Ann. Inst. H. Poincaré
Anal. Non Linéaire 39 (2022), 1235–1260
DOI 10.4171/AIHPC/30

Lyapunov functions and finite-time stabilization in optimal time for homogeneous linear and quasilinear hyperbolic systems

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Abstract. Hyperbolic systems in one-dimensional space are frequently used in the modeling of many physical systems. In our recent works we introduced time-independent feedbacks leading to finite stabilization in optimal time of homogeneous linear and quasilinear hyperbolic systems. In this work we present Lyapunov's functions for these feedbacks and use estimates for Lyapunov's functions to rediscover the finite stabilization results.

1. Introduction

Hyperbolic systems in one-dimensional space are frequently used in the modeling of many systems such as traffic flow ([1]), heat exchangers ([39]), fluids in open channels ([15,18,22,23]), transmission lines ([14]), and phase transition ([20]). In our recent works ([10,11]), we introduced time-independent feedbacks leading to finite stabilization in optimal time of homogeneous linear and quasilinear hyperbolic systems. In this work we present Lyapunov's functions for these feedbacks and use estimates for Lyapunov's functions to rediscover the finite stabilization results. More precisely, we are concerned about the following homogeneous, quasilinear, hyperbolic system in one-dimensional space:

$$\partial_t w(t,x) = \Sigma(x, w(t,x))\partial_x w(t,x) \quad \text{for } (t,x) \in [0,+\infty) \times (0,1). \tag{1.1}$$

Here, $w = (w_1, \dots, w_n)^{\mathsf{T}} \colon [0,+\infty) \times (0,1) \to \mathbb{R}^n$ and $\Sigma(\cdot,\cdot)$ is an $(n \times n)$ real matrix-valued function defined in $[0,1] \times \mathbb{R}^n$. We assume that $\Sigma(\cdot,\cdot)$ has $m \geq 1$ distinct positive eigenvalues and $k = n - m \geq 1$ distinct negative eigenvalues. We also assume that, maybe after a change of variables, $\Sigma(x,y)$ for $x \in [0,1]$ and $y \in \mathbb{R}^n$ is of the form

$$\Sigma(x,y) = \operatorname{diag}(-\lambda_1(x,y), \dots, -\lambda_k(x,y), \lambda_{k+1}(x,y), \dots, \lambda_{k+m}(x,y)), \tag{1.2}$$

where

$$-\lambda_1(x,y) < \cdots < -\lambda_k(x,y) < 0 < \lambda_{k+1}(x,y) < \cdots \lambda_{k+m}(x,y). \tag{1.3}$$

2020 Mathematics Subject Classification. 35F05, 35F15, 93C20, 93D05, 93D15, 93D23.
Keywords. Hyperbolic systems, boundary controls, Lyapunov functions, backstepping, finite-time stabilization, optimal time.

Throughout the paper we assume that

$$\lambda_i \text{ and } \partial_y \lambda_i \text{ are of class } C^1 \text{ with respect to } x \text{ and } y \text{ for } 1 \le i \le n = k + m. \tag{1.4}$$

Denote

$$w_- = (w_1, \dots, w_k)^\mathsf{T} \quad tand \quad w_+ = (w_{k+1}, \dots, w_{k+m})^\mathsf{T}.$$

The following types of boundary conditions and controls are considered. The boundary condition at $x = 0$ is given by

$$w_-(t, 0) = \mathrm{B}(w_+(t, 0)) \quad \text{for } t \ge 0, \tag{1.5}$$

for some

$$\mathrm{B} \in (C^2(\mathbb{R}^m))^k \quad \text{with } \mathrm{B}(0) = 0,$$

and the boundary control at $x = 1$ is

$$w_+(t, 1) = (W_{k+1}, \dots, W_{k+m})^\mathsf{T}(t) \quad \text{for } t \ge 0, \tag{1.6}$$

where $W_{k+1}, \dots, W_{k+m}$ are controls.

Set

$$\tau_i = \int_0^1 \frac{1}{\lambda_i(x, 0)}\, dx \quad \text{for } 1 \le i \le n. \tag{1.7}$$

The exact controllability, the null-controllability, and the boundary stabilization of hyperbolic systems in one dimension have been widely investigated in the literature for almost half a century; see, for example, [3] and the references therein. Concerning the exact controllability and the null-controllability related to (1.5) and (1.6), the pioneer works date back to Rauch and Taylor ([35]) and Russell ([36]) for linear inhomogeneous systems. In the quasilinear case with $m \ge k$, the null-controllability was established for $m \ge k$ by Li in [31, Theorem 3.2] (see also [32]). These results hold for time $\tau_k + \tau_{k+1}$.

Concerning the stabilization of (1.1), many works are concerned with boundary conditions of the specific form

$$\begin{pmatrix} w_-(t, 0) \\ w_+(t, 1) \end{pmatrix} = G \begin{pmatrix} w_+(t, 1) \\ w_-(t, 0) \end{pmatrix}, \tag{1.8}$$

where $G\colon \mathbb{R}^n \to \mathbb{R}^n$ is a suitable smooth vector field. Three approaches have been proposed to deal with (1.8). The first one is based on the characteristic method. This method was investigated in the framework of the C^1-norm ([21, 30]). The second one is based on Lyapunov functions ([4–7, 17, 29]). The third one is via the delay equations and was investigated in the framework of the $W^{2,p}$-norm with $p \ge 1$ ([9]). Surprisingly, the stability criterion in the nonlinear setting depends on the norm considered ([9]). Required assumptions impose some restrictions on the magnitude of the coupling coefficients when dealing with inhomogeneous systems.

Another way to stabilize (1.1) is to use the backstepping approach. This was first proposed by Coron et al. ([13]) for 2×2 inhomogeneous system ($m = k = 1$). Later this approach was extended and can now be applied for general pairs (m, k) in the linear case ([2, 8, 10, 12, 16, 27]). In [13], the authors obtained feedbacks leading to finite stabilization in time $\tau_1 + \tau_2$ with $m = k = 1$. In [27], the authors considered the case where Σ is constant and obtained feedback laws for null-controllability at time $\tau_k + \sum_{l=1}^{m} \tau_{k+l}$. Later ([2, 8]), feedbacks leading to finite stabilization in time $\tau_k + \tau_{k+1}$ were derived.

Set, as in [10, 11],

$$T_{\text{opt}} :=\leq \begin{cases} \max\{\tau_1 + \tau_{m+1}, \dots, \tau_k + \tau_{m+k}, \tau_{k+1}\} & \text{if } m \geq k, \\ \max\{\tau_{k+1-m} + \tau_{k+1}, \tau_{k+2-m} + \tau_{k+2}, \dots, \tau_k + \tau_{k+m}\} & \text{if } m < k. \end{cases} \tag{1.9}$$

Define

$$\mathcal{B} := \big\{B \in \mathbb{R}^{k\times m} \text{ such that (1.11) holds for } 1 \leq i \leq \min\{m-1, k\}\big\}, \tag{1.10}$$

where

$$\begin{aligned} &\text{the } i \times i \text{ matrix formed from the last } i \text{ columns} \\ &\text{and the last } i \text{ rows of } B \text{ is invertible.} \end{aligned} \tag{1.11}$$

Using the backstepping approach, we established null-controllability for linear inhomogeneous systems for the optimal time T_{opt} under the condition $B := \nabla \mathrm{B}(0) \in \mathcal{B}$ ([10, 12]) (see also [11] for the nonlinear, homogeneous case). This condition is very natural for obtaining null-controllability at T_{opt}, which roughly speaking allows us to use the l controls $W_{k+m-l+1}, \dots, W_{k+m}$ to control the l directions $w_{k-l+1}, \dots, w_k$ for $1 \leq l \leq \min\{k, m\}$ (the possibility of implementing l controls corresponding to the fastest positive speeds to control l components corresponding to the lowest negative speeds[1]). The optimality of T_{opt} was given in [10] (see also [37]). Related exact controllability results can also be found in [10, 12, 26, 28]. It is easy to see that $\mathcal{B}$ is an open subset of the set of (real) $k \times m$ matrices and the Hausdorff dimension of its complement is $\min\{k, m-1\}$.

We previously obtained time-independent feedbacks leading to finite stabilization for the optimal time T_{opt} of the system (1.1), (1.5), and (1.6) when $B \in \mathcal{B}$ in the linear case ([10]), and in the nonlinear case ([11]). In this paper we introduce Lyapunov functions for these feedbacks. As a consequence of our estimate of the decay rate of solutions via the Lyapunov functions (Theorems 1.1 and 3.1), we are able to rediscover finite stabilization results in optimal time ([10, 11]).

