k=-1}^{\infty}\mathcal{Y}_k,$$

where $\widetilde{\varphi}_k = \sum_{l\geq -1, |l-k|\leq 4}\varphi_l$. We separately consider the two cases: $2^k \geq 1/\varepsilon$ and $2^k \leq 1/\varepsilon$. If $2^k \geq 1/\varepsilon$, we have

$$|\mathcal{Y}_k| \leq \left(\int b^\varepsilon(\tfrac{u}{|u|}\cdot\sigma)|(\widetilde{\varphi}_k h)(u)|^2\,\mathrm{d}u\,\mathrm{d}\sigma\right)^{\frac{1}{2}}$$
$$\times\left(\int b^\varepsilon(\tfrac{u}{|u|}\cdot\sigma)(|(\varphi_k f)(u^+)|^2 + |(\varphi_k f)(|u|\tfrac{u^+}{|u^+|})|^2)\,\mathrm{d}u\,\mathrm{d}\sigma\right)^{\frac{1}{2}}.$$

By the changes of variable $u \to u^+$ and $u \to w = |u|\frac{u^+}{|u^+|}$ respectively, we have $|\mathcal{Y}_k| \lesssim \varepsilon^{-2s}|\widetilde{\varphi}_k h|_{L^2}|\varphi_k f|_{L^2}$. Taking the sum over $2^k \geq 1/\varepsilon$, we get

$$\left|\sum_{2^k\geq 1/\varepsilon}\mathcal{Y}_k\right| \lesssim \sum_{2^k\geq 1/\varepsilon}\varepsilon^{-2s}|\widetilde{\varphi}_k h|_{L^2}|\varphi_k f|_{L^2} \lesssim |W^\varepsilon h|_{L^2}|W^\varepsilon f|_{L^2}.$$

If $2^k \leq 1/\varepsilon$, by Proposition A.2 and dyadic decomposition in frequency space, we have

$$\begin{aligned}
\mathcal{Y}_k &= \int b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)(\widehat{\widetilde{\varphi}_k h}(\xi^+) - \widehat{\widetilde{\varphi}_k h}(|\xi|\tfrac{\xi^+}{|\xi^+|}))\overline{\widehat{\varphi_k f}}(\xi)\,\mathrm{d}\xi\,\mathrm{d}\sigma \\
&= \sum_{l=-1}^{\infty}\int b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)((\varphi_l\widehat{\widetilde{\varphi}_k h})(\xi^+) - (\varphi_l\widehat{\widetilde{\varphi}_k h})(|\xi|\tfrac{\xi^+}{|\xi^+|}))(\widetilde{\varphi}_l\overline{\widehat{\varphi_k f}})(\xi)\,\mathrm{d}\xi\,\mathrm{d}\sigma \\
&:= \sum_{l=-1}^{\infty}\mathcal{Y}_{k,l}.
\end{aligned}$$

If $2^l \geq 1/\varepsilon$, we have $|\mathcal{Y}_{k,l}| \lesssim \varepsilon^{-2s}|\varphi_l\widehat{\widetilde{\varphi}_k h}|_{L^2}|\widetilde{\varphi}_l\widehat{\varphi_k f}|_{L^2}$ and thus

$$\sum_{2^l\geq 1/\varepsilon}|\mathcal{Y}_{k,l}| \lesssim \sum_{2^l\geq 1/\varepsilon}\varepsilon^{-2s}|\varphi_l\widehat{\widetilde{\varphi}_k h}|_{L^2}|\widetilde{\varphi}_l\widehat{\varphi_k f}|_{L^2} \lesssim |W^\varepsilon(D)\widetilde{\varphi}_k h|_{L^2}|W^\varepsilon(D)\varphi_k f|_{L^2}.$$

Then by (A.1) we have $\sum_{2^k\leq 1/\varepsilon, 2^l\geq 1/\varepsilon}|\mathcal{Y}_{k,l}| \lesssim |W^\varepsilon(D)h|_{L^2}|W^\varepsilon(D)f|_{L^2}$. If $2^l \leq 1/\varepsilon$, we have

$$\begin{aligned}
\mathcal{Y}_{k,l} &= \int b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)\mathrm{1}_{\theta\geq 2^{-\frac{k+l}{2}}}((\varphi_l\widehat{\widetilde{\varphi}_k h})(\xi^+) - (\varphi_l\widehat{\widetilde{\varphi}_k h})(|\xi|\tfrac{\xi^+}{|\xi^+|}))(\widetilde{\varphi}_l\overline{\widehat{\varphi_k f}})(\xi)\,\mathrm{d}\xi\,\mathrm{d}\sigma \\
&\quad + \int b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)\mathrm{1}_{\theta\leq 2^{-\frac{k+l}{2}}}((\varphi_l\widehat{\widetilde{\varphi}_k h})(\xi^+) - (\varphi_l\widehat{\widetilde{\varphi}_k h})(|\xi|\tfrac{\xi^+}{|\xi^+|}))(\widetilde{\varphi}_l\overline{\widehat{\varphi_k f}})(\xi)\,\mathrm{d}\xi\,\mathrm{d}\sigma \\
&:= \mathcal{Y}_{k,l,1} + \mathcal{Y}_{k,l,2}.
\end{aligned}$$

Since $\int b^\varepsilon(\cos\theta)\mathrm{1}_{\theta\geq 2^{-k/2-l/2}}\,\mathrm{d}\sigma \lesssim 2^{s(k+l)}$ we have $|\mathcal{Y}_{k,l,1}| \lesssim 2^{s(k+l)}|\varphi_l\widehat{\widetilde{\varphi}_k h}|_{L^2}|\widetilde{\varphi}_l\widehat{\varphi_k f}|_{L^2}$ and thus

$$\begin{aligned}
\sum_{\substack{2^k\leq 1/\varepsilon,\\ 2^l\leq 1/\varepsilon}}|\mathcal{Y}_{k,l,1}| &\leq \Big(\sum_{\substack{2^k\leq 1/\varepsilon,\\ 2^l\leq 1/\varepsilon}} 2^{2sl}|\varphi_l\widehat{\widetilde{\varphi}_k h}|^2_{L^2}\Big)^{\frac{1}{2}}\Big(\sum_{\substack{2^k\leq 1/\varepsilon,\\ 2^l\leq 1/\varepsilon}} 2^{2sk}|\widetilde{\varphi}_l\widehat{\varphi_k f}|^2_{L^2}\Big)^{\frac{1}{2}} \\
&\lesssim \Big(\sum_{2^k\leq 1/\varepsilon}|W^\varepsilon(D)\widetilde{\varphi}_k h|^2_{L^2}\Big)^{\frac{1}{2}}\Big(\sum_{2^k\leq 1/\varepsilon}2^{2sk}|\varphi_k f|^2_{L^2}\Big)^{\frac{1}{2}} \\
&\lesssim |W^\varepsilon(D)h|_{L^2}|W^\varepsilon f|_{L^2},
\end{aligned}$$

where we use (A.1) in the last inequality. Recalling that $\xi^+ = \frac{\xi+|\xi|\sigma}{2}$, by Taylor expansion we have

$$(\varphi_l\widehat{\widetilde{\varphi}_k h})(\xi^+) - (\varphi_l\widehat{\widetilde{\varphi}_k h})(|\xi|\tfrac{\xi^+}{|\xi^+|}) = \Big(1-\frac{1}{\cos\frac{\theta}{2}}\Big)\int_0^1(\nabla\varphi_l\widehat{\widetilde{\varphi}_k h})(\xi^+(\kappa))\cdot\xi^+\,\mathrm{d}\kappa,$$

where $\xi^+(\kappa) = (1-\kappa)|\xi|\frac{\xi^+}{|\xi^+|} + \kappa\xi^+$ and $\cos\theta = \frac{\xi}{|\xi|}\cdot\sigma$, from which we get

$$\begin{aligned}
|\mathcal{Y}_{k,l,2}| = \Big|\int_{[0,1]\times\mathbb{R}^3\times\mathbb{S}^2} & b^\varepsilon(\tfrac{\xi}{|\xi|}\cdot\sigma)\Big(1-\frac{1}{\cos\frac{\theta}{2}}\Big)\mathrm{1}_{\theta\leq 2^{-\frac{k+l}{2}}}(\widetilde{\varphi}_l\overline{\widehat{\varphi_k f}})(\xi) \\
&\times(\nabla\varphi_l\widehat{\widetilde{\varphi}_k h})(\xi^+(\kappa))\cdot\xi^+\,\mathrm{d}\kappa\,\mathrm{d}\xi\,\mathrm{d}\sigma\Big|
\end{aligned}$$

$$\lesssim \left(\int_0^{2^{-\frac{k+l}{2}}} \int_{\mathbb{R}^3} \theta^{1-2s} |(\widehat{\tilde{\varphi}_l \varphi_k f})(\xi)|^2 \,\mathrm{d}\theta \,\mathrm{d}\xi\right)^{\frac{1}{2}}$$

$$\times \left(\int_0^{2^{-\frac{k+l}{2}}} \int_{\mathbb{R}^3} \theta^{1-2s} |\eta|^2 |(\nabla \widehat{\varphi_l \tilde{\varphi}_k h})(\eta)|^2 \,\mathrm{d}\theta \,\mathrm{d}\eta\right)^{\frac{1}{2}}$$

$$\lesssim 2^{s(k+l)/2} |\widehat{\tilde{\varphi}_l \varphi_k f}|_{L^2} \left(2^{-(2-s)(k+l)} \int |\eta|^2 |(\nabla \widehat{\varphi_l \tilde{\varphi}_k h})(\eta)|^2 \,\mathrm{d}\eta\right)^{\frac{1}{2}},$$

where we use the change of variable $\xi \to \eta = \xi^+(\kappa)$ and the fact that $|1 - \frac{1}{\cos\frac{\theta}{2}}| \lesssim \theta^2$. Recalling (2.12) and (2.14), we have

$$|\mathcal{Y}_{k,l,2}| \lesssim 2^{(s-1)(k+l)} |\widehat{\tilde{\varphi}_l \varphi_k f}|_{L^2} |\widehat{\tilde{\varphi}_l \tilde{\varphi}_k h}|_{L^2} + 2^{s(k+l)-k} |\widehat{\tilde{\varphi}_l \varphi_k f}|_{L^2} |\varphi_l \widehat{v \tilde{\varphi}_k h}|.$$

By the Cauchy–Schwarz inequality and (A.1) we get

$$\sum_{\substack{2^k \le 1/\varepsilon,\\ 2^l \le 1/\varepsilon}} |\mathcal{Y}_{k,l,2}| \lesssim (|W^\varepsilon h|_{L^2} + |W^\varepsilon(D) h|_{L^2})(|W^\varepsilon f|_{L^2} + |W^\varepsilon(D) f|_{L^2}).$$

Patching together all the above results, we conclude that

$$|\mathcal{Y}^{\varepsilon,0}(h, f)| \lesssim (|W^\varepsilon h|_{L^2} + |W^\varepsilon(D) h|_{L^2})(|W^\varepsilon f|_{L^2} + |W^\varepsilon(D) f|_{L^2}). \tag{2.29}$$

Step 2: $\gamma \neq 0$. For simplicity, denote $w = |u| \frac{u^+}{|u^+|}$; then $W_{\gamma/2}(u) = W_{\gamma/2}(w)$. Note the identity

$$\begin{aligned}\langle u\rangle^\gamma h(u)(f(u^+) - f(w)) &= (W_{\gamma/2} h)(u)((W_{\gamma/2} f)(u^+) - (W_{\gamma/2} f)(w)) \\ &\quad + (W_{\gamma/2} h)(u)(W_{\gamma/2} f)(u^+)(W_{\gamma/2}(w) W_{-\gamma/2}(u^+) - 1);\end{aligned}$$

thus

$$\begin{aligned}&\mathcal{Y}^{\varepsilon,\gamma}(h, f) = \mathcal{Y}^{\varepsilon,0}(W_{\gamma/2} h, W_{\gamma/2} f) + \mathcal{A}, \\ &\mathcal{A} := \int b^\varepsilon(\tfrac{u}{|u|} \cdot \sigma)(W_{\gamma/2} h)(u)(W_{\gamma/2} f)(u^+)(W_{\gamma/2}(w) W_{-\gamma/2}(u^+) - 1) \,\mathrm{d}u \,\mathrm{d}\sigma.\end{aligned}$$

Using $|W_{\gamma/2}(u) W_{-\gamma/2}(u^+) - 1| \lesssim \sin^2\frac{\theta}{2}$ and the change of variable $u \to u^+$, we have

$$\begin{aligned}|\mathcal{A}| &\le \left(\int b^\varepsilon(\tfrac{u}{|u|} \cdot \sigma) |(W_{\gamma/2} h)(u)|^2 |W_{\gamma/2}(w) W_{-\gamma/2}(u^+) - 1| \,\mathrm{d}u \,\mathrm{d}\sigma\right)^{\frac{1}{2}} \\ &\quad \times \left(\int b^\varepsilon(\tfrac{u}{|u|} \cdot \sigma) |(W_{\gamma/2} f)(u^+)|^2 |W_{\gamma/2}(w) W_{-\gamma/2}(u^+) - 1| \,\mathrm{d}u \,\mathrm{d}\sigma\right)^{\frac{1}{2}} \\ &\lesssim |W_{\gamma/2} h|_{L^2} |W_{\gamma/2} f|_{L^2}.\end{aligned}$$

We then use (2.29) to handle $\mathcal{Y}^{\varepsilon,0}(W_{\gamma/2} h, W_{\gamma/2} f)$ and finish the proof. ∎

Remark 2.3. Denote

$$\mathcal{X}^{\varepsilon,\gamma}(h,f) := \int b^{\varepsilon}(\tfrac{u}{|u|}\cdot\sigma)|u|^{\gamma}(1-\phi(u))h(u)(f(u^{+}) - f(|u|\tfrac{u^{+}}{|u^{+}|}))\,\mathrm{d}u\,\mathrm{d}\sigma;$$

then

$$|\mathcal{X}^{\varepsilon,\gamma}(h,f)| \lesssim (|W^{\varepsilon}W_{\gamma/2}h|_{L^2} + |W^{\varepsilon}(D)W_{\gamma/2}h|_{L^2}) \\ \times (|W^{\varepsilon}W_{\gamma/2}f|_{L^2} + |W^{\varepsilon}(D)W_{\gamma/2}f|_{L^2}).$$

Indeed, since

$$|u|^{\gamma}(1-\phi(u)) = \langle u\rangle^{\gamma}(|u|^{\gamma}\langle u\rangle^{-\gamma} - 1)(1-\phi(u)) + \langle u\rangle^{\gamma}(1-\phi(u)),$$

we have

$$\mathcal{X}^{\varepsilon,\gamma}(h,f) = \mathcal{Y}^{\varepsilon,\gamma}((|\cdot|^{\gamma}\langle\cdot\rangle^{-\gamma} - 1)(1-\phi)h, f) + \mathcal{Y}^{\varepsilon,\gamma}((1-\phi)h, f).$$

Then the result follows from Lemma 2.6 and (A.2).

Lemma 2.7. *Recall* $\mathcal{R}^{\varepsilon,\gamma}_{*,g}(h) = \int b^{\varepsilon}(\cos\theta)\langle v - v_*\rangle^{\gamma} g_*(h' - h)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v$ *defined in* (1.26). *If* $g \geq 0$, *then*

$$\mathcal{R}^{\varepsilon,\gamma}_{*,g}(h) \lesssim \mathcal{R}^{\varepsilon,0}_{gW_{|\gamma|}}(W_{\gamma/2}h) + |g|_{L^1_{|\gamma+2|}}|h|^2_{L^2_{\gamma/2}}.$$

Proof. Let $H = W_{\gamma/2}h$; then

$$(h' - h)^2 = (H'W'_{-\gamma/2} - HW_{-\gamma/2})^2 \lesssim W'_{-\gamma}(H' - H)^2 + (W'_{-\gamma/2} - W_{-\gamma/2})^2 H^2.$$

Observing that $\langle v'\rangle^{-\gamma} \lesssim \langle v' - v_*\rangle^{-\gamma}\langle v_*\rangle^{|\gamma|} \sim \langle v - v_*\rangle^{-\gamma}\langle v_*\rangle^{|\gamma|}$, we have

$$\mathcal{R}^{\varepsilon,\gamma}_{*,g}(h) \lesssim \mathcal{R}^{\varepsilon,0}_{gW_{|\gamma|}}(W_{\gamma/2}h) \\ + \int b^{\varepsilon}(\cos\theta)\langle v - v_*\rangle^{\gamma}(\langle v'\rangle^{-\gamma/2} - \langle v\rangle^{-\gamma/2})^2 g_* H^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v.$$

By Taylor expansion, one has $(W'_{-\gamma/2} - W_{-\gamma/2})^2 \lesssim \int_0^1 \langle v(\kappa)\rangle^{-\gamma-2}\langle v - v_*\rangle^2 \sin^2\frac{\theta}{2}\,\mathrm{d}\kappa$. Note that $\langle v - v_*\rangle^{\gamma+2} \sim \langle v(\kappa) - v_*\rangle^{\gamma+2} \lesssim \langle v(\kappa)\rangle^{\gamma+2}\langle v_*\rangle^{|\gamma+2|}$. Then we have

$$\mathcal{R}^{\varepsilon,\gamma}_{*,g}(h) \lesssim \mathcal{R}^{\varepsilon,0}_{gW_{|\gamma|}}(W_{\gamma/2}h) + \int b^{\varepsilon}(\cos\theta)\theta^2\langle v_*\rangle^{|\gamma+2|} g_* H^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v,$$

which yields the desired result. ∎

Now we are in a position to prove the following upper bound for $Q^{\varepsilon}_{\geq 0}$.

Proposition 2.6. *It holds that*

$$|\langle Q^{\varepsilon}_{\geq 0}(g,h), f\rangle_v| \lesssim |g|_{L^1_{|\gamma|+2}}|h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2}.$$

Proof. Define the translation operator T_{v_*} by $(T_{v_*}f)(u) = f(v_* + u)$ for $v_*, u \in \mathbb{R}^3$. By the change of variable $v \to u = v - v_*$ and geometric decomposition, we have $\langle Q^\varepsilon_{\geq 0}(g,h), f\rangle_v = \mathcal{D}_1 + \mathcal{D}_2$, where

$$\mathcal{D}_1 := \int b^\varepsilon(\tfrac{u}{|u|}\cdot\sigma)|u|^\gamma(1-\phi(u))g_*(T_{v_*}h)(u) \times ((T_{v_*}f)(u^+) - (T_{v_*}f)(|u|\tfrac{u^+}{|u^+|}))\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}u,$$

$$\mathcal{D}_2 := \int b^\varepsilon(\tfrac{u}{|u|}\cdot\sigma)|u|^\gamma(1-\phi(u))g_* \times (T_{v_*}h)(u)((T_{v_*}f)(|u|\tfrac{u^+}{|u^+|}) - (T_{v_*}f)(u))\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}u.$$

Step 1*: Estimate of* $\mathcal{D}_1$. By Remark 2.3 we have

$$|\mathcal{D}_1| \lesssim \int |g_*|(|W^\varepsilon W_{\gamma/2}T_{v_*}h|_{L^2} + |W^\varepsilon(D)W_{\gamma/2}T_{v_*}h|_{L^2}) \times (|W^\varepsilon W_{\gamma/2}T_{v_*}f|_{L^2} + |W^\varepsilon(D)W_{\gamma/2}T_{v_*}f|_{L^2})\,\mathrm{d}v_*.$$

It is easy to check that

$$|W^\varepsilon W_{\gamma/2}T_{v_*}h|_{L^2} \lesssim W^\varepsilon(v_*)W_{|\gamma|/2}(v_*)|W^\varepsilon W_{\gamma/2}h|_{L^2}. \tag{2.30}$$

By Lemma A.1 we have

$$\begin{aligned}|W^\varepsilon(D)W_{\gamma/2}T_{v_*}h|_{L^2} &\lesssim |W_{\gamma/2}W^\varepsilon(D)T_{v_*}h|_{L^2} + |T_{v_*}h|_{H^{s-1}_{\gamma/2-1}}\\ &\lesssim W_{|\gamma|/2}(v_*)(|W_{\gamma/2}W^\varepsilon(D)h|_{L^2} + |h|_{L^2_{\gamma/2-1}})\\ &\lesssim W_{|\gamma|/2}(v_*)|W^\varepsilon(D)W_{\gamma/2}h|_{L^2}.\end{aligned} \tag{2.31}$$

Thus we get the following estimate of $\mathcal{D}_1$:

$$|\mathcal{D}_1| \lesssim |g|_{L^1_{|\gamma|+2}}(|W^\varepsilon(D)W_{\gamma/2}h|_{L^2} + |W^\varepsilon W_{\gamma/2}h|_{L^2}) \times (|W^\varepsilon(D)W_{\gamma/2}f|_{L^2} + |W^\varepsilon W_{\gamma/2}f|_{L^2}).$$

Step 2*: Estimate of* $\mathcal{D}_2$. Let $u = r\tau$ and $\varsigma = \frac{\tau+\sigma}{|\tau+\sigma|}$; then $\frac{u}{|u|}\cdot\sigma = 2(\tau\cdot\varsigma)^2 - 1$ and $|u|\frac{u^+}{|u^+|} = r\varsigma$. In the change of variable $(u,\sigma)\to(r,\tau,\varsigma)$, it holds that $\mathrm{d}u\,\mathrm{d}\sigma = 4(\tau\cdot\varsigma)r^2\,\mathrm{d}r\,\mathrm{d}\tau\,\mathrm{d}\varsigma$. Then

$$\begin{aligned}\mathcal{D}_2 &= 4\int r^\gamma(1-\phi(r))b^\varepsilon(2(\tau\cdot\varsigma)^2-1)(T_{v_*}h)(r\tau)\\ &\quad\times((T_{v_*}f)(r\varsigma) - (T_{v_*}f)(r\tau))(\tau\cdot\varsigma)r^2\,\mathrm{d}r\,\mathrm{d}\tau\,\mathrm{d}\varsigma\,\mathrm{d}v_*\\ &= 2\int r^\gamma(1-\phi(r))b^\varepsilon(2(\tau\cdot\varsigma)^2-1)((T_{v_*}h)(r\tau) - (T_{v_*}h)(r\varsigma))\\ &\quad\times((T_{v_*}f)(r\varsigma) - (T_{v_*}f)(r\tau))(\tau\cdot\varsigma)r^2\,\mathrm{d}r\,\mathrm{d}\tau\,\mathrm{d}\varsigma\,\mathrm{d}v_*\\ &= -\frac{1}{2}\int b^\varepsilon(\tfrac{u}{|u|}\cdot\sigma)|u|^\gamma(1-\phi(u))g_*((T_{v_*}h)(|u|\tfrac{u^+}{|u^+|}) - (T_{v_*}h)(u))\\ &\quad\times((T_{v_*}f)(|u|\tfrac{u^+}{|u^+|}) - (T_{v_*}f)(u))\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}u.\end{aligned}$$

Then by the Cauchy–Schwarz inequality and the fact that $|u|^\gamma(1-\phi(u)) \lesssim \langle u\rangle^\gamma$, we have

$$\begin{aligned}|\mathcal{D}_2| \lesssim & \left(\int b^\varepsilon(\tfrac{u}{|u|}\cdot\sigma)\langle u\rangle^\gamma |g_*|((T_{v_*}h)(|u|\tfrac{u^+}{|u^+|}) - (T_{v_*}h)(u))^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}u\right)^{\frac{1}{2}} \\ & \times\left(\int b^\varepsilon(\tfrac{u}{|u|}\cdot\sigma)\langle u\rangle^\gamma |g_*|((T_{v_*}f)(|u|\tfrac{u^+}{|u^+|}) - (T_{v_*})f(u))^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}u\right)^{\frac{1}{2}} \\ := & (\mathcal{D}_{2,h})^{\frac{1}{2}}(\mathcal{D}_{2,f})^{\frac{1}{2}}.\end{aligned}$$

Note that $\mathcal{D}_{2,h}$ and $\mathcal{D}_{2,f}$ have exactly the same structure. It suffices to focus on $\mathcal{D}_{2,f}$. Since

$$\begin{aligned}((T_{v_*}f)(|u|\tfrac{u^+}{|u^+|}) - (T_{v_*}f)(u))^2 \leq & \, 2((T_{v_*}f)(|u|\tfrac{u^+}{|u^+|}) - (T_{v_*}f)(u^+))^2 \\ & + 2((T_{v_*}f)(u^+) - (T_{v_*}f)(u))^2,\end{aligned}$$

we have

$$\begin{aligned}\mathcal{D}_{2,f} \lesssim & \int b^\varepsilon(\tfrac{u}{|u|}\cdot\sigma)\langle u\rangle^\gamma |g_*|((T_{v_*}f)(|u|\tfrac{u^+}{|u^+|}) - (T_{v_*}f)(u^+))^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}u \\ & + \int b^\varepsilon(\tfrac{u}{|u|}\cdot\sigma)\langle u\rangle^\gamma |g_*|((T_{v_*}f)(u^+) - (T_{v_*}f)(u))^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}u \\ := & \, \mathcal{D}_{2,f,1} + \mathcal{D}_{2,f,2}.\end{aligned}$$

By Lemma 2.2 and the facts (2.30) and (2.31), we have

$$\mathcal{D}_{2,f,1} = \int |g_*|\mathcal{Z}^{\varepsilon,\gamma}(T_{v_*}f)\,\mathrm{d}v_* \lesssim |g|_{L^1_{|\gamma|+2}}(|W^\varepsilon(D)W_{\gamma/2}f|^2_{L^2} + |W^\varepsilon W_{\gamma/2}f|^2_{L^2}).$$

Thanks to Lemma 2.7 we have

$$\mathcal{D}_{2,f,2} = \mathcal{R}^{\varepsilon,\gamma}_{*,|g|}(f) \lesssim \mathcal{R}^{\varepsilon,0}_{|g|W_{|\gamma|}}(W_{\gamma/2}f) + |g|_{L^1_{|\gamma|+2}}|f|^2_{L^2_{\gamma/2}}.$$

Thanks to the estimate $\mathcal{R}^{\varepsilon,0}_g(f) \lesssim |g|_{L^1}\mathcal{R}^{\varepsilon,0}_\mu(f) + |g|_{L^1_2}|W^\varepsilon(D)f|^2_{L^2}$ (see [10, Lemma 3.3]), using (2.17) to get $\mathcal{R}^{\varepsilon,0}_\mu(f) \lesssim |W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})f|^2_{L^2} + |W^\varepsilon(D)f|^2_{L^2} + |W^\varepsilon f|^2_{L^2}$, we have

$$\mathcal{D}_{2,f,2} \lesssim |g|_{L^1_{|\gamma|+2}}(|W^\varepsilon((-\Delta_{\mathbb{S}^2})^{\frac{1}{2}})W_{\gamma/2}f|^2_{L^2} + |W^\varepsilon(D)W_{\gamma/2}f|^2_{L^2} + |W^\varepsilon W_{\gamma/2}f|^2_{L^2}).$$

Therefore we have $\mathcal{D}_{2,f} \lesssim |g|_{L^1_{|\gamma|+2}}|f|^2_{\varepsilon,\gamma/2}$, which yields

$$|\mathcal{D}_2| \lesssim |g|_{L^1_{|\gamma|+2}}|h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2}.$$

We finish the proof by patching together the estimates for $\mathcal{D}_1$ and $\mathcal{D}_2$. ■

Recalling (2.27), by Propositions 2.5 and 2.6, we are led to the following theorem.

