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Adaptation to a heterogeneous patchy environment with non-local dispersion

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Abstract. In this work, we provide an asymptotic analysis of the solutions to an elliptic integro-differential equation. This equation describes the evolutionary equilibria of a phenotypically structured population, subject to selection, mutation, and both local and non-local dispersion in a spatially heterogeneous, possibly patchy, environment. Considering small effects of mutations, we provide an asymptotic description of the equilibria of the phenotypic density. This asymptotic description involves a Hamilton–Jacobi equation with constraint coupled with an eigenvalue problem. Based on such analysis, we characterize some qualitative properties of the phenotypic density at equilibrium depending on the heterogeneity of the environment. In particular, we show that when the heterogeneity of the environment is low, the population concentrates around a single phenotypic trait leading to a unimodal phenotypic distribution. On the contrary, a strong fragmentation of the environment leads to multi-modal phenotypic distributions.

1. Introduction

1.1. The model and motivations

We are interested in the study of evolutionary equilibria of phenotypically structured populations in spatially heterogeneous and possibly patchy environments. Understanding the interplay between heterogeneous selection, migration, and mutation is a major objective of evolutionary biology theory and could lead to a better understanding of the speciation process and the evolutionary response to global change [34]. Joint evolution and spatial dispersion have to be considered in the study of species that need to adapt to climatic change, or in epidemiology, where pathogenic viruses or bacteria may propagate in a population that has been partially vaccinated or treated with antibiotics [20,21]. The mathematical problem then amounts to describing both phenotypic and spatial structure of the population, and connects to key questions in evolutionary ecology about the evolution of species ranges (where are the individuals?) and niches (which phenotypes are observed within the species?).

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We investigate particularly the effect of fragmented environment, considering non-local dispersion. The non-local dispersion may have antagonistic effects on the population dynamics. On the one hand, it may allow the population to reach new favorable geographic regions which are not accessible by a local diffusion. On the other hand, it may also impede local adaptation by bringing individuals with locally maladapted traits from other regions. While the role of the non-local dispersion and the fragmentation of the environment is significant in many situations, as in the adaptation of forest trees to climate change because of the effect of the wind on the seeds or the pollens, very few theoretical works take it into account [22].

More precisely, we provide in this work an asymptotic analysis of the equilibria of a non-local parabolic Lotka–Volterra-type equation, modeling the interplay between mutation, selection, local and non-local dispersion in possibly fragmented environments. The equation under study is

$$\begin{cases} -\sigma_\theta \partial_{\theta\theta} n - \sigma_x \partial_{xx} n + \sigma_x L n = n(R(x,\theta) - \kappa\rho(x)) & \text{in } \Omega \times]-A, A[, \\ \rho(x) = \int_{]-A,A[} n(x,\theta)\, d\theta & \text{in } \Omega, \\ Ln(x,\theta) = \int_\Omega [n(x,\theta) - n(y,\theta)] K(x-y)\, dy & \text{in } \Omega \times]-A, A[, \\ \partial_{\nu_x} n(x,\theta) = 0 \text{ on } \partial\Omega \times]-A, A[, \quad \partial_{\nu_\theta} n(x, \pm A) = 0 \text{ on } \Omega \times \{\pm A\}, \end{cases} \tag{1}$$

with Ω a bounded subset of $\mathbb{R}$, representing the spatial domain. Here, $n(x,\theta)$ stands for the density of a population at equilibrium at position x with a phenotypical trait θ. The term $R(x,\theta)$ corresponds to the intrinsic growth rate of individuals of phenotype θ at position x. The term $\rho(x)$ corresponds to the total size of the population at position x. Via the term $\kappa\rho$ in the right-hand side of (E), we take into account a mortality rate due to the uniform competition between the individuals at the same position, with intensity κ. The trait of the parent is transmitted to the offspring. However, the trait can be modified due to the mutations which we model by a Laplace term with respect to θ. We also consider that the species is subject to a local and a non-local dispersion in the space variable x. Indeed, in addition to a classical local dispersion term modeled by a Laplace term with respect to x, we also take into account a non-local dispersion modeled by the integral operator L, assuming that the individuals can jump from position x to position y with a rate $K(x-y)$. Finally, we have denoted by ∂_{ν_x}, ∂_{ν_θ} the exterior derivatives with respect to the variables x and θ. The Neumann boundary condition with respect to x models the fact that the species cannot leave the domain. The Neumann boundary condition with respect to θ means that the mutants cannot be born with a trait in $]-A, A[^c$.