To keep the notation simple in the introduction, from now on we will only discuss the linear setting, i.e., $\Sigma(x, y) = \Sigma(x)$ (so $\lambda_i(x, y) = \lambda_i(x)$) and $\mathrm{B}(\cdot) = B\cdot$ (recall that $B = \nabla \mathrm{B}(0)$). The nonlinear setting will be discussed in Section 3. The boundary condition at $x = 0$ becomes

$$w_-(t, 0) = B w_+(t, 0) \quad \text{for } t \geq 0. \tag{1.12}$$

[1] The i direction ($1 \leq i \leq n$) is called positive (resp. negative) if λ_i is positive (resp. negative).

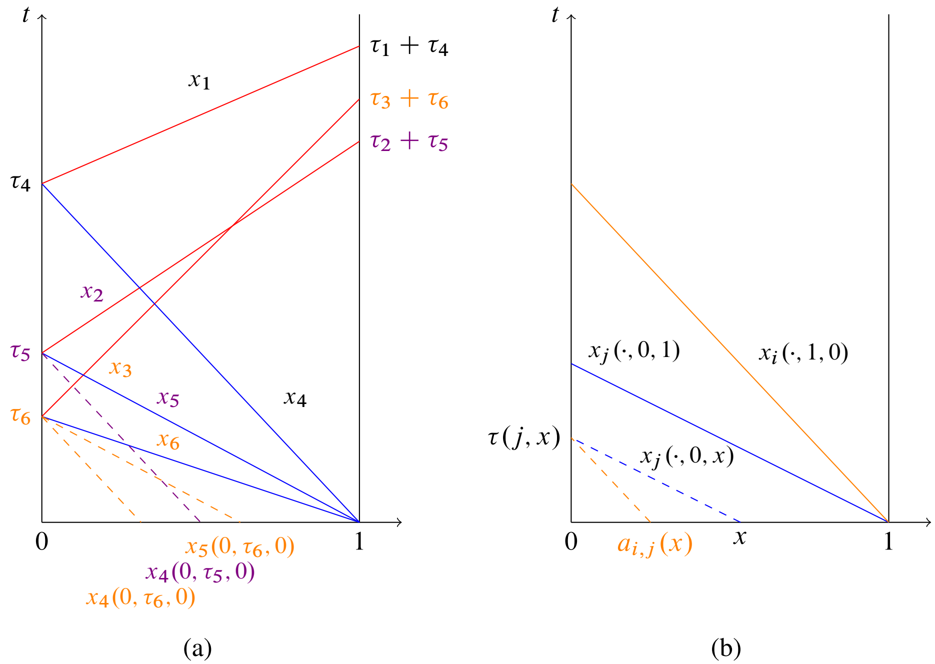

Figure 1. (a) $k = m = 3$, Σ is constant, $x_1 = x_1(\cdot, \tau_4, 0)$, $x_2 = x_2(\cdot, \tau_5, 0)$, $x_3 = x_3(\cdot, \tau_4, 0)$, $x_4 = x_4(\cdot, 0, 1)$, $x_5 = x_5(\cdot, 0, 1)$, and $x_6 = x_6(\cdot, 0, 1)$. (b) $k + 1 \le i < j \le k + m$ and Σ is constant.

We first introduce/recall some notation. Extend λ_i in $\mathbb{R}$ with $1 \le i \le k + m$ by $\lambda_i(0)$ for $x < 0$ and $\lambda_i(1)$ for $x > 1$. For $(s, \xi) \in [0, T] \times [0, 1]$, define $x_i(t, s, \xi)$ for $t \in \mathbb{R}$ by

$$\frac{d}{dt} x_i(t, s, \xi) = \lambda_i(x_i(t, s, \xi)) \text{ and } x_i(s, s, \xi) = \xi \quad \text{if } 1 \le i \le k, \tag{1.13}$$

and

$$\frac{d}{dt} x_i(t, s, \xi) = -\lambda_i(x_i(t, s, \xi)) \text{ and } x_i(s, s, \xi) = \xi \quad \text{if } k + 1 \le i \le k + m \tag{1.14}$$

(see Figure 1).

For $x \in [0, 1]$, and $k + 1 \le j \le k + m$, let $\tau(j, x) \in \mathbb{R}_+$ be such that

$$x_j(\tau(j, x), 0, x) = 0,$$

and set $k + 1 \le i < j \le k + m$,

$$a_{i,j}(x) = x_i(0, \tau(j, x), 0) \tag{1.15}$$

(see Figure 1 (b)). It is clear that $\tau(j, 1) = \tau_j$ for $k + 1 \le j \le k + m$.

We now recall the feedback in [10]. We first consider the case $m \geq k$. Using (1.11) with $i = 1$, one can derive that $w_k(t,0) = 0$ if and only if

$$w_{m+k}(t,0) = M_k(w_{k+1}, \dots, w_{m+k-1})^{\mathsf{T}}(t,0), \tag{1.16}$$

for some constant matrix M_k of size $1 \times (m-1)$. Using (1.11) with $i = 2$, one can derive that $w_k(t,0) = w_{k-1}(t,0) = 0$ if and only if (1.16) and

$$w_{m+k-1}(t,0) = M_{k-1}(w_{k+1}, \dots, w_{m+k-2})^{\mathsf{T}}(t,0) \tag{1.17}$$

hold for some constant matrix M_{k-1} of size $1 \times (m-2)$ by the Gaussian elimination method etc. Finally, using (1.11) with $i = k$, one can derive that $w_k(t,0) = w_{k-1}(t,0) \cdots = w_1(t,0) = 0$ if and only if (1.16), (1.17), ..., and

$$w_{m+1}(t,0) = M_1(w_{k+1}, \dots, w_m)^{\mathsf{T}}(t,0) \tag{1.18}$$

hold for some constant matrix M_1 of size $1 \times (m-k)$ by applying (1.11) with $i = k$ and using the Gaussian elimination method when $m > k$. When $m = k$, a similar fact holds with $M_1 = 0$.

The feedback is then given as follows:

$$w_{m+k}(t,1) = M_k\big(w_{k+1}(t, x_{k+1}(0,\tau_{m+k},0)), \dots, w_{k+m-1}(t, x_{k+m-1}(0,\tau_{m+k},0))\big)^{\mathsf{T}}, \tag{1.19}$$

$$w_{m+k-1}(t,1) = M_{k-1}\big(w_{k+1}(t, x_{k+1}(0,\tau_{m+k-1},0)), \dots, w_{k+m-2}(t, x_{k+m-2}(0,\tau_{m+k-1},0))\big)^{\mathsf{T}}, \tag{1.20}$$

$$\vdots$$

$$w_{m+1}(t,1) = M_1\big(w_{k+1}(t, x_{k+1}(0,\tau_{m+1},0)), \dots, w_m(t, x_{m+1}(0,\tau_{m+1},0))\big)^{\mathsf{T}}, \tag{1.21}$$

and

$$w_j(t,1) = 0 \quad \text{for } k+1 \leq j \leq m \tag{1.22}$$

(see Figure 1 (a)).[2]

We next deal with the case $m < k$. The construction in this case is based on the construction given in the case $m = k$. The feedback is then given by

$$w_{k+m}(t,1) = M_k\big(w_{k+1}(t, x_{k+1}(0,\tau_{k+m},0)), \dots, w_{k+m-1}(t, x_{k+m-1}(0,\tau_{k+m},0))\big), \tag{1.23}$$

[2] In [10] we use $x_i(-\tau_j,0,0)$ with $k+1 \leq i < j \leq k+m$ in the feedback above. Nevertheless, $x_i(-\tau_j,0,0) = x_i(0,\tau_j,0)$.

$$w_{k+m-1}(t,1) = M_{k-1}\big(w_{k+1}(t, x_{k+1}(0, \tau_{k+m-1}, 0)), \dots, \\ w_{k+m-2}(t, x_{k+m-2}(0, \tau_{k+m-1}, 0))\big), \tag{1.24}$$

$$\vdots$$

$$w_{k+2}(t,1) = M_{k+2-m}\big(w_{k+1}(t, x_{k+1}(0, \tau_{k+m-1}, 0))\big), \tag{1.25}$$

$$w_{k+1}(t,1) = M_{k+1-m}, \tag{1.26}$$

with the convention $M_{k+1-m} = 0$.

Remark 1.1. The well-posedness of (1.1) with $\Sigma(x,y) = \Sigma(x)$, (1.5), with the feedback given above for $w_0 \in [L^\infty(0,1)]^n$ is given by [10, Lemma 3.2]. More precisely, for $w_0 \in [L^\infty(0,1)]^n$ and $T > 0$, there exists a unique broad solution $w \in [L^\infty((0,T)\times[0,1])]^n \cap [C([0,T]; L^2(0,1))]^n \cap [C([0,1]; L^2(0,T))]^n$. The broad solutions are defined in [10, Definition 3.1]. The proof is based on a fixed point argument using the norm

$$\|w\| = \sup_{1\le i\le n} \operatorname*{ess\,sup}_{(\tau,\xi)\in(0,T)\times(0,1)} e^{-L_1\tau - L_2\xi}|w_i(\tau,\xi)|,$$

where L_1, L_2 are two large positive numbers with L_1 much larger than L_2.