Theorem 2.2. *For any $\eta > 0$, the following estimates are valid:*

- *If $\gamma > -\frac{3}{2}$, then $|\langle Q^{\epsilon}(g,h), f\rangle_v| \lesssim (|g|_{L^2_{|\gamma|}} + |g|_{L^1_{|\gamma|+2}})|h|_{\epsilon,\gamma/2}|f|_{\epsilon,\gamma/2}$.*
- *If $\gamma = -\frac{3}{2}$, then*

  $$|\langle Q^{\epsilon}(g,h), f\rangle_v| \lesssim (|g|_{L^1_{|\gamma|}} + |g|_{H^{s_1}_{|\gamma|}})|W^{\epsilon}(D)h|_{H^{s_2}_{\gamma/2}}|f|_{\epsilon,\gamma/2} + |g|_{L^1_{|\gamma|+2}}|h|_{\epsilon,\gamma/2}|f|_{\epsilon,\gamma/2}.$$

  Here $(s_1, s_2) = (0, \eta)$ or $(\eta, 0)$.
- *If $-3 < \gamma < -\frac{3}{2}$, then*

  $$|\langle Q^{\epsilon}(g,h), f\rangle_v| \lesssim |g|_{H^{s_1}_{|\gamma|}}|W^{\epsilon}(D)h|_{H^{s_2}_{\gamma/2}}|W^{\epsilon}(D)f|_{H^{s_3}_{\gamma/2}} + |g|_{L^1_{|\gamma|+2}}|h|_{\epsilon,\gamma/2}|f|_{\epsilon,\gamma/2}.$$

  Here the constants $s_1, s_2, s_3 \geq 0$ verify either $s_1 + s_2 + s_3 = -\gamma - \frac{3}{2}$, $s_2 + s_3 > 0$ or $s_1 = -\gamma - \frac{3}{2} + \eta$, $s_2 = s_3 = 0$.

2.2.2. Upper bounds of $\mathcal{I}(g,h,f)$. For ease of notation, we abbreviate $\mathcal{I}(g,h,f)$ as $\mathcal{I}$. We first do some rearranging. Noting that

$$\begin{aligned}(\mu^{\frac{1}{2}})'_* - \mu^{\frac{1}{2}}_* &= ((\mu^{\frac{1}{4}})'_* + \mu^{\frac{1}{4}}_*)((\mu^{\frac{1}{4}})'_* - \mu^{\frac{1}{4}}_*) \\ &= ((\mu^{\frac{1}{8}})'_* + \mu^{\frac{1}{8}}_*)^2((\mu^{\frac{1}{8}})'_* - \mu^{\frac{1}{8}}_*)^2 + 2\mu^{\frac{1}{4}}_*((\mu^{\frac{1}{4}})'_* - \mu^{\frac{1}{4}}_*),\end{aligned}$$

and $h = (h - h') + h'$, recalling (2.26) we have

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \tag{2.32}$$

$$\mathcal{I}_1 := \int b^{\epsilon}(\cos\theta)|v - v_*|^{\gamma}((\mu^{\frac{1}{8}})'_* + \mu^{\frac{1}{8}}_*)^2((\mu^{\frac{1}{8}})'_* - \mu^{\frac{1}{8}}_*)^2 g_* h f' \, d\sigma \, dv_* \, dv, \tag{2.33}$$

$$\mathcal{I}_2 := 2\int b^{\epsilon}(\cos\theta)|v - v_*|^{\gamma}((\mu^{\frac{1}{4}})'_* - \mu^{\frac{1}{4}}_*)(\mu^{\frac{1}{4}}g)_*(h - h')f' \, d\sigma \, dv_* \, dv, \tag{2.34}$$

$$\mathcal{I}_3 := 2\int b^{\epsilon}(\cos\theta)|v - v_*|^{\gamma}((\mu^{\frac{1}{4}})'_* - \mu^{\frac{1}{4}}_*)(\mu^{\frac{1}{4}}g)_* h' f' \, d\sigma \, dv_* \, dv. \tag{2.35}$$

We derive some upper bounds for $\mathcal{I}(g,h,f)$ in the following proposition.

Proposition 2.7. *For any $\eta > 0$, the following estimates are valid:*

- *If $\gamma > -\frac{3}{2}$, then $|\mathcal{I}(g,h,f)| \lesssim |g|_{L^2}|h|_{\epsilon,\gamma/2}|W^{\epsilon}f|_{L^2_{\gamma/2}}$.*
- *If $\gamma = -\frac{3}{2}$, then*

  $$\begin{aligned}|\mathcal{I}(g,h,f)| \lesssim\ & |\mu^{\frac{1}{8}}g|_{H^{s_1}}(|W^{\epsilon}(D)h|_{H^{s_2}_{\gamma/2}} + |h|_{\epsilon,\gamma/2})|W^{\epsilon}f|_{L^2_{\gamma/2}} \\ &+ |g|_{L^2}|h|_{\epsilon,\gamma/2}|W^{\epsilon}f|_{L^2_{\gamma/2}},\end{aligned}$$

  where $(s_1, s_2) = (0, \eta)$ or $(\eta, 0)$.
- *If $-3 < \gamma < -\frac{3}{2}$, then*

  $$\begin{aligned}|\mathcal{I}(g,h,f)| \lesssim\ & |\mu^{\frac{1}{8}}g|_{H^{s_1}}(|W^{\epsilon}(D)h|_{H^{s_2}_{\gamma/2}} + |h|_{\epsilon,\gamma/2})|W^{\epsilon}f|_{L^2_{\gamma/2}} \\ &+ |g|_{L^2}|h|_{\epsilon,\gamma/2}|W^{\epsilon}f|_{L^2_{\gamma/2}},\end{aligned}$$

where $s_1, s_2 \geq 0$ *are constants verifying either* $s_1 + 2s_2 = -\gamma - \frac{3}{2}$, $s_2 > 0$ *or* $s_1 = -\gamma - \frac{3}{2} + \eta$, $s_2 = 0$.

Proof. In the proof, we will constantly use the following fact:

$$((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2 \lesssim \min\{1, |v-v_*|^2 \sin^2 \tfrac{\theta}{2}\} \sim \min\{1, |v'-v_*|^2 \sin^2 \tfrac{\theta}{2}\} \sim \min\{1, |v-v'_*|^2 \sin^2 \tfrac{\theta}{2}\}. \tag{2.36}$$

We estimate $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ one by one.

Step 1*: Estimate of* $\mathcal{I}_1$. Recalling (2.33), we use ϕ defined in (1.19) to separate the relative velocity into two parts:

$$\begin{aligned}\mathcal{I}_1 &= \int b^\varepsilon(\cos\theta)|v-v_*|^\gamma(1-\phi(v-v_*))((\mu^{\frac{1}{8}})'_* + \mu_*^{\frac{1}{8}})^2((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2 \\ &\qquad \times g_* h f' \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ &\quad + \int b^\varepsilon(\cos\theta)|v-v_*|^\gamma\phi(v-v_*)((\mu^{\frac{1}{8}})'_* + \mu_*^{\frac{1}{8}})^2((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2 g_* h f' \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ &:= \mathcal{I}_{1,1} + \mathcal{I}_{1,2}.\end{aligned}$$

Estimate of $\mathcal{I}_{1,1}$. Note that $|v-v_*| \sim \langle v-v_*\rangle$ in $\mathcal{I}_{1,1}$. By the Cauchy–Schwarz inequality we have

$$\begin{aligned}|\mathcal{I}_{1,1}| &\lesssim \left(\int b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma((\mu^{\frac{1}{8}})'_* + \mu_*^{\frac{1}{8}})^2((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2 g_*^2 h^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{\frac{1}{2}} \\ &\quad \times \left(\int b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma((\mu^{\frac{1}{8}})'_* + \mu_*^{\frac{1}{8}})^2((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2 (f^2)' \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{\frac{1}{2}} \\ &:= (\mathcal{I}_{1,1,1})^{\frac{1}{2}}(\mathcal{I}_{1,1,2})^{\frac{1}{2}}.\end{aligned}$$

We claim that

$$\begin{aligned}\mathcal{A} &:= \int b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma((\mu^{\frac{1}{8}})'_* + \mu_*^{\frac{1}{8}})^2((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2 \,\mathrm{d}\sigma \\ &\lesssim (W^\varepsilon)^2(v)\langle v\rangle^\gamma,\end{aligned} \tag{2.37}$$

which yields $\mathcal{I}_{1,1,1} \lesssim |g|^2_{L^2}|W^\varepsilon h|^2_{L^2_{\gamma/2}}$.

To prove (2.37), we notice that

$$\begin{aligned}\mathcal{A} &\lesssim \int b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma \mu_*^{\frac{1}{4}}((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2 \,\mathrm{d}\sigma \\ &\quad + \int b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma (\mu^{\frac{1}{4}})'_*((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2 \,\mathrm{d}\sigma \\ &:= \mathcal{A}_1 + \mathcal{A}_2.\end{aligned}$$

By Proposition A.1 and the fact (2.10), we get $\mathcal{A}_1 \lesssim \langle v-v_*\rangle^\gamma \mu_*^{\frac{1}{4}}(W^\varepsilon)^2(v-v_*) \lesssim (W^\varepsilon)^2(v)\langle v\rangle^\gamma$. As for $\mathcal{A}_2$, thanks to $|v-v_*|\sim|v-v'_*|$ and thus $\langle v-v_*\rangle^\gamma \lesssim \langle v-v'_*\rangle^\gamma \lesssim \langle v\rangle^\gamma\langle v'_*\rangle^{|\gamma|}$, we have

$$\mathcal{A}_2 \lesssim \langle v\rangle^\gamma \int b^\varepsilon(\cos\theta)(\mu^{\frac{1}{8}})'_* \min\{1,|v-v_*|^2\sin^2\tfrac{\theta}{2}\}\,d\sigma.$$

If $|v-v_*|\geq 10|v|$, then $|v'_*|=|v'_*-v+v|\geq|v'_*-v|-|v|\geq(1/\sqrt{2}-1/10)|v-v_*|\geq \frac{1}{5}|v-v_*|$ and thus $(\mu^{\frac{1}{8}})'_*\lesssim \mu^{\frac{1}{200}}(v-v_*)$, which yields

$$\mathcal{A}_2 \lesssim \langle v\rangle^\gamma \mu^{\frac{1}{200}}(v-v_*)(W^\varepsilon)^2(v-v_*) \lesssim \langle v\rangle^\gamma.$$

If $|v-v_*|\leq 10|v|$, by Proposition A.1 we have

$$\mathcal{A}_2 \lesssim \langle v\rangle^\gamma \int b^\varepsilon(\cos\theta)\min\{1,|v|^2\sin^2\tfrac{\theta}{2}\}\,d\sigma \lesssim (W^\varepsilon)^2(v)\langle v\rangle^\gamma.$$

We have finished the proof of (2.37). We now consider $\mathcal{I}_{1,1,2}$. By the change of variable $(v,v_*)\to(v',v'_*)$ we have

$$\begin{aligned}
\mathcal{I}_{1,1,2} &= \int b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma((\mu^{\frac{1}{8}})'_*+\mu_*^{\frac{1}{8}})^2((\mu^{\frac{1}{8}})'_*-\mu_*^{\frac{1}{8}})^2 f^2\,d\sigma\,dv_*\,dv \\
&\leq 2\int b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma \mu_*^{\frac{1}{4}}((\mu^{\frac{1}{8}})'_*-\mu_*^{\frac{1}{8}})^2 f^2\,d\sigma\,dv_*\,dv \\
&\quad + 2\int b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma(\mu^{\frac{1}{4}})'_*((\mu^{\frac{1}{8}})'_*-\mu_*^{\frac{1}{8}})^2 f^2\,d\sigma\,dv_*\,dv \\
&:= \mathcal{I}_{1,1,2,1}+\mathcal{I}_{1,1,2,2}.
\end{aligned}$$

With the help of (2.36), Proposition A.1 and (2.10), we have

$$\begin{aligned}
\mathcal{I}_{1,1,2,1} &\lesssim \int b^\varepsilon(\cos\theta)\langle v-v_*\rangle^\gamma \mu_*^{\frac{1}{4}}\min\{1,|v-v_*|^2\sin^2\tfrac{\theta}{2}\} f^2\,d\sigma\,dv_*\,dv \\
&\lesssim \int \langle v\rangle^\gamma\langle v_*\rangle^{|\gamma|}(W^\varepsilon)^2(v)(W^\varepsilon)^2(v_*)\mu_*^{\frac{1}{4}} f^2\,dv_*\,dv \lesssim |W^\varepsilon f|^2_{L^2_{\gamma/2}}.
\end{aligned}$$

By the fact that $|v-v_*|\sim|v-v'_*|$ and the change of variable $v_*\to v'_*$ we have

$$\begin{aligned}
\mathcal{I}_{1,1,2,2} &\lesssim \int b^\varepsilon(\cos\theta)\langle v-v'_*\rangle^\gamma(\mu^{\frac{1}{4}})'_*\min\{1,|v-v'_*|^2\sin^2\tfrac{\theta}{2}\} f^2\,d\sigma\,dv'_*\,dv \\
&\lesssim |W^\varepsilon f|^2_{L^2_{\gamma/2}}.
\end{aligned}$$

Therefore we have $\mathcal{I}_{1,1,2}\lesssim |W^\varepsilon f|^2_{L^2_{\gamma/2}}$. Patching together the estimates of $\mathcal{I}_{1,1,1}$ and $\mathcal{I}_{1,1,2}$, we have

$$\mathcal{I}_{1,1} \lesssim |g|_{L^2}|W^\varepsilon h|_{L^2_{\gamma/2}}|W^\varepsilon f|_{L^2_{\gamma/2}}. \tag{2.38}$$

Estimate of $\mathcal{I}_{1,2}$. By the Cauchy–Schwarz inequality we have

$$|\mathcal{I}_{1,2}| \lesssim \Big(\int b^\varepsilon(\cos\theta)|v-v_*|^\gamma \phi(v-v_*)((\mu^{\frac{1}{8}})'_* + \mu_*^{\frac{1}{8}})^2((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2 \times |g_*|h^2 \, d\sigma \, dv_* \, dv\Big)^{\frac{1}{2}}$$
$$\times \Big(\int b^\varepsilon(\cos\theta)|v-v_*|^\gamma \phi(v-v_*)((\mu^{\frac{1}{8}})'_* + \mu_*^{\frac{1}{8}})^2((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2 \times |g_*|(f^2)' \, d\sigma \, dv_* \, dv\Big)^{\frac{1}{2}}$$
$$:= (\mathcal{I}_{1,2,1})^{\frac{1}{2}}(\mathcal{I}_{1,2,2})^{\frac{1}{2}}.$$

Note that the support of function ϕ is $B_{\frac{4}{3}}$. When $|v-v_*| \le \frac{4}{3}$, it holds that $|v_*| \ge |v| - \frac{4}{3}$ and $|v'_*| \ge |v| - |v-v'_*| \ge |v| - |v-v_*| \ge |v| - \frac{4}{3}$, which imply that $((\mu^{\frac{1}{8}})'_* + \mu_*^{\frac{1}{8}})^2 \lesssim \mu^{\frac{1}{8}}$. By (2.36), Proposition A.1, the Cauchy–Schwarz inequality and the assumption $\gamma + 2 > -1$, one has

$$\mathcal{I}_{1,2,1} \lesssim \int |v-v_*|^{\gamma+2}\phi(v-v_*)\mu^{\frac{1}{8}}|g_*|h^2 \, d\sigma \, dv_* \, dv \lesssim |g|_{L^2}|\mu^{\frac{1}{16}}h|^2_{L^2}.$$

By the change of variable $v \to v'$, we can similarly derive that $\mathcal{I}_{1,2,2} \lesssim |g|_{L^2}|\mu^{\frac{1}{16}} f|^2_{L^2}$. Patching together the estimates of $\mathcal{I}_{1,2,1}$ and $\mathcal{I}_{1,2,2}$, we arrive at

$$|\mathcal{I}_{1,2}| \lesssim |g|_{L^2}|\mu^{\frac{1}{16}}h|_{L^2}|\mu^{\frac{1}{16}} f|_{L^2}.$$

From this, together with estimate (2.38) of $\mathcal{I}_{1,1}$, we obtain

$$|\mathcal{I}_1| \lesssim |g|_{L^2}|W^\varepsilon h|_{L^2_{\gamma/2}}|W^\varepsilon f|_{L^2_{\gamma/2}}.$$

Step 2: Estimate of $\mathcal{I}_2$. Recalling (2.34), by the Cauchy–Schwarz inequality we have

$$\mathcal{I}_2 \lesssim \Big(\int b^\varepsilon(\cos\theta)|v-v_*|^\gamma|(\mu^{\frac{1}{4}}g)_*|(h-h')^2 \, d\sigma \, dv_* \, dv\Big)^{\frac{1}{2}}$$
$$\times \Big(\int b^\varepsilon(\cos\theta)|v-v_*|^\gamma((\mu^{\frac{1}{4}})'_* - \mu_*^{\frac{1}{4}})^2|(\mu^{\frac{1}{4}}g)_*|(f^2)' \, d\sigma \, dv_* \, dv\Big)^{\frac{1}{2}}$$
$$:= (\mathcal{I}_{2,1})^{\frac{1}{2}}(\mathcal{I}_{2,2})^{\frac{1}{2}}.$$

Estimate of $\mathcal{I}_{2,1}$. Noticing that $(h-h')^2 = (h^2)' - h^2 - 2h(h'-h)$, we have

$$\mathcal{I}_{2,1} = \mathcal{I}_{2,1,1} - 2\langle Q^\varepsilon(|\mu^{\frac{1}{4}}g|, h), h\rangle,$$
$$\mathcal{I}_{2,1,1} := \int b^\varepsilon(\cos\theta)|v-v_*|^\gamma|(\mu^{\frac{1}{4}}g)_*|((h^2)' - h^2) \, d\sigma \, dv_* \, dv.$$

By the cancellation lemma in [1], one has $\mathcal{I}_{2,1,1} = C(\varepsilon)\int |v-v_*|^\gamma|(\mu^{\frac{1}{4}}g)_*|h^2 \, dv_* \, dv$ with $|C(\varepsilon)| \lesssim 1$. Thus, by Lemma 2.5 and Theorem 2.2,

- if $\gamma > -\frac{3}{2}$, then $|\mathcal{I}_{2,1}| \lesssim |\mu^{\frac{1}{8}}g|_{L^2}|h|^2_{\varepsilon,\gamma/2}$;

- if $\gamma = -\frac{3}{2}$, then $|\mathcal{I}_{2,1}| \lesssim |\mu^{\frac{1}{8}} g|_{H^{s_1}} |W^\varepsilon(D)h|_{H^{s_2}_{\gamma/2}} |h|_{\varepsilon,\gamma/2} + |\mu^{\frac{1}{8}} g|_{L^2} |h|^2_{\varepsilon,\gamma/2}$, where $(s_1, s_2) = (0, \eta)$ or $(\eta, 0)$;
- if $-3 < \gamma < -\frac{3}{2}$, then $|\mathcal{I}_{2,1}| \lesssim |\mu^{\frac{1}{8}} g|_{H^{s_1}} |W^\varepsilon(D)h|^2_{H^{s_2}_{\gamma/2}} + |\mu^{\frac{1}{8}} g|_{L^2} |h|^2_{\varepsilon,\gamma/2}$, where $s_1, s_2 \geq 0$ verify either $s_1 + 2s_2 = -\gamma - \frac{3}{2}$, $s_2 > 0$ or $s_1 = -\gamma - \frac{3}{2} + \eta$, $s_2 = 0$.

Estimate of $\mathcal{I}_{2,2}$. We separate the relative velocity $|v - v_*|$ into two regions by introducing the cutoff function ϕ. If $|v - v_*| \gtrsim 1$, by the change of variable $v \to v'$, the estimate is the same as that for $\mathcal{I}_{1,1,1}$. If $|v - v_*| \lesssim 1$, the estimate is exactly the same as that for $\mathcal{I}_{1,2,2}$. We conclude that $\mathcal{I}_{2,2} \lesssim |\mu^{\frac{1}{8}} g|_{L^2} |W^\varepsilon f|^2_{L^2_{\gamma/2}}$. Patching together the estimates of $\mathcal{I}_{2,1}$ and $\mathcal{I}_{2,2}$, we get

$$\begin{aligned} \gamma > -\frac{3}{2}: &\quad |\mathcal{I}_2| \lesssim |\mu^{\frac{1}{8}} g|_{L^2} |h|_{\varepsilon,\gamma/2} |W^\varepsilon f|_{L^2_{\gamma/2}}, \\ -3 < \gamma \leq -\frac{3}{2}: &\quad |\mathcal{I}_2| \lesssim |\mu^{\frac{1}{8}} g|_{H^{s_1}} (|W^\varepsilon(D)h|_{H^{s_2}_{\gamma/2}} + |h|_{\varepsilon,\gamma/2}) |W^\varepsilon f|_{L^2_{\gamma/2}}. \end{aligned}$$

Step 3*: Estimate of* $\mathcal{I}_3$. Recalling (2.35), by the change of variables $(v, v_*) \to (v', v'_*)$ and $(v, v_*, \sigma) \to (v_*, v, -\sigma)$, we have

$$\mathcal{I}_3 = 2 \int b^\varepsilon(\cos\theta) |v - v_*|^\gamma (\mu^{\frac{1}{4}} - (\mu^{\frac{1}{4}})')(\mu^{\frac{1}{4}} g)' h_* f_* \, d\sigma \, dv_* \, dv.$$

For ease of notation, let

$$\begin{aligned} E_1 &= \{(v, v_*, \sigma) \mid |v - v_*| \geq \tfrac{1}{\varepsilon}\}, \\ E_2 &= \{(v, v_*, \sigma) \mid |v - v_*| \leq \tfrac{1}{\varepsilon},\ \sin\tfrac{\theta}{2} \geq |v - v_*|^{-1}\}, \\ E_3 &= \{(v, v_*, \sigma) \mid |v - v_*| \leq \tfrac{1}{\varepsilon},\ \sin\tfrac{\theta}{2} \leq |v - v_*|^{-1}\}. \end{aligned}$$

Then $\mathcal{I}_3$ can be decomposed into three parts $\mathcal{I}_{3,1}$, $\mathcal{I}_{3,2}$ and $\mathcal{I}_{3,3}$ which correspond to E_1, E_2 and E_3 respectively.

Estimate of $\mathcal{I}_{3,1}$. By the change of variable $v \to v'$ and the fact that $|v' - v_*| \geq |v - v_*|/\sqrt{2}$, we have

$$\begin{aligned} |\mathcal{I}_{3,1}| &\lesssim \int b^\varepsilon(\cos\theta) |v' - v_*|^\gamma 1_{|v' - v_*| \geq (\sqrt{2}\varepsilon)^{-1}} |(\mu^{\frac{1}{4}} g)' h_* f_*| \, d\sigma \, dv_* \, dv' \\ &\lesssim \varepsilon^{-2s} \int |v' - v_*|^\gamma 1_{|v' - v_*| \geq (\sqrt{2}\varepsilon)^{-1}} |(\mu^{\frac{1}{4}} g)' h_* f_*| \, dv_* \, dv'. \end{aligned}$$

On one hand, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} &\varepsilon^{-2s} \int |v' - v_*|^\gamma 1_{|v' - v_*| \geq (\sqrt{2}\varepsilon)^{-1}} |(\mu^{\frac{1}{4}} g)'| \, dv' \\ &\leq |\mu^{\frac{1}{8}} g|_{L^2} \varepsilon^{-2s} \left(\int |v' - v_*|^{2\gamma} 1_{|v' - v_*| \geq (\sqrt{2}\varepsilon)^{-1}} (\mu^{\frac{1}{4}})' \, dv' \right)^{\frac{1}{2}} \\ &\lesssim |\mu^{\frac{1}{8}} g|_{L^2} \varepsilon^{-2s} \langle v_* \rangle^\gamma, \end{aligned} \tag{2.39}$$

where we use the fact that $\langle v'-v_*\rangle^{2\gamma} \lesssim \langle v'\rangle^{|2\gamma|}\langle v_*\rangle^{2\gamma}$. On the other hand, using $\varepsilon^{-1} \lesssim |v'-v_*|$, by the Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
&\varepsilon^{-2s}\int |v'-v_*|^{\gamma}\mathrm{1}_{|v'-v_*|\geq(\sqrt{2}\varepsilon)^{-1}}|(\mu^{\frac{1}{4}}g)'|\,\mathrm{d}v' \\
&\quad\lesssim \int |v'-v_*|^{\gamma+2s}|(\mu^{\frac{1}{4}}g)'|\,\mathrm{d}v' \leq |\mu^{\frac{1}{8}}g|_{L^2}\Big(\int |v'-v_*|^{2\gamma+4s}(\mu^{\frac{1}{4}})'\,\mathrm{d}v'\Big)^{\frac{1}{2}} \\
&\quad\lesssim |\mu^{\frac{1}{8}}g|_{L^2}\langle v_*\rangle^{\gamma+2s},
\end{aligned}
\tag{2.40}
$$

where we use (2.7). With estimates (2.39) and (2.40) in hand, we have

$$
|\mathcal{I}_{3,1}| \lesssim |\mu^{\frac{1}{8}}g|_{L^2}|W^{\varepsilon}h|_{L^2_{\gamma/2}}|W^{\varepsilon}f|_{L^2_{\gamma/2}}.
$$

Estimate of $\mathcal{I}_{3,2}$. Thanks to $\frac{\sqrt{2}}{2}|v-v_*| \leq |v'-v_*| \leq |v-v_*|$ and the change of variable $v \to v'$, we get

$$
\begin{aligned}
|\mathcal{I}_{3,2}| &\lesssim \int b^{\varepsilon}(\cos\theta)\mathrm{1}_{\sin\frac{\theta}{2}\geq(\sqrt{2}|v'-v_*|)^{-1}}|v'-v_*|^{\gamma}\mathrm{1}_{|v'-v_*|\leq 1/\varepsilon}|(\mu^{\frac{1}{4}}g)'h_*f_*|\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v' \\
&\lesssim \int |v'-v_*|^{\gamma+2s}\mathrm{1}_{|v'-v_*|\leq 1/\varepsilon}|(\mu^{\frac{1}{4}}g)'h_*f_*|\,\mathrm{d}v_*\,\mathrm{d}v'.
\end{aligned}
\tag{2.41}
$$

On one hand, similar to the argument in (2.40), we have

$$
\int |v'-v_*|^{\gamma+2s}\mathrm{1}_{|v'-v_*|\leq 1/\varepsilon}|(\mu^{\frac{1}{4}}g)'|\,\mathrm{d}v' \lesssim |\mu^{\frac{1}{8}}g|_{L^2}\langle v_*\rangle^{\gamma+2s}. \tag{2.42}
$$

On the other hand, if $|v_*| \geq 2/\varepsilon$, then $|v'| \geq |v_*| - |v'-v_*| \geq |v_*|/2 \geq 1/\varepsilon$, which implies $\mu' \leq \mu_*^{\frac{1}{4}} \lesssim e^{-\frac{1}{2\varepsilon^2}}$. Then we deduce that

$$
\begin{aligned}
&\mathrm{1}_{|v_*|\geq\frac{2}{\varepsilon}}\int |v'-v_*|^{\gamma+2s}\mathrm{1}_{|v'-v_*|\leq 1/\varepsilon}|(\mu^{\frac{1}{4}}g)'|\,\mathrm{d}v' \\
&\quad\lesssim \mathrm{1}_{|v_*|\geq\frac{2}{\varepsilon}}|\mu^{\frac{1}{8}}g|_{L^2}\Big(\int |v'-v_*|^{2\gamma+4s}\mathrm{1}_{|v'-v_*|\leq 1/\varepsilon}(\mu^{\frac{1}{4}})'\,\mathrm{d}v'\Big)^{\frac{1}{2}} \\
&\quad\lesssim \mathrm{1}_{|v_*|\geq\frac{2}{\varepsilon}}|\mu^{\frac{1}{8}}g|_{L^2}\mu_*^{\frac{1}{64}}(\varepsilon^{-1})^{\gamma+2s+\frac{3}{2}}e^{-\frac{1}{32\varepsilon^2}} \lesssim \mathrm{1}_{|v_*|\geq\frac{2}{\varepsilon}}|\mu^{\frac{1}{8}}g|_{L^2}\mu_*^{\frac{1}{64}}.
\end{aligned}
\tag{2.43}
$$

With estimates (2.42) and (2.43) in hand, we have

$$
|\mathcal{I}_{3,2}| \lesssim |\mu^{\frac{1}{8}}g|_{L^2}|W^{\varepsilon}h|_{L^2_{\gamma/2}}|W^{\varepsilon}f|_{L^2_{\gamma/2}}.
$$

Estimate of $\mathcal{I}_{3,3}$. By Taylor expansion, one has

$$
\begin{aligned}
\mu^{\frac{1}{4}} - (\mu^{\frac{1}{4}})' &= (\nabla\mu^{\frac{1}{4}})(v')\cdot(v-v') \\
&\qquad + \kappa\int_0^1 \big((\nabla^2\mu^{\frac{1}{4}})(v(\kappa)) : (v-v')\otimes(v-v')\big)\,\mathrm{d}\kappa.
\end{aligned}
\tag{2.44}
$$

Given a Boltzmann kernel $B(v-v_*,\sigma) = B(|v-v_*|,\cos\theta)$ and a suitable function $F:\mathbb{R}^3 \to \mathbb{R}^3$, for any fixed $v_* \in \mathbb{R}^3$, it holds that

$$\int B(|v-v_*|,\cos\theta)(v-v')\cdot F(v')\,\mathrm{d}\sigma\,\mathrm{d}v = 0. \tag{2.45}$$

By (2.44) and (2.45), we have

$$\begin{aligned}|\mathcal{I}_{3,3}| &= \bigg|\int_{E_3\times[0,1]} b^\varepsilon(\cos\theta)|v-v_*|^\gamma \mathrm{1}_{|v-v_*|\le\frac{1}{\varepsilon},\sin\frac{\theta}{2}\le|v-v_*|^{-1}} \\ &\qquad \times \kappa\big((\nabla^2\mu^{\frac14})(v(\kappa)) : (v-v')\otimes(v-v')\big)(\mu^{\frac14}g)'h_* f_*\,\mathrm{d}\kappa\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\bigg| \\ &\lesssim \int b^\varepsilon(\cos\theta)|v'-v_*|^{\gamma+2}\sin^2\frac{\theta}{2}\mathrm{1}_{|v'-v_*|\le\frac{1}{\varepsilon},\sin\frac{\theta}{2}\le|v'-v_*|^{-1}}|(\mu^{\frac14}g)' \\ &\qquad \times h_* f_*|\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v' \\ &\lesssim \int |v'-v_*|^{\gamma+2s}\mathrm{1}_{|v'-v_*|\le\frac{1}{\varepsilon}}|(\mu^{\frac14}g)'h_* f_*|\,\mathrm{d}v_*\,\mathrm{d}v'.\end{aligned}$$

Copying the argument applied to (2.41), we have $|\mathcal{I}_{3,3}| \lesssim |\mu^{\frac18}g|_{L^2}|W^\varepsilon h|_{L^2_{\gamma/2}}|W^\varepsilon f|_{L^2_{\gamma/2}}$.