We are in particular interested in a regime where the mutations have small effects. To study such a situation, we set $\sigma_\theta = \varepsilon^2$, with ε a small parameter. We also set $\sigma_x = \kappa = 1$ to reduce the amount of notation. The equation on the population density, denoted now by n_ε, is then written

$$\begin{cases} -\varepsilon^2 \partial_{\theta\theta} n_\varepsilon - \partial_{xx} n_\varepsilon + L n_\varepsilon = n_\varepsilon (R(x,\theta) - \rho_\varepsilon(x)) & \text{in } \Omega \times]-A, A[, \\ \rho_\varepsilon(x) = \displaystyle\int_{]-A,A[} n_\varepsilon(x,\theta)\, d\theta & \text{in } \Omega, \\ L n_\varepsilon(x,\theta) = \displaystyle\int_\Omega [n_\varepsilon(x,\theta) - n_\varepsilon(y,\theta)] K(x-y)\, dy & \text{in } \Omega \times]-A, A[, \\ \partial_{\nu_x} n_\varepsilon(x,\theta) = 0 \text{ on } \partial\Omega \times]-A, A[, \quad \partial_{\nu_\theta} n_\varepsilon(x, \pm A) = 0 \text{ on } \Omega \times \{\pm A\}. \end{cases} \tag{E}$$

Note that when the mutation rate σ_θ is small compared to the dispersion rate σ_x, equation (1) can be brought to the equation above via a change of variables. Our objective is to provide an asymptotic analysis of n_ε as the parameter ε becomes vanishingly small.

Several questions motivate our analysis. Can we determine extinction and survival criteria for this model? How would the population be spatially distributed at equilibrium? Would all the spatial domain be exploited close to its local carrying capacity or would we observe formation of clusters in certain zones of the environment? How would the population be distributed phenotypically? Would the population be adapted locally everywhere, or would we observe emergence of dominant traits? More specifically, would we observe emergence of generalist traits being adapted to an average environment, or specialist traits being adapted to certain zones of the environment? What will be the impact of the fragmentation of the habitat on the phenotypical distribution of the population? Would it lead to the emergence of specialist traits?

1.2. The state of the art

Related models and the questions of spatial distribution and ecological niches were studied from a numerical point of view in the biological literature, considering local dispersion and a connected domain (see for instance [14, 31]). In particular, in [31] formation of clusters was observed in a closely related model. While the authors suggested that the formation of such clusters would be a result of the bounded domain or small mutational effects, they did not provide any analytical support for such a hypothesis. This type of model, again in the context of local dispersion, was later derived from stochastic individual-based models by Champagnat and Méléard [12], where further numerical studies were provided (see also [3] for a preliminary analysis of this model by Arnold, Desvillettes, and Prévost). More recently, Alfaro, Coville, and Raoul [2] studied a closely related model, considering again local dispersion but in unbounded domains. They proved propagation phenomena and existence of traveling front solutions for parabolic equations close to (E) (see also the related works [1, 9]). In the context of their study with unbounded domain, one expects indeed that the population would propagate and get adapted locally in every position in space attaining its local carrying capacity, which is in contrast with what we observe in bounded domains, in particular with what we obtain in the present work. However, in this model to our knowledge, there is as yet no result characterizing rigorously the population distribution at the back of the front (see however the work of Berestycki, Jin, and Silvestre in [5] in a particular case with spatially homogeneous growth rate).

A large number of articles have also studied a closely related equation known as the "cane-toads" model, where the growth rate is independent of the trait, but the trait influences the ability of dispersal leading to a θ coefficient in front of the diffusion term in space (see for instance [6–8, 33]). This equation is motivated by the propagation of cane toads in Australia by taking into account the role of a phenotypical trait: the size of the legs of the toads. Closer to our work, the steady states of a "cane-toads"-type model, in the regime of small mutations, were studied by Perthame and Souganidis [30] and by Lam and Lou [23]. In another related project, a model where similarly to (E) the growth rate, and not the dispersion rate, depends on the phenotype, but considering a discrete spatial structure, was studied by Mirrahimi and Gandon [27, 28]. In these works, an asymptotic analysis of the steady states in the regime of small mutations was provided. In particular, it was shown that the presence of spatial heterogeneity can lead to polymorphic situations, which is the emergence of several dominant traits in the population.