Concerning these feedbacks, we have the following theorem.

Theorem 1.1. *Let $m, k \ge 1$, and $w_0 \in [L^\infty(0,1)]^n$, and assume that $\Sigma(x,y) = \Sigma(x)$ and $\mathrm{B}(\cdot) = B\cdot$.*[3] *There exists a constant $C \ge 1$, depending only on B and Σ, such that for all $q \ge 1$ and $\Lambda \ge 1$, it holds that*

$$\|w(t,\cdot)\|_{L^q(0,1)} \le C e^{\Lambda(T_{\text{opt}}-t)}\|w(0,\cdot)\|_{L^q(0,1)} \quad \text{for } t \ge 0, \tag{1.27}$$

where w is the solution of (1.1) with $w(0,\cdot) = w_0$ satisfying the feedback (1.19)–(1.22) when $m \ge k$ and (1.23)–(1.26) when $m < k$. As a consequence, we have

$$\|w(t,\cdot)\|_{L^\infty(0,1)} \le C e^{\Lambda(T_{\text{opt}}-t)}\|w(t=0,\cdot)\|_{L^\infty(0,1)} \quad \text{for } t \ge 0. \tag{1.28}$$

As a consequence of Theorem 1.1, finite stabilization in the optimal time T_{opt} is achieved by taking $\Lambda \to +\infty$ since C is independent of Λ. The spirit of deriving appropriate information for the L^∞-norm from the one associated to the L^q-norm was also considered in [4]. The proof of Theorem 1.1 is based on considering the following Lyapunov function. Let $q \ge 1$ and, with $\ell = \max\{m,k\}$, let $\mathcal{V}: [L^q(0,1)]^n \to \mathbb{R}$ be defined by

$$\mathcal{V}(v) = \sum_{i=1}^{\ell} \int_0^1 p_i(x)|v_i(x)|^q\,dx \\ + \sum_{\substack{i \\ \ell+1\le m+i \\ \le k+m}} \int_0^1 p_{m+i}(x)\big|v_{m+i}(x) - M_i\big(v_{k+1}(a_{k+1,m+i}(x)), \dots, \\ v_{m+i-1}(a_{m+i-1,m+i}(x))\big)\big|^q\,dx, \tag{1.29}$$

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[3] Recall that $B = \nabla \mathrm{B}(0)$.

where

$$p_i(x) = \lambda_i^{-1}(x)e^{-q\Lambda\int_0^x \lambda_i^{-1}(s)\,ds+q\Lambda\int_0^1 \lambda_i^{-1}(s)\,ds} \quad \text{for } 1 \le i \le k, \tag{1.30}$$

$$p_i(x) = \Gamma^q\lambda_i^{-1}(x)e^{q\Lambda\int_0^x \lambda_i^{-1}(s)\,ds} \quad \text{for } k+1 \le i \le \ell, \tag{1.31}$$

$$p_{m+i}(x) = \Gamma^q\lambda_{m+i}^{-1}(x)e^{q\Lambda\int_0^x \lambda_{m+i}^{-1}(s)\,ds+q\Lambda\int_0^1 \lambda_i^{-1}(s)\,ds} \quad \text{for } \ell+1 \le m+i \le m+k, \tag{1.32}$$

for some large positive constant $\Gamma \ge 1$ depending only on Σ and B (it is independent of Λ and q).

Remark 1.2. Our Lyapunov functions are explicit. This is useful to study the robustness of our feedback laws with respect to disturbances. The use of Lyapunov functions is a classical tool to study the robustness of feedback laws for control systems in finite dimensions (see, for example, [33, Sections 4.6, 4.7, 5.5.2, 11.7]). For one-dimensional hyperbolic systems, Lyapunov functions are used in particular for the study of a classical robustness property called the input-to-state stability (ISS); see, for example, [19,25,34,38].

The paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. The nonlinear setting is considered in Section 3. The main result there is Theorem 3.1, which is a variant of Theorem 1.1. In the appendix we will establish a lemma that is used in the proofs of Theorems 1.1 and 3.1.

2. Analysis for the linear setting – proof of Theorem 1.1

This section, containing two subsections, is devoted to the proof of Theorem 1.1. The first subsection concerns the case $m \ge k$ and the second the case $m < k$.

2.1. Proof of Theorem 1.1 for $m \ge k$

One can check that $a_{i,j}$ is of class C^1 since Λ is of class C^1 (see, for example, [24, Chapter V]). We claim that, for $k+1 \le i < j \le k+m$ and for $x \in [0,1]$,

$$a'_{i,j}(x) = \lambda_i(a_{i,j}(x))/\lambda_j(x). \tag{2.1}$$

Indeed, by the characteristic method and the definitions of $a_{i,j}$ and $\tau(j,\cdot)$ (see also Figure 1 (b)), we have

$$a_{i,j}(x_j(t,0,x)) = x_i(t,\tau(j,x),0) \quad \text{for } 0 \le t \le \tau(j,x).$$

Taking the derivative with respect to t gives

$$a'_{i,j}(x_j(t,0,x))\partial_t x_j(t,0,x) = \partial_t x_i(t,\tau(j,x),0).$$

This implies, by the definition of x_i and x_j,

$$a'_{i,j}(x_j(t,0,x))\lambda_j(x_j(t,0,x)) = \lambda_i(x_i(t,\tau(j,x),0)).$$

Considering $t = 0$, we obtain (2.1).

As a consequence of (2.1), we have

$$\partial_x \big(w_i(t, a_{i,j}(x)) \big) = \frac{\lambda_i(a_{i,j}(x))}{\lambda_j(x)} \partial_x w_i(t, a_{i,j}(x)). \tag{2.2}$$

Identity (2.2) is one of the key ingredients in deriving properties for $\frac{d}{dt}\mathcal{V}(w(t,\cdot))$, which will be done next.

In what follows, we assume that w is smooth. The general case will follow by a standard approximation argument. Set

$$\begin{aligned} S_{m+i}(t,x) = {} & \lambda_{m+i}(x)\partial_x w_{m+i}(t,x) \\ & - M_i \big(\lambda_{k+1}(a_{k+1,m+i}(x))\partial_x w_{k+1}(t, a_{k+1,m+i}(x)), \dots, \\ & \qquad \lambda_{m+i-1}(a_{m+i-1,m+i}(x))\partial_x w_{m+i-1}(t, a_{m+i-1,m+i}(x)) \big), \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} T_{m+i}(t,x) = {} & w_{m+i}(t,x) \\ & - M_i \big(w_{k+1}(t, a_{k+1,m+i}(x)), \dots, w_{m+i-1}(t, a_{m+i-1,m+i}(x)) \big). \end{aligned} \tag{2.4}$$

Since M_i is constant, it follows from the definition of $\mathcal{V}(v)$ and (1.1) that, for $t \geq 0$,

$$\frac{d}{dt}\mathcal{V}(w(t,\cdot)) = \mathcal{U}_1(t) + \mathcal{U}_2(t), \tag{2.5}$$

where

$$\begin{aligned} \mathcal{U}_1(t) = {} & -\sum_{i=1}^{k} \int_0^1 p_i(x)\lambda_i(x)\partial_x |w_i(t,x)|^q \, dx \\ & + \sum_{i=k+1}^{m} \int_0^1 p_i(x)\lambda_i(x)\partial_x |w_i(t,x)|^q \, dx \end{aligned} \tag{2.6}$$

and

$$\mathcal{U}_2(t) = \sum_{i=1}^{k} \int_0^1 q p_{m+i}(x) S_{m+i}(t,x) |T_{m+i}(t,x)|^{q-2} T_{m+i}(t,x) \, dx. \tag{2.7}$$

We next consider $\mathcal{U}_1$. An integration by parts yields

$$\begin{aligned} \mathcal{U}_1(t) = {} & \sum_{i=1}^{k} \int_0^1 (\lambda_i p_i)'(x) |w_i(t,x)|^q \, dx - \sum_{i=k+1}^{m} \int_0^1 (\lambda_i p_i)'(x) |w_i(t,x)|^q \, dx \\ & - \sum_{i=1}^{k} \lambda_i(x) p_i(x) |w_i(t,x)|^q \Big|_0^1 + \sum_{i=k+1}^{m} \lambda_i(x) p_i(x) |w_i(t,x)|^q \Big|_0^1. \end{aligned} \tag{2.8}$$

Using the feedback (1.22) and the boundary condition (1.5), we obtain

$$\begin{aligned}\mathcal{U}_1(t) &= \sum_{i=1}^{k}\int_0^1 (\lambda_i p_i)'(x)|w_i(t,x)|^q\,dx - \sum_{i=k+1}^{m}\int_0^1 (\lambda_i p_i)'(x)|w_i(t,x)|^q\,dx \\ &\quad - \sum_{i=1}^{k}\lambda_i(1)p_i(1)|w_i(t,1)|^q + \sum_{i=1}^{k}\lambda_i(0)p_i(0)|(Bw_+)_i(t,0)|^q \\ &\quad - \sum_{i=k+1}^{m}\lambda_i(0)p_i(0)|w_i(t,0)|^q. \end{aligned} \tag{2.9}$$