Patching together the above estimates of $\mathcal{I}_{3,1}$, $\mathcal{I}_{3,2}$ and $\mathcal{I}_{3,3}$, we have

$$|\mathcal{I}_3| \lesssim |\mu^{\frac18}g|_{L^2}|W^\varepsilon h|_{L^2_{\gamma/2}}|W^\varepsilon f|_{L^2_{\gamma/2}}.$$

Recalling (2.32), the proposition follows from the above estimates of $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$. ■

2.2.3. Upper bounds for the nonlinear term $\Gamma^\varepsilon(g,h)$. We are ready to give estimates of the inner product $\langle\Gamma^\varepsilon(g,h), f\rangle_v$.

Theorem 2.3. *For any $\eta > 0$, the following estimates are valid:*

- *If $\gamma > -\frac32$, then $|\langle\Gamma^\varepsilon(g,h), f\rangle_v| \lesssim |g|_{L^2}|h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2}$.*
- *If $\gamma = -\frac32$, then*

$$\begin{aligned}|\langle\Gamma^\varepsilon(g,h), f\rangle_v| &\lesssim |\mu^{\frac18}g|_{H^{s_1}}(|W^\varepsilon(D)h|_{H^{s_2}_{\gamma/2}} + |h|_{\varepsilon,\gamma/2})|f|_{\varepsilon,\gamma/2} \\ &\quad + |g|_{L^2}|h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2},\end{aligned}$$

where $(s_1,s_2) = (0,\eta)$ or $(\eta,0)$.

- *If $-3 < \gamma < -\frac32$, then*

$$\begin{aligned}|\langle\Gamma^\varepsilon(g,h), f\rangle_v| &\lesssim |\mu^{\frac18}g|_{H^{s_1}}(|W^\varepsilon(D)h|_{H^{s_2}_{\gamma/2}} + |h|_{\varepsilon,\gamma/2})|f|_{\varepsilon,\gamma/2} \\ &\quad + |g|_{L^2}|h|_{\varepsilon,\gamma/2}|f|_{\varepsilon,\gamma/2},\end{aligned}$$

where the constants $s_1, s_2 \ge 0$ verify either $s_1 + s_2 = -\gamma - \frac32$, $s_2 > 0$ or $s_1 = -\gamma - \frac32 + \eta$, $s_2 = 0$.

As a direct application we have

$$|\langle \Gamma^\varepsilon(\mu^{\frac{1}{2}}, f), f\rangle_v| \lesssim |f|^2_{\varepsilon,\gamma/2}. \tag{2.46}$$

Proof. Recalling (2.25), the estimates of $|\langle \Gamma^\varepsilon(g,h), f\rangle_v|$ follow directly from Theorem 2.2 and Proposition 2.7. By taking $s_2 = 0$, we get (2.46). ∎

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. On one hand, by Theorem 2.1, we derived that $\langle \mathcal{L}^\varepsilon f, f\rangle_v + |f|^2_{L^2_{\gamma/2}} \gtrsim |f|^2_{\varepsilon,\gamma/2}$. On the other hand, recalling (1.8), by (2.24) and (2.46), we have $\langle \mathcal{L}^\varepsilon f, f\rangle_v \lesssim |f|^2_{\varepsilon,\gamma/2}$, which ends the proof. ∎

2.3. Upper bound of commutator between weight function and the nonlinear term

In this subsection we want to prove the following lemma.

Lemma 2.8. *Let $l \geq 2$. The following commutator estimates are valid:*

- *If $\gamma \geq -2$, then* $|\langle \Gamma^\varepsilon(g, W_l h) - W_l\Gamma^\varepsilon(g,h), f\rangle_v| \lesssim |g|_{L^2}|h|_{L^2_{l+\gamma/2}}|f|_{\varepsilon,\gamma/2}$.
- *If $-3 < \gamma < -2$, then*

$$\begin{aligned}|\langle \Gamma^\varepsilon(g, W_l h) - W_l\Gamma^\varepsilon(g,h), f\rangle_v| &\lesssim |g|_{L^2}|h|_{L^2_{l+\gamma/2}}|f|_{\varepsilon,\gamma/2} \\ &\quad + |\mu^{\frac{1}{32}}g|_{H^{s_1}}|\mu^{\frac{1}{32}}h|_{H^{s_2}}|f|_{\varepsilon,\gamma/2},\end{aligned}$$

where the constants $s_1, s_2 \geq 0$ verify $s_1 + s_2 = -\gamma/2 - 1$.

This lemma is a consequence of Lemmas 2.9 and 2.10 by recalling (2.25). We first prove the commutator estimate for Q^ε.

Lemma 2.9. *Let $l \geq 2$. The following commutator estimates are valid:*

- *If $\gamma \geq -2$, then* $|\langle Q^\varepsilon(\mu^{\frac{1}{2}}g, W_l h) - W_l Q^\varepsilon(\mu^{\frac{1}{2}}g,h), f\rangle_v| \lesssim |\mu^{\frac{1}{32}}g|_{L^2}|h|_{L^2_{l+\gamma/2}}|f|_{\varepsilon,\gamma/2}$.
- *If $-3 < \gamma < -2$, then*

$$\begin{aligned}&|\langle Q^\varepsilon(\mu^{\frac{1}{2}}g, W_l h) - W_l Q^\varepsilon(\mu^{\frac{1}{2}}g,h), f\rangle_v| \\ &\quad\lesssim (|\mu^{\frac{1}{32}}g|_{L^2}|h|_{L^2_{l+\gamma/2}} + |\mu^{\frac{1}{32}}g|_{H^{s_1}}|\mu^{\frac{1}{32}}h|_{H^{s_2}})|f|_{\varepsilon,\gamma/2},\end{aligned}$$

where the constants $s_1, s_2 \geq 0$ verify $s_1 + s_2 = -\gamma/2 - 1$.

Proof. Recall that $B^{\varepsilon,\gamma} = |v - v_*|^\gamma b^\varepsilon(\cos\theta)$ and note that

$$\begin{aligned}\langle Q^\varepsilon(\mu^{\frac{1}{2}}g, W_l h) - W_l Q^\varepsilon(\mu^{\frac{1}{2}}g,h), f\rangle_v &= \int B^{\varepsilon,\gamma}(W_l - W_l')\mu_*^{\frac{1}{2}}g_* h f'\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ &= \int B^{\varepsilon,\gamma}(W_l - W_l')\mu_*^{\frac{1}{2}}g_* h(f' - f)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ &\quad + \int B^{\varepsilon,\gamma}(W_l - W_l')\mu_*^{\frac{1}{2}}g_* h f\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ &:= \mathcal{A}_1 + \mathcal{A}_2.\end{aligned}$$

Step 1*: Estimate of* $\mathcal{A}_1$. By the Cauchy–Schwarz inequality, we have

$$|\mathcal{A}_1| \leq \left(\int B^{\varepsilon,\gamma}\mu_*^{\frac{1}{2}}(f'-f)^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{\frac{1}{2}}\left(\int B^{\varepsilon,\gamma}(W_l-W_l')^2\mu_*^{\frac{1}{2}}g_*^2h^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{\frac{1}{2}}$$
$$:= (\mathcal{A}_{1,1})^{\frac{1}{2}}(\mathcal{A}_{1,2})^{\frac{1}{2}}.$$

By the estimate of $\mathcal{I}_{2,1}$ in the proof of Proposition 2.7, we have $\mathcal{A}_{1,1} \lesssim |f|^2_{\varepsilon,\gamma/2}$. Since $|\nabla W_l| \lesssim W_{l-1}$, we derive

$$\int B^{\varepsilon,\gamma}(W_l-W_l')^2\,\mathrm{d}\sigma \lesssim \int b^{\varepsilon}(\cos\theta)\sin^2\tfrac{\theta}{2}|v-v_*|^{\gamma+2}\langle v\rangle^{2l-2}\langle v_*\rangle^{2l-2}\,\mathrm{d}\sigma$$
$$\lesssim |v-v_*|^{\gamma+2}\langle v\rangle^{2l-2}\langle v_*\rangle^{2l-2}, \tag{2.47}$$

which gives

$$\mathcal{A}_{1,2} \lesssim \int |v-v_*|^{\gamma+2}\langle v\rangle^{2l-2}\langle v_*\rangle^{2l-2}\mu_*^{\frac{1}{2}}g_*^2h^2\,\mathrm{d}v_*\,\mathrm{d}v.$$

If $\gamma+2 \geq 0$, then $|v-v_*|^{\gamma+2} \lesssim \langle v\rangle^{\gamma+2}\langle v_*\rangle^{\gamma+2}$ and thus $\mathcal{A}_{1,2} \lesssim |\mu^{\frac{1}{16}}g|^2_{L^2}|h|^2_{L^2_{l+\gamma/2}}$. If $\gamma+2<0$, we make the following decomposition:

$$\mathcal{A}_{1,2} \lesssim \int |v-v_*|^{\gamma+2}1_{|v-v_*|\leq 1}\langle v\rangle^{2l-2}\langle v_*\rangle^{2l-2}\mu_*^{\frac{1}{2}}g_*^2h^2\,\mathrm{d}v_*\,\mathrm{d}v$$
$$+\int |v-v_*|^{\gamma+2}1_{|v-v_*|\geq 1}\langle v\rangle^{2l-2}\langle v_*\rangle^{2l-2}\mu_*^{\frac{1}{2}}g_*^2h^2\,\mathrm{d}v_*\,\mathrm{d}v$$
$$:= \mathcal{A}_{1,2,1}+\mathcal{A}_{1,2,2}.$$

When $|v-v_*|\leq 1$, there holds $|v_*|\geq |v|-1$, thus $|v_*|^2 \geq \frac{|v|^2}{2}-1$ and $\mu_* \lesssim \mu^{\frac{1}{2}}$. Therefore we get $\langle v\rangle^{2l-2}\langle v_*\rangle^{2l-2}\mu_*^{\frac{1}{2}} \lesssim \langle v\rangle^{2l-2}\langle v_*\rangle^{2l-2}\mu_*^{\frac{1}{8}}\mu^{\frac{1}{8}} \lesssim \mu_*^{\frac{1}{16}}\mu^{\frac{1}{16}}$, which yields

$$\mathcal{A}_{1,2,1} \lesssim \int |v-v_*|^{\gamma+2}1_{|v-v_*|\leq 1}\mu_*^{\frac{1}{16}}\mu^{\frac{1}{16}}g_*^2h^2\,\mathrm{d}v_*\,\mathrm{d}v \lesssim |\mu^{\frac{1}{32}}g|^2_{H^{s_1}}|\mu^{\frac{1}{32}}h|^2_{H^{s_2}}, \tag{2.48}$$

where in the last inequality we use the Hardy–Littlewood–Sobolev inequality and the Sobolev embedding theorem if $s_1 \in (0,-(\gamma+2)/2)$ and Hardy's inequality if $s_1=0$ or $s_1=-(\gamma+2)/2$.

When $|v-v_*|\geq 1$, there holds $|v-v_*|^{\gamma+2} \sim \langle v-v_*\rangle^{\gamma+2} \lesssim \langle v\rangle^{\gamma+2}\langle v_*\rangle^{|\gamma+2|}$, which yields

$$\mathcal{A}_{1,2,2} \lesssim \int \langle v\rangle^{2l+\gamma}\langle v_*\rangle^{2l-2+|\gamma+2|}\mu_*^{\frac{1}{2}}g_*^2h^2\,\mathrm{d}v_*\,\mathrm{d}v \lesssim |\mu^{\frac{1}{32}}g|^2_{L^2}|h|^2_{L^2_{l+\gamma/2}}.$$

Patching together the above estimates,

- if $\gamma+2\geq 0$, then $|\mathcal{A}_1| \lesssim |\mu^{\frac{1}{16}}g|_{L^2}|h|_{L^2_{l+\gamma/2}}|f|_{\varepsilon,\gamma/2}$;
- if $\gamma+2<0$, then $|\mathcal{A}_1| \lesssim (|\mu^{\frac{1}{32}}g|_{H^{s_1}}|\mu^{\frac{1}{32}}h|_{H^{s_2}}+|\mu^{\frac{1}{32}}g|_{L^2}|h|_{L^2_{l+\gamma/2}})|f|_{\varepsilon,\gamma/2}$.

Step 2: Estimate of $\mathcal{A}_2$. By Taylor expansion, one has

$$W_l' - W_l = (\nabla W_l)(v) \cdot (v' - v) + \int_0^1 (1-\kappa)\big((\nabla^2 W_l)(v(\kappa)) : (v'-v) \otimes (v'-v)\big)\,\mathrm{d}\kappa.$$

Thus we have

$$\begin{aligned}
\mathcal{A}_2 = &-\int B^{\varepsilon,\gamma} (\nabla W_l)(v) \cdot (v'-v) \mu_*^{\frac{1}{2}} g_* h f \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
&-\int B^{\varepsilon,\gamma} (1-\kappa)\big((\nabla^2 W_l)(v(\kappa)) : (v'-v) \otimes (v'-v)\big) \mu_*^{\frac{1}{2}} g_* h f \,\mathrm{d}\kappa\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\
:= &\ \mathcal{A}_{2,1} + \mathcal{A}_{2,2}.
\end{aligned}$$

Estimate of $\mathcal{A}_{2,1}$. Thanks to the fact that there exists a constant $C(\varepsilon)$ with $|C(\varepsilon)| \lesssim 1$ such that

$$\int b^\varepsilon(\cos\theta)(v'-v)\,\mathrm{d}\sigma = -(v-v_*)\int b^\varepsilon(\cos\theta)\sin^2\tfrac{\theta}{2}\,\mathrm{d}\sigma = -(v-v_*)C(\varepsilon). \quad (2.49)$$

From this, together with $|\nabla W_l| \lesssim W_{l-1}$, we have

$$\begin{aligned}
|\mathcal{A}_{2,1}| \lesssim &\int |v-v_*|^{\gamma+1} \langle v\rangle^{l-1} \mu_*^{\frac{1}{2}} |g_* h f| \,\mathrm{d}v_*\,\mathrm{d}v \\
\lesssim &\int |v-v_*|^{\gamma+1} 1_{|v-v_*|\le 1} \langle v\rangle^{l-1} \mu_*^{\frac{1}{2}} |g_* h f| \,\mathrm{d}v_*\,\mathrm{d}v \\
&+ \int |v-v_*|^{\gamma+1} 1_{|v-v_*|\ge 1} \langle v\rangle^{l-1} \mu_*^{\frac{1}{2}} |g_* h f| \,\mathrm{d}v_*\,\mathrm{d}v \\
:= &\ \mathcal{A}_{2,1,1} + \mathcal{A}_{2,1,2}.
\end{aligned}$$

When $|v-v_*| \le 1$, as before, one has $\langle v\rangle^{l-1}\mu_*^{\frac{1}{2}} \lesssim \langle v\rangle^{l-1}\mu_*^{\frac{1}{8}}\mu^{\frac{1}{8}} \lesssim \mu_*^{\frac{1}{16}}\mu^{\frac{1}{16}}$. Thus, by the Cauchy–Schwarz inequality we have

$$\begin{aligned}
\mathcal{A}_{2,1,1} \lesssim &\int |v-v_*|^{\gamma+1} 1_{|v-v_*|\le 1} \mu_*^{\frac{1}{16}} \mu^{\frac{1}{16}} |g_* h f| \,\mathrm{d}v_*\,\mathrm{d}v \\
\le &\left(\int |v-v_*|^{\gamma+2} 1_{|v-v_*|\le 1} \mu_*^{\frac{1}{16}} \mu^{\frac{1}{16}} g_*^2 h^2 \,\mathrm{d}v_*\,\mathrm{d}v\right)^{\frac{1}{2}} \\
&\times \left(\int |v-v_*|^{\gamma} 1_{|v-v_*|\le 1} \mu_*^{\frac{1}{16}} \mu^{\frac{1}{16}} f^2 \,\mathrm{d}v_*\,\mathrm{d}v\right)^{\frac{1}{2}} \\
\lesssim &\left(\int |v-v_*|^{\gamma+2} 1_{|v-v_*|\le 1} \mu_*^{\frac{1}{16}} \mu^{\frac{1}{16}} g_*^2 h^2 \,\mathrm{d}v_*\,\mathrm{d}v\right)^{\frac{1}{2}} |\mu^{\frac{1}{32}} f|_{L^2}.
\end{aligned}$$

If $\gamma \ge -2$, we directly get $\mathcal{A}_{2,1,1} \lesssim |\mu^{\frac{1}{32}} g|_{L^2} |\mu^{\frac{1}{32}} h|_{L^2} |\mu^{\frac{1}{32}} f|_{L^2}$. If $-3 < \gamma < -2$, recalling (2.48), we have $\mathcal{A}_{2,1,1} \lesssim |\mu^{\frac{1}{32}} g|_{H^{s_1}} |\mu^{\frac{1}{32}} h|_{H^{s_2}} |\mu^{\frac{1}{32}} f|_{L^2}$. Using $|v-v_*|^{\gamma+1} 1_{|v-v_*|\ge 1} \lesssim \langle v\rangle^{\gamma+1} \langle v_*\rangle^{|\gamma+1|}$, we have $\mathcal{A}_{2,1,2} \lesssim |\mu^{\frac{1}{32}} g|_{L^2} |h|_{L^2_{l+\gamma/2}} |f|_{L^2_{\gamma/2}}$.

Estimate of $\mathcal{A}_{2,2}$. Since $|(\nabla^2 W_l)(v(\kappa))| \lesssim \langle v(\kappa)\rangle^{l-2} \lesssim \langle v\rangle^{l-2}\langle v_*\rangle^{l-2}$ and $|v'-v|^2 = \sin^2\frac{\theta}{2}|v-v_*|^2$, we have

$$|\mathcal{A}_{2,2}| \lesssim \int b^\varepsilon(\cos\theta)\sin^2\tfrac{\theta}{2}|v-v_*|^{\gamma+2}\langle v\rangle^{l-2}\langle v_*\rangle^{l-2}\mu_*^{\frac12}|g_* h f|\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v$$
$$\lesssim \int |v-v_*|^{\gamma+2}\langle v\rangle^{l-2}\mu_*^{\frac18}|g_* h f|\,\mathrm{d}v_*\,\mathrm{d}v.$$

Thanks to $\gamma+2>-1$, using the Cauchy–Schwarz inequality and (2.7), we have $\int |v-v_*|^{\gamma+2}\mu_*^{\frac18}|g_*|\,\mathrm{d}v_* \lesssim \langle v\rangle^{\gamma+2}|\mu^{\frac{1}{16}}g|_{L^2}$ and thus $|\mathcal{A}_{2,2}| \lesssim |\mu^{\frac{1}{16}}g|_{L^2}|h|_{L^2_{l+\gamma/2}}|f|_{L^2_{\gamma/2}}$. Patching together the estimates of $\mathcal{A}_{2,1,1}$, $\mathcal{A}_{2,1,2}$ and $\mathcal{A}_{2,2}$, we conclude as follows:

- if $\gamma \ge -2$, then $|\mathcal{A}_2| \lesssim |\mu^{\frac{1}{32}}g|_{L^2}|h|_{L^2_{l+\gamma/2}}|f|_{L^2_{\gamma/2}}$;
- if $-3<\gamma<-2$, then $|\mathcal{A}_2| \lesssim (|\mu^{\frac{1}{32}}g|_{H^{s_1}}|\mu^{\frac{1}{32}}h|_{H^{s_2}} + |\mu^{\frac{1}{32}}g|_{L^2}|h|_{L^2_{l+\gamma/2}})|f|_{L^2_{\gamma/2}}$.

The lemma follows by patching together the estimates of $\mathcal{A}_1$ and $\mathcal{A}_2$. ∎

The next lemma gives the commutator estimate for $\mathcal{I}(g,h,f)$.

Lemma 2.10. *Let $l\ge 1$, there holds*

$$|\mathcal{I}(g, W_l h, f) - \mathcal{I}(g,h,W_l f)| \lesssim |g|_{L^2}|h|_{L^2_{l+\gamma/2}}|W^\varepsilon f|_{L^2_{\gamma/2}}.$$

Proof. By the definition of $\mathcal{I}(g,h,f)$ and the fact that $(\mu^{\frac12})'_* - \mu_*^{\frac12} = ((\mu^{\frac14})'_* - \mu_*^{\frac14})^2 + 2\mu_*^{\frac14}((\mu^{\frac14})'_* - \mu_*^{\frac14})$, we have

$$\begin{aligned}\mathcal{I}(g,W_l h,f) - \mathcal{I}(g,h,W_l f) &= \int B^{\varepsilon,\gamma}((\mu^{\frac12})'_* - \mu_*^{\frac12})(W_l - W_l')g_* h f'\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &= \int B^{\varepsilon,\gamma}((\mu^{\frac14})'_* - \mu_*^{\frac14})^2(W_l - W_l')g_* h f'\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &\quad + 2\int B^{\varepsilon,\gamma}\mu_*^{\frac14}((\mu^{\frac14})'_* - \mu_*^{\frac14})(W_l - W_l')g_* h f'\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &:= \mathcal{A}_1 + 2\mathcal{A}_2.\end{aligned}$$

Step 1*: Estimate of* $\mathcal{A}_1$. By the Cauchy–Schwarz inequality we have

$$\begin{aligned}|\mathcal{A}_1| \le &\left(\int B^{\varepsilon,\gamma}((\mu^{\frac14})'_* - \mu_*^{\frac14})^2(f^2)'\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{\frac12}\\ &\times\left(\int B^{\varepsilon,\gamma}((\mu^{\frac14})'_* - \mu_*^{\frac14})^2(W_l - W_l')^2 g_*^2 h^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{\frac12} := (\mathcal{A}_{1,1})^{\frac12}(\mathcal{A}_{1,2})^{\frac12}.\end{aligned}$$

By the change of variables $(v,v_*)\to(v'_*,v')$ and the proof of the upper bound in Proposition 2.1, we have

$$\mathcal{A}_{1,1} = \int B^{\varepsilon,\gamma}((\mu^{\frac14})' - \mu^{\frac14})^2 f_*^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \lesssim |W^\varepsilon f|^2_{L^2_{\gamma/2}}.$$

Thanks to

$$((\mu^{\frac{1}{4}})'_* - \mu_*^{\frac{1}{4}})^2 = ((\mu^{\frac{1}{8}})'_* + \mu_*^{\frac{1}{8}})^2((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2 \leq 2((\mu^{\frac{1}{4}})'_* + \mu_*^{\frac{1}{4}})((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2,$$

we have

$$\begin{aligned}\mathcal{A}_{1,2} &\lesssim \int B^{\varepsilon,\gamma}\mu_*^{\frac{1}{4}}((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2(W_l - W_l')^2 g_*^2 h^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ &\quad + \int B^{\varepsilon,\gamma}(\mu^{\frac{1}{4}})'_*((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2(W_l - W_l')^2 g_*^2 h^2 \,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \\ &:= \mathcal{A}_{1,2,1} + \mathcal{A}_{1,2,2}.\end{aligned}$$

Thanks to the facts $|v - v'_*| \sim |v - v_*|$ and

$$(W_l - W_l')^2 \lesssim \min\{\sin^2\tfrac{\theta}{2}|v - v'_*|^2\langle v\rangle^{2l-2}\langle v'_*\rangle^{2l-2}, \sin^2\tfrac{\theta}{2}\langle v\rangle^{2l}\langle v'_*\rangle^{2l}\}, \tag{2.50}$$

$$((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2 \lesssim \min\{\sin^2\tfrac{\theta}{2}|v - v'_*|^2, 1\}, \tag{2.51}$$

we assert

$$\mathcal{B} := \int B^{\varepsilon,\gamma}(\mu^{\frac{1}{4}})'_*((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2(W_l - W_l')^2 \,\mathrm{d}\sigma \lesssim \langle v\rangle^{2l+\gamma}, \tag{2.52}$$

which yields $\mathcal{A}_{1,2,2} \lesssim |g|^2_{L^2}|h|^2_{L^2_{l+\gamma/2}}$. In fact, by (2.50) and (2.51), on one hand, there holds

$$\mathcal{B} \lesssim \int b^\varepsilon(\cos\theta)\sin^4\tfrac{\theta}{2}|v - v'_*|^{\gamma+4}(\mu^{\frac{1}{4}})'_*\langle v\rangle^{2l-2}\langle v'_*\rangle^{2l-2}\,\mathrm{d}\sigma.$$

When $|v - v_*| \leq 1$, there holds $|v - v'_*| \leq 1$, $|v - v'_*|^{\gamma+4} \leq 1$ and $\langle v\rangle \sim \langle v'_*\rangle$, thus $\langle v\rangle^{2l-2} \lesssim \langle v\rangle^{2l+\gamma}\langle v'_*\rangle^{-2-\gamma}$, which yields

$$\begin{aligned}\mathcal{B} &\lesssim \int b^\varepsilon(\cos\theta)\sin^4\tfrac{\theta}{2}(\mu^{\frac{1}{4}})'_*\langle v\rangle^{2l+\gamma}\langle v'_*\rangle^{2l-4-\gamma}\,\mathrm{d}\sigma \\ &\lesssim \int b^\varepsilon(\cos\theta)\sin^4\tfrac{\theta}{2}\langle v\rangle^{2l+\gamma}\,\mathrm{d}\sigma \lesssim \langle v\rangle^{2l+\gamma}.\end{aligned}$$

By (2.50) and (2.51), on the other hand, there holds

$$\mathcal{B} \lesssim \int b^\varepsilon(\cos\theta)\sin^2\tfrac{\theta}{2}|v - v'_*|^{\gamma}(\mu^{\frac{1}{4}})'_*\langle v\rangle^{2l}\langle v'_*\rangle^{2l}\,\mathrm{d}\sigma.$$

When $|v - v_*| \geq 1$, there holds $|v - v'_*|^\gamma \sim \langle v - v'_*\rangle^\gamma \lesssim \langle v\rangle^\gamma\langle v'_*\rangle^{|\gamma|}$, which yields

$$\begin{aligned}\mathcal{B} &\lesssim \int b^\varepsilon(\cos\theta)\sin^2\tfrac{\theta}{2}(\mu^{\frac{1}{4}})'_*\langle v\rangle^{2l+\gamma}\langle v'_*\rangle^{2l+|\gamma|}\,\mathrm{d}\sigma \\ &\lesssim \int b^\varepsilon(\cos\theta)\sin^2\tfrac{\theta}{2}\langle v\rangle^{2l+\gamma}\,\mathrm{d}\sigma \lesssim \langle v\rangle^{2l+\gamma}.\end{aligned}$$

Now estimate (2.52) is proved. Note that (2.50) and (2.51) are still valid if v'_* is replaced by v_* on the right-hand sides. Then similarly to (2.52), we can prove

$$\int B^{\varepsilon,\gamma}\mu_*^{\frac{1}{4}}((\mu^{\frac{1}{8}})'_* - \mu_*^{\frac{1}{8}})^2(W_l - W_l')^2\,\mathrm{d}\sigma \lesssim \langle v\rangle^{2l+\gamma}\mu_*^{\frac{1}{8}},$$

which yields $\mathcal{A}_{1,2,1} \lesssim |\mu^{\frac{1}{16}} g|^2_{L^2} |h|^2_{L^2_{l+\gamma/2}}$. Patching together the estimates of $\mathcal{A}_{1,2,1}$ and $\mathcal{A}_{1,2,2}$, we arrive at $\mathcal{A}_{1,2} \lesssim |g|^2_{L^2} |h|^2_{L^2_{l+\gamma/2}}$. Patching together the estimates of $\mathcal{A}_{1,1}$ and $\mathcal{A}_{1,2}$, we conclude that $|\mathcal{A}_1| \lesssim |g|_{L^2} |h|_{L^2_{l+\gamma/2}} |W^\varepsilon f|_{L^2_{\gamma/2}}$.