In this work we will use an approach based on Hamilton–Jacobi equations, which is adapted to study the small mutation regime (ε small). A closely related approach was first introduced in [17, 18], by Friedlin using probabilistic techniques and by Evans and Souganidis using deterministic tools, to study the propagation phenomena in reaction-diffusion equations. In the context of models from evolutionary biology and in the regime of small mutations, this method was suggested by Dieckmann, Jabin, Mishler, and Perthame [13]. In [29], Barles and Perthame provide the first rigorous results within this approach and obtain a concentration phenomena considering homogeneous environments: as the mutational effects become small, the solution converges to a Dirac mass. In this case, the population at equilibrium is monomorphic (there is a single dominant trait in the population). We quote [4] which extends the main results of [29]. This approach was then widely extended to study more general models with heterogeneity. In particular, in the context of the space heterogeneous environments, the works [9, 27, 30, 33] are within this framework. However, the analyses provided in these previous works do not allow problem (E) to be studied. The closest work is [9], which studies the propagation phenomenon in an unbounded domain, considering a different rescaling. Note also that none of the previous works considered a non-local dispersion operator, which adds significant difficulties to the analysis.

In an ecological context, fragmented environments and non-local spatial dispersion phenomena were studied by Léculier, Mirrahimi, and Roquejoffre [24] and by Léculier and Roquejoffre [25]. Neither work takes into account any phenotypical structure. In [24], the authors study invasion phenomena in a Fisher–KPP equation involving a fractional Laplacian arising in a fragmented periodic environment with Dirichlet exterior conditions. In [25], the authors study the existence and uniqueness of bounded positive steady states in a Fisher–KPP equation involving a fractional Laplacian in general fragmented environment with Dirichlet exterior conditions. One of the perspectives of the present work is to study models with other operators of dispersion, as the fractional Laplacian $(-\partial_{xx})^\alpha$, instead of $-\partial_{xx} + L$, and considering Dirichlet exterior conditions.

1.3. The assumptions and the notation

The domain $\Omega \subset \mathbb{R}$ is assumed to be bounded and composed of one or several connected components:

$$\Omega = \bigcup_{i=1}^{m}]a_i, b_i[\quad \text{with } a_1 < b_1 < a_2 < \cdots < a_m < b_m. \tag{H1}$$

We assume that the growth rate verifies

$$R \in C^1(\overline{\Omega} \times [-A, A]) \quad \text{and} \quad \|R\|_{W^{1,\infty}(\Omega\times]-A,A[)} < C_R. \tag{H2}$$

Example 1. A typical example of a growth rate is written

$$R(x,\theta) = r - g(bx - \theta)^2.$$

In this example, r is the maximal growth rate. The above quadratic term indicates that the optimal trait at position x is given by $\theta_o = bx$. The term b is the gradient of the environment: it indicates how fast the optimal trait varies as a function of position in space. Moreover, g corresponds to the selection pressure. If g increases, the habitats becomes more hostile for unsuitable individuals.

We make the following assumptions on K:

$$\begin{aligned} &K \in C^1(\Omega), \quad K > 0, \quad K(x) = K(-x), \\ &0 < c_K < K < C_K, \quad \text{and} \quad |\partial_x K| < C_K. \end{aligned} \tag{H3}$$

We introduce here two eigenvalue problems associated to equation (E): let $\lambda(\theta,\rho)$ be the principal eigenvalue of the operator $-\partial_{xx} - L - [R(\cdot,\theta) - \rho]\mathrm{Id}$ and μ_ε be the principal eigenvalue of the operator $-\partial_{xx} - \varepsilon^2\partial_{\theta\theta} - L - R$ with Neumann boundary conditions, i.e.

$$\begin{cases} -\partial_{xx}\psi^\theta + L(\psi^\theta) - [R(\cdot,\theta) - \rho]\psi^\theta = \lambda(\theta,\rho)\psi^\theta & \text{in } \Omega, \\ \partial_{\nu_x}\psi^\theta = 0 & \text{in } \partial\Omega, \end{cases} \tag{2}$$

and

$$\begin{cases} -\partial_{xx}\xi_\varepsilon - \varepsilon^2\partial_{\theta\theta}\xi_\varepsilon + L\xi_\varepsilon - R\xi_\varepsilon = \mu_\varepsilon\xi_\varepsilon & \text{in } \Omega \times]-A, A[, \\ \partial_{\nu_x}\xi_\varepsilon = \partial_{\nu_\theta}\xi_\varepsilon = 0 & \text{on } \partial(\Omega \times]-A, A[). \end{cases} \tag{3}$$

Throughout the article, we consider that the principal eigenfunctions (such as ψ^θ or ξ_ε) are taken positive with L^2 norms equal to 1. The existence and some properties of λ are proved in Appendix A (the existence of μ follows similar arguments to that of λ, therefore we leave it to the reader).