We next deal with $\mathcal{U}_2$. Using (2.2), we derive from the definition of S_{m+i} that

$$\begin{aligned} S_{m+i}(t,x) &= \lambda_{m+i}(x)\partial_x w_{m+i}(t,x) \\ &\quad - \lambda_{m+i}(x)M_i\big(\partial_x\big(w_{k+1}(t,a_{k+1,m+i}(x))\big),\dots, \\ &\qquad \partial_x\big(w_{m+i-1}(t,a_{m+i-1,m+i}(x))\big)\big), \end{aligned} \tag{2.10}$$

which yields, since M_i is constant,

$$S_{m+i}(t,x) = \lambda_{m+i}(x)\partial_x T_{m+i}(t,x). \tag{2.11}$$

Combining (2.7) and (2.11) and integrating by parts yield

$$\begin{aligned}\mathcal{U}_2(t) &= -\sum_{i=1}^{k}\int_0^1 (\lambda_{m+i}p_{m+i})'(x)|T_{m+i}(t,x)|^q \\ &\quad + \sum_{i=1}^{k}\lambda_{m+i}(x)p_{m+i}(x)|T_{m+i}(t,x)|^q\Big|_0^1. \end{aligned} \tag{2.12}$$

By the feedback laws (1.19)–(1.21), the boundary term on the right-hand side of (2.12) is

$$-\sum_{i=1}^{k}\lambda_{m+i}(0)p_{m+i}(0)\big|w_{m+i}(t,0) - M_i(w_{k+1}(t,0),\dots,w_{m+i-1}(t,0))\big|^q.$$

One then has

$$\begin{aligned}\mathcal{U}_2(t) &= -\sum_{i=1}^{k}\int_0^1 (\lambda_{m+i}p_{m+i})'(x)|T_{m+i}(t,x)|^q \qquad (2.13)\\ &\quad -\sum_{i=1}^{k}\lambda_{m+i}(0)p_{m+i}(0)\big|w_{m+i}(t,0) - M_i(w_{k+1}(t,0),\dots,w_{m+i-1}(t,0))\big|^q. \end{aligned}$$

From (2.9) and (2.13), we obtain

$$\mathcal{U}_1(t) + \mathcal{U}_2(t) = \mathcal{W}_1(t) + \mathcal{W}_2(t), \tag{2.14}$$

where

$$\begin{aligned}\mathcal{W}_1(t) = &-\sum_{i=1}^{k}\lambda_i(1)p_i(1)|w_i(t,1)|^q + \sum_{i=1}^{k}\lambda_i(0)p_i(0)|(Bw_+)_i(t,0)|^q \\ &- \sum_{i=k+1}^{m}\lambda_i(0)p_i(0)|w_i(t,0)|^q \\ &- \sum_{i=1}^{k}\lambda_{m+i}(0)p_{m+i}(0)\big|w_{m+i}(t,0) - M_i(w_{k+1}(t,0),\dots,w_{m+i-1}(t,0))\big|^q,\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\mathcal{W}_2(t) = &\sum_{i=1}^{k}\int_0^1(\lambda_i p_i)'(x)|w_i(t,x)|^q\,dx - \sum_{i=k+1}^{m}\int_0^1(\lambda_i p_i)'(x)|w_i(t,x)|^q\,dx \\ &- \sum_{i=1}^{k}\int_0^1(\lambda_{m+i}p_{m+i})'(x)\big|w_{m+i}(t,x) - M_i\big(w_{k+1}(t,a_{k+1,m+i}(x)),\dots, \\ &\qquad w_{m+i-1}(t,a_{m+i-1,m+i}(x))\big)\big|^q\,dx.\end{aligned} \tag{2.16}$$

On the other hand, (1.30), (1.31), and (1.32) imply

$$(\lambda_i p_i)' = -q\Lambda p_i \quad \text{for } 1 \le i \le k, \tag{2.17}$$

$$(\lambda_i p_i)' = q\Lambda p_i \quad \text{for } k+1 \le i \le k+m. \tag{2.18}$$

Using (2.17) and (2.18), we derive from (2.16) that

$$\mathcal{W}_2(t) = -q\Lambda\mathcal{V}(t). \tag{2.19}$$

We have, by the Gaussian elimination process,

$$\sum_{i=j}^{k}\big|w_{m+i}(t,0) - M_i(w_{k+1}(t,0),\dots,w_{m+i-1}(t,0))\big| \ge C\sum_{i=j}^{k}|(Bw_+)_i(t,0)|$$

for $j = k$, then $j = k-1,\dots$, and finally for $j = 1$. Using the fact that

$$\int_0^1 \lambda_{i_1}^{-1}(s)\,ds < \int_0^1 \lambda_{i_2}^{-1}(s)\,ds \quad \text{for } 1 \le i_1 < i_2 \le k,$$

and, for $a_i \ge 0$ with $1 \le i \le j \le k$ and $1 \le q < +\infty$,

$$\Big(\sum_{i=1}^{j}a_i\Big)^q \le C^q\sum_{i=1}^{j}a_i^q$$

for some positive constant C independent of q and a_i, we derive from (1.30) and (1.32) that, for large Γ (the largeness of Γ depends only on B, k, and l; it is in particular independent of Λ and q),

$$\sum_{i=1}^{k} \lambda_{m+i}(0) p_{m+i}(0) \big| w_{m+i}(t,0) - M_i(w_{k+1}(t,0), \ldots, w_{m+i-1}(t,0)) \big|^q \\ \geq \sum_{i=1}^{k} \lambda_i(0) p_i(0) |(Bw_+)_i(t,0)|^q.$$

It follows from (2.15) that

$$\mathcal{W}_1(t) \leq 0. \tag{2.20}$$

Combining (2.5), (2.14), (2.19), and (2.20) yields

$$\frac{d}{dt}\mathcal{V}(w(t,\cdot)) \leq -q\Lambda \mathcal{V}(w(t,\cdot)).$$

This implies

$$\mathcal{V}(w(t,\cdot)) \leq e^{-q\Lambda t}\mathcal{V}(w(0,\cdot)). \tag{2.21}$$

Set

$$A = \sup_{\substack{1\leq i\leq n \\ x\in(0,1)}} p_i(x) \quad \text{and} \quad a = \inf_{\substack{1\leq i\leq n \\ x\in(0,1)}} p_i(x), \tag{2.22}$$

and define, for $v \in [L^2(0,1)]^n$,

$$\|v\|_{\mathcal{V}}^q = \int_0^1 \sum_{i=1}^{m} |v_i(x)|^q \, dx \\ + \int_0^1 \sum_{i=1}^{k} \big| v_{m+i}(x) - M_i\big(v_{k+1}(a_{k+1,m+i}(x)), \ldots, \\ v_{m+i-1}(a_{m+i-1,m+i}(x))\big)\big|^q \, dx. \tag{2.23}$$

Using (1.30), (1.31), (1.32), and the definition of T_{opt} (1.9), one can check that

$$A/a \leq C^q e^{q\Lambda T_{\text{opt}}} \tag{2.24}$$

for some positive constant C depending only on Γ and Σ. It follows that

$$\|w(t,\cdot)\|_{\mathcal{V}}^q \overset{(2.22),(2.23)}{\leq} \frac{1}{a}\mathcal{V}(w(t,\cdot)) \overset{(2.21)}{\leq} \frac{1}{a} e^{-q\Lambda t}\mathcal{V}(w(0,\cdot)) \\ \overset{(2.22),(2.23)}{\leq} \frac{A}{a} e^{-q\Lambda t}\|w_0\|_{\mathcal{V}}^q \overset{(2.24)}{\leq} C^q e^{q\Lambda(T_{\text{opt}}-t)}\|w_0\|_{\mathcal{V}}^q.$$

Since $\|v\|_{\mathcal{V}} \sim \|v\|_{L^q(0,1)}$ for $v \in [L^q(0,1)]^n$ by Lemma A.1, assertion (1.27) follows.