Step 2: *Estimate of* $\mathcal{A}_2$. By the Cauchy–Schwarz inequality we have

$$|\mathcal{A}_2| \leq \left(\int B^{\varepsilon,\gamma} \mu_*^{\frac{1}{4}} ((\mu^{\frac{1}{4}})'_* - \mu_*^{\frac{1}{4}})^2 |g_*| (f^2)' \, d\sigma \, dv_* \, dv \right)^{\frac{1}{2}}$$
$$\times \left(\int B^{\varepsilon,\gamma} \mu_*^{\frac{1}{4}} (W_l - W_l')^2 |g_*| h^2 \, d\sigma \, dv_* \, dv \right)^{\frac{1}{2}} := (\mathcal{A}_{2,1})^{\frac{1}{2}} (\mathcal{A}_{2,2})^{\frac{1}{2}}.$$

Estimate of $\mathcal{A}_{2,1}$. By the change of variable $v \to v'$, we have

$$\mathcal{A}_{2,1} \lesssim \int b^\varepsilon(\cos\theta) |v - v_*|^\gamma \mu_*^{\frac{1}{4}} ((\mu^{\frac{1}{4}})'_* - \mu_*^{\frac{1}{4}})^2 |g_*| f^2 \, d\sigma \, dv_* \, dv.$$

By (2.36) and Proposition A.1 we get

$$\begin{aligned} \mathcal{A}_{2,1} &\lesssim \int (1_{|v-v_*|\leq\sqrt{2}} |v - v_*|^{\gamma+2} + \langle v - v_* \rangle^\gamma (W^\varepsilon)^2 (v - v_*)) \mu_*^{\frac{1}{4}} |g_*| f^2 \, dv_* \, dv \\ &\lesssim \int (|v - v_*|^{\gamma+2} \mu_*^{\frac{1}{8}} \mu^{\frac{1}{16}} + \langle v \rangle^\gamma (W^\varepsilon)^2 (v) \mu_*^{\frac{1}{8}}) |g_*| f^2 \, dv_* \, dv \\ &\lesssim |\mu^{\frac{1}{16}} g|_{L^2} |W^\varepsilon f|^2_{L^2_{\gamma/2}}, \end{aligned}$$

where we use the fact that $\mu_*^{\frac{1}{4}} \lesssim \mu_*^{\frac{1}{8}} \mu^{\frac{1}{16}}$ when $|v - v_*| \leq \sqrt{2}$ and the estimate

$$\begin{aligned} \int |v - v_*|^{\gamma+2} \mu_*^{\frac{1}{8}} |g_*| \, dv_* &\leq \left(\int |v - v_*|^{2\gamma+4} \mu_*^{\frac{1}{8}} \, dv_* \right)^{\frac{1}{2}} \left(\int \mu_*^{\frac{1}{8}} g_*^2 \, dv_* \right)^{\frac{1}{2}} \\ &\lesssim \langle v \rangle^{\gamma+2} |\mu^{\frac{1}{16}} g|_{L^2} \end{aligned} \tag{2.53}$$

given by the Cauchy–Schwarz inequality and (2.7).

Estimate of $\mathcal{A}_{2,2}$. Recalling (2.47) we have

$$\begin{aligned} \mathcal{A}_{2,2} &\lesssim \int |v - v_*|^{\gamma+2} \langle v \rangle^{2l-2} \langle v_* \rangle^{2l-2} \mu_*^{\frac{1}{4}} |g_*| h^2 \, dv_* \, dv \\ &\lesssim \int |v - v_*|^{\gamma+2} \langle v \rangle^{2l-2} \mu_*^{\frac{1}{8}} |g_*| h^2 \, dv_* \, dv \lesssim |\mu^{\frac{1}{16}} g|_{L^2} |h|^2_{L^2_{l+\gamma/2}}, \end{aligned}$$

where we use (2.53). Putting together the estimates of $\mathcal{A}_{2,1}$ and $\mathcal{A}_{2,2}$, we arrive at

$$|\mathcal{A}_2| \lesssim |\mu^{\frac{1}{16}} g|_{L^2} |h|_{L^2_{l+\gamma/2}} |W^\varepsilon f|_{L^2_{\gamma/2}}.$$

Patching together the estimates of $\mathcal{A}_1$ and $\mathcal{A}_2$, we finish the proof. ∎

3. Diversity of longtime behavior of the semigroup

In this section we will give the proof of Theorem 1.2. We begin with a technical lemma for a commutator estimate.

Lemma 3.1. *Let* $-2s \leq \gamma < 0$, $j \in \mathbb{N}$, $2^j \geq 1/\varepsilon$. *Let* $\chi_M(v) := \chi(v/M)$ *with* $(\chi, M) = (\phi, 1/\varepsilon)$ *or* $(\chi, M) = (1-\phi, 1/\varepsilon)$ *or* $(\chi, M) = (\psi, 2^j)$. *Here* ϕ *and* ψ *are defined in* (1.19). *For any* $0 < \eta < 1$, *it holds that*

$$|\langle[\Gamma^\varepsilon(g,\cdot),\chi_M]h, f\chi_M\rangle_v| \lesssim \eta^{-1}\varepsilon^{2s}(|g|^2_{L^2}|h|^2_{\varepsilon,\gamma/2}+|g|^2_{\varepsilon,\gamma/2}|h|^2_{L^2})+\eta|f\chi_M|^2_{\varepsilon,\gamma/2}, \quad (3.1)$$

$$|\langle[\mathcal{L}^\varepsilon,\chi_M]f, f\chi_M\rangle_v| \lesssim \eta^{-1}\varepsilon^{2s}|f|^2_{\varepsilon,\gamma/2} + \eta|f\chi_M|^2_{\varepsilon,\gamma/2}. \quad (3.2)$$

Proof. In this proof, we denote $\mathcal{I}(g,h,f) := \langle[\Gamma^\varepsilon(g,\cdot),\chi_M]h, f\chi_M\rangle_v = \langle\Gamma^\varepsilon(g,h\chi_M) - \chi_M\Gamma^\varepsilon(g,h), f\chi_M\rangle_v$. Direct calculation gives

$$\mathcal{I}(g,h,f) = \int B^{\varepsilon,\gamma}[(g\mu^{\frac12})_* + g_*((\mu^{\frac12})'_* - \mu^{\frac12}_*)]h(f\chi_M)'(-(\chi_M)' + \chi_M)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v.$$

By the Cauchy–Schwarz inequality and the change of variable $(v,v_*) \to (v'_*, v')$, we get that

$$\begin{aligned}|\mathcal{I}(g,h,f)| \lesssim &\left(\int B^{\varepsilon,\gamma} g_*^2 h^2(\mu_*^{\frac12} + (\mu^{\frac12})'_*)((\chi_M)' - \chi_M)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{\frac12}\\ &\times\left(\int B^{\varepsilon,\gamma}[\mu_*^{\frac12}((f\chi_M)' - f\chi_M)^2 + (f\chi_M)^2_*((\mu^{\frac14})' - \mu^{\frac14})^2]\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right)^{\frac12}\\ &+\left|\int B^{\varepsilon,\gamma}(g\mu^{\frac12})_* h f\chi_M((\chi_M)' - \chi_M)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\right|.\end{aligned}$$

By the estimate of $\mathcal{I}_{2,1}$ in the proof of Proposition 2.7, it holds that

$$\int B^{\varepsilon,\gamma}\mu_*^{\frac12}((f\chi_M)' - f\chi_M)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \lesssim |f\chi_M|^2_{\varepsilon,\gamma/2}.$$

By the proof of the upper bound in Proposition 2.1 we have

$$\int B^{\varepsilon,\gamma}(f\chi_M)^2_*((\mu^{\frac14})' - \mu^{\frac14})^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v \lesssim |W^\varepsilon f\chi_M|^2_{L^2_{\gamma/2}} \lesssim |f\chi_M|^2_{\varepsilon,\gamma/2}.$$

By these two estimates, we have

$$|\mathcal{I}(g,h,f)| \lesssim \eta|f\chi_M|^2_{\varepsilon,\gamma/2} + \eta^{-1}\mathcal{J}(g,h) + |\mathcal{K}(g,h,f)|, \quad (3.3)$$

where

$$\mathcal{J}(g,h) := \int B^{\varepsilon,\gamma} g_*^2 h^2(\mu_*^{\frac12} + (\mu^{\frac12})'_*)((\chi_M)' - \chi_M)^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v,$$

$$\mathcal{K}(g,h,f) := \int B^{\varepsilon,\gamma}(g\mu^{\frac12})_* h f\chi_M((\chi_M)' - \chi_M)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v.$$

We will now estimate $\mathcal{J}(g,h)$ and $\mathcal{K}(g,h,f)$.

Estimate of $\mathcal{J}(g,h)$. We separate the integration domain of $\mathcal{J}(g,h)$ into three regions: $\{|v_*| \leq \delta M\}$, $\{|v_*| \geq \delta M, |v| \leq \delta|v_*|\}$ and $\{|v_*| \geq \delta M, |v| \geq \delta|v_*|\}$ where $\delta = \frac{1}{100}$ throughout the proof. We first consider the region $\{|v_*| \leq \delta M\}$. By Taylor expansion,

$$\chi_M(v') - \chi_M(v) = \int_0^1 (\nabla\chi_M)(v(\kappa)) \cdot (v' - v)\,\mathrm{d}\kappa,$$

where $v(\kappa) = v + \kappa(v' - v)$. By the support of $\nabla\chi$, one has $\frac{M}{10} \leq |v(\kappa)| \leq 10M$. Therefore $\chi_M(v') - \chi_M(v)$ is supported in $|v| \sim |v(\kappa)| \sim |v - v_*| \sim M$. In the region $\{|v_*| \geq \delta M, |v| \leq \delta|v_*|\}$, we deduce that $|v_*| \sim |v - v_*| \sim |v - v'_*| \sim |v'_*|$. In the region $\{|v_*| \geq \delta M, |v| \geq \delta|v_*|\}$, there holds $|v| \geq \delta^2 M$. Putting together all the facts, since $\chi \leq 1$ and $|\nabla\chi_M| \lesssim M^{-1}$, we get

$$\begin{aligned}|\chi_M(v') - \chi_M(v)|^2 &\lesssim \mathrm{1}_{|v_*|\leq\delta M}\mathrm{1}_{|v|\sim|v-v_*|\sim M} M^{-2}\theta^2|v - v_*|^2 \\ &\quad + (\mathrm{1}_{|v_*|\geq\delta M}\mathrm{1}_{|v'_*|\sim|v_*|\sim|v-v_*|}\mathrm{1}_{|v|\leq\delta|v_*|} \\ &\quad + \mathrm{1}_{|v_*|\geq\delta M}\mathrm{1}_{|v|\geq\delta^2 M}\mathrm{1}_{|v|\geq\delta|v_*|}) \min\big\{1, M^{-2}|v - v_*|^2\theta^2\big\}.\end{aligned}$$

From this, together with Proposition A.1 and thanks to the factor $\mu_*^{\frac{1}{2}} + (\mu^{\frac{1}{2}})'_*$, we have for any $a \geq 0$,

$$\begin{aligned}\mathcal{J}(g,h) &\lesssim |g\mathrm{1}_{|\cdot|\leq\delta M}|^2_{L^2_{-a}}|h\mathrm{1}_{|\cdot|\sim M}|^2_{L^2_{\gamma/2+a}} + e^{-\delta^3 M^2}|W^\varepsilon g\mathrm{1}_{|\cdot|\geq\delta M}|^2_{L^2_{\gamma/2+a}}|h|^2_{L^2_{-a}} \\ &\quad + M^{-2s}|g\mathrm{1}_{|\cdot|\geq\delta M}|^2_{L^2_{-a}}|W^\varepsilon h\mathrm{1}_{|\cdot|\geq\delta^2 M}|^2_{L^2_{a+\gamma/2}}. \qquad (3.4)\end{aligned}$$

Estimate of $\mathcal{K}(g,h,f)$. We decompose the integration domain of $\mathcal{K}(g,h,f)$ into two regions: $\{|v_*| \leq \delta M\}$ and $\{|v_*| \geq \delta M\}$. Correspondingly, $\mathcal{K}(g,h,f) = \mathcal{K}_1(g,h,f) + \mathcal{K}_2(g,h,f)$.

We first deal with $\mathcal{K}_1(g,h,f)$ whose integration domain is $\{|v_*| \leq \delta M\}$. In this case, recall that $\chi_M(v') - \chi_M(v)$ is supported in $|v| \sim |v - v_*| \sim M$. By (2.49) and Taylor expansion,

$$\begin{aligned}\chi_M(v') - \chi_M(v) &= (\nabla\chi_M)(v) \cdot (v - v') \\ &\quad + \int_0^1 (1-\kappa)(\nabla^2\chi_M)(v(\kappa)) : (v' - v) \otimes (v' - v)\,\mathrm{d}\kappa, \qquad (3.5)\end{aligned}$$

we infer that $|\int B^{\varepsilon,\gamma}(\chi_M(v') - \chi_M(v))\,\mathrm{d}\sigma| \lesssim \mathrm{1}_{|v_*|\leq\delta M}\mathrm{1}_{|v|\sim|v-v_*|\sim M}\langle v\rangle^\gamma$, which yields that

$$\begin{aligned}|\mathcal{K}_1(g,h,f)| &\lesssim |g\mu^{\frac{1}{2}}\mathrm{1}_{|\cdot|\leq\delta M}|_{L^1}|h\mathrm{1}_{|\cdot|\sim M}|_{L^2_{\gamma/2}}|f\chi_M|_{L^2_{\gamma/2}} \\ &\lesssim \varepsilon^s|g|_{L^2}|W^\varepsilon h|_{L^2_{\gamma/2}}|f\chi_M|_{L^2_{\gamma/2}}. \qquad (3.6)\end{aligned}$$

We turn to estimate $\mathcal{K}_2(g,h,f)$ in which $|v_*| \geq \delta M$. When $(\chi, M) = (\phi, 1/\varepsilon)$, the support of χ_M is in the ball $B_{\delta^{-1}M}$. In this case, we have

$$\begin{aligned}
&\mathcal{K}_2(g,h,f) = \mathcal{K}_{2,1} + \mathcal{K}_{2,2}, && (3.7)\\
&\mathcal{K}_{2,1} := \int B^{\varepsilon,\gamma} 1_{|v_*|\geq \delta M} 1_{\sin\frac{\theta}{2} \leq |v-v_*|^{-1}} (g\mu^{\frac{1}{2}})_* 1_{|v|\leq \delta^{-1}M} h f \chi_M ((\chi_M)' - \chi_M)\, \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v,\\
&\mathcal{K}_{2,2} := \int B^{\varepsilon,\gamma} 1_{|v_*|\geq \delta M} 1_{\sin\frac{\theta}{2} \geq |v-v_*|^{-1}} (g\mu^{\frac{1}{2}})_* 1_{|v|\leq \delta^{-1}M} h f \chi_M ((\chi_M)' - \chi_M)\, \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v.
\end{aligned}$$

By Taylor expansion (3.5) and (2.49), one has

$$\begin{aligned}
|\mathcal{K}_{2,1}| &\lesssim \left| \int |v-v_*|^{\gamma} 1_{|v_*|\geq \delta M} 1_{|v|\leq \delta^{-1}M} |(g\mu^{\frac{1}{2}})_* h f \chi_M| (|v-v_*|^{2s} + |v-v_*|^{2s-1})\, \mathrm{d}v_*\, \mathrm{d}v \right|\\
&\lesssim e^{-\delta^3 M^2} (|g\mu^{\frac{1}{4}} 1_{|\cdot|\geq \delta M}|_{L^1_{1+\gamma+2s}} + |g\mu^{\frac{1}{4}} 1_{|\cdot|\geq \delta M}|_{L^2_{1+\gamma+2s}})\\
&\quad \times |h 1_{|\cdot|\leq \delta^{-1}M}|_{L^2_{\gamma/2+s}} |f\chi_M|_{L^2_{\gamma/2+s}}\\
&\lesssim \varepsilon^s |g|_{L^2} |W^{\varepsilon} h|_{L^2_{\gamma/2}} |W^{\varepsilon} f \chi_M|_{L^2_{\gamma/2}}. \qquad (3.8)
\end{aligned}$$

For $\mathcal{K}_{2,2}$, since $\gamma + 2s \geq 0$, it is not difficult to check that

$$\begin{aligned}
|\mathcal{K}_{2,2}| &\lesssim e^{-\delta^3 M^2} |g\mu^{\frac{1}{4}} 1_{|\cdot|\geq \delta M}|_{L^1} |h 1_{|\cdot|\leq \delta^{-1}M}|_{L^2_{\gamma/2+s}} |f\chi_M|_{L^2_{\gamma/2+s}}\\
&\lesssim \varepsilon^s |g|_{L^2} |W^{\varepsilon} h|_{L^2_{\gamma/2}} |W^{\varepsilon} f \chi_M|_{L^2_{\gamma/2}}. \qquad (3.9)
\end{aligned}$$

When $(\chi, M) = (1-\phi, 1/\varepsilon)$ or $(\chi, M) = (\psi, 2^j)$, the support of χ_M is outside the ball $B_{\delta M}$ and so

$$\mathcal{K}_2(g,h,f) = \int B^{\varepsilon,\gamma} 1_{|v_*|\geq \delta M} 1_{|v|\geq \delta M} (g\mu^{\frac{1}{2}})_* h f \chi_M ((\chi_M)' - \chi_M)\, \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v.$$

When $|v-v_*| \geq 1$, then $|v-v_*|^{\gamma} \sim \langle v-v_* \rangle^{\gamma} \lesssim \langle v \rangle^{\gamma} \langle v_* \rangle^{|\gamma|}$. Using $\int b^{\varepsilon}(\cos\theta)\, \mathrm{d}\sigma \lesssim \varepsilon^{-2s}$, we get

$$\begin{aligned}
&\left| \int B^{\varepsilon,\gamma} 1_{|v_*|\geq \delta M} 1_{|v|\geq \delta M} 1_{|v-v_*|\geq 1} (g\mu^{\frac{1}{2}})_* h f \chi_M ((\chi_M)' - \chi_M)\, \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v \right|\\
&\quad \lesssim \varepsilon^{-2s} e^{-\delta^3 M^2} \int \langle v \rangle^{\gamma} 1_{|v_*|\geq \delta M} 1_{|v|\geq \delta M} |(g\mu^{\frac{1}{4}})_* h f \chi_M|\, \mathrm{d}v_*\, \mathrm{d}v\\
&\quad \lesssim \varepsilon^s |g|_{L^2} |W^{\varepsilon} h|_{L^2_{\gamma/2}} |W^{\varepsilon} f \chi_M|_{L^2_{\gamma/2}}. \qquad (3.10)
\end{aligned}$$

When $|v-v_*| \leq 1$, then $\mu_*^{\frac{1}{2}} \lesssim \mu_*^{\frac{1}{8}} \mu^{\frac{1}{8}}$. We can use decomposition (3.7) and similar arguments to get

$$\begin{aligned}
&\left| \int B^{\varepsilon,\gamma} 1_{|v_*|\geq \delta M} 1_{|v|\geq \delta M} 1_{|v-v_*|\leq 1} (g\mu^{\frac{1}{2}})_* h f \chi_M ((\chi_M)' - \chi_M)\, \mathrm{d}\sigma\, \mathrm{d}v_*\, \mathrm{d}v \right|\\
&\quad \lesssim \varepsilon^s |g|_{L^2} |W^{\varepsilon} h|_{L^2_{\gamma/2}} |W^{\varepsilon} f \chi_M|_{L^2_{\gamma/2}}. \qquad (3.11)
\end{aligned}$$

Patching together (3.8) and (3.9), patching together (3.10) and (3.11), for the three pairs of (χ, M) we conclude that

$$|\mathcal{K}_2(g,h,f)| \lesssim \varepsilon^s |g|_{L^2} |W^\varepsilon h|_{L^2_{\gamma/2}} |W^\varepsilon f \chi_M|_{L^2_{\gamma/2}}. \tag{3.12}$$

Patching together (3.6) and (3.12) we get

$$|\mathcal{K}(g,h,f)| \lesssim \varepsilon^s |g|_{L^2} |W^\varepsilon h|_{L^2_{\gamma/2}} |W^\varepsilon f \chi_M|_{L^2_{\gamma/2}}. \tag{3.13}$$

Plugging (3.4) and (3.13) into (3.3), we get (3.1).

Recalling (1.8), plugging (3.4) and (3.13) into (3.3), by taking $(g,h) = (\mu^{\frac{1}{2}}, f)$, we get

$$|\langle [\mathcal{L}_1^\varepsilon, \chi_M] f, f\chi_M\rangle_v| = |\mathcal{I}(\mu^{\frac{1}{2}}, f, f)| \lesssim \eta^{-1}\varepsilon^{2s} |f|^2_{\varepsilon,\gamma/2} + \eta |f\chi_M|^2_{\varepsilon,\gamma/2}. \tag{3.14}$$

Using (2.24) we get

$$\begin{aligned}
|\langle [\mathcal{L}_2^\varepsilon, \chi_M] f, f\chi_M\rangle_v| &= |\langle \mathcal{L}_2^\varepsilon \chi_M f, f\chi_M\rangle_v - \langle \mathcal{L}_2^\varepsilon f, \chi_M f \chi_M\rangle_v| \\
&= |\langle \mathcal{L}_2^\varepsilon \chi_M f, (1-\chi_M) f\chi_M\rangle_v - \langle \mathcal{L}_2^\varepsilon (1-\chi_M) f, \chi_M f \chi_M\rangle_v| \\
&\lesssim |\chi_M f|_{L^2_{\gamma/2}} |(1-\chi_M) f\chi_M|_{L^2_{\gamma/2}} + |(1-\chi_M) f|_{L^2_{\gamma/2}} |\chi_M f\chi_M|_{L^2_{\gamma/2}} \\
&\lesssim \varepsilon^s |W^\varepsilon f|_{L^2_{\gamma/2}} |W^\varepsilon f\chi_M|_{L^2_{\gamma/2}} \\
&\lesssim \eta^{-1}\varepsilon^{2s} |f|^2_{\varepsilon,\gamma/2} + \eta |f\chi_M|^2_{\varepsilon,\gamma/2}.
\end{aligned} \tag{3.15}$$

Patching together (3.14) and (3.15), we arrive at (3.2). ■

In the rest of this section, we set $f(t) = e^{-\mathcal{L}^\varepsilon t} f_0$ with $f_0 \in \mathcal{N}^\perp$. Then f verifies that $f(t) \in \mathcal{N}^\perp$ for any $t \geq 0$ and solves

$$\partial_t f + \mathcal{L}^\varepsilon f = 0, \quad f|_{t=0} = f_0. \tag{3.16}$$

Now we are in a position to prove (1.29) and (1.32) in Theorem 1.2.

Proof of Theorem 1.2 (part 1: (1.29) and (1.32)). We first prove (1.29). Since $f(t) = e^{-\mathcal{L}^\varepsilon t} f_0 \in \mathcal{N}^\perp$, by Proposition 2.4, there is a universal constant $\lambda_1 > 0$ such that $\frac{\mathrm{d}}{\mathrm{d}t}|f|^2_{L^2} + \lambda_1 |f|^2_{\varepsilon,\gamma/2} \leq 0$ and thus for any $t \geq 0$,

$$|f(t)|^2_{L^2} + \lambda_1 \int_0^t |f(\tau)|^2_{\varepsilon,\gamma/2} \,\mathrm{d}\tau \leq |f_0|^2_{L^2}. \tag{3.17}$$

Recall that $f^l(v) = \phi(\varepsilon v) f(v)$ and $f^h = f - f^l$. Recalling (3.16) we have

$$\begin{aligned}
\partial_t f^l + \mathcal{L}^\varepsilon f^l &= [\mathcal{L}^\varepsilon, \phi(\varepsilon \cdot)] f, \\
\partial_t f^h + \mathcal{L}^\varepsilon f^h &= [\mathcal{L}^\varepsilon, 1 - \phi(\varepsilon \cdot)] f.
\end{aligned}$$

Thanks to Theorem 1.1, Proposition 2.4 and that $|f^h|_{\varepsilon,\gamma/2} \gtrsim \varepsilon^{-s}|f^h|_{L^2_{\gamma/2}}$, we have

$$\langle \mathcal{L}^\varepsilon f^l, f^l\rangle_v \gtrsim |f^l|^2_{\varepsilon,\gamma/2} - C|f^l|^2_{L^2_{\gamma/2}}, \tag{3.18}$$

$$\langle \mathcal{L}^\varepsilon f^l, f^l\rangle_v \gtrsim |(\mathbb{I}-\mathbb{P})f^l|^2_{\varepsilon,\gamma/2} \geq |(\mathbb{I}-\mathbb{P})f^l|^2_{L^2_{\gamma/2}} \geq \frac{1}{2}|f^l|^2_{L^2_{\gamma/2}} - |\mathbb{P}f^l|^2_{L^2_{\gamma/2}}, \tag{3.19}$$

$$\langle \mathcal{L}^\varepsilon f^h, f^h\rangle_v \gtrsim |f^h|^2_{\varepsilon,\gamma/2} - C\varepsilon^{2s}|f^h|^2_{\varepsilon,\gamma/2}. \tag{3.20}$$

By (3.18), (3.19) and the identity $\mathbb{P}(f^l + f^h) = \mathbb{P}f = 0$, we derive that

$$\begin{aligned}\langle \mathcal{L}^\varepsilon f^l, f^l\rangle_v &\gtrsim |f^l|^2_{\varepsilon,\gamma/2} - C|\mathbb{P}f^l|^2_{L^2_{\gamma/2}} = |f^l|^2_{\varepsilon,\gamma/2} - C|\mathbb{P}f^h|^2_{L^2_{\gamma/2}} \\ &\geq |f^l|^2_{\varepsilon,\gamma/2} - C\varepsilon^{2s}|f^h|^2_{\varepsilon,\gamma/2}. \end{aligned}\tag{3.21}$$

Thanks to (3.2) in Lemma 3.1, (3.21) and (3.20), for some universal constant $\lambda_2 > 0$ we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|f^l|^2_{L^2} + \lambda_2|f^l|^2_{\varepsilon,\gamma/2} &\lesssim \varepsilon^{2s}(|f^h|^2_{\varepsilon,\gamma/2} + |f|^2_{\varepsilon,\gamma/2}) \lesssim \varepsilon^{2s}|f|^2_{\varepsilon,\gamma/2},\\ \frac{\mathrm{d}}{\mathrm{d}t}|f^h|^2_{L^2} + \lambda_2|f^h|^2_{\varepsilon,\gamma/2} &\lesssim \varepsilon^{2s}(|f^h|^2_{\varepsilon,\gamma/2} + |f|^2_{\varepsilon,\gamma/2}) \lesssim \varepsilon^{2s}|f|^2_{\varepsilon,\gamma/2}.\end{aligned}$$

Since $\gamma \geq -2s$, then $\lambda_2|f^l|^2_{\varepsilon,\gamma/2} \geq c|f^l|^2_{L^2}$ for some universal constant $c > 0$. Recalling (3.17), we get (1.29) by using Grönwall's inequality.