We make the following assumption:

$$\exists\, \theta_0 \in \,]-A, A[\text{ such that } \quad \min_{\theta \in]-A,A[} \lambda(\theta, 0) = \lambda(\theta_0, 0) < 0. \tag{H4}$$

Lemma 1. *Under assumptions* (H1)–(H4) *we have*

$$\mu_\varepsilon \xrightarrow[\varepsilon \to 0]{} \lambda(\theta_0, 0).$$

It follows obviously that

$$\exists\, \varepsilon_0 > 0, \ \forall \varepsilon \in \,]0, \varepsilon_0[, \quad \mu_\varepsilon < \frac{\lambda(\theta_0, 0)}{2} < 0. \tag{4}$$

We postpone the proof of Lemma 1 to Appendix B and we make the hypothesis that (4) holds true.

1.4. The results and the strategy

First, we prove the following theorem which provides conditions for existence or non-existence of a solution of (E) for all small values ε.

Theorem 1. *Under assumptions* (H1)–(H4) *for all* $\varepsilon \in \,]0, \varepsilon_0[$ *there exists a non-trivial positive bounded solution* n_ε *of* (E). *If Assumption* (H4) *does not hold and* $\lambda(\theta_0, 0) > 0$, *then there exists* $\varepsilon_0 > 0$ *small enough such that for all* $\varepsilon < \varepsilon_0$, *there does not exist a positive solution* n_ε *to* (E).

We expect indeed that in a dynamic version of (E), the solution would converge in long time to a non-trivial stationary solution, that is a solution to (E), when such a non-trivial steady state exists, and to 0 otherwise. Admitting such a property, the theorem above provides us with conditions of survival and extinction of the population. The survival condition means indeed that there exists at least one trait θ such that $\lambda(\theta, 0) < 0$, so that such a trait is viable in the absence of competition.

The proof of Theorem 1 follows that of Lam and Lou [23, Theorem 2.1] which treats the case of local diffusion. This proof relies on a topological degree argument. In Section 4 we provide the additional arguments which allow the proof of [23] to be adapted to the non-local operator L.

Next we perform the Hopf–Cole transformation

$$n_\varepsilon(x, \theta) = e^{\frac{u_\varepsilon(x,\theta)}{\varepsilon}}. \tag{5}$$

This is the usual first step in the Hamilton–Jacobi approach (see [4,13,29]). The main idea in this approach is to first study the limit of u_ε as $\varepsilon \to 0$, and next to obtain, from this limit, information on the limit of the phenotypic density n_ε. The advantage of this transformation is that the limit of u_ε is usually a continuous function that solves a Hamilton–Jacobi equation, while the limit of n_ε is a measure. Performing such a change of variable, we find that u_ε is a solution to

$$\begin{cases} -\frac{1}{\varepsilon}\partial_{xx}u_\varepsilon - \frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} - \varepsilon\partial_{\theta\theta}u_\varepsilon - |\partial_\theta u_\varepsilon|^2 \\ \quad + \int_\Omega \Big[1 - e^{\frac{u_\varepsilon(y)-u_\varepsilon(x)}{\varepsilon}}\Big] K(x-y)\,dy = R(x,\theta) - \rho_\varepsilon(x) \quad \text{in } \Omega \times]-A, A[, \\ \rho_\varepsilon(x) = \int_{]-A,A[} n_\varepsilon(x,\theta)\,d\theta \qquad \text{in }]-A, A[, \\ \partial_{\nu_x} u_\varepsilon(x,\theta) = 0 \text{ on } \partial\Omega \times]-A, A[, \quad \partial_{\nu_\theta} u_\varepsilon(x, \pm A) = 0 \text{ in } \Omega. \end{cases} \tag{E_{HC}}$$

We prove the following:

Theorem 2. *Under assumptions (H1)–(H4), as $\varepsilon \to 0$ along subsequences, the following hold:*

(1) *ρ_ε converges uniformly to a function $\rho \in C^1(\Omega)$ with*

$$0 < c \le \rho \le C.$$

(2) *u_ε converges uniformly to a continuous function u with u a viscosity solution of*

$$\begin{cases} -|\partial_\theta u(\theta)|^2 = -\lambda(\theta, \rho), \\ \max u(\theta) = 0, \\ \partial_{\nu_\theta} u(\pm A) = 0, \end{cases} \tag{6}$$

where $\lambda(\theta, \rho)$ is the principal eigenvalue introduced in (2). Moreover, the limit u depends only on θ.