It is clear that (1.28) is a consequence of (1.27) by taking $q \to +\infty$. ∎

2.2. Proof of Theorem 1.1 for $m < k$

The proof of Theorem 1.1 for $m < k$ is similar to that for $m \geq k$. Indeed, one has

$$\mathcal{W}_2(t) = -\Lambda \mathcal{V}. \tag{2.25}$$

We have, by the Gaussian elimination process, for $k + 1 \leq m + j \leq m + k$,

$$\begin{aligned}\sum_{\substack{i \\ m+j \leq m+i \\ \leq m+k}} &\big| w_{m+i}(t,0) - M_i(w_{k+1}(t,0), \ldots, w_{m+i-1}(t,0)) \big| \\ &\geq C \sum_{\substack{i \\ m+j \leq m+i \\ \leq m+k}} |(Bw_+)_i(t,0)|.\end{aligned}$$

and, for $1 \leq j \leq k - m$,

$$\sum_{\substack{i \\ k+1 \leq m+i \\ \leq m+k}} \big| w_{m+i}(t,0) - M_i(w_{k+1}(t,0), \ldots, w_{m+i-1}(t,0)) \big| \geq C |(Bw_+)_j(t,0)|.$$

Using the fact

$$\int_0^1 \lambda_{i_1}^{-1}(s)\, ds < \int_0^1 \lambda_{i_2}^{-1}(s)\, ds \quad \text{for } 1 \leq i_1 < i_2 \leq k,$$

we derive from (1.30) and (1.32) that, for large Γ (the largeness of Γ depends only on B, k, and l; it is in particular independent of Λ and q),

$$\begin{aligned}\sum_{\substack{i \\ k+1 \leq m+i \\ \leq m+k}} &\lambda_{m+i}(0) p_{m+i}(0) \big| w_{m+i}(t,0) - M_i(w_{k+1}(t,0), \ldots, w_{m+i-1}(t,0)) \big|^q \\ &\geq \sum_{i=1}^{k} \lambda_i(0) p_i(0) |(Bw_+)_i(t,0)|^q.\end{aligned}$$

One can then derive that

$$\mathcal{W}_1(t) \leq 0. \tag{2.26}$$

Combining (2.25) and (2.26) yields

$$\frac{d}{dt}\mathcal{V}(t) \leq -\Lambda \mathcal{V}(t).$$

The conclusion now follows as in the proof of Theorem 1.1 for $m \geq k$. The details are omitted. ∎

3. On the nonlinear setting

The following result was established in [11].

Proposition 3.1. *Assume that* $\nabla B(0) \in \mathcal{B}$*. Then, for any* $T > T_{\rm opt}$*, there exist* $\varepsilon > 0$ *and a time-independent feedback control for* (1.1)*,* (1.5)*, and* (1.6) *such that if the compatibility conditions (at* $x = 0$*)* (3.1) *and* (3.2) *below hold for* $w(0, \cdot)$*,*

$$(\|w(0,\cdot)\|_{C^1([0,1])} < \varepsilon) \Rightarrow (w(T,\cdot) = 0).$$

In what follows, we denote, for $x \in [0, 1]$ and $y \in \mathbb{R}^n$,

$$\Sigma_-(x, y) = \operatorname{diag}(-\lambda_1(x, y), \dots, -\lambda_k(x, y)),$$
$$\Sigma_+(x, y) = \operatorname{diag}(\lambda_{k+1}(x, y), \dots, \lambda_n(x, y)).$$

The compatibility conditions considered in Theorem 3.1 are

$$w_-(0,0) = \mathrm{B}(w_+(0,0)) \tag{3.1}$$

and

$$\Sigma_-(0, w(0,0))\partial_x w_-(0,0) = \nabla \mathrm{B}(w_+(0,0))\Sigma_+(0, w(0,0))\partial_x w_+(0,0). \tag{3.2}$$

We next describe the feedback given in the proof of Proposition 3.1 in [11]. Let x_j be defined as

$$\frac{d}{dt}x_j(t,s,\xi) = \lambda_j\big(x_j(t,s,\xi), w(t, x_j(t,s,\xi))\big) \text{ and } x_j(s,s,\xi) = \xi \quad \text{for } 1 \le j \le k,$$

and

$$\frac{d}{dt}x_j(t,s,\xi) = -\lambda_j\big(x_j(t,s,\xi), w(t, x_j(t,s,\xi))\big) \text{ and } x_j(s,s,\xi) = \xi \quad \text{for } k+1 \le j \le k+m.$$

At this stage we do not explicitly mention the domain of x_j. Later, we will only consider flows in regions where the solution w is well defined.

To arrange the compatibility of our controls, we also introduce auxiliary variables satisfying autonomous dynamics. Set $\delta = T - T_{\rm opt} > 0$. For $t \ge 0$, let, for $k+1 \le j \le k+m$,

$$\zeta_j(0) = w_{0,j}(1), \quad \zeta_j'(0) = \lambda_j(1, w_0(1))w_{0,j}'(1), \quad \zeta_j(t) = 0 \quad \text{for } t \ge \delta/2 \tag{3.3}$$

and

$$\eta_j(0) = 1, \quad \eta_j'(0) = 0, \quad \eta_j(t) = 0 \quad \text{for } t \ge \delta/2. \tag{3.4}$$

We first deal with the case $m \ge k$. Consider the last equation of (1.5). Impose the condition $w_k(t,0) = 0$. Using (1.11) with $i = 1$ and the implicit function theorem, one can then write the last equation of (1.5) in the form

$$w_{m+k}(t,0) = M_k(w_{k+1}(t,0), \dots, w_{m+k-1}(t,0)), \tag{3.5}$$

for some C^2 nonlinear map M_k from U_k into $\mathbb{R}$ for some neighborhood U_k of $0 \in \mathbb{R}^{m-1}$ with $M_k(0) = 0$ provided that $|w_+(t,0)|$ is sufficiently small.

Consider the last two equations of (1.5) and impose the condition $w_k(t,0) = w_{k-1}(t,0) = 0$. Using (1.11) with $i = 2$ and the Gaussian elimination approach, one can then write these two equations in the form (3.5) and

$$w_{m+k-1}(t,0) = M_{k-1}(w_{k+1}(t,0), \dots, w_{m+k-2}(t,0)), \tag{3.6}$$

for some C^2 nonlinear map M_{k-1} from U_{k-1} into $\mathbb{R}$ for some neighborhood U_{k-1} of $0 \in \mathbb{R}^{m-2}$ with $M_{k-1}(0) = 0$ provided that $|w_+(t,0)|$ is sufficiently small, etc. Finally, consider the k equations of (1.5) and impose the condition $w_k(t,0) = \dots = w_1(t,0) = 0$. Using (1.11) with $i = k$ and the Gaussian elimination approach, one can then write these k equations in the form (3.5), (3.6), ..., and

$$w_{m+1}(t,0) = M_1(w_{k+1}(t,0), \dots, w_m(t,0)), \tag{3.7}$$

for some C^2 nonlinear map M_1 from U_1 into $\mathbb{R}$ for some neighborhood U_1 of $0 \in \mathbb{R}^{m-k}$ with $M_1(0) = 0$ provided that $|w_+(t,0)|$ is sufficiently small for $m > k$. When $m = k$, we just define $M_1 = 0$.

We are ready to construct a feedback law for the null-controllability at time T. Let t_{m+k} be such that

$$x_{m+k}(t + t_{m+k}, t, 1) = 0.$$

It is clear that t_{m+k} depends only on the current state $w(t,\cdot)$. Let $D_{m+k} = D_{m+k}(t) \subset \mathbb{R}^2$ be the open set whose boundary is $\{t\} \times [0,1]$, $[t, t + t_{m+k}] \times \{0\}$, and $\{(s, x_{m+k}(s,t,1));\ s \in [t, t+t_{m+k}]\}$. Then D_{m+k} depends only on the current state as well. This implies

$$x_{k+1}(t, t + t_{m+k}, 0), \dots, x_{k+m-1}(t, t+t_{m+k}, 0)$$
$$\text{are well defined by the current state } w(t,\cdot).$$

As a consequence, the feedback

$$\begin{aligned} w_{m+k}(t,1) = \zeta_{m+k}(t) &\\ + (1 - \eta_{m+k}(t)) M_k\big(&w_{k+1}(t, x_{k+1}(t, t+t_{m+k}, 0)), \dots, \\ &w_{k+m-1}(t, x_{k+m-1}(t, t+t_{m+k}, 0))\big) \end{aligned} \tag{3.8}$$

is well defined by the current state $w(t,\cdot)$.

We then consider system (1.1), (1.5), and the feedback (3.8). Let t_{m+k-1} be such that

$$x_{m+k-1}(t + t_{m+k-1}, t, 1) = 0.$$

It is clear that t_{m+k-1} depends only on the current state $w(t,\cdot)$ and the feedback law (3.8). Let $D_{m+k-1} = D_{m+k-1}(t) \subset \mathbb{R}^2$ be the open set whose boundary is $\{t\} \times [0,1]$, $[t, t + t_{m+k-1}] \times \{0\}$, and $\{(s, x_{m+k-1}(s,t,1));\ s \in [t, t + t_{m+k-1}]\}$. Then D_{m+k-1} depends only on the current state. This implies

$$x_{k+1}(t, t + t_{m+k-1}, 0), \dots, x_{k+m-2}(t, t + t_{m+k-1}, 0)$$
$$\text{are well defined by the current state } w(t,\cdot).$$

As a consequence, the feedback

$$\begin{aligned} w_{m+k-1}(t,1) = \zeta_{m+k-1}(t) &\\ + (1 - \eta_{m+k-1}(t)) M_{k-1}\big(w_{k+1}(t, x_{k+1}(t, t + t_{m+k-1}, 0)), \dots, &\\ w_{k+m-2}(t, x_{k+m-2}(t, t + t_{m+k-1}, 0))\big) & \end{aligned} \tag{3.9}$$

is well defined by the current state $w(t,\cdot)$.