Next we want to prove (1.32). Recalling (3.16), applying the operator $\mathcal{P}_j$ with $2^j \geq 1/\varepsilon$ we have

$$\partial_t \mathcal{P}_j f + \mathcal{L}^\varepsilon \mathcal{P}_j f = [\mathcal{L}^\varepsilon, \psi(2^{-j}\cdot)]f.$$

Thanks to Theorem 1.1 and Lemma 3.1, for some constant $C_0 > 0$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathcal{P}_j f|^2_{L^2} + C_0|\mathcal{P}_j f|^2_{\varepsilon,\gamma/2} \gtrsim -\varepsilon^{2s}|f|^2_{\varepsilon,\gamma/2}.$$

Observe that $|W^\varepsilon \mathcal{P}_j f|^2_{L^2_{\gamma/2}} \sim \varepsilon^{-2s}2^{j\gamma}|\mathcal{P}_j f|^2_{L^2}$ and

$$|W^\varepsilon(D)W_{\gamma/2}\mathcal{P}_j f|^2_{L^2} + |W^\varepsilon((-\triangle_{\mathbb{S}^2})^{\frac{1}{2}})W_{\gamma/2}\mathcal{P}_j f|^2_{L^2} \lesssim \varepsilon^{-2s}2^{j\gamma}|\mathcal{P}_j f|^2_{L^2}.$$

We are led to

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathcal{P}_j f|^2_{L^2} \gtrsim -\varepsilon^{2s}|f|^2_{\varepsilon,\gamma/2} - \varepsilon^{-2s}2^{j\gamma}|\mathcal{P}_j f|^2_{L^2}.$$

From this, together with (3.17), we get $|\mathcal{P}_j f(t)|^2_{L^2} \geq |\mathcal{P}_j f_0|^2_{L^2} - C\varepsilon^{-2s}2^{j\gamma}t - C\varepsilon^{2s}$, which yields (1.32) for $t \in [0, C^{-1}\eta 2^{-j\gamma}\varepsilon^{2s}]$. ∎

To complete the proof of Theorem 1.2, we need the following proposition.

Proposition 3.1. *Let $c, c_1, c_2, q, Y_0 > 0$ be five positive constants. Consider the ordinary differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}Y + c_1 Y_1 + c_2 Y_2^{1+\frac{1}{q}} \leq 0, \quad Y|_{t=0} = Y_0, \tag{3.22}$$

where $Y \le c(Y_1 + Y_2)$ *and* $Y, Y_1, Y_2 \ge 0$. *Depending on the value of* Y_0, *we have the following two estimates:*

(1) *If* $Y_0 > 2c(c_1/c_2)^q$, *let* t_* *be the time such that* $Y(t_*) = 2c(c_1/c_2)^q$, *then for any* $t \ge 0$,

$$Y(t) \le 1_{t<t_*} Y_0 \exp\Big(-\frac{c_1}{2c}t\Big) + 1_{t\ge t_*} \frac{Y_*}{(1 + C_1(t - t_*))^q}, \tag{3.23}$$

where $Y_* = Y(t_*), C_1 = \frac{c_1}{2cq}$. *Moreover, the critical time verifies* $t_* \le \frac{2c}{c_1} \ln \frac{Y_0}{Y_*}$.

(2) *If* $Y_0 \le 2c(c_1/c_2)^q$, *then for any* $t \ge 0$,

$$Y(t) \le \frac{Y_0}{(1 + C_2 t)^q}, \tag{3.24}$$

where $C_2 = \frac{c_2}{2cq}(\frac{Y_0}{2c})^{1/q}$.

Proof. It is easy to check that $Y(t)$ is a strictly decreasing function before it vanishes. When $Y_0 > 2c(c_1/c_2)^q$, since $Y(t_*) = 2c(c_1/c_2)^q$, we have

$$c_1 \frac{Y(t_*)}{2c} = c_2 \Big(\frac{Y(t_*)}{2c}\Big)^{1+\frac{1}{q}}. \tag{3.25}$$

Since $Y \le c(Y_1 + Y_2)$, one has $\max\{Y_1, Y_2\} \ge \frac{1}{2c} Y$. Then we deduce

$$\begin{aligned} c_1 Y_1 + c_2 Y_2^{1+\frac{1}{q}} &\ge \max\big\{c_1 Y_1, c_2 Y_2^{1+\frac{1}{q}}\big\} \\ &\ge \min\Big\{c_1 \frac{Y}{2c}, c_2 \Big(\frac{Y}{2c}\Big)^{1+\frac{1}{q}}\Big\} = \begin{cases} c_1 \dfrac{Y}{2c}, & t < t_*, \\ c_2 \Big(\dfrac{Y}{2c}\Big)^{1+\frac{1}{q}}, & t \ge t_*. \end{cases} \end{aligned} \tag{3.26}$$

Note that the last equality employs (3.25), which is also the reason for our choice of t_*. When $t < t_*$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} Y + \frac{c_1}{2c} Y \le 0 \Rightarrow Y(t) \le Y_0 \exp\Big(-\frac{c_1}{2c}t\Big). \tag{3.27}$$

On the interval $[t_*, \infty)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} Y + c_2 \Big(\frac{Y}{2c}\Big)^{1+\frac{1}{q}} \le 0 \Rightarrow Y(t) \le \frac{Y(t_*)}{(1 + C_1(t - t_*))^q}, \tag{3.28}$$

with $C_1 = \frac{c_2}{2cq}(\frac{Y(t_*)}{2c})^{1/q} = \frac{c_1}{2cq}$. Patching together (3.27) and (3.28), we conclude (3.23). Since $Y(t)$ is a strictly decreasing function before it vanishes, we have

$$2c(c_1/c_2)^q = Y(t_*) \le \lim_{t\to t_*^-} Y(t) \le Y_0 \exp\Big(-\frac{c_1}{2c}t_*\Big) \Rightarrow t_* \le \frac{2c}{c_1} \ln \frac{Y_0}{Y_*}.$$

If $Y_0 \le 2c(c_1/c_2)^q$, then for any $t \ge 0$,

$$c_1 \frac{Y(t)}{2c} \ge c_2 \Big(\frac{Y(t)}{2c}\Big)^{1+\frac{1}{q}}.$$

Similar to (3.26), we have $c_1 Y_1 + c_2 Y_2^{1+\frac{1}{q}} \geq c_2(\frac{Y}{2c})^{1+\frac{1}{q}}$ on the interval $[0, \infty)$ and thus

$$\frac{\mathrm{d}}{\mathrm{d}t} Y + c_2\Big(\frac{Y}{2c}\Big)^{1+\frac{1}{q}} \leq 0,$$

which yields (3.24). The proof is complete now. ∎

The following example shows that the structure of estimate (3.23) is sharp for (3.22).

Example 3.1. Take $c = c_1 = q = Y(0) = 1$ and $c_2 = \varepsilon^{-2s}$ in Proposition 3.1. Then (3.22) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t} Y + Y_1 + \varepsilon^{-2s} Y_2^2 = 0, \quad Y|_{t=0} = 1,$$

where $Y_1 + Y_2 \geq Y$ and $Y_1, Y_2, Y \geq 0$. Here $\varepsilon > 0$ is sufficiently small. Let us impose $Y_1 + Y_2 = Y$ and $Y_1 = \varepsilon^{-2s} Y_2^2$. Then

$$e^{-6t} 1_{t<t_*} + \varepsilon^{2s} \frac{1}{4 + 3(t - t_*)} 1_{t \geq t_*} \leq Y(t) \leq e^{-t/4} 1_{t<t_*} + \varepsilon^{2s} \frac{1}{4 + (t - t_*)} 1_{t \geq t_*}, \tag{3.29}$$

where t_* is the critical time such that $Y(t_*) = \frac{1}{4}\varepsilon^{2s}$ verifying $t_* \sim \ln \frac{1}{Y(t_*)}$.

Proof. Note that we can solve $\varepsilon^{-2s} Y_2^2 + Y_2 = Y$ to get $Y_2 = \frac{-1+\sqrt{1+4\varepsilon^{-2s}Y}}{2\varepsilon^{-2s}}$ and thus

$$Y_1 + \varepsilon^{-2s} Y_2^2 = 2\varepsilon^{-2s} Y_2^2 = \frac{1 + 2\varepsilon^{-2s} Y - \sqrt{1 + 4\varepsilon^{-2s} Y}}{\varepsilon^{-2s}}.$$

Now let $X = \varepsilon^{-2s} Y$. Then we have the following ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t} X + 1 + 2X - \sqrt{1 + 4X} = 0, \quad X|_{t=0} = \varepsilon^{-2s}.$$

If we set $f(x) = 1 + 2x - \sqrt{1 + 4x}$, then one has $f'(x) = 2 - 2(1 + 4x)^{-\frac{1}{2}}$, $f''(x) = 4(1 + 4x)^{-\frac{3}{2}}$, $f^{(3)}(x) = -24(1 + 4x)^{-5/2}$, $f^{(4)}(x) = 240(1 + 4x)^{-7/2}$. By Taylor expansion, one has

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2} x^2 + \frac{f^{(3)}(0)}{6} x^3 + \frac{1}{6} \int_0^x (x - t)^3 f^{(4)}(t)\, \mathrm{d}t \\ &= 2x^2 - 4x^3 + \frac{1}{6} \int_0^x (x - t)^3 f^{(4)}(t)\, \mathrm{d}t. \end{aligned}$$

Since $0 \leq f^{(4)}(t) \leq 240$, we have $2x^2 - 4x^3 \leq f(x) \leq 2x^2 - 4x^3 + 10x^4$. If $x \leq \frac{1}{4}$, then $4x^3 \leq x^2$ and $10x^4 \leq x^2$, which gives

$$x^2 \leq 1 + 2x - \sqrt{1 + 4x} \leq 3x^2, \quad x \leq \tfrac{1}{4}. \tag{3.30}$$

Let $g(x) = f(x) - x/4$; if $x \geq \frac{1}{4}$, then $g'(x) = \frac{7}{4} - 2(1 + 4x)^{-\frac{1}{2}} \geq \frac{7}{4} - \sqrt{2} > 0$, which yields

$$g(x) \geq g(\tfrac{1}{4}) = \tfrac{3}{2} - \sqrt{2} - \tfrac{1}{16} > 0 \;\Rightarrow\; f(x) \geq x/4.$$

If $x \geq \frac{1}{4}$, then $1 + 2x \leq 6x$. Therefore we have

$$x/4 \leq 1 + 2x - \sqrt{1+4x} \leq 6x, \quad x \geq \tfrac{1}{4}. \tag{3.31}$$

Note that t_* is the critical time such that $X(t_*) = \frac{1}{4}$; then by (3.31) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}X + X/4 \leq \frac{\mathrm{d}}{\mathrm{d}t}X + 1 + 2X - \sqrt{1+4X} = 0 \leq \frac{\mathrm{d}}{\mathrm{d}t}X + 6X, \quad t \leq t_*,$$

which yields $-6 \leq \frac{\mathrm{d}}{\mathrm{d}t}\ln X \leq -\frac{1}{4}, \quad t \leq t_*$. Integrating over $[0, t]$ and recalling $X(0) = \varepsilon^{-2s}$, we have

$$\varepsilon^{-2s}\exp(-6t) \leq X(t) \leq \varepsilon^{-2s}\exp(-t/4), \quad t \leq t_*. \tag{3.32}$$

By (3.30), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}X + X^2 \leq \frac{\mathrm{d}}{\mathrm{d}t}X + 1 + 2X - \sqrt{1+4X} = 0 \leq \frac{\mathrm{d}}{\mathrm{d}t}X + 3X^2, \quad t \geq t_*,$$

which indicates

$$-3 \leq \frac{\mathrm{d}}{\mathrm{d}t}\Big(-\frac{1}{X}\Big) \leq -1, \quad t \geq t_*.$$

Integrating over $[t_*, t]$, we have

$$\frac{1}{4 + 3(t - t_*)} \leq X(t) \leq \frac{1}{4 + (t - t_*)}, \quad t \geq t_*. \tag{3.33}$$

Recalling that $X = \varepsilon^{-2s}Y$, patching together (3.32) and (3.33), we get (3.29). By (3.32), recalling $X(t_*) = \frac{1}{4}$, we have

$$\frac{-2s\ln\varepsilon - \ln\frac{1}{4}}{6} \leq t_* \leq 4(-2s\ln\varepsilon - \ln\tfrac{1}{4}),$$

which yields $t_* \sim -2s\ln\varepsilon \sim \ln\frac{1}{Y(t_*)}$ since ε is small enough. ■

We are ready to complete the proof of Theorem 1.2.

Proof of Theorem 1.2 *(part 2:* (1.30) *and* (1.31)*).* By Theorem 1.1, Lemma 2.8 and (2.24), for $l \geq 2$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|f|^2_{L^2_l} + \lambda_3|f|^2_{\varepsilon,\gamma/2+l} \lesssim |f|^2_{L^2_{l+\gamma/2}},$$

for some universal constant $\lambda_3 > 0$. Observe that

$$|f|^2_{L^2_{l+\gamma/2}} \lesssim |f^h|^2_{L^2_{l+\gamma/2}} + \eta|f^l|^2_{L^2_{\gamma/2+l+s}} + C_\eta|f^l|^2_{L^2_{\gamma/2+s}}.$$

By taking η small enough, when $\varepsilon > 0$ is small enough we infer that $\frac{\mathrm{d}}{\mathrm{d}t}|f|^2_{L^2_l} \lesssim |f^l|^2_{L^2_{\gamma/2+s}} \lesssim |f|^2_{\varepsilon,\gamma/2}$. Recalling (3.17), we have $|f(t)|^2_{L^2_l} \lesssim |f_0|^2_{L^2_l}$ for any $t \geq 0$. Recalling that $\frac{\mathrm{d}}{\mathrm{d}t}|f|^2_{L^2} + \lambda_1|f|^2_{\varepsilon,\gamma/2} \leq 0$ and using the interpolation inequality

$$|f|_{L^2} \leq |f|^{\frac{q}{q+1}}_{L^2_{\gamma/2}}|f|^{\frac{1}{q+1}}_{L^2_{-\gamma q/2}},$$

since $\gamma + 2s \geq 0$, for some universal constants $\widetilde{C}_1, \widetilde{C}_2$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}|f|_{L^2}^2 + \widetilde{C}_1|f^l|_{L^2}^2 + \widetilde{C}_2|f_0|_{L^2_{-\gamma q/2}}^{-2/q}\varepsilon^{-2s}|f^h|_{L^2}^{2+\frac{2}{q}} \leq 0.$$

Let $Y(t) := |f(t)|_{L^2}^2$, $Y_1(t) := |f^l(t)|_{L^2}^2$, $Y_2(t) := |f^h(t)|_{L^2}^2$. We obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}Y + c_1 Y_1 + c_2 Y_2^{1+\frac{1}{q}} \leq 0,$$

where $c_1 = \widetilde{C}_1$, $c_2 = \widetilde{C}_2|f_0|_{L^2_{-\gamma q/2}}^{-2/q}\varepsilon^{-2s}$. Noting that $Y \leq 2(Y_1 + Y_2)$, by taking $c = 2$ in Proposition 3.1, we get (1.30) and (1.31) with

$$C_0 = \sqrt{\widetilde{C}_1/\widetilde{C}_2}, \quad \lambda_0 = \frac{\widetilde{C}_1}{4}, \quad C_1 = \frac{\widetilde{C}_2}{4}. \tag{3.34}$$

Now the proof of Theorem 1.2 is complete. ∎

4. Nonlinear Boltzmann equation in the perturbation framework

In this section we will prove Theorem 1.3. In Section 4.1 we establish global well-posedness and propagation of regularity for the Cauchy problem (1.9). In Section 4.2 we derive global dynamics by using Proposition 3.1. Section 4.3 is devoted to the global asymptotic formula (1.43).

4.1. Global well-posedness and propagation of regularity

The main task is to provide a priori estimates for equation (1.9). We start with the following linear equation:

$$\partial_t f + v \cdot \nabla_x f + \mathscr{L}^\varepsilon f = g, \quad t > 0,\ x \in \mathbb{T}^3,\ v \in \mathbb{R}^3. \tag{4.1}$$

Here g is given and f is unknown.

4.1.1. Estimate for the linear equation. Suppose f is a solution to (4.1). Recalling (1.21), we set $f_1 := \mathbb{P}f = (a + b \cdot v + c|v|^2)\mu^{\frac{1}{2}}$ and $f_2 := f - \mathbb{P}f$. We derive the a priori estimate for (4.1) in the following proposition.

Proposition 4.1. *Let $N \geq 1$ and f be a solution to (4.1). Then for M large enough, it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{N,M}(f) + \frac{1}{2}(|\nabla_x(a,b,c)|_{H_x^{N-1}}^2 + \|f_2\|_{H_x^N L^2_{\varepsilon,\gamma/2}}^2)$$
$$\lesssim \sum_{|\alpha|\leq N}|(\partial^\alpha g, \partial^\alpha f)| + \sum_{|\alpha|\leq N-1}\sum_{j=1}^{13}\int_{\mathbb{T}^3}|\langle\partial^\alpha g, e_j\rangle_v|^2\,\mathrm{d}x, \tag{4.2}$$

where

$$\mathcal{E}_{N,M}(f) := M\|f\|^2_{H^N_x L^2} + \mathcal{I}_N(f),$$
$$\frac{1}{2}M\|f\|^2_{H^N_x L^2} \leq \mathcal{E}_{N,M}(f) \leq 2M\|f\|^2_{H^N_x L^2}. \tag{4.3}$$

Here $\mathcal{I}_N(f)$ *is defined in* (4.8) *and* $\{e_j\}_{1\leq j\leq 13}$ *are defined explicitly by*

$$e_1 = \mu^{\frac{1}{2}}, \quad e_2 = v_1\mu^{\frac{1}{2}}, \quad e_3 = v_2\mu^{\frac{1}{2}}, \quad e_4 = v_3\mu^{\frac{1}{2}}, \quad e_5 = v_1^2\mu^{\frac{1}{2}},$$
$$e_6 = v_2^2\mu^{\frac{1}{2}}, \quad e_7 = v_3^2\mu^{\frac{1}{2}}, \quad e_8 = v_1v_2\mu^{\frac{1}{2}}, \quad e_9 = v_2v_3\mu^{\frac{1}{2}},$$
$$e_{10} = v_3v_1\mu^{\frac{1}{2}}, \quad e_{11} = |v|^2v_1\mu^{\frac{1}{2}}, \quad e_{12} = |v|^2v_2\mu^{\frac{1}{2}}, \quad e_{13} = |v|^2v_3\mu^{\frac{1}{2}}.$$

The proof of Proposition 4.1 will be postponed a while. We first recall some basics of macro–micro decomposition. By (1.21) the macro part f_1 is given by

$$f_1(t,x,v) = (a(t,x) + b(t,x)\cdot v + c(t,x)|v|^2)\mu^{\frac{1}{2}}, \tag{4.4}$$

which solves

$$\partial_t f_1 + v\cdot\nabla_x f_1 = -\partial_t f_2 + l + g, \tag{4.5}$$

where $l = -v\cdot\nabla_x f_2 - \mathcal{L}^\varepsilon f_2$.

Let $A = (a_{ij})_{1\leq i\leq 13, 1\leq j\leq 13}$ be the 13×13 matrix defined by $a_{ij} = \langle e_i, e_j\rangle_v$ and y be the column vector with 13 components $\partial_t a$, $\{\partial_t b_i + \partial_i a\}_{1\leq i\leq 3}$, $\{\partial_t c + \partial_i b_i\}_{1\leq i\leq 3}$, $\{\partial_i b_j + \partial_j b_i\}_{1\leq i<j\leq 3}$, $\{\partial_i c\}_{1\leq i\leq 3}$. Let e be the column vector with 13 components $\{e_j\}_{j=1}^{13}$. Plugging (4.4) into (4.5), we get

$$e\cdot y = -\partial_t f_2 + l + g. \tag{4.6}$$

Define a column vector $z = (z_i)_{i=1}^{13} := (\langle -\partial_t f_2 + l + g, e_i\rangle_v)_{i=1}^{13}$. Taking the inner product between (4.6) and the column vector e in the space $L^2(\mathbb{R}^3_v)$, one has $Ay = z$. For simplicity, we define the following column vectors:

$$\tilde{f} = (\tilde{f}^{(0)}, \{\tilde{f}^{(1)}_i\}_{1\leq i\leq 3}, \{\tilde{f}^{(2)}_i\}_{1\leq i\leq 3}, \{\tilde{f}^{(2)}_{ij}\}_{1\leq i<j\leq 3}, \{\tilde{f}^{(3)}_i\}_{1\leq i\leq 3})^{\mathsf{T}} := A^{-1}\langle f_2, e\rangle_v,$$
$$\tilde{l} = (l^{(0)}, \{l^{(1)}_i\}_{1\leq i\leq 3}, \{l^{(2)}_i\}_{1\leq i\leq 3}, \{l^{(2)}_{ij}\}_{1\leq i<j\leq 3}, \{l^{(3)}_i\}_{1\leq i\leq 3})^{\mathsf{T}} := A^{-1}\langle l, e\rangle_v,$$
$$\tilde{g} = (g^{(0)}, \{g^{(1)}_i\}_{1\leq i\leq 3}, \{g^{(2)}_i\}_{1\leq i\leq 3}, \{g^{(2)}_{ij}\}_{1\leq i<j\leq 3}, \{g^{(3)}_i\}_{1\leq i\leq 3})^{\mathsf{T}} := A^{-1}\langle g, e\rangle_v.$$

Here T denotes vector transpose. Then the equation $Ay = z$ is equivalent to

$$y = A^{-1}z = -\partial_t\tilde{f} + \tilde{l} + \tilde{g}. \tag{4.7}$$

Following the notation in [6], let us define the temporal energy functional $\mathcal{I}_N(f)$ as

$$\mathcal{I}_N(f) := \sum_{|\alpha|\leq N-1}\sum_{i=1}^{3}(\mathcal{I}^a_{\alpha,i}(f) + \mathcal{I}^b_{\alpha,i}(f) + \mathcal{I}^c_{\alpha,i}(f) + \mathcal{I}^{ab}_{\alpha,i}(f)), \tag{4.8}$$

where

$$\mathcal{I}^a_{\alpha,i}(f) := \langle \partial^\alpha \tilde{f}^{(1)}_i, \partial_i \partial^\alpha a\rangle_x, \quad \mathcal{I}^c_{\alpha,i}(f) := \langle \partial^\alpha \tilde{f}^{(3)}_i, \partial_i \partial^\alpha c\rangle_x,$$
$$\mathcal{I}^{ab}_{\alpha,i}(f) := \langle \partial_i \partial^\alpha a, \partial^\alpha b_i\rangle_x, \tag{4.9}$$

$$\mathcal{I}^b_{\alpha,i}(f) := -\sum_{j\neq i}\langle \partial^\alpha \tilde{f}^{(2)}_j, \partial_i \partial^\alpha b_i\rangle_x + \sum_{j\neq i}\langle \partial^\alpha \tilde{f}^{(2)}_{ji}, \partial_j \partial^\alpha b_i\rangle_x$$
$$+ 2\langle \partial^\alpha \tilde{f}^{(2)}_i, \partial_i \partial^\alpha b_i\rangle_x. \tag{4.10}$$

In the following we give a lemma for the dissipation of (a, b, c).

Lemma 4.1. *Let $N \geq 1$. Recall that $e = \{e_j\}_{j=1}^{13}$. There exists a constant $C > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_N(f) + \frac{1}{2}|\nabla_x(a,b,c)|^2_{H^{N-1}_x} \leq C\left(\|f_2\|^2_{H^N_x L^2_{\varepsilon,\gamma/2}} + \sum_{|\alpha|\leq N-1}\int |\langle \partial^\alpha g, e\rangle_v|^2\,\mathrm{d}x\right). \tag{4.11}$$

The proof of Lemma 4.1 will be given in the appendix. Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. Applying ∂^α to equation (4.1), taking the inner product with $\partial^\alpha f$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha f\|^2_{L^2} + (\mathcal{L}^\varepsilon \partial^\alpha f, \partial^\alpha f) = (\partial^\alpha g, \partial^\alpha f).$$

Thanks to Proposition 2.4, for some constant $c_0 > 0$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{H^N_x L^2} + c_0\|f_2\|^2_{H^N_x L^2_{\varepsilon,\gamma/2}} \leq \sum_{|\alpha|\leq N}|(\partial^\alpha g, \partial^\alpha f)|. \tag{4.12}$$

Then (4.2) follows by making a suitable combination of (4.12) and (4.11). More precisely, one can multiply (4.12) by a large constant and then add the resultant to (4.11). Inequality (4.3) is a direct result of the fact that $|\mathcal{I}_N(f)| \lesssim \|f\|^2_{H^N_x L^2}$. ■

4.1.2. A priori estimate in $H^N_x L^2$. In this subsection we derive the a priori estimate in $H^N_x L^2$ for solutions to the Cauchy problem (1.9). We apply Proposition 4.1 by taking $g = \Gamma^\varepsilon(f, f)$. For ease of notation, let us define the energy and dissipation functionals

$$\mathcal{E}_N(f) := \|f\|^2_{H^N_x L^2}, \quad \mathcal{D}_N(f) := \|f\|^2_{H^N_x L^2_{\varepsilon,\gamma/2}}.$$

The a priori estimate in $H^N_x L^2$ can be concluded as follows.

Theorem 4.1. *Let $-\frac{3}{2} < \gamma < 0$, $N \geq 2$. There exists $\delta_0 > 0$ independent of ε such that if a solution f^ε to the Cauchy problem* (1.9) *satisfies $\sup_{0\leq t\leq T}\mathcal{E}_2(f^\varepsilon(t)) \leq \delta_0$ for some $0 < T \leq \infty$, then*

$$\sup_{t\in[0,T]}\mathcal{E}_N(f^\varepsilon(t)) + \int_0^T \mathcal{D}_N(f^\varepsilon(s))\,\mathrm{d}s \leq C(\mathcal{E}_N(f_0)),$$

where $C(\cdot)$ is a continuous increasing function verifying $C(0) = 0$. When $N = 2$, $C(x) \lesssim x$.

Proof. Let $(a^\varepsilon(t,x), b^\varepsilon(t,x), c^\varepsilon(t,x))$ be the macroscopic components of $f^\varepsilon(t,x,\cdot)$ defined through (1.22). Let $f_1^\varepsilon := \mathbb{P} f^\varepsilon(t,x,v) = (a^\varepsilon(t,x) + b^\varepsilon(t,x)\cdot v + c^\varepsilon(t,x)|v|^2)\mu^{\frac{1}{2}}$, $f_2^\varepsilon := f^\varepsilon - f_1^\varepsilon$. Thanks to (1.6) and (1.11), we have $\int (a^\varepsilon(t,x), b^\varepsilon(t,x), c^\varepsilon(t,x))\,\mathrm{d}x = 0$. By the Poincaré inequality, $|(a^\varepsilon, b^\varepsilon, c^\varepsilon)|_{H^N_x} \sim |\nabla_x(a^\varepsilon, b^\varepsilon, c^\varepsilon)|_{H^{N-1}_x}$ and thus for some universal constant $c_0 > 0$,

$$\frac{1}{2}(|\nabla_x(a^\varepsilon, b^\varepsilon, c^\varepsilon)|^2_{H^{N-1}_x} + \|f_2^\varepsilon\|^2_{H^N_x L^2_{\varepsilon,\gamma/2}}) \geq c_0\|f^\varepsilon\|^2_{H^N_x L^2_{\varepsilon,\gamma/2}}. \tag{4.13}$$

By Proposition 4.1, we need to estimate the quantities $|(\partial^\alpha \Gamma^\varepsilon(f^\varepsilon, f^\varepsilon), \partial^\alpha f^\varepsilon)|$ and $\int |\langle \partial^\alpha \Gamma^\varepsilon(f^\varepsilon, f^\varepsilon), e\rangle_v|^2\,\mathrm{d}x$ for $|\alpha| \leq N$. In this sequel, we denote by $\hat{f}$ the Fourier transform of f with respect to the x variable. Observe that

$$(\Gamma^\varepsilon(g,h), f) = \sum_{k,m\in\mathbb{Z}^3} \langle \Gamma^\varepsilon(\hat{g}(k), \hat{h}(m-k)), \hat{f}(m)\rangle_v.$$

From this, together with Theorem 2.3, we get

$$|(\Gamma^\varepsilon(\partial^\alpha_x g, \partial^\beta_x h), f)| \lesssim \sum_{k,m\in\mathbb{Z}^3} |k|^{|\alpha|}|m-k|^{|\beta|}|\hat{g}(k)|_{L^2}|\hat{h}(m-k)|_{\varepsilon,\gamma/2}|\hat{f}(m)|_{\varepsilon,\gamma/2}.$$

From this, we derive that for $a, b \geq 0$ with $a + b > \frac{3}{2}$,

$$|(\Gamma^\varepsilon(\partial^\alpha_x g, \partial^\beta_x h), f)| \lesssim \|g\|_{H^{|\alpha|+a}_x L^2}\|h\|_{H^{|\beta|+b}_x L^2_{\varepsilon,\gamma/2}}\|f\|_{L^2_{\varepsilon,\gamma/2}}. \tag{4.14}$$

As a result, for $|\alpha| \leq N$,

$$\begin{aligned}|(\partial^\alpha \Gamma^\varepsilon(g,h), f)| &\lesssim \|g\|_{H^2_x L^2}\|h\|_{H^N_x L^2_{\varepsilon,\gamma/2}}\|f\|_{L^2_{\varepsilon,\gamma/2}} \\ &\quad + 1_{N\geq 3}\|g\|_{H^N_x L^2}\|h\|_{H^{N-1}_x L^2_{\varepsilon,\gamma/2}}\|f\|_{L^2_{\varepsilon,\gamma/2}}.\end{aligned} \tag{4.15}$$

Taking the sum over $|\alpha| \leq N$ we have

$$\begin{aligned}\sum_{|\alpha|\leq N} |(\partial^\alpha \Gamma^\varepsilon(f^\varepsilon, f^\varepsilon), \partial^\alpha f^\varepsilon)| &\lesssim \sqrt{\mathcal{E}_2(f^\varepsilon)}\mathcal{D}_N(f^\varepsilon) \\ &\quad + 1_{N\geq 3}\sqrt{\mathcal{E}_N(f^\varepsilon)}\sqrt{\mathcal{D}_{N-1}(f^\varepsilon)}\sqrt{\mathcal{D}_N(f^\varepsilon)}.\end{aligned} \tag{4.16}$$

Thanks to Theorem 2.3, estimate (4.14), similar to (4.15) and (4.16), we have for $|\alpha| \leq N$, $1 \leq j \leq 13$,

$$\int |\langle \partial^\alpha \Gamma^\varepsilon(f^\varepsilon, f^\varepsilon), e_j\rangle_v|^2\,\mathrm{d}x \lesssim \mathcal{E}_2(f^\varepsilon)\mathcal{D}_N(f^\varepsilon) + 1_{N\geq 3}\mathcal{E}_N(f^\varepsilon)\mathcal{D}_{N-1}(f^\varepsilon). \tag{4.17}$$

Recall that $\mathcal{E}_{N,M}(f^\varepsilon) \sim \mathcal{E}_N(f^\varepsilon)$ by (4.3). Invoking Proposition 4.1, using (4.13), (4.16) and (4.17), for any $0 < \eta < 1$ we arrive at

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{N,M}(f^\varepsilon) + c_0\mathcal{D}_N(f^\varepsilon) &\lesssim (\sqrt{\mathcal{E}_2(f^\varepsilon)} + \mathcal{E}_2(f^\varepsilon) + 1_{N\geq 3}\eta)\mathcal{D}_N(f^\varepsilon) \\ &\quad + 1_{N\geq 3}\eta^{-1}\mathcal{E}_N(f^\varepsilon)\mathcal{D}_{N-1}(f^\varepsilon).\end{aligned} \tag{4.18}$$

For $N = 2$, if δ_0 is sufficiently small, under the condition $\sup_{0\le t\le T} \mathcal{E}_2(f^\varepsilon(t)) \le \delta_0$ we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{2,M}(f^\varepsilon) + \frac{c_0}{2}\mathcal{D}_2(f^\varepsilon) \le 0, \\ &\sup_{t\in[0,T]} \mathcal{E}_2(f^\varepsilon(t)) + \int_0^T \mathcal{D}_2(f^\varepsilon(s))\,\mathrm{d}s \lesssim \mathcal{E}_2(f_0). \end{aligned} \tag{4.19}$$

For $N \ge 3$, taking η small enough, by the smallness assumption $\sup_{0\le t\le T} \mathcal{E}_2(f^\varepsilon(t)) \le \delta_0$, (4.18) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{N,M}(f^\varepsilon) + \frac{c_0}{4}\mathcal{D}_N(f^\varepsilon) \lesssim \mathcal{E}_N(f^\varepsilon)\mathcal{D}_{N-1}(f^\varepsilon).$$

Then we can get the desired result by using mathematical induction. ■

4.1.3. Propagation of the weighted Sobolev regularity $H_x^N L_l^2$. We aim to prove the following proposition.

Proposition 4.2. *Let $-\frac{3}{2} < \gamma < 0$, $l \ge 2$, $N \ge 2$. There exists $\delta_0 > 0$ independent of ε such that if a solution f^ε to the Cauchy problem (1.9) satisfies $\sup_{0\le t\le T} \mathcal{E}_2(f^\varepsilon(t)) \le \delta_0$ for some $0 < T \le \infty$, then*

$$\sup_{t\in[0,T]} \|f^\varepsilon(t)\|^2_{H_x^N L_l^2} + \int_0^T \|f^\varepsilon(s)\|^2_{H_x^N L^2_{\varepsilon,l+\gamma/2}}\,\mathrm{d}s \le C(\|f_0\|^2_{H_x^N L_l^2}),$$

where $C(\cdot)$ is a continuous increasing function verifying $C(0) = 0$. When $N = 2$, $C(x) \lesssim x$.

Proof. We omit the superscript ε in f^ε to write

$$\partial_t f + v\cdot\nabla_x f + \mathcal{L}^\varepsilon f = \Gamma^\varepsilon(f, f). \tag{4.20}$$

Applying $W_l\partial^\alpha$ to both sides of (4.20) we have

$$\partial_t W_l\partial^\alpha f + v\cdot\nabla_x W_l\partial^\alpha f + W_l\mathcal{L}^\varepsilon\partial^\alpha f = W_l\partial^\alpha\Gamma^\varepsilon(f, f).$$

Taking the inner product with $W_l\partial^\alpha f$ and taking the sum over $|\alpha| \le N$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{H_x^N L_l^2} + \sum_{|\alpha|\le N}(W_l\mathcal{L}^\varepsilon\partial^\alpha f, W_l\partial^\alpha f) = \sum_{|\alpha|\le N}(W_l\partial^\alpha\Gamma^\varepsilon(f, f), W_l\partial^\alpha f).$$

By Theorem 1.1, Lemma 2.8 and the condition $\gamma/2 + l \ge 0$, for some constant $c_0 > 0$ we have

$$\sum_{|\alpha|\le N}(W_l\mathcal{L}^\varepsilon\partial^\alpha f, W_l\partial^\alpha f) \ge c_0\|f\|^2_{H_x^N L^2_{\varepsilon,l+\gamma/2}} - C\|f\|^2_{H_x^N L^2_{l+\gamma/2}}.$$

Observe that

$$\begin{aligned} \sum_{|\alpha|\le N}(W_l\partial^\alpha\Gamma^\varepsilon(f, f), W_l\partial^\alpha f) &= \sum_{|\alpha|\le N}(W_l\partial^\alpha\Gamma^\varepsilon(f, f) - \partial^\alpha\Gamma^\varepsilon(f, W_l f), W_l\partial^\alpha f) \\ &\quad + \sum_{|\alpha|\le N}(\partial^\alpha\Gamma^\varepsilon(f, W_l f), W_l\partial^\alpha f). \end{aligned}$$

With the help of the proof of (4.15), Theorem 2.3 and Lemma 2.8 imply that

$$\sum_{|\alpha|\le N} |(W_l\partial^\alpha\Gamma^\varepsilon(f,f), W_l\partial^\alpha f)|$$
$$\lesssim \sqrt{\mathcal{E}_2(f)}\|f\|^2_{H^N_x L^2_{\varepsilon,l+\gamma/2}} + 1_{N\ge3}\sqrt{\mathcal{E}_N(f)}\|f\|_{H^{N-1}_x L^2_{\varepsilon,l+\gamma/2}}\|f\|_{H^N_x L^2_{\varepsilon,l+\gamma/2}}.$$

Putting together the above results and using the condition $\sup_{0\le t\le T}\mathcal{E}_2(f^\varepsilon(t)) \le \delta_0$ with δ_0 small enough, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{H^N_x L^2_l} + \frac{c_0}{2}\|f\|^2_{H^N_x L^2_{\varepsilon,l+\gamma/2}} \lesssim \|f\|^2_{H^N_x L^2_{l+\gamma/2}} + 1_{N\ge3}\mathcal{E}_N(f)\|f\|^2_{H^{N-1}_x L^2_{\varepsilon,l+\gamma/2}}.$$

It is not difficult to check that

$$\begin{aligned}\|f\|_{H^N_x L^2_{l+\gamma/2}} &\le \|f^l\|_{H^N_x L^2_{l+\gamma/2}} + \|f^h\|_{H^N_x L^2_{l+\gamma/2}}\\ &\le \eta\|f^l\|_{H^N_x L^2_{l+\gamma/2+s}} + C_\eta\|f^l\|_{H^N_x L^2_{\gamma/2+s}} + \varepsilon^s\|\varepsilon^{-s}f^h\|_{H^N_x L^2_{l+\gamma/2}}\\ &\lesssim (\eta+\varepsilon^s)\|f\|^2_{H^N_x L^2_{\varepsilon,l+\gamma/2}} + C_\eta\mathcal{D}_N(f).\end{aligned}$$

Taking η small enough, when ε is small we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{H^N_x L^2_l} + \frac{c_0}{4}\|f\|^2_{H^N_x L^2_{\varepsilon,l+\gamma/2}} \lesssim \mathcal{D}_N(f) + 1_{N\ge3}\mathcal{E}_N(f)\|f\|^2_{H^{N-1}_x L^2_{\varepsilon,l+\gamma/2}}.$$

For $N = 2$ the desired result is easily obtained thanks to Theorem 4.1. For $N \ge 3$ we use mathematical induction to get the desired result. ■

4.1.4. Propagation of full regularity. We first give a useful lemma.

Lemma 4.2. *Let $l_2 \ge l_1 \ge 0$, $m \ge 0$, $l \in \mathbb{R}$. For any $\eta > 0$, there is a constant C_η such that*

$$|f|^2_{H^m_l} \lesssim (\eta+\varepsilon^{2s})|W^\varepsilon(D)f|^2_{H^m_l} + C_\eta|f|^2_{L^2_l}, \quad |f|_{L^2_{\varepsilon,l_1}} \lesssim |f|_{L^2_{\varepsilon,l_2}}.$$

Proof. Recall (1.20). By the interpolation inequality, it is easy to check that

$$|f|^2_{H^m_l} \lesssim |f^\phi|^2_{H^m_l} + |f_\phi|^2_{H^m_l} \lesssim |f^\phi|^2_{H^m_l} + \eta|f_\phi|^2_{H^{m+s}_l} + C_\eta|f_\phi|^2_{L^2_l}.$$

Then the first result follows from Lemma A.2. The second result follows from the definition of $|\cdot|_{L^2_{\varepsilon,l}}$ in (1.24) and Lemma A.2. ■

We are ready to prove propagation of full regularity.

Proposition 4.3. *Suppose $-\frac{3}{2} < \gamma < 0$, $N \ge 2$. Recall the weight functions (1.34) and the functionals (1.35), (1.36), (1.37). There exists $\delta_0 > 0$ independent of ε such that if a solution f^ε to the Cauchy problem (1.9) satisfies $\sup_{0\le t\le T}\mathcal{E}_2(f^\varepsilon(t)) \le \delta_0$ for some $0 < T \le \infty$, then*

$$\sup_{t\in[0,T]}\mathcal{E}^{N,J}(f^\varepsilon(t)) + \int_0^T \mathcal{D}^{N,J}(f^\varepsilon(\tau))\,\mathrm{d}\tau \le C(\mathcal{E}^{N,J}(f_0)),$$

where $C(\cdot)$ is a continuous increasing function verifying $C(0) = 0$.

Proof. Since we have the control of $\dot{\mathcal{E}}^{m,0}(f)$ for $0 \le m \le N$ by Proposition 4.2, we will focus on the estimate of $\dot{\mathcal{E}}^{k-j,j}(f)$ with $1 \le k \le N$, $1 \le j \le k$. We denote

$$\Gamma^\varepsilon(g,h;\beta)(v) := \int B^\varepsilon(v-v_*,\sigma)(\partial_\beta \mu^{\frac{1}{2}})_*(g'_*h' - g_*h)\,\mathrm{d}\sigma\,\mathrm{d}v_*.$$

With this notation one has

$$\partial^\alpha_\beta \Gamma^\varepsilon(g,h) = \sum_{\substack{\beta_0+\beta_1+\beta_2=\beta,\\ \alpha_1+\alpha_2=\alpha}} C_\beta^{\beta_0,\beta_1,\beta_2} C_\alpha^{\alpha_1,\alpha_2} \Gamma^\varepsilon(\partial^{\alpha_1}_{\beta_1} g, \partial^{\alpha_2}_{\beta_2} h;\beta_0).$$

It is easy to check that for any fixed β, $\Gamma^\varepsilon(g,h;\beta)$ shares the same upper bound and commutator estimates as those for $\Gamma^\varepsilon(g,h)$. Recall that $\mathcal{L}^\varepsilon g = -\Gamma^\varepsilon(\mu^{\frac{1}{2}}, g) - \Gamma^\varepsilon(g, \mu^{\frac{1}{2}})$. Thus,

$$\partial^\alpha_\beta \mathcal{L}^\varepsilon g = \mathcal{L}^\varepsilon \partial^\alpha_\beta g - \sum_{\substack{\beta_0+\beta_1+\beta_2=\beta,\\ \beta_2<\beta}} C_\beta^{\beta_0,\beta_1,\beta_2}[\Gamma^\varepsilon(\partial_{\beta_1}\mu^{\frac{1}{2}}, \partial^\alpha_{\beta_2} g;\beta_0) + \Gamma^\varepsilon(\partial^\alpha_{\beta_2} g, \partial_{\beta_1}\mu^{\frac{1}{2}};\beta_0)]. \quad (4.21)$$

Let $1 \le k \le N$, $1 \le j \le k$. Taking two indexes α and β such that $|\alpha| = k-j$, $|\beta| = j$, $\beta = (\beta^1, \beta^2, \beta^3)$, applying $W_q\partial^\alpha_\beta$ to both sides of (4.20), we obtain

$$\partial_t W_q \partial^\alpha_\beta f + v\cdot\nabla_x W_q\partial^\alpha_\beta f + \sum_{i=1}^3 W_q \beta^i \partial^{\alpha+e_i}_{\beta-e_i} f + W_q\partial^\alpha_\beta \mathcal{L}^\varepsilon f = W_q\partial^\alpha_\beta \Gamma^\varepsilon(f,f).$$

Here $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. Let $W_q = W_{l_j}$. Taking the inner product with $W_q\partial^\alpha_\beta f$, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial^\alpha_\beta f\|^2_{L^2_q} + \sum_{i=1}^3 \beta^i (W_q\partial^{\alpha+e_i}_{\beta-e_i} f, W_q\partial^\alpha_\beta f) + (W_q\partial^\alpha_\beta \mathcal{L}^\varepsilon f, W_q\partial^\alpha_\beta f)$$
$$= (W_q\partial^\alpha_\beta\Gamma^\varepsilon(f,f), W_q\partial^\alpha_\beta f).$$

Let us give the estimates term by term.

(i) *Estimate of* $(W_q\partial^{\alpha+e_i}_{\beta-e_i} f, W_q\partial^\alpha_\beta f)$. It is not difficult to check that

$$|(W_q\partial^{\alpha+e_i}_{\beta-e_i} f, W_q\partial^\alpha_\beta f)| \lesssim \|W_q W_{-\gamma/2}\partial^{\alpha+e_i}_{\beta-e_i} f\|_{L^2}\|W_q W_{\gamma/2}\partial^\alpha_\beta f\|_{L^2}$$
$$\lesssim \eta\dot{\mathcal{D}}^{k-j,j}(f) + \eta^{-1}\dot{\mathcal{D}}^{k-j+1,j-1}(f),$$

where we use (1.34).

(ii) *Estimate of* $(W_q\partial^\alpha_\beta\mathcal{L}^\varepsilon f, W_q\partial^\alpha_\beta f)$. Thanks to (4.21), Theorems 1.1 and 2.3 and Lemma 2.8, for some universal constant $c_0 > 0$ we have

$$(W_q\partial^\alpha_\beta\mathcal{L}^\varepsilon f, W_q\partial^\alpha_\beta f) \ge c_0\|\partial^\alpha_\beta f\|^2_{L^2_{\varepsilon,q+\gamma/2}} - C\|\partial^\alpha_\beta f\|^2_{L^2_{q+\gamma/2}} - C\|f\|^2_{H^{k-j}_x H^{j-1}_{\varepsilon,q+\gamma/2}}.$$

By Lemma 4.2 and our assumption for W_{l_j} in (1.34), the above inequality can be rewritten as

$$(W_q\partial^\alpha_\beta\mathcal{L}^\varepsilon f, W_q\partial^\alpha_\beta f) \geq c_0\|\partial^\alpha_\beta f\|^2_{L^2_{\varepsilon,q+\gamma/2}} - (\eta+\varepsilon^{2s})\dot{\mathcal{D}}^{k-j,j}(f) - C_\eta\dot{\mathcal{D}}^{k-j,0}(f) - C\mathcal{D}^{k-1}(f).$$

(iii) *Estimate of* $(W_q\partial^\alpha_\beta\Gamma^\varepsilon(f,f), W_q\partial^\alpha_\beta f)$. It is easy to check that

$$\begin{aligned}(W_q\partial^\alpha_\beta\Gamma^\varepsilon(f,f), W_q\partial^\alpha_\beta f) &= (W_q\Gamma^\varepsilon(f,\partial^\alpha_\beta f), W_q\partial^\alpha_\beta f)\\ &+ \sum_{\substack{\beta_0+\beta_1+\beta_2=\beta,\\ \alpha_1+\alpha_2=\alpha,\\ |\alpha_2|+|\beta_2|\leq k-1}} C^{\beta_0,\beta_1,\beta_2}_\beta C^{\alpha_1,\alpha_2}_\alpha (W_q\Gamma^\varepsilon(\partial^{\alpha_1}_{\beta_1}f, \partial^{\alpha_2}_{\beta_2}f;\beta_0), W_q\partial^\alpha_\beta f).\end{aligned}$$

By Theorem 2.3 and Lemma 2.8, for $a,b\geq 0$, $a+b=2$, we have

$$|(W_q\Gamma^\varepsilon(g,h), W_q f)| \lesssim \|g\|_{H^a_x L^2}\|h\|_{H^b_x L^2_{\varepsilon,q+\gamma/2}}\|f\|_{L^2_{\varepsilon,q+\gamma/2}},$$

which gives for any $0<\eta<1$,

$$|(W_q\Gamma^\varepsilon(f,\partial^\alpha_\beta f), W_q\partial^\alpha_\beta f)| \lesssim \mathcal{E}_2^{\frac{1}{2}}(f)\dot{\mathcal{D}}^{k-j,j}(f) \lesssim (\eta+\eta^{-1}\mathcal{E}_2(f))\dot{\mathcal{D}}^{k-j,j}(f).$$

It remains to estimate $A := (W_q\Gamma^\varepsilon(\partial^{\alpha_1}_{\beta_1}f, \partial^{\alpha_2}_{\beta_2}f;\beta_0), W_q\partial^\alpha_\beta f)$ where $|\alpha_2|+|\beta_2|\leq k-1$. We consider three cases.

Case 1*:* $k=1$. There are only two situations: $(|\alpha_1|,|\beta_1|)=(0,0)$ or $(0,1)$. Then we have

$$\begin{aligned}|A| &\lesssim (\|\partial_\beta f\|_{L^2}+\|f\|_{L^2})\|f\|_{H^2_x L^2_{\varepsilon,q+\gamma/2}}\|\partial_\beta f\|_{L^2_{\varepsilon,q+\gamma/2}}\\ &\lesssim \eta^{-1}(\dot{\mathcal{E}}^{0,1}(f)+1)\mathcal{D}_2(W_{l_0}f) + (\eta+\eta^{-1}\mathcal{E}_2(f))\dot{\mathcal{D}}^{0,1}(f).\end{aligned}$$

Case 2*:* $k=2$. We divide the estimate into two subcases: $|\alpha_2|+|\beta_2|=1$ and $|\alpha_2|+|\beta_2|=0$.

Subcase 2.1*:* $|\alpha_2|+|\beta_2|=1$. Note that $(|\alpha_2|,|\beta_2|)=(1,0)$ or $(|\alpha_2|,|\beta_2|)=(0,1)$. If $(|\alpha_2|,|\beta_2|)=(1,0)$, we get that $j=1$ and $(|\alpha_1|,|\beta_1|)=(0,1)$ or $(0,0)$. Then we have

$$|A| \lesssim \eta^{-1}\mathcal{E}_2(f)\|W_q f\|^2_{H^1_x L^2_{\varepsilon,\gamma/2}} + \eta^{-1}\|f\|^2_{H^1_x\dot{H}^1}\|W_q f\|^2_{H^2_x L^2_{\varepsilon,\gamma/2}} + \eta\|\partial^\alpha_\beta f\|^2_{L^2_{\varepsilon,q+\gamma/2}}.$$

If $(|\alpha_2|,|\beta_2|)=(0,1)$, then we have $(|\alpha_1|,|\beta_1|)=(2-j,j-1)$ or $(2-j,j-2)$ if $j\geq 2$. These imply that

$$|A| \lesssim \eta^{-1}(\mathcal{E}_2(f)+\dot{\mathcal{E}}^{2-j+1,j-1}(f))\|W_q f\|^2_{H^1_x\dot{H}^1_{\varepsilon,\gamma/2}} + \eta\|\partial^\alpha_\beta f\|^2_{L^2_{\varepsilon,q+\gamma/2}}.$$

Subcase 2.2*:* $|\alpha_2|+|\beta_2|=0$. We deduce that $(|\alpha_1|,|\beta_1|)=(2-j,j)$ or $(2-j,j-1)$ or $(2-j,j-2)$ if $j\geq 2$. Then we arrive at

$$|A| \lesssim \eta^{-1}(\|f\|^2_{\dot{H}^{2-j}_x\dot{H}^j}+\mathcal{E}^1(f))\mathcal{D}_2(W_q f) + \eta\|\partial^\alpha_\beta f\|^2_{L^2_{\varepsilon,q+\gamma/2}}.$$

Case 3*:* $k \geq 3$. We consider four subcases.

Subcase 3.1*:* $|\alpha_2| + |\beta_2| = k-1$. Note that $(|\alpha_2|, |\beta_2|) = (k-j-1, j)$ or $(k-j, j-1)$. We derive

$$\begin{aligned}|A| &\lesssim \eta^{-1}\mathcal{E}_2(f)(\|W_q f\|^2_{\dot{H}^{k-j}_x \dot{H}^j_{\varepsilon,\gamma/2}} + \|W_q f\|^2_{\dot{H}^{k-j-1}\dot{H}^j_{\varepsilon,\gamma/2}}) \\ &\quad + \eta^{-1}\|f\|^2_{H^2_x H^1}\|W_q f\|^2_{H^{k-j}_x H^{j-1}_{\varepsilon,\gamma/2}} + \eta\|\partial^\alpha_\beta f\|^2_{L^2_{\varepsilon,q+\gamma/2}} \\ &\lesssim (\eta^{-1}\mathcal{E}_2(f)+\eta)\dot{\mathcal{D}}^{k-j,j}(f) + \eta^{-1}\mathcal{D}^{k-1}(f)(\dot{\mathcal{E}}^{2,1}(f)+\mathcal{E}^2(f)).\end{aligned}$$

Subcase 3.2*:* $|\alpha_2| + |\beta_2| = k-2$ *and* $|\beta_2| = j$. By taking $(a,b) = (1,1)$, we get

$$|A| \lesssim \eta^{-1}(\dot{\mathcal{E}}^{3,0}(f) + \mathcal{E}^2(f))\mathcal{D}^{k-1}(f) + \eta\dot{\mathcal{D}}^{k-j,j}(f).$$

Subcase 3.3*:* $|\alpha_2| + |\beta_2| = k-2$ *and* $|\beta_2| \leq j-1$. Observing that $|\alpha_1| + |\beta_1| \leq 2$ and $|\beta_0| + |\beta_1| \geq 1$, we have

$$A| \lesssim \eta^{-1}(\dot{\mathcal{E}}^{3,0}(f) + \dot{\mathcal{E}}^{2,1}(f) + \dot{\mathcal{E}}^{1,2}(f)1_{j\geq 2} + \mathcal{E}^2(f))\mathcal{D}^{k-1}(f) + \eta\dot{\mathcal{D}}^{k-j,j}(f).$$

Subcase 3.4*:* $|\alpha_2| + |\beta_2| \leq k-3$. It is not difficult to see that

$$|A| \lesssim \eta^{-1}(\dot{\mathcal{E}}^{k-j,j}(f) + \mathcal{E}^{k-1}(f))\mathcal{D}^{k-1}(f) + \eta\dot{\mathcal{D}}^{k-j,j}(f).$$

Now we patch together the above estimates to derive that

(1) if $k=1$, then

$$\begin{aligned}&|(W_q\partial^\alpha_\beta\Gamma^\varepsilon(f,f), W_q\partial^\alpha_\beta f)| \\ &\quad\lesssim \eta^{-1}(\dot{\mathcal{E}}^{0,1}(f)+1)\mathcal{D}_2(W_{l_0}f) + (\eta+\eta^{-1}\mathcal{E}_2(f))\dot{\mathcal{D}}^{0,1}(f);\end{aligned}$$

(2) if $k=2$, then

$$\begin{aligned}&|(W_q\partial^\alpha_\beta\Gamma^\varepsilon(f,f), W_q\partial^\alpha_\beta f)| \\ &\quad\lesssim (\eta+\eta^{-1}\mathcal{E}_2(f))\dot{\mathcal{D}}^{2-j,j}(f) \\ &\qquad + \eta^{-1}(\dot{\mathcal{E}}^{2-j,j}(f) + \dot{\mathcal{E}}^{1,1}(f) + \mathcal{E}^1(f))\mathcal{D}_2(W_{l_0}f) \\ &\qquad + \eta^{-1}(\mathcal{E}_2(f) + \dot{\mathcal{E}}^{2-j+1,j-1}(f))(\dot{\mathcal{D}}^{1,1}(f) + \mathcal{D}^1(f));\end{aligned}$$

(3) if $k \geq 3$, then

$$\begin{aligned}&|(W_q\partial^\alpha_\beta\Gamma^\varepsilon(f,f), W_q\partial^\alpha_\beta f)| \\ &\quad\lesssim (\eta+\eta^{-1}\mathcal{E}_2(f))\dot{\mathcal{D}}^{k-j,j}(f) \\ &\qquad + \eta^{-1}\mathcal{D}^{k-1}(f)(\dot{\mathcal{E}}^{2,1}(f) + \dot{\mathcal{E}}^{3,0}(f) + \dot{\mathcal{E}}^{1,2}(f)1_{j\geq 2} + \mathcal{E}^{k-1}(f) \\ &\qquad\qquad + \dot{\mathcal{E}}^{k-j,j}(f)).\end{aligned}$$

To get the estimate of $\mathcal{E}^1(f)$, it remains to consider $\dot{\mathcal{E}}^{0,1}$. Taking $k = j = 1$, by the above estimates, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathcal{E}}^{0,1}(f) + \frac{1}{2}c_0\dot{\mathcal{D}}^{0,1}(f) \lesssim C_\eta(\dot{\mathcal{E}}^{0,1}(f) + 1)\mathcal{D}_2(W_{l_0}f) + (\eta + \varepsilon^{2s} + \eta^{-1}\mathcal{E}_2(f))\dot{\mathcal{D}}^{0,1}(f).$$

Taking η small enough and since $\sup_{0\le t\le T}\mathcal{E}_2(f(t)) \le \delta_0$ with δ_0 small enough, if ε is small enough, by Proposition 4.2 and Grönwall's inequality, we conclude that

$$\sup_{t\in[0,T]}\mathcal{E}^1(f(t)) + \int_0^T \mathcal{D}^1(f(\tau))\,\mathrm{d}\tau \le C(\|f_0\|_{H^2_xL^2_{l_0}}, \mathcal{E}^1(f_0)). \tag{4.22}$$

To prove the propagation of $\mathcal{E}^2(f)$, we need to consider the energy $\dot{\mathcal{E}}^{2-j,j}$ with $j = 1, 2$. Taking $k = 2$, $j = 1$, it is not difficult to conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathcal{E}}^{1,1}(f) + \frac{1}{2}c_0\dot{\mathcal{D}}^{1,1}(f) \lesssim \mathcal{D}^1(f) + (1 + \mathcal{E}^1(f))\mathcal{D}_2(W_{l_0}f) + \dot{\mathcal{E}}^{1,1}(f)\mathcal{D}_2(W_{l_0}f).$$

Then by Grönwall's inequality, Proposition 4.2 and (4.22), we get

$$\sup_{t\in[0,T]}\dot{\mathcal{E}}^{1,1}(f(t)) + \int_0^T \dot{\mathcal{D}}^{1,1}(f(\tau))\,\mathrm{d}\tau \le C(\mathcal{E}^{2,1}(f_0)). \tag{4.23}$$

Taking $k = 2$, $j = 2$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathcal{E}}^{0,2}(f) + \frac{1}{2}c_0\dot{\mathcal{D}}^{0,2}(f) \lesssim{}& \dot{\mathcal{D}}^{1,1}(f) + \mathcal{D}^1(f)\\ &+ (\dot{\mathcal{E}}^{0,2}(f) + \dot{\mathcal{E}}^{1,1}(f) + \mathcal{E}^1(f))\mathcal{D}_2(W_{l_0}f)\\ &+ (\mathcal{E}_2(f) + \dot{\mathcal{E}}^{1,1}(f))(\dot{\mathcal{D}}^{1,1}(f) + \mathcal{D}^1(f)).\end{aligned}$$

Then by Grönwall's inequality, Proposition 4.2, (4.22) and (4.23), we get

$$\sup_{t\in[0,T]}\dot{\mathcal{E}}^{0,2}(f(t)) + \int_0^T \dot{\mathcal{D}}^{0,2}(f(\tau))\,\mathrm{d}\tau \le C(\mathcal{E}^{2,2}(f_0)).$$

In other words, for $0 \le J \le 2$, we have $\sup_{t\in[0,T]}\mathcal{E}^{2,J}(f(t)) + \int_0^T \mathcal{D}^{2,J}(f(\tau))\,\mathrm{d}\tau \le C(\mathcal{E}^{2,J}(f_0))$.