We continue this process and reach the system (1.1), (1.5), (3.8), ...,

$$\begin{aligned} w_{m+2}(t,1) = \zeta_{m+2}(t) &\\ + (1 - \eta_{m+2}(t)) M_2\big(w_{k+1}(t, x_{k+1}(t, t + t_{m+2}, 0)), \dots, &\\ w_{m+1}(t, x_{m+1}(t, t + t_{m+2}, 0))\big). & \end{aligned} \tag{3.10}$$

Let t_{m+1} be such that

$$x_{m+1}(t + t_{m+1}, t, 1) = 0.$$

It is clear that t_{m+1} depends only on the current state $w(t,\cdot)$ and the feedback law (3.8), ..., (3.10). Let $D_{m+1} = D_{m+1}(t) \subset \mathbb{R}^2$ be the open set whose boundary is $\{t\} \times [0,1]$, $[t, t + t_{m+1}] \times \{0\}$, and $\{(s, x_{m+1}(s,t,1));\ s \in [t, t + t_{m+1}]\}$. Then D_{m+1} depends only on the current state. This implies

$$x_{k+1}(t, t + t_{m+1}, 0), \dots, x_m(t, t + t_{m+1}, 0)$$
$$\text{are well defined by the current state } w(t,\cdot).$$

As a consequence, the feedback

$$\begin{aligned} w_{m+1}(t,1) = \zeta_{m+1}(t) &\\ + (1 - \eta_{m+1}(t)) M_1\big(w_{k+1}(t, x_{k+1}(t, t + t_{m+1}, 0)), \dots, &\\ w_m(t, x_m(t, t + t_{m+1}, 0))\big) & \end{aligned} \tag{3.11}$$

is well defined by the current state $w(t,\cdot)$.

To complete the feedback for the system, we consider, for $k + 1 \le j \le m$,

$$w_j(t,1) = \zeta_j(t). \tag{3.12}$$

We next consider the case $k > m$. The feedback law is then given as

$$\begin{aligned} w_{m+k}(t,1) &= \zeta_{m+k}(t) \\ &\quad + (1-\eta_{m+k}(t))M_k\big(w_{k+1}(t, x_{k+1}(t, t+t_{m+k}, 0)), \dots, \\ &\qquad\qquad w_{k+m-1}(t, x_{k+m-1}(t, t+t_{m+k}, 0))\big), \end{aligned} \tag{3.13}$$

$$\vdots$$

$$\begin{aligned} w_{k+2}(t,1) &= \zeta_{k+2}(t) \\ &\quad + (1-\eta_{k+2}(t))M_{k+2-m}\big(w_{k+1}(t, x_{k+1}(t, t+t_{k+2}, 0))\big), \end{aligned} \tag{3.14}$$

$$w_{k+1}(t,1) = \zeta_{k+1}(t) + (1-\eta_{k+1}(t))M_{k+1-m}, \tag{3.15}$$

with the convention $M_{k+1-m} = 0$.

Remark 3.1. The feedbacks above are *time independent* and the well-posedness of the control system is established in [11, Lemma 2.2] for small initial data.

To introduce the Lyapunov function, as in the linear setting, for $k+1 \le i < j \le k+m$, and for $x \in [0,1]$, $t \ge \delta/2$, let $\tau(j,t,x)$ be such that

$$x_j(\tau(j,t,x), t, x) = 0,$$

and define

$$a_{i,j}(t,x) = a_{i,j}(x, w(t,\cdot)) = x_i(t, \tau(j,t,x), 0).$$

In the last identities, by convention we considered $x_i(t,\tau(j,t,x),0)$ as a function of t and x denoted by $a_{i,j}(t,x)$ or a function of x and $w(t,\cdot)$ denoted by $a_{i,j}(x, w(t,\cdot))$.

Set

$$\mathcal{H} = \big\{v \in [C^1([0,1])]^n;\ v \text{ satisfies the compatibility conditions at 0 and 1}\big\}.$$

Let $q \ge 1$ and let $\mathcal{V}\colon \mathcal{H} \to \mathbb{R}$ $(q \ge 1)$ be defined by

$$\mathcal{V}(v) = \widehat{\mathcal{V}}(v) + \widetilde{\mathcal{V}}(v). \tag{3.16}$$

Here, with $\ell = \max\{m,k\}$,

$$\begin{aligned} \widehat{\mathcal{V}}(v) &= \sum_{i=1}^{\ell} \int_0^1 p_i(x)|v_i(x)|^q \, dx \\ &\quad + \sum_{\substack{i \\ \ell+1 \le m+i \\ \le k+m}} \int_0^1 p_{m+i}(x)\big|v_{m+i}(x) - M_i\big(v_{k+1}(a^v_{k+1,m+i}(x,v)), \dots, \\ &\qquad\qquad v_{m+i-1}(a^v_{m+i-1,m+i}(x,v))\big)\big|^q \, dx \end{aligned} \tag{3.17}$$

and

$$\widetilde{\mathcal{V}}(v) = \sum_{i=1}^{\ell} \int_0^1 p_i(x) |\partial_t v(0,x)|^q \, dx \tag{3.18}$$
$$+ \sum_{\substack{i \\ \ell+1 \le m+i \\ \le k+m}} \int_0^1 p_{m+i}(x) \big| \partial_t v_{m+i}(0,x) - \partial_t \big(M_i \big(v_{k+1}(t, a^v_{k+1,m+i}(t,x)), \ldots, v_{m+i-1}(t, a^v_{m+i-1,m+i}(t,x)) \big) \big)_{t=0} \big|^q \, dx.$$

Here $v(t,\cdot)$ is the corresponding solution with $v(0,\cdot) = v$ and $a^v_{k+j,m+i}$ is defined as $a_{k+j,m+i}$ with $w(t,\cdot)$ replaced by $v(t,\cdot)$. We also define here

$$p_i(x) = \lambda_i^{-1}(x,0) e^{-q\Lambda \int_0^x \lambda_i^{-1}(s,0)\,ds + q\Lambda \int_0^1 \lambda_i^{-1}(s,0)\,ds} \quad \text{for } 1 \le i \le k, \tag{3.19}$$
$$p_i(x) = \Gamma^q \lambda_i^{-1}(x,0) e^{q\Lambda \int_0^x \lambda_i^{-1}(s,0)\,ds} \quad \text{for } k+1 \le i \le \ell, \tag{3.20}$$
$$p_{m+i}(x) = \Gamma^q \lambda_{m+i}^{-1}(x,0) e^{q\Lambda \int_0^x \lambda_{m+i}^{-1}(s,0)\,ds + q\Lambda \int_0^1 \lambda_i^{-1}(s,0)\,ds} \quad \text{for } \ell+1 \le m+i \le m+k, \tag{3.21}$$

for some large positive constant $\Gamma \ge 1$ depending only on Σ and B (it is independent of Λ and q).

Concerning the feedback given above, we have the following theorem.

Theorem 3.1. *Let $m, k \ge 1$. There exists a constant $C \ge 1$, depending only on* B *and Σ such that for $\Lambda \ge 1$ and for $T > T_{\rm opt}$, there exists $\varepsilon > 0$ such that if the compatibility conditions (at $x = 0$)* (3.1) *and* (3.2) *hold for $w(0,\cdot)$, and $\|w(0,\cdot)\|_{C^1([0,1])} < \varepsilon$, we have, for $t \ge \delta/2$ with $\delta = T - T_{\rm opt}$,*

$$\|w(t,\cdot)\|_{W^{1,q}(0,1)} \le C e^{\Lambda(T_{\rm opt} - t)} \big(\|w(0,\cdot)\|_{W^{1,q}(0,1)} + \|\zeta\|_{C^1} + \|\eta\|_{C^1} \|w(0,\cdot)\|_{W^{1,q}(0,1)} \big), \tag{3.22}$$

where w is the solution of (1.1) *with $w(0,\cdot) = w_0$ satisfying* (3.8)–(3.12) *when $m \ge k$ and* (3.13)–(3.15) *when $m < k$, where ζ_j and η_j are given in* (3.3) *and* (3.4). *As a consequence, we have*

$$\|w(t,\cdot)\|_{C^1([0,1])} \le C e^{\Lambda(T_{\rm opt} - t)} \big(\|w(0,\cdot)\|_{C^1([0,1])} + \|\zeta\|_{C^1} + \|\eta\|_{C^1} \|w(0,\cdot)\|_{C^1([0,1])} \big). \tag{3.23}$$

Proof. We first claim that, for $k+1 \le i < j \le k+m$ and $x \in [0,1]$,

$$\lambda_i \big(a_{i,j}(t,x), w(t, a_{i,j}(t,x)) \big) + \partial_t a_{i,j}(t,x) = \lambda_j(x, w(t,x)) \partial_x a_{i,j}(t,x). \tag{3.24}$$

Indeed, by the characteristic, we have

$$a_{i,j}(s, x_j(s,t,x)) = x_i(s, \tau(j,t,x), 0) \quad \text{for } t \le s \le \tau(j,t,x).$$

Taking the derivative with respect to s yields, for $t \le s \le \tau(j,t,x)$,

$$\begin{aligned}\partial_t a_{i,j}(s, x_j(s,t,x)) + \partial_s x_j(s,t,x)\partial_x a_{i,j}(s, x_j(s,t,x)) \\ = \partial_s x_i(s, \tau(j,t,x), 0).\end{aligned}$$

Considering $s = t$ and using the definition of the flows, we obtain the claim.