Now we shall use mathematical induction to complete the proof. We assume that the result in the proposition holds for $0 \le J \le N \le n$ with $n \ge 2$. For $0 \le J \le N = n+1$, since $J = 0$ is handled in Proposition 4.2, we begin with the propagation of $\dot{\mathcal{E}}^{n,1}(f)$. From the above inequalities, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathcal{E}}^{n,1}(f) + \frac{1}{2}c_0\dot{\mathcal{D}}^{n,1}(f) \lesssim (1 + \dot{\mathcal{E}}^{n,1}(f) + \mathcal{E}^n(f) + \dot{\mathcal{E}}^{3,0}(f) + \dot{\mathcal{E}}^{2,1}(f))\mathcal{D}^n(f) + \dot{\mathcal{D}}^{n+1,0}(f),$$

which yields that $\sup_{t\in[0,T]} \mathcal{E}^{n+1,1}(f(t)) + \int_0^T \mathcal{D}^{n+1,1}(f(\tau))\,\mathrm{d}\tau \le C(\mathcal{E}^{n+1,1}(f_0))$ thanks to Grönwall's inequality. For $j \ge 2$ we derive that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\dot{\mathcal{E}}^{n+1-j,j}(f) + \frac{1}{2}c_0\dot{\mathcal{D}}^{n+1-j,j}(f) &\\ \lesssim (1 + \dot{\mathcal{E}}^{n+1-j,j}(f) + \mathcal{E}^{n}(f) + \dot{\mathcal{E}}^{3,0}(f) + \dot{\mathcal{E}}^{2,1}(f) + \dot{\mathcal{E}}^{1,2}(f))\mathcal{D}^{n}(f) &\\ + \dot{\mathcal{D}}^{n+2-j,j-1}(f).\end{aligned}$$

Using mathematical induction to index j, we get for $2 \le j \le J$,

$$\sup_{t\in[0,T]} \dot{\mathcal{E}}^{n+1-j,j}(f(t)) + \int_0^T \dot{\mathcal{D}}^{n+1-j,j}(f(\tau))\,\mathrm{d}\tau \le C(\mathcal{E}^{n+1,J}(f_0)),$$

which completes the inductive argument for n. We end the proof of the proposition. ■

Proof of Theorem 1.3 (part 1: global well-posedness and propagation of regularity). By a standard continuity argument, the global well-posedness in $H^2_x L^2$ follows from the a priori estimate in Theorem 4.1 and the local well-posedness result (see [9] for instance). The propagation results (1.38) and (1.39) follow directly from Propositions 4.2 and 4.3. ■

4.2. Global dynamics

We now give the proof of the second part of Theorem 1.3.

Proof of Theorem 1.3 (part 2: global dynamics). We first prove (1.42). It is easy to check that $\mathcal{P}_j f^\varepsilon$ verifies

$$\partial_t \mathcal{P}_j f^\varepsilon + v\cdot\nabla_x \mathcal{P}_j f^\varepsilon + \mathcal{L}^\varepsilon \mathcal{P}_j f^\varepsilon = [\mathcal{L}^\varepsilon, \mathcal{P}_j] f^\varepsilon + \mathcal{P}_j \Gamma^\varepsilon(f^\varepsilon, f^\varepsilon).$$

Thanks to Theorem 1.1, Lemma 3.1 and (4.14), for some $C_0 > 0$ one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{P}_j f^\varepsilon\|^2_{L^2} \ge -C_0(\|\mathcal{P}_j f^\varepsilon\|^2_{L^2_{\varepsilon,\gamma/2}} + \varepsilon^{2s}\|f^\varepsilon\|^2_{L^2_{\varepsilon,\gamma/2}} + \varepsilon^{2s}\|f^\varepsilon\|^2_{H^2_x L^2}\|f^\varepsilon\|^2_{L^2_{\varepsilon,\gamma/2}} &\\ + \|f^\varepsilon\|_{H^2_x L^2}\|\mathcal{P}_j f^\varepsilon\|^2_{L^2_{\varepsilon,\gamma/2}}).\end{aligned}$$

By Theorem 4.1 for the case $N = 2$, we have

$$\sup_{t\ge 0} \mathcal{E}_2(f^\varepsilon(t)) + \int_0^\infty \mathcal{D}_2(f^\varepsilon(s))\,\mathrm{d}s \lesssim \mathcal{E}_2(f_0) \le \delta_0.$$

Recalling that $\|\mathcal{P}_j f^\varepsilon\|^2_{L^2_{\varepsilon,\gamma/2}} \lesssim \varepsilon^{-2s}2^{j\gamma}\|\mathcal{P}_j f^\varepsilon\|^2_{L^2} \lesssim \varepsilon^{-2s}2^{j\gamma}\delta_0$, we have

$$\|\mathcal{P}_j f^\varepsilon(t)\|^2_{L^2} \ge \|\mathcal{P}_j f_0\|^2_{L^2} - C\varepsilon^{-2s}2^{j\gamma}\delta_0 t - C\delta_0\varepsilon^{2s},$$

which yields (1.42).

We will now prove (1.40) and (1.41). By the interpolation inequality

$$|f|_{L^2} \lesssim |f|_{L^2_{\gamma/2}}^{\frac{q}{q+1}} |f|_{L^2_{-q\gamma/2}}^{\frac{1}{q+1}}$$

and the facts $\sup_{t\geq 0} \|f^\varepsilon(t)\|_{H^2_x L^2_l} \lesssim \|f_0\|_{H^2_x L^2_l}$ from Proposition 4.2 and $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{2,M}(f^\varepsilon) + \frac{c_0}{2}\mathcal{D}_2(f^\varepsilon) \leq 0$ from (4.19), for some universal constants $\widetilde{C}_1, \widetilde{C}_2 > 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{2,M}(f^\varepsilon) + \widetilde{C}_1 \|f^l\|^2_{H^2_x L^2} + \widetilde{C}_2 \|f_0\|^{-2/q}_{H^2_x L^2_{-q\gamma/2}} \varepsilon^{-2s} \|f^h\|^{2(1+\frac{1}{q})}_{H^2_x L^2} \leq 0.$$

Let $Y(t) = \mathcal{E}_{2,M}(f^\varepsilon(t))$, $Y_1(t) = \|f^l(t)\|^2_{H^2_x L^2}$, $Y_2(t) = \|f^h(t)\|^2_{H^2_x L^2}$; then

$$\frac{\mathrm{d}}{\mathrm{d}t} Y + c_1 Y_1 + c_2 Y_2^{1+\frac{1}{q}} \leq 0,$$

where $c_1 = \widetilde{C}_1$, $c_2 = \widetilde{C}_2 \|f_0\|^{-2/q}_{H^2_x L^2_{-q\gamma/2}} \varepsilon^{-2s}$. By (4.3), $\frac{1}{4}M(Y_1 + Y_2) \leq Y \leq 4M(Y_1 + Y_2)$. By taking $c = 4M$ and applying Proposition 3.1, we can define

$$C_q = 8(\widetilde{C}_1/\widetilde{C}_2)^q, \quad \lambda_0 = \frac{\widetilde{C}_1}{8M},$$
$$C(f_0) = q^{-1}(8M)^{-1-\frac{1}{q}} (\mathcal{E}_{2,M}(f_0))^{\frac{1}{q}} \widetilde{C}_2 \varepsilon^{-2s} \|f_0\|^{-2/q}_{H^2_x L^2_{-q\gamma/2}}, \tag{4.24}$$

to get (1.40) and (1.41). ∎

4.3. Global asymptotic formula

In this subsection we want to prove (1.43). Let f^ε and f^0 be the solutions to (1.9) and (1.10) respectively with the same initial data f_0. Let $F^\varepsilon_R := \varepsilon^{2-2s}(f^\varepsilon - f^0)$, which solves

$$\partial_t F^\varepsilon_R + v \cdot \nabla_x F^\varepsilon_R + \mathcal{L}^0 F^\varepsilon_R = \varepsilon^{2s-2}((\mathcal{L}^0 - \mathcal{L}^\varepsilon) f^\varepsilon + (\Gamma^\varepsilon - \Gamma^0)(f^\varepsilon, f^0))$$
$$+ \Gamma^\varepsilon(f^\varepsilon, F^\varepsilon_R) + \Gamma^0(F^\varepsilon_R, f^0). \tag{4.25}$$

We first derive an estimate on the operator difference $\Gamma^0 - \Gamma^\varepsilon$.

Lemma 4.3. *Let $\gamma > -3$. It holds that*

$$|\langle(\Gamma^0 - \Gamma^\varepsilon)(g, h), f\rangle_v| \lesssim \varepsilon^{2-2s} |g|_{L^2} |h|_{H^2_{\gamma/2+2}} |f|_{L^2_{\gamma/2}}.$$

Proof. By direct calculation we have

$$\langle(\Gamma^0 - \Gamma^\varepsilon)(g, h), f\rangle_v = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4,$$

where

$$\mathcal{A}_1 := \int (b - b^\varepsilon)(\cos\theta)|v - v_*|^\gamma ((\mu^{\frac{1}{2}})'_* - \mu^{\frac{1}{2}}_*) g_* h' f' \,\mathrm{d}\sigma \,\mathrm{d}v_* \,\mathrm{d}v,$$

$$\mathcal{A}_2 := \int (b - b^\varepsilon)(\cos\theta)|v - v_*|^\gamma ((\mu^{\frac{1}{2}})'_* - \mu^{\frac{1}{2}}_*) g_* (h - h') f' \,\mathrm{d}\sigma \,\mathrm{d}v_* \,\mathrm{d}v,$$

$$\mathcal{A}_3 := \int (b - b^\varepsilon)(\cos\theta)|v - v_*|^\gamma \mu_*^{\frac{1}{2}} g_* (h - h') f' \, d\sigma \, dv_* \, dv,$$
$$\mathcal{A}_4 := \int (b - b^\varepsilon)(\cos\theta)|v - v_*|^\gamma \mu_*^{\frac{1}{2}} g_* (h' f' - h f) \, d\sigma \, dv_* \, dv.$$

Note that $b - b^\varepsilon$ is supported for $\sin\frac{\theta}{2} \leq \frac{4}{3}\varepsilon$ and so

$$\int (b - b^\varepsilon)(\cos\theta) \sin^2\frac{\theta}{2} \, d\sigma \lesssim \int_0^{4\varepsilon/3} t^{1-2s} \, dt \lesssim \varepsilon^{2-2s}. \tag{4.26}$$

Estimate of $\mathcal{A}_1$. By the change of variable $(v, v_*) \to (v'_*, v')$ we have

$$\mathcal{A}_1 = \int (b - b^\varepsilon)(\cos\theta)|v - v_*|^\gamma (\mu^{\frac{1}{2}} - (\mu^{\frac{1}{2}})') g' h_* f_* \, d\sigma \, dv_* \, dv.$$

By Taylor expansion one has

$$\mu^{\frac{1}{2}} - (\mu^{\frac{1}{2}})' = (\nabla \mu^{\frac{1}{2}})(v') \cdot (v - v') + \int_0^1 (1-\kappa)\big((\nabla^2 \mu^{\frac{1}{2}})(v(\kappa)) : (v - v') \otimes (v - v')\big) \, d\kappa,$$

where $v(\kappa) = v' + \kappa(v - v')$. Recalling (2.45), we have

$$\begin{aligned} |\mathcal{A}_1| &= \bigg| \int (b - b^\varepsilon)(\cos\theta)|v - v_*|^\gamma (1-\kappa)\big((\nabla^2 \mu^{\frac{1}{2}})(v(\kappa)) : (v - v') \otimes (v - v')\big) \\ &\qquad \times g' h_* f_* \, d\kappa \, d\sigma \, dv_* \, dv \bigg| \\ &\lesssim \varepsilon^{2-2s} \bigg(\int \langle v_* \rangle^{\gamma+4} (g^2)' h_*^2 \, dv_* \, dv' \bigg)^{\frac{1}{2}} \\ &\quad \times \bigg(\int |v(\kappa) - v_*|^\gamma \mu^{\frac{1}{8}}(v(\kappa)) f_*^2 \, d\kappa \, dv_* \, dv(\kappa) \bigg)^{\frac{1}{2}} \\ &\lesssim \varepsilon^{2-2s} |g|_{L^2} |h|_{L^2_{\gamma/2+2}} |f|_{L^2_{\gamma/2}}, \end{aligned}$$

where we use the changes of variable $v \to v'$ and $v \to v(\kappa)$, and estimate (4.26).

Estimate of $\mathcal{A}_2$. By the Cauchy–Schwarz inequality and the change of variables $(v, v_*) \to (v', v'_*)$, we have

$$\begin{aligned} |\mathcal{A}_2| &\leq \bigg(\int (b - b^\varepsilon)(\cos\theta)|v - v_*|^{\gamma+2} g_*^2 (h - h')^2 ((\mu^{\frac{1}{4}})'_* + \mu_*^{\frac{1}{4}})^2 \, d\sigma \, dv_* \, dv \bigg)^{\frac{1}{2}} \\ &\quad \times \bigg(\int (b - b^\varepsilon)(\cos\theta)|v - v_*|^{\gamma-2} ((\mu^{\frac{1}{4}})'_* - \mu_*^{\frac{1}{4}})^2 f^2 \, d\sigma \, dv_* \, dv \bigg)^{\frac{1}{2}} \\ &:= (\mathcal{A}_{2,1})^{\frac{1}{2}} \times (\mathcal{A}_{2,2})^{\frac{1}{2}}. \end{aligned}$$

By Taylor expansion, $h - h' = \int_0^1 (\nabla h)(v(\kappa)) \cdot (v - v') \, d\kappa$ where $v(\kappa) = v' + \kappa(v - v')$. By the change of variable $v \to v(\kappa)$ and (4.26), we get

$$\mathcal{A}_{2,1} \lesssim \varepsilon^{2-2s} \int \langle v(\kappa) \rangle^{\gamma+4} g_*^2 |(\nabla h)(v(\kappa))|^2 \, dv_* \, dv(\kappa) \, d\kappa \lesssim \varepsilon^{2-2s} |g|_{L^2}^2 |h|_{H^1_{\gamma/2+2}}^2.$$

Using $((\mu^{\frac{1}{4}})'_* - \mu_*^{\frac{1}{4}})^2 \lesssim ((\mu^{\frac{1}{4}})'_* + \mu_*^{\frac{1}{4}}) \sin^2 \frac{\theta}{2} |v - v_*|^2$, the change of variable $v_* \to v'_*$ and (4.26), we have

$$\mathcal{A}_{2,2} \lesssim \varepsilon^{2-2s} \int |v - v_*|^\gamma \mu_*^{\frac{1}{4}} f^2 \, \mathrm{d}v_* \, \mathrm{d}v \lesssim \varepsilon^{2-2s} |f|^2_{L^2_{\gamma/2}}.$$

Patching together the estimates of $\mathcal{A}_{2,1}$ and $\mathcal{A}_{2,2}$, we have

$$|\mathcal{A}_2| \lesssim \varepsilon^{2-2s} |g|_{L^2} |h|_{H^1_{\gamma/2+2}} |f|_{L^2_{\gamma/2}}.$$

Estimate of $\mathcal{A}_3$. By Taylor expansion, one has

$$h - h' = (\nabla h)(v') \cdot (v - v') + \int_0^1 (1-\kappa)\big((\nabla^2 h)(v(\kappa)) : (v - v') \otimes (v - v')\big) \, \mathrm{d}\kappa,$$

where $v(\kappa) = v' + \kappa(v - v')$. From this, together with (2.45), we have

$$\begin{aligned}
|\mathcal{A}_3| &= \bigg| \int (b - b^\varepsilon)(\cos\theta) |v - v_*|^\gamma \mu_*^{\frac{1}{2}} g_* (1-\kappa)\big((\nabla^2 h)(v(\kappa)) : (v - v') \otimes (v - v')\big) \\
&\qquad \times f' \mathrm{d}\kappa \, \mathrm{d}\sigma \, \mathrm{d}v_* \, \mathrm{d}v \bigg| \\
&\lesssim \varepsilon^{2-2s} \left(\int |v(\kappa) - v_*|^{\gamma+4} \mu_*^{\frac{1}{2}} g_*^2 |(\nabla^2 h)(v(\kappa))|^2 \, \mathrm{d}\kappa \, \mathrm{d}v_* \, \mathrm{d}v(\kappa) \right)^{\frac{1}{2}} \\
&\quad \times \left(\int |v' - v_*|^\gamma \mu_*^{\frac{1}{2}} |f'|^2 \, \mathrm{d}v_* \, \mathrm{d}v' \right)^{\frac{1}{2}} \lesssim \varepsilon^{2-2s} |g|_{L^2} |h|_{H^2_{\gamma/2+2}} |f|_{L^2_{\gamma/2}}.
\end{aligned}$$

Estimate of $\mathcal{A}_4$. By the cancellation lemma and Lemma 2.5, we have

$$|\mathcal{A}_4| \lesssim \varepsilon^{2-2s} \int |v - v_*|^\gamma \mu_*^{\frac{1}{2}} |g_* h f| \, \mathrm{d}v_* \, \mathrm{d}v \lesssim \varepsilon^{2-2s} |g|_{L^2} |h|_{H^2_{\gamma/2}} |f|_{L^2_{\gamma/2}}.$$

The lemma then follows by patching together the above estimates. ■

We are ready to prove (1.43).

Proof of Theorem 1.3 (part 3: asymptotic formula). Recalling (4.25) we set

$$g = \varepsilon^{2s-2}((\mathcal{L}^0 - \mathcal{L}^\varepsilon) f^\varepsilon + (\Gamma^\varepsilon - \Gamma^0)(f^\varepsilon, f^0)) + \Gamma^\varepsilon(f^\varepsilon, F_R^\varepsilon) + \Gamma^0(F_R^\varepsilon, f^0).$$

By applying Proposition 4.1 with the previous nonlinear term g, using (4.13) for F_R^ε, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_{N,M}(F_R^\varepsilon) + c_0 \|F_R^\varepsilon\|^2_{H_x^N L^2_{0,\gamma/2}} \\
&\qquad \lesssim \sum_{|\alpha| \le N} |(\partial^\alpha g, \partial^\alpha F_R^\varepsilon)| + \sum_{|\alpha| \le N-1} \sum_{j=1}^{13} \int |\langle \partial^\alpha g, e_j \rangle_v|^2 \, \mathrm{d}x.
\end{aligned}$$

By Theorem 2.3 with $\varepsilon = 0$ and (4.15), we have

$$
\begin{aligned}
&|(\partial^\alpha \Gamma^0(F_R^\varepsilon, f^0), \partial^\alpha F_R^\varepsilon)| + |(\partial^\alpha \Gamma^\varepsilon(f^\varepsilon, F_R^\varepsilon), \partial^\alpha F_R^\varepsilon)| \\
&\quad \lesssim (\|F_R^\varepsilon\|_{H_x^N L^2} \|f^0\|_{H_x^N L^2_{0,\gamma/2}} + \|f^\varepsilon\|_{H_x^2 L^2} \|F_R^\varepsilon\|_{H_x^N L^2_{0,\gamma/2}} \\
&\qquad + 1_{N\geq 3} \|f^\varepsilon\|_{H_x^N L^2} \|F_R^\varepsilon\|_{H_x^{N-1} L^2_{0,\gamma/2}}) \|F_R^\varepsilon\|_{H_x^N L^2_{0,\gamma/2}}.
\end{aligned}
$$

By Lemma 4.3 we have

$$
\begin{aligned}
&\varepsilon^{2s-2} |(\partial^\alpha (\Gamma^\varepsilon - \Gamma^0)(f^\varepsilon, f^0), \partial^\alpha F_R^\varepsilon)| + \varepsilon^{2s-2} |(\partial^\alpha (\mathscr{L}^0 - \mathscr{L}^\varepsilon) f^\varepsilon, \partial^\alpha F_R^\varepsilon)| \\
&\quad \lesssim (\|f^\varepsilon\|_{H_x^N L^2} \|f^0\|_{H_x^N H^2_{\gamma/2+2}} + \|f^\varepsilon\|_{H_x^N H^2_{\gamma/2+2}}) \|F_R^\varepsilon\|_{H_x^N L^2_{0,\gamma/2}}.
\end{aligned}
$$

Recalling (4.17), for any $|\alpha| \leq N$ we have

$$
\begin{aligned}
&\int (|\langle \partial^\alpha \Gamma^\varepsilon(f^\varepsilon, F_R^\varepsilon), e_j \rangle_v|^2 + |\langle \partial^\alpha \Gamma^0(F_R^\varepsilon, f^0), e_j \rangle_v|^2)\, \mathrm{d}x \\
&\quad \lesssim \|f^\varepsilon\|^2_{H_x^2 L^2} \|F_R^\varepsilon\|^2_{H_x^N L^2_{0,\gamma/2}} + 1_{N\geq 3} \|f^\varepsilon\|^2_{H_x^N L^2} \|F_R^\varepsilon\|^2_{H_x^{N-1} L^2_{0,\gamma/2}} \\
&\qquad + \|f^0\|^2_{H_x^N L^2_{0,\gamma/2}} \|F_R^\varepsilon\|^2_{H_x^N L^2}.
\end{aligned}
$$

By Lemma 4.3 we get

$$
\begin{aligned}
&\varepsilon^{2s-2} \int |\langle \partial^\alpha (\Gamma^\varepsilon - \Gamma^0)(f^\varepsilon, f^0), e_j \rangle_v|^2 \,\mathrm{d}x + \varepsilon^{2s-2} \int |\langle \partial^\alpha (\mathscr{L}^0 - \mathscr{L}^\varepsilon) f^\varepsilon, e_j \rangle_v|^2 \,\mathrm{d}x \\
&\quad \lesssim \|f^\varepsilon\|^2_{H_x^N L^2} \|f^0\|^2_{H_x^N H^2_{\gamma/2+2}} + \|f^\varepsilon\|^2_{H_x^N H^2_{\gamma/2+2}}.
\end{aligned}
$$

Patching together the above results, since $\sup_{t\geq 0} \|f^\varepsilon(t)\|_{H_x^2 L^2} \lesssim \|f_0\|_{H_x^2 L^2} \leq \delta_0$ with δ_0 small enough, we arrive at

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}_{N,M}(F_R^\varepsilon) + \frac{c_0}{2} \|F_R^\varepsilon\|^2_{H_x^N L^2_{0,\gamma/2}} &\lesssim 1_{N\geq 3} \|f^\varepsilon\|^2_{H_x^N L^2} \|F_R^\varepsilon\|^2_{H_x^{N-1} L^2_{0,\gamma/2}} \\
&\quad + \|f^0\|^2_{H_x^N L^2_{0,\gamma/2}} \|F_R^\varepsilon\|^2_{H_x^N L^2} \\
&\quad + \|f^\varepsilon\|^2_{H_x^N L^2} \|f^0\|^2_{H_x^N H^2_{\gamma/2+2}} + \|f^\varepsilon\|^2_{H_x^N H^2_{\gamma/2+2}}.
\end{aligned}
$$

By (4.3), recall that $\mathcal{E}_{N,M}(\cdot) \sim \|\cdot\|^2_{H_x^N L^2}$. Thanks to Proposition 4.3, we derive that

$$
\int_0^\infty (\|f^\varepsilon(\tau)\|^2_{H_x^N H^2_{\gamma/2+2}} + \|f^0(\tau)\|^2_{H_x^N H^2_{\gamma/2+2}} + \|f^0(\tau)\|^2_{H_x^N L^2_{0,\gamma/2}})\, \mathrm{d}\tau \leq C(\mathcal{E}^{N+2,2}(f_0)),
$$

which yields when $N = 2$,

$$
\sup_{t\geq 0} \|F_R^\varepsilon(t)\|^2_{H_x^2 L^2} + \int_0^\infty \|F_R^\varepsilon(\tau)\|^2_{H_x^2 L^2_{0,\gamma/2}} \,\mathrm{d}\tau \leq C(\mathcal{E}^{4,2}(f_0)).
$$

From this, together with mathematical induction, for $N \geq 3$ we will get

$$
\sup_{t\geq 0} \|F_R^\varepsilon(t)\|^2_{H_x^N L^2} + \int_0^\infty \|F_R^\varepsilon(\tau)\|^2_{H_x^N L^2_{0,\gamma/2}} \,\mathrm{d}\tau \leq C(\mathcal{E}^{N+2,2}(f_0)),
$$

which ends the proof of (1.43) and completes the proof of Theorem 1.3. ■

A. Appendix

We first give the definition of the symbol class $S^m_{1,0}$.

Definition A.1. A smooth function $a(v,\xi)$ is said to be a symbol of type $S^m_{1,0}$ if $a(v,\xi)$ verifies for any $\alpha,\beta \in \mathbb{N}^3$,

$$|(\partial^\alpha_\xi \partial^\beta_v a)(v,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|},$$

where $C_{\alpha,\beta}$ is a constant depending only on α and β.

The following result is an estimate of the commutator between a pseudo-differential operator $M(D)$ and a multiplication operator Φ.