As a consequence of (3.24), we have

$$\begin{aligned}&\partial_x \big(w_i(t, a_{i,j}(t,x))\big) \\ &\quad = \frac{\lambda_i\big(a_{i,j}(t,x), w(t, a_{i,j}(t,x))\big) + \partial_t a_{i,j}(t,x)}{\lambda_j(x, w(t,x))} \partial_x w_i(t, a_{i,j}(t,x)).\end{aligned} \tag{3.25}$$

Identity (3.25) is a variant of (2.2) for the nonlinear setting and plays a role in our analysis.

We next consider only the case $m \ge k$. The case $m < k$ can be proved similarly to Theorem 1.1. We will assume that the solutions are of class C^2. The general case can be established via a density argument as in [4, p. 1475] and [3, Comments 4.6, pp. 127–128].

We first deal with $\widehat{\mathcal{V}}$. We have, for $t \ge \delta/2$,

$$\begin{aligned}\frac{d}{dt}\widehat{\mathcal{V}}(w(t,\cdot)) = &- \sum_{i=1}^{k} \int_0^1 p_i(x)\lambda_i(x, w(t,x))\partial_x |w_i(t,x)|^q \, dx \\ &+ \sum_{i=k+1}^{m} \int_0^1 p_i(x)\lambda_i(x, w(t,x))\partial_x |w_i(t,x)|^q \, dx \\ &+ \sum_{i=1}^{k} \int_0^1 q p_{m+i}(x)\partial_t T_{m+i}(t,x)|T_{m+i}(t,x)|^{q-2} T_{m+i}(t,x)\, dx,\end{aligned} \tag{3.26}$$

where

$$\begin{aligned}T_{m+i}(t,x) = &\; w_{m+i}(t,x) \\ &- M_i\big(w_{k+1}(t, a_{k+1,m+i}(t,x)), \dots, w_{m+i-1}(t, a_{m+i-1,m+i}(t,x))\big).\end{aligned} \tag{3.27}$$

Using (3.25) and noting that, for $k+1 \le i \le j \le k+m$,

$$\partial_t w_i(t, a_{i,j}(t,x)) = \lambda_i\big(a_{i,j}(t,x), w(t, a_{i,j}(t,x))\big)\partial_x w_i(t, a_{i,j}(t,x)),$$

one can prove that

$$\partial_t T_{m+i}(t,x) = \lambda_{m+i}(x, w(t,x))\partial_x T_{m+i}(t,x). \tag{3.28}$$

Using (3.28) and integrating by parts, as in (2.14), we obtain

$$\frac{d}{dt}\widehat{\mathcal{V}}(w(t,\cdot)) = \widehat{\mathcal{W}}_1(t) + \widehat{\mathcal{W}}_2(t), \tag{3.29}$$

where

$$
\begin{aligned}
\widehat{\mathcal{W}}_1(t) = &-\sum_{i=1}^{k} \lambda_i(1, w(t,1)) p_i(1) |w_i(t,1)|^q \\
&+ \sum_{i=1}^{k} \lambda_i(0, w(t,0)) p_i(0) |(Bu_+)_i(t,0)|^q \\
&- \sum_{i=k+1}^{m} \lambda_i(0, w(t,0)) p_i(0) |w_i(t,0)|^q \\
&- \sum_{i=1}^{k} \lambda_{m+i}(0, w(t,0)) p_{m+i}(0) \\
&\quad \times \left| w_{m+i}(t,0) - M_i(w_{k+1}(t,0), \ldots, w_{m+i-1}(t,0)) \right|^q
\end{aligned} \tag{3.30}
$$

and

$$
\begin{aligned}
\widehat{\mathcal{W}}_2(t) = &\sum_{i=1}^{k} \int_0^1 \big(\lambda_i(x, w(t,x)) p_i(x)\big)_x |w_i(t,x)|^q \, dx \\
&- \sum_{i=k+1}^{m} \int_0^1 \big(\lambda_i(x, w(t,x)) p_i(x)\big)_x |w_i(t,x)|^q \, dx \\
&- \sum_{i=1}^{k} \int_0^1 \big(\lambda_{m+i}(x, w(t,x)) p_{m+i}(x)\big)_x \\
&\quad \times \left| w_{m+i}(t,x) \, dx - M_i\big(w_{k+1}(t, a_{k+1,m+i}(t,x)), \ldots, \right. \\
&\qquad \left. w_{m+i-1}(t, a_{m+i-1,m+i}(t,x))\big) \right|^q dx.
\end{aligned} \tag{3.31}
$$

As in the proof of Theorem 1.1, we also have, for large Γ and $|w(t,0)|$ sufficiently small,

$$
\begin{aligned}
&\sum_{i=1}^{k} \lambda_{m+i}(0, w(t,0)) p_{m+i}(0) \left| w_{m+i}(t,0) - M_i\big(w_{k+1}(t,0), \ldots, w_{m+i-1}(t,0)\big) \right|^2 \\
&\qquad \geq \sum_{i=1}^{k} \lambda_i(0, w(t,0)) p_i(0) |(Bw_+)_i(t,0)|^2.
\end{aligned}
$$

This implies

$$
\widehat{\mathcal{W}}_1(t) \leq 0. \tag{3.32}
$$

Concerning $\widehat{\mathcal{W}}_2(t)$, we write

$$
\lambda_i(x, w(t,x)) p_i(x) = \frac{\lambda_i(x, w(t,x))}{\lambda_i(x,0)} \lambda_i(x,0) p_i(x).
$$

Note that, since Σ and $\partial_y \Sigma$ are of class C^1,

$$\Big|\frac{\lambda_i(x, w(t,x))}{\lambda_i(x,0)} - 1\Big| + \Big|\partial_x\Big(\frac{\lambda_i(x, w(t,x))}{\lambda_i(x,0)}\Big)\Big| \le C(\varepsilon, \delta),$$

a quantity which goes to 0 if $\varepsilon \to 0$ for fixed δ.

Using (3.19) and (3.21), we obtain

$$\widehat{W}_2(t) \le -q\Lambda(1 - C(\varepsilon,\delta))\widehat{\mathcal{V}}(t). \tag{3.33}$$

Combining (3.29), (3.32), and (3.33) yields

$$\frac{d}{dt}\widehat{\mathcal{V}}(t) \le -q(\Lambda - C(\varepsilon,\delta))\widehat{\mathcal{V}}(t) \quad \text{for } t \ge \delta/2. \tag{3.34}$$

We next investigate $\widetilde{\mathcal{V}}$. By (3.18), we have, for $t \ge \delta/2$,

$$\begin{aligned} \widetilde{\mathcal{V}}(w(t,x)) = &\sum_{i=1}^{k} \int_0^1 p_i(x)|\partial_t w(t,x)|^q \, dx \\ &+ \sum_{\substack{i \\ k+1 \le m+i \\ \le k+m}} \int_0^1 p_{m+i}(x) \\ &\qquad \times \big|\partial_t w_{m+i}(t,x) - \big(M_i\big(w_{k+1}(t, a_{k+1,m+i}(t,x)), \dots, \\ &\qquad\qquad w_{m+i-1}(t, a_{m+i-1,m+i}(t,x))\big)\big)_t\big|^q \, dx. \end{aligned} \tag{3.35}$$

Using (3.28), we have

$$\begin{aligned} &\frac{d}{dt}\widetilde{\mathcal{V}}(w(t,\cdot)) \\ &\quad = -\sum_{i=1}^{k} \int_0^1 p_i(x)\lambda_i(x, w(t,x))\partial_x|\partial_t w_i(t,x)|^q \, dx \\ &\quad + \sum_{i=1}^{k} \int_0^1 \frac{q p_i(x)}{\lambda_i(x, w(t,x))}\partial_y\lambda_i(x, w(t,x))\partial_t w(t,x)|\partial_t w_i(t,x)|^q \, dx \\ &\quad + \sum_{i=k+1}^{m} \int_0^1 p_i(x)\lambda_i(x, w(t,x))\partial_x|\partial_t w_i(t,x)|^q \, dx \\ &\quad + \sum_{i=k+1}^{m} \int_0^1 \frac{q p_i(x)}{\lambda_i(x, w(t,x))}\partial_y\lambda_i(x, w(t,x))\partial_t w(t,x)|\partial_t w_i(t,x)|^q \, dx \\ &\quad + \sum_{i=1}^{k} \int_0^1 p_{m+i}(x)\lambda_{m+i}(x, w(t,x))\partial_x(|\partial_t T_{m+i}(t,x)|^q) \, dx \\ &\quad + \sum_{i=1}^{k} \int_0^1 \frac{q p_{m+i}(x)}{\lambda_{m+i}(x, w(t,x))}\partial_y\lambda_{m+i}(x, w(t,x))\partial_t w(t,x)|\partial_t T_{m+i}(t,x)|^q \, dx. \end{aligned}$$

Set

$$\widetilde{\mathcal{W}}_3(t) = \sum_{i=1}^{k} \int_0^1 \frac{qp_i(x)}{\lambda_i(x, w(t,x))} \partial_y \lambda_i(x, w(t,x)) \partial_t w(t,x) |\partial_t w(t,x)|^q \, dx \tag{3.36}$$
$$+ \sum_{i=k+1}^{m} \int_0^1 \frac{qp_i(x)}{\lambda_i(x, w(t,x))} \partial_y \lambda_i(x, w(t,x)) \partial_t w(t,x) |\partial_t w(t,x)|^q \, dx$$
$$+ \sum_{i=1}^{k} \int_0^1 \frac{qp_{m+i}(x)}{\lambda_{m+i}(x, w(t,x))} \partial_y \lambda_{m+i}(x, w(t,x)) \partial_t w(t,x) |\partial_t T_{m+i}(t,x)|^q \, dx.$$

An integration by parts yields

$$\frac{d}{dt} \widetilde{\mathcal{V}}(w(t,\cdot)) = \widetilde{\mathcal{W}}_1(t) + \widetilde{\mathcal{W}}_2(t) + \widetilde{\mathcal{W}}_3(t), \tag{3.37}$$

where

$$\widetilde{\mathcal{W}}_1(t) = -\sum_{i=1}^{k} \lambda_i(1, w(t,1)) p_i(1) |\partial_t w_i(t,1)|^q$$
$$+ \sum_{i=1}^{k} \lambda_i(0, w(t,0)) p_i(0) |\partial_t (Bu_+)_i(t,0)|^q$$
$$- \sum_{i=k+1}^{m} \lambda_i(0, w(t,0)) p_i(0) |\partial_t w_i(t,0)|^q$$
$$- \sum_{i=1}^{k} \lambda_{m+i}(0, w(t,0)) p_{m+i}(0)$$
$$\times \left| \partial_t w_{m+i}(t,0) - \big(M_i(w_{k+1}(t,0), \ldots, w_{m+i-1}(t,0))\big)_t \right|^q \tag{3.38}$$

and

$$\widetilde{\mathcal{W}}_2(t) = \sum_{i=1}^{k} \int_0^1 \big(\lambda_i(x, w(t,x)) p_i(x)\big)_x |\partial_t w_i(t,x)|^q \, dx$$
$$- \sum_{i=k+1}^{m} \int_0^1 \big(\lambda_i(x, w(t,x)) p_i(x)\big)_x |\partial_t w_i(t,x)|^q \, dx$$
$$- \sum_{i=1}^{k} \int_0^1 \big(\lambda_{m+i}(x, w(t,x)) p_{m+i}(x)\big)_x$$
$$\times \left| \partial_t w_{m+i}(t,x) - \big(M_i\big(w_{k+1}(t, a_{k+1,m+i}(t,x)), \ldots, \right.$$
$$\left. w_{m+i-1}(t, a_{m+i-1,m+i}(t,x))\big)\big)_t \right|^q dx. \tag{3.39}$$

As before, we have

$$\widetilde{\mathcal{W}}_1(t) + \widetilde{\mathcal{W}}_2(t) \le -q\Lambda(1 - C(\varepsilon, \delta)) \widetilde{\mathcal{V}}. \tag{3.40}$$

One can check that

$$\widetilde{W}_3 \le C(\varepsilon, \delta) q \widetilde{\mathcal{V}}. \tag{3.41}$$

From (3.37), (3.40), and (3.41), we derive that

$$\frac{d}{dt} \widetilde{\mathcal{V}}(t) \le -q\Lambda(1 - C(\varepsilon, \delta)) \widetilde{\mathcal{V}}. \tag{3.42}$$

Combining (3.34) and (3.42) yields

$$\frac{d}{dt} \mathcal{V}(t) \le -q\Lambda(1 - C(\varepsilon, \delta)) \mathcal{V}.$$

The conclusion now follows as in the linear case after taking ε sufficiently small, replacing $\Lambda(1 - C\varepsilon)$ by Λ, and noting that, for $0 \le t \le \delta/2$,

$$\|w(t, \cdot)\|_{C^1([0,1])} \le C(\|w(0, \cdot)\|_{C^1([0,1])} + \|\zeta\|_{C^1} + \|\eta\|_{C^1} \|w(0, \cdot)\|_{C^1([0,1])}).$$

We also note here that the conclusion (A.3) of Lemma A.1 also holds for nonlinear maps M_i of class C^1 with $M_i(0) = 0$ provided that $\|v\|_{C^1([0,1)}$ is sufficiently small. The details are omitted. ■

A. A useful lemma

Lemma A.1. *Let m, $k \ge 1$. For $k + 1 \le i < j \le k + m$, let $b_{i,j} \colon [0, 1] \to [0, 1]$ be of class C^1 such that*

$$c_1 \le |b'_{i,j}(x)| \le c_2 \quad \text{for } x \in (0, 1), \tag{A.1}$$

for some positive constants c_1 and c_2. Set $\ell = \max\{k, m\}$. For $\ell + 1 \le m + i \le m + k$, let $M_i \in \mathbb{R}^{1 \times (m+1-k-i)}$. Define, for $v \in [L^q(0, 1)]^n$,

$$\begin{aligned} |||v|||^q = &\sum_{i=1}^{\ell} \int_0^1 |v_i(x)|^q \, dx \\ &+ \sum_{\substack{i \\ \ell+1 \le m+i \\ \le k+m}} \int_0^1 \Big| v_{m+i}(x) - M_i\big(v_{k+1}(b_{k+1,m+i}(x)), \dots, \\ &\qquad v_{m+i-1}(b_{m+i-1,m+i}(x))\big)\Big|^q \, dx. \end{aligned} \tag{A.2}$$

We have

$$\lambda^{-1} \|v\|_{L^q(0,1)} \le |||v||| \le \lambda \|v\|_{L^q(0,1)}, \tag{A.3}$$

for some $\lambda \ge 1$ depending only on k, m, c_1, and c_2, and M_i; it is independent of q.

Proof. We only consider the case $m \ge k$. The other case can be proved similarly. It is clear that

$$|||v||| \le C\|v\|_{L^q(0,1)}. \tag{A.4}$$

On the other hand, using the inequality, for $\xi_1, \xi_2 \in \mathbb{R}^d$ with $d \geq 1$,

$$|\xi_1|^q + |\xi_2 - \xi_1|^q \geq C^{-q}(|\xi_1|^q + |\xi_2|^q),$$

we have, for $1 \leq i \leq k$,

$$\int_0^1 \big|v_{m+i}(x) - M_i\big(v_{k+1}(b_{k+1,m+i}(x)), \ldots, v_{m+i-1}(b_{m+i-1,m+i}(x))\big)\big|^q \, dx + \sum_{k+1 \leq j \leq m+i-1} \int_0^1 |v_i(b_{j,m+i}(x))|^q \, dx \geq C^{-q} \int_0^1 |v_{m+i}(x)|^q \, dx. \tag{A.5}$$

Using (A.1), by a change of variables we obtain, for $k+1 \leq i < j \leq m+k$,

$$\int_0^1 |v_i(b_{i,j}(x))|^q \, dx \leq C \int_0^1 |v_i(x)|^q \, dx. \tag{A.6}$$

From (A.5) and (A.6), we deduce that

$$\sum_{i=1}^k \int_0^1 \big|v_{m+i}(x) - M_i\big(v_{k+1}(b_{k+1,m+i}(x)), \ldots, v_{m+i-1}(b_{m+i-1,m+i}(x))\big)\big|^q \, dx + \sum_{i=k+1}^m \int_0^1 |v_i(x)|^q \, dx \geq C^{-q} \int_0^1 \sum_{i=k+1}^n |v_i(x)|^q \, dx. \tag{A.7}$$

The conclusion then follows from (A.4) and (A.7). ∎

Acknowledgments. H.-M. Nguyen thanks Fondation des Sciences Mathématiques de Paris (FSMP) for the Chaire d'excellence which allowed him to visit Laboratoire Jacques Louis Lions and Mines ParisTech, during which time this work was done.

Funding. The authors were partially supported by ANR Finite4SoS grant no. ANR-15-CE23-0007.

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Zbl 1040.93031 MR 1925036

Received 7 July 2020; accepted 28 September 2021.

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