Lemma A.1 ([10]). *Let $l,s,r \in \mathbb{R}$, $M = M(\xi) \in S^r_{1,0}$ and $\Phi = \Phi(\xi) \in S^l_{1,0}$. Then there exists a constant C such that*

$$|[M(D),\Phi]f|_{H^s} \le C|f|_{H^{r+s-1}_{l-1}}.$$

As an application of Lemma A.1, since $W^\varepsilon = W^\varepsilon(\xi) \in S^s_{1,0}$, $2^k\varphi_k = 2^k\varphi_k(\xi) \in S^1_{1,0}$ with $0 < s < 1$, we have

$$\begin{aligned}\sum_{k\ge -1}^{\infty} |W^\varepsilon(D)\varphi_k f|^2_{L^2} &= \sum_{k\ge -1}^{\infty} 2^{-2k}|W^\varepsilon(D)2^k\varphi_k f|^2_{L^2}\\ &\lesssim \sum_{k\ge -1}^{\infty} 2^{-2k}(|2^k\varphi_k W^\varepsilon(D)f|^2_{L^2} + |f|^2_{H^{s-1}})\\ &\lesssim |W^\varepsilon(D)f|^2_{L^2}. \end{aligned} \tag{A.1}$$

Lemma A.2 ([12]). *Let $W^\varepsilon_q(v) := \phi(\varepsilon v)\langle v\rangle^q + \varepsilon^{-q}(1-\phi(\varepsilon v))$. Let $l \in \mathbb{R}$, $m,q \ge 0$. It holds that*

$$|f|_{H^m_l} \sim |f^\phi|_{H^m_l} + |f_\phi|_{H^m_l}, \quad |W^\varepsilon_q(D)W_l f|_{H^m} \sim |W^\varepsilon_q(D)f|_{H^m_l}.$$

Let $\Phi = \Phi(\xi) \in S^l_{1,0}$. Assume that $B^\varepsilon(\xi)$ verifies $|B^\varepsilon(\xi)| \le W^\varepsilon_q(\xi)$ and $|\partial^\alpha B^\varepsilon(\xi)| \le W^\varepsilon_{(q-|\alpha|)^+}(\xi)$ for any index $\alpha \in \mathbb{N}^3$; then

$$|\Phi B^\varepsilon(D)f|_{H^m} + |B^\varepsilon(D)\Phi f|_{H^m} \lesssim |W^\varepsilon_q(D)W_l f|_{H^m}. \tag{A.2}$$

Proposition A.1. *Let $A^\varepsilon(\xi) := \int b^\varepsilon(\frac{\xi}{|\xi|}\cdot\sigma)\min\{|\xi|^2\sin^2\frac{\theta}{2},1\}\,d\sigma$; then*

$$A^\varepsilon(\xi) \sim |\xi|^2 1_{|\xi|\le\sqrt{2}} + 1_{|\xi|\ge\sqrt{2}}(W^\varepsilon)^2(\xi) \lesssim (W^\varepsilon)^2(\xi).$$

Proof. Recalling (1.3) we first get

$$A^\varepsilon(\xi) = 2\pi\int_0^{\pi/2} \sin\theta b(\cos\theta)(1-\phi(\sin\tfrac{\theta}{2}/\varepsilon))\min\{|\xi|^2\sin^2\tfrac{\theta}{2},1\}\,d\theta.$$

By the change of variable $t = \sin\frac{\theta}{2}$ we have

$$A^\varepsilon(\xi) \sim \int_0^{\frac{\sqrt{2}}{2}} t^{-1-2s}(1-\phi(t/\varepsilon))\min\{|\xi|^2t^2, 1\}\,\mathrm{d}t$$
$$= |\xi|^{2s}\int_0^{\frac{\sqrt{2}}{2}|\xi|} t^{-1-2s}(1-\phi(\varepsilon^{-1}t|\xi|^{-1}))\min\{t^2, 1\}\,\mathrm{d}t.$$

By the definition of ϕ we have

$$|\xi|^{2s}\int_{\frac{4}{3}\varepsilon|\xi|}^{\frac{\sqrt{2}}{2}|\xi|} t^{-1-2s}\min\{t^2, 1\}\,\mathrm{d}t \lesssim A^\varepsilon(\xi) \lesssim |\xi|^{2s}\int_{\frac{3}{4}\varepsilon|\xi|}^{\frac{\sqrt{2}}{2}|\xi|} t^{-1-2s}\min\{t^2, 1\}\,\mathrm{d}t.$$

Now we focus on the quantity $I(\xi) := |\xi|^{2s}\int_{c\varepsilon|\xi|}^{\sqrt{2}|\xi|/2} t^{-1-2s}\min\{t^2, 1\}\,\mathrm{d}t$ for a constant $\frac{3}{4} \le c \le \frac{4}{3}$ and small ε. For instance, we assume $\varepsilon < \frac{1}{10}$.

(1) If $|\xi| \le \sqrt{2}$, then $I(\xi) = |\xi|^{2s}\int_{c\varepsilon|\xi|}^{\sqrt{2}|\xi|/2} t^{1-2s}\,\mathrm{d}t \sim (1-s)^{-1}|\xi|^2$.

(2) If $\sqrt{2} < |\xi| \le (c\varepsilon)^{-1}$, then

$$I(\xi) = |\xi|^{2s}\left(\int_{c\varepsilon|\xi|}^{1} t^{1-2s}\,\mathrm{d}t + \int_1^{\frac{\sqrt{2}}{2}|\xi|} t^{-1-2s}\,\mathrm{d}t\right)$$
$$\sim (1-s)^{-1}|\xi|^{2s}(1-(c\varepsilon|\xi|)^{2-2s}) + |\xi|^{2s}(1-(\sqrt{2}|\xi|^{-1})^{2s}).$$

(3) If $|\xi| \ge (c\varepsilon)^{-1}$, then $I(\xi) = |\xi|^{2s}\int_{c\varepsilon|\xi|}^{\sqrt{2}|\xi|/2} t^{-1-2s}\,\mathrm{d}t \sim \varepsilon^{-2s}$.

The desired result follows from the above estimates. ∎

Proposition A.2. *Let h, f be real-valued functions. It holds that*

$$\int_{\mathbb{S}^2\times\mathbb{R}^3} b(\tfrac{u}{|u|}\cdot\sigma)h(u)(f(u^+) - f(\tfrac{|u|}{|u^+|}u^+))\,\mathrm{d}\sigma\,\mathrm{d}u$$
$$= \int_{\mathbb{S}^2\times\mathbb{R}^3} b(\tfrac{\xi}{|\xi|}\cdot\sigma)(\hat{h}(\xi^+) - \hat{h}(\tfrac{|\xi|}{|\xi^+|}\xi^+))\bar{\hat{f}}(\xi)\,\mathrm{d}\sigma\,\mathrm{d}\xi.$$

Here $u^+ = \frac{u+|u|\sigma}{2}$, $\xi^+ = \frac{\xi+|\xi|\sigma}{2}$.

Proof. Let $F(u) := \int_{\mathbb{S}^2} b(\frac{u}{|u|}\cdot\sigma)f(\frac{|u|}{|u^+|}u^+)\,\mathrm{d}\sigma$. By Plancherel's equality we have

$$\int_{\mathbb{S}^2\times\mathbb{R}^3} b(\tfrac{u}{|u|}\cdot\sigma)h(u)f(\tfrac{|u|}{|u^+|}u^+)\,\mathrm{d}\sigma\,\mathrm{d}u = \int_{\mathbb{R}^3} h(u)F(u)\,\mathrm{d}u = \int_{\mathbb{R}^3}\hat{h}(\xi)\bar{\hat{F}}(\xi)\,\mathrm{d}\xi.$$

Next we compute the Fourier transform $\widehat{F}$ of F. By definition, we have

$$\widehat{F}(\xi) = \int_{\mathbb{R}^3} e^{-\mathrm{i}u\cdot\xi}F(u)\,\mathrm{d}u = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} e^{-\mathrm{i}u\cdot\xi}e^{\mathrm{i}\frac{|u|}{|u^+|}u^+\cdot\eta}b(\tfrac{u}{|u|}\cdot\sigma)\hat{f}(\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}u.$$

Noticing that $\frac{|u|}{|u^+|}u^+ \cdot \eta = \frac{1}{2}((\frac{u}{|u|}\cdot\sigma + 1)/2)^{-\frac{1}{2}}(u\cdot\eta + |u|\,|\eta|\frac{\eta}{|\eta|}\cdot\sigma)$ and the fact that $\int_{\mathbb{S}^2} b_1(\kappa\cdot\sigma)b_2(\tau\cdot\sigma)\,\mathrm{d}\sigma = \int_{\mathbb{S}^2} b_1(\tau\cdot\sigma)b_2(\kappa\cdot\sigma)\,\mathrm{d}\sigma$, one has

$$\begin{aligned}\widehat{F}(\xi) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{S}^2\times\mathbb{R}^3\times\mathbb{R}^3} e^{-\mathrm{i}u\cdot\xi}e^{\mathrm{i}\frac{|\eta|}{|\eta^+|}\eta^+\cdot u} b(\tfrac{u}{|u|}\cdot\sigma)\hat{f}(\eta)\,\mathrm{d}\sigma\,\mathrm{d}\eta\,\mathrm{d}u \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{S}^2\times\mathbb{R}^3} b(\tfrac{\eta}{|\eta|}\cdot\sigma)\hat{f}(\eta)\delta(\xi = \tfrac{|\eta|}{|\eta^+|}\eta^+)\,\mathrm{d}\sigma\,\mathrm{d}\eta,\end{aligned}$$

which yields

$$\int_{\mathbb{S}^2\times\mathbb{R}^3} b(\tfrac{u}{|u|}\cdot\sigma)h(u)f(\tfrac{|u|}{|u^+|}u^+)\,\mathrm{d}\sigma\,\mathrm{d}u = \int_{\mathbb{S}^2\times\mathbb{R}^3} b(\tfrac{\xi}{|\xi|}\cdot\sigma)\hat{h}(\tfrac{|\xi|}{|\xi^+|}\xi^+)\bar{\hat{f}}(\xi)\,\mathrm{d}\sigma\,\mathrm{d}\xi.$$

A similar argument can be applied to the remainder term and then we get the desired result. ∎

Lemma A.3. *Let $\mathcal{F}$ be the Fourier transform; then $\mathcal{F}W^\varepsilon((-\triangle_{\mathbb{S}^2})^{\frac{1}{2}}) = W^\varepsilon((-\triangle_{\mathbb{S}^2})^{\frac{1}{2}})\mathcal{F}$.*

Proof. By definition in (1.23), if $\xi = \rho\tau$, we have

$$\begin{aligned}\mathcal{F}\big(W^\varepsilon((-\triangle_{\mathbb{S}^2})^{\frac{1}{2}})f\big)(\xi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} W^\varepsilon\big((l(l+1))^{\frac{1}{2}}\big)\mathcal{F}(Y_l^m f_l^m)(\xi) \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} W^\varepsilon\big((l(l+1))^{\frac{1}{2}}\big)Y_l^m(\tau)W_l^m(\rho),\end{aligned}$$

where we use the fact that $\mathcal{F}(Y_l^m f_l^m)(\xi) = Y_l^m(\tau)W_l^m(\rho)$ for some function W_l^m. Note that $(\mathcal{F}f)(\xi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^m(\tau)W_l^m(\rho)$, which yields

$$\begin{aligned}W^\varepsilon((-\triangle_{\mathbb{S}^2})^{\frac{1}{2}})(\mathcal{F}f)(\xi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} W^\varepsilon\big((l(l+1))^{\frac{1}{2}}\big)Y_l^m(\tau)W_l^m(\rho) \\ &= \mathcal{F}\big(W^\varepsilon((-\triangle_{\mathbb{S}^2})^{\frac{1}{2}})f\big)(\xi),\end{aligned}$$

and ends the proof of the lemma. ∎

In the rest of this appendix, we aim to prove Lemma 4.1. For some of the details, [6] is a good reference. Note that (4.7) is equivalent to

$$\begin{aligned}\partial_t a &= -\partial_t \tilde{f}^{(0)} + l^{(0)} + g^{(0)}, \\ \partial_t b_i + \partial_i a &= -\partial_t \tilde{f}_i^{(1)} + l_i^{(1)} + g_i^{(1)}, \quad 1\le i\le 3, \\ \partial_t c + \partial_i b_i &= -\partial_t \tilde{f}_i^{(2)} + l_i^{(2)} + g_i^{(2)}, \quad 1\le i\le 3, \qquad \text{(A.3)} \\ \partial_i b_j + \partial_j b_i &= -\partial_t \tilde{f}_{ij}^{(2)} + l_{ij}^{(2)} + g_{ij}^{(2)}, \quad 1\le i< j\le 3, \qquad \text{(A.4)} \\ \partial_i c &= -\partial_t \tilde{f}_i^{(3)} + l_i^{(3)} + g_i^{(3)}, \quad 1\le i\le 3.\end{aligned}$$

Based on equations (A.3) and (A.4), it is easy to derive the following proposition.

Proposition A.3. *For $j = 1, 2, 3$, the macroscopic component b_j satisfies*

$$-\triangle_x b_j - \partial_j^2 b_j = \sum_{i\neq j} \partial_j(-\partial_t \tilde{f}_i^{(2)} + l_i^{(2)} + g_i^{(2)}) - \sum_{i\neq j} \partial_i(-\partial_t \tilde{f}_{ij}^{(2)} + l_{ij}^{(2)} + g_{ij}^{(2)}) - 2\partial_j(-\partial_t \tilde{f}_j^{(2)} + l_j^{(2)} + g_j^{(2)}). \tag{A.5}$$

The functions $\tilde{f}, \tilde{l}, \tilde{g}$ can be controlled as follows.

Proposition A.4. *It holds that*

$$\sum_{|\alpha|\leq N} |\partial^\alpha \tilde{f}|_{L_x^2}^2 \lesssim \|f_2\|^2_{H_x^N L^2_{\varepsilon,\gamma/2}}, \qquad \sum_{|\alpha|\leq N-1} |\partial^\alpha \tilde{l}|_{L_x^2}^2 \lesssim \|f_2\|^2_{H_x^N L^2_{\varepsilon,\gamma/2}},$$
$$\sum_{|\alpha|\leq N-1} |\partial^\alpha \tilde{g}|_{L_x^2}^2 \lesssim \sum_{|\alpha|\leq N-1} \int |\langle \partial^\alpha g, e\rangle_v|^2 \,\mathrm{d}x.$$

Proof. The first one easily follows by recalling $\tilde{f} = A^{-1}\langle f_2, e\rangle_v$ and using $|\langle \partial^\alpha f_2, e\rangle_v| \lesssim |\mu^{\frac{1}{4}}\partial^\alpha f_2|_{L^2}$. Recalling $\tilde{l} = -A^{-1}\langle v\cdot\nabla_x f_2 + \mathscr{L}^\varepsilon f_2, e\rangle_v$, noting $|\alpha| \leq N-1$, using the upper bound in Theorem 1.1, we get the second inequality. The third one is obvious by recalling $\tilde{g} = A^{-1}\langle g, e\rangle_v$. ∎

The next lemma gives macroscopic conservation laws.

Lemma A.4. *The macroscopic components (a, b, c) satisfy the following system of equations:*

$$\partial_t a - \tfrac{1}{2}\nabla_x\cdot\langle \mu^{\frac{1}{2}}|v|^2 v, f_2\rangle_v = \tfrac{1}{2}\langle (5-|v|^2)\mu^{\frac{1}{2}}, g\rangle_v.$$
$$\partial_t b + \nabla_x(a+5c) + \nabla_x\cdot\langle \mu^{\frac{1}{2}} v\otimes v, f_2\rangle_v = \langle v\mu^{\frac{1}{2}}, g\rangle_v.$$
$$\partial_t c + \tfrac{1}{3}\nabla_x\cdot b + \tfrac{1}{6}\nabla_x\cdot\langle \mu^{\frac{1}{2}}|v|^2 v, f_2\rangle_v = \tfrac{1}{6}\langle (|v|^2-3)\mu^{\frac{1}{2}}, g\rangle_v.$$

Proof. Multiply both sides of equation (4.1) by the collision invariants $\mu^{\frac{1}{2}}\{1, v_i, |v|^2\}$, then integrate over $\mathbb{R}^3$ to get equations for the inner products $\langle \mu^{\frac{1}{2}}, f\rangle_v$, $\langle \mu^{\frac{1}{2}} v_i, f\rangle_v$, $\langle \mu^{\frac{1}{2}}|v|^2, f\rangle_v$. Recalling (1.22), make suitable combinations to get the desired equations. ∎

The previous lemma yields the following one.

Lemma A.5. *The following two estimates are valid:*

$$\sum_{|\alpha|\leq N-1} |\partial^\alpha \partial_t a|_{L_x^2}^2 \lesssim \|f_2\|^2_{H_x^N L^2_{\varepsilon,\gamma/2}} + \sum_{|\alpha|\leq N-1} \int |\langle \partial^\alpha g, e\rangle_v|^2 \,\mathrm{d}x,$$
$$\sum_{|\alpha|\leq N-1} |\partial^\alpha \partial_t (b,c)|_{L_x^2}^2 \lesssim |\nabla_x(a,b,c)|^2_{H_x^{N-1}} + \|f_2\|^2_{H_x^N L^2_{\varepsilon,\gamma/2}} + \sum_{|\alpha|\leq N-1} \int |\langle \partial^\alpha g, e\rangle_v|^2 \,\mathrm{d}x.$$

Now we are ready to prove Lemma 4.1.

Proof of Lemma 4.1. For $|\alpha| \le N-1$, by applying ∂^α to equation (A.5) for b_j, then taking the inner product with $\partial^\alpha b_j$, one has

$$|\nabla_x \partial^\alpha b_j|^2_{L^2_x} + |\partial_j \partial^\alpha b_j|^2_{L^2_x} = \left\langle \sum_{i\neq j} \partial_j \partial^\alpha (-\partial_t \tilde{f}^{(2)}_i + l^{(2)}_i + g^{(2)}_i), \partial^\alpha b_j \right\rangle_x$$
$$- \left\langle \sum_{i\neq j} \partial_i \partial^\alpha (-\partial_t \tilde{f}^{(2)}_{ij} + l^{(2)}_{ij} + g^{(2)}_{ij}), \partial^\alpha b_j \right\rangle_x$$
$$- 2\langle \partial_j \partial^\alpha (-\partial_t \tilde{f}^{(2)}_j + l^{(2)}_j + g^{(2)}_j), \partial^\alpha b_j \rangle_x.$$

By integration by parts, the time derivative can be transferred to $\partial^\alpha b_j$; recalling (4.10), one has

$$|\nabla_x \partial^\alpha b_j|^2_{L^2_x} + |\partial_j \partial^\alpha b_j|^2_{L^2_x} = -\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{I}^b_{\alpha,j}(f)$$
$$+ \left\langle \sum_{i\neq j} \partial_j \partial^\alpha \tilde{f}^{(2)}_i, \partial_t \partial^\alpha b_j \right\rangle_x - \left\langle \sum_{i\neq j} \partial_i \partial^\alpha \tilde{f}^{(2)}_{ij}, \partial_t \partial^\alpha b_j \right\rangle_x$$
$$- 2\langle \partial_j \partial^\alpha \tilde{f}^{(2)}_j, \partial_t \partial^\alpha b_j \rangle_x + \left\langle \sum_{i\neq j} \partial_j \partial^\alpha (l^{(2)}_i + g^{(2)}_i), \partial^\alpha b_j \right\rangle_x$$
$$- \left\langle \sum_{i\neq j} \partial_i \partial^\alpha (l^{(2)}_{ij} + g^{(2)}_{ij}), \partial^\alpha b_j \right\rangle_x$$
$$- 2\langle \partial_j \partial^\alpha (l^{(2)}_j + g^{(2)}_j), \partial^\alpha b_j \rangle_x.$$

By the Cauchy–Schwarz inequality one has

$$\left\langle \sum_{i\neq j} \partial_j \partial^\alpha \tilde{f}^{(2)}_i, \partial_t \partial^\alpha b_j \right\rangle_x + \left\langle \sum_{i\neq j} \partial_i \partial^\alpha \tilde{f}^{(2)}_{ij}, \partial_t \partial^\alpha b_j \right\rangle_x - 2\langle \partial_j \partial^\alpha \tilde{f}^{(2)}_j, \partial_t \partial^\alpha b_j \rangle_x$$
$$\le \eta \sum_{|\alpha|\le N-1} |\partial^\alpha \partial_t (a,b,c)|^2_{L^2_x} + \frac{1}{4\eta} \sum_{|\alpha|\le N} |\partial^\alpha \tilde{f}|^2_{L^2_x}.$$

Via integrating by parts, by the Cauchy–Schwarz inequality, one has

$$\left\langle \sum_{i\neq j} \partial_j \partial^\alpha (l^{(2)}_i + g^{(2)}_i), \partial^\alpha b_j \right\rangle_x - \left\langle \sum_{i\neq j} \partial_i \partial^\alpha (l^{(2)}_{ij} + g^{(2)}_{ij}), \partial^\alpha b_j \right\rangle_x$$
$$- 2\langle \partial_j \partial^\alpha (l^{(2)}_j + g^{(2)}_j), \partial^\alpha b_j \rangle_x$$
$$= -\left\langle \sum_{i\neq j} \partial^\alpha (l^{(2)}_i + g^{(2)}_i), \partial_j \partial^\alpha b_j \right\rangle_x + \left\langle \sum_{i\neq j} \partial^\alpha (l^{(2)}_{ij} + g^{(2)}_{ij}), \partial_i \partial^\alpha b_j \right\rangle_x$$
$$+ 2\langle \partial^\alpha (l^{(2)}_j + g^{(2)}_j), \partial_j \partial^\alpha b_j \rangle_x$$
$$\le \eta |\nabla_x (a,b,c)|^2_{H^{N-1}_x} + \frac{1}{\eta} \sum_{|\alpha|\le N-1} |\partial^\alpha \tilde{l}|^2_{L^2_x} + \frac{1}{\eta} \sum_{|\alpha|\le N-1} |\partial^\alpha \tilde{g}|^2_{L^2_x}.$$

Taking the sum over $1 \le j \le 3$, by Proposition A.4 and Lemma A.5 we get

$$|\nabla_x \partial^\alpha b|^2_{L^2_x} + \frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=1}^{3} \mathcal{I}^b_{\alpha,j}(f) \le \eta |\nabla_x (a,b,c)|^2_{H^{N-1}_x} + \frac{C}{\eta}\Big(\|f_2\|^2_{H^N_x L^2_{\varepsilon,\gamma/2}} + \sum_{|\alpha|\le N-1} \int |\langle \partial^\alpha g, e\rangle_v|^2 \,\mathrm{d}x \Big).$$

Similar techniques can be used to deal with $|\nabla_x \partial^\alpha c|^2_{L^2_x}$ and $|\nabla_x \partial^\alpha a|^2_{L^2_x}$. Recalling (4.9) we have

$$|\nabla_x \partial^\alpha c|^2_{L^2_x} + \frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=1}^{3} \mathcal{I}^c_{\alpha,j}(f) + |\nabla_x \partial^\alpha a|^2_{L^2_x} + \frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=1}^{3} (\mathcal{I}^a_{\alpha,j}(f) + \mathcal{I}^{ab}_{\alpha,j}(f)) \le \eta |\nabla_x (a,b,c)|^2_{H^{N-1}_x} + \frac{C}{\eta}\Big(\|f_2\|^2_{H^N_x L^2_{\varepsilon,\gamma/2}} + \sum_{|\alpha|\le N-1} \int |\langle \partial^\alpha g, e\rangle_v|^2 \,\mathrm{d}x \Big).$$

Patching together the above estimates and taking the sum over $|\alpha| \le N-1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{I}_N(f) + |\nabla_x (a,b,c)|^2_{H^{N-1}_x} \le \eta |\nabla_x (a,b,c)|^2_{H^{N-1}_x} + \frac{C}{\eta}\Big(\|f_2\|^2_{H^N_x L^2_{\varepsilon,\gamma/2}} + \sum_{|\alpha|\le N-1} \int |\langle \partial^\alpha g, e\rangle_v|^2 \,\mathrm{d}x \Big).$$

Taking $\eta = \frac{1}{2}$, the lemma then follows. ■

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References

[1] R. Alexandre, L. Desvillettes, C. Villani, and B. Wennberg, Entropy dissipation and long-range interactions. *Arch. Ration. Mech. Anal.* **152** (2000), no. 4, 327–355 Zbl 0968.76076 MR 1765272

[2] R. Alexandre, F. Hérau, and W.-X. Li, Global hypoelliptic and symbolic estimates for the linearized Boltzmann operator without angular cutoff. *J. Math. Pures Appl. (9)* **126** (2019), 1–71 Zbl 1450.35084 MR 3950012

[3] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang, The Boltzmann equation without angular cutoff in the whole space: II, Global existence for hard potential. *Anal. Appl. (Singap.)* **9** (2011), no. 2, 113–134 Zbl 1220.35110 MR 2793203

[4] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang, The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential. *J. Funct. Anal.* **262** (2012), no. 3, 915–1010 Zbl 1232.35110 MR 2863853

[5] E. A. Carlen, M. C. Carvalho, and X. Lu, On strong convergence to equilibrium for the Boltzmann equation with soft potentials. *J. Stat. Phys.* **135** (2009), no. 4, 681–736 Zbl 1191.76087 MR 2546739

[6] R. Duan, On the Cauchy problem for the Boltzmann equation in the whole space: global existence and uniform stability in $L^2_\xi(H^N_x)$. *J. Differential Equations* **244** (2008), no. 12, 3204–3234 Zbl 1150.82019 MR 2420519

[7] P. T. Gressman and R. M. Strain, Global classical solutions of the Boltzmann equation without angular cut-off. *J. Amer. Math. Soc.* **24** (2011), no. 3, 771–847 Zbl 1248.35140 MR 2784329

[8] P. T. Gressman and R. M. Strain, Sharp anisotropic estimates for the Boltzmann collision operator and its entropy production. *Adv. Math.* **227** (2011), no. 6, 2349–2384 Zbl 1234.35173 MR 2807092

[9] Y. Guo, Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Ration. Mech. Anal.* **169** (2003), no. 4, 305–353 Zbl 1044.76056 MR 2013332

[10] L.-B. He, Sharp bounds for Boltzmann and Landau collision operators. *Ann. Sci. Éc. Norm. Supér. (4)* **51** (2018), no. 5, 1253–1341 Zbl 1428.35266 MR 3942041

[11] L.-B. He and J.-C. Jiang, On the global dynamics of the inhomogeneous Boltzmann equations without angular cutoff: Hard potentials and Maxwellian molecules. 2017, arXiv:1710.00315

[12] L.-B. He, J.-C. Jiang, and Y.-L. Zhou, On the cutoff approximation for the Boltzmann equation with long-range interaction. *J. Stat. Phys.* **181** (2020), no. 5, 1817–1905 Zbl 1460.35249 MR 4179789

[13] L.-B. He, Z.-A. Yao, and Y.-L. Zhou, Asymptotic analysis of the Boltzmann equation with very soft potentials from angular cutoff to non-cutoff. *Commun. Math. Sci.* **19** (2021), no. 2, 287–324 Zbl 07440962 MR 4250283

[14] N. Lerner, Y. Morimoto, K. Pravda-Starov, and C.-J. Xu, Phase space analysis and functional calculus for the linearized Landau and Boltzmann operators. *Kinet. Relat. Models* **6** (2013), no. 3, 625–648 Zbl 1282.35270 MR 3063534

[15] C. Mouhot, Explicit coercivity estimates for the linearized Boltzmann and Landau operators. *Comm. Partial Differential Equations* **31** (2006), no. 7-9, 1321–1348 Zbl 1101.76053 MR 2254617

[16] C. Mouhot and R. M. Strain, Spectral gap and coercivity estimates for linearized Boltzmann collision operators without angular cutoff. *J. Math. Pures Appl. (9)* **87** (2007), no. 5, 515–535 Zbl 1388.76338 MR 2322149

[17] Y. P. Pao, Boltzmann collision operator with inverse-power intermolecular potentials. I, II. *Comm. Pure Appl. Math.* **27** (1974), 407–428; **27** (1974), 559–581 Zbl 0304.45010 MR 636407

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