(3) *n_ε converges to a measure n in the sense of measures. Moreover,*

$$\operatorname{supp} n \subset \Omega \times \{u(\theta) = 0\} \subset \Omega \times \{\lambda(\theta, \rho) = 0\}. \tag{7}$$

The theorem above allows us to characterize the phenotypic density n, at the limit as $\varepsilon \to 0$, via the Hamilton–Jacobi equation with constraint (6) coupled with the eigenvalue problem (2) and the inclusion properties (7). We expect indeed that n would be the sum of Dirac masses in θ as follows:

$$n(x,\theta) = \sum_{i=1}^{d} \rho_i(x)\delta(\theta - \theta_i).$$

In Section 2 we will use the information obtained above to characterize the phenotypic density n in some particular situations. On the one hand, we will identify a situation where the phenotypic density at the limit will be a single Dirac mass in θ corresponding to a monomorphic population. On the other hand, we will show that a strong fragmentation of the environment will lead to polymorphic situations.

Let us present briefly below the main ingredients to prove (6). We first provide heuristic arguments to understand how the Hamilton–Jacobi equation in (6) can be recovered. To this end, we perform asymptotic developments of u_ε and ρ_ε with respect to the powers of ε, i.e.

$$u_\varepsilon(x,\theta) = u_0(x,\theta) + \varepsilon u_1(x,\theta) + o(\varepsilon) \quad \text{and} \quad \rho_\varepsilon(x) = \rho_0(x) + o_\varepsilon(1).$$

Next we implement such asymptotic developments into (E_{HC}) to obtain

$$\begin{aligned}\frac{1}{\varepsilon}&\Big(-\partial_{xx}u_0 - 2|\partial_x u_0 \partial_x u_1| - \frac{|\partial_x u_0|^2}{\varepsilon}\Big)\\ &+ \int_\Omega [1 - e^{\frac{u_0(y,\theta)-u_0(x,\theta)}{\varepsilon} + u_1(y,\theta) - u_1(x,\theta) + o_\varepsilon(1)}] K(x-y)\,dy\\ &- \partial_{xx}u_1 - |\partial_x u_1|^2 - |\partial_\theta u_0|^2 - [R - \rho_0] + o_\varepsilon(1) = 0.\end{aligned}$$

We then organize the equation by powers of ε. Keeping the terms of order ε^{-2}, we find

$$\partial_x u_0(x,\theta) = 0 \;\Rightarrow\; u_0(x,\theta) = u_0(\theta).$$

Moreover, keeping the terms of order ε^0, we deduce that

$$\begin{aligned}-|\partial_\theta u_0(\theta)|^2 = [R(x,\theta) - \rho_0(x)] + \partial_{xx}u_1(x,\theta) + |\partial_x u_1(x,\theta)|^2\\ - \int_\Omega [1 - e^{u_1(y,\theta)-u_1(x,\theta)}] K(x-y)\,dy.\end{aligned} \tag{8}$$

Setting $\widetilde{\psi} = \exp(u_1)$ and substituting it in the equation above we obtain

$$-\partial_{xx}\widetilde{\psi} + L(\widetilde{\psi}) - [R(x,\theta) - \rho_0(x)]\widetilde{\psi} = |\partial_\theta u_0(\theta)|^2 \widetilde{\psi}.$$

Since $-|\partial_\theta u_0(\theta)|^2$ does not depend on x and $\widetilde{\psi}$ is a positive function, the equation above suggests that $\widetilde{\psi}$ and $-|\partial_\theta u_0(\theta)|^2$ are respectively the principal eigenfunction and eigenvalue introduced in (2). This leads in particular to the Hamilton–Jacobi equation in (6).

From a technical point of view, the convergence of $(u_\varepsilon)_{\varepsilon>0}$ is proved using the Arzelà–Ascoli theorem and a perturbed test function argument (see [15]). In order to apply the Arzelà–Ascoli theorem, we provide first some regularity estimates on u_ε. We prove in particular, using Bernstein's method, that the first derivatives are bounded. These bounds rely on the establishment of Harnack-type inequalities. More precisely, we prove the following regularity results on u_ε.

Theorem 3. *Under assumptions* (H1)–(H4)*, the following results hold true:*

(1) [Harnack inequality] *There exists a constant $C > 0$ (independent of the choice of ε) such that for all intervals $I \subset]-A, A[$ with $|I| = \varepsilon$, there holds*

$$\sup_{(x,\theta)\in\Omega\times I} n_\varepsilon(x,\theta) \le C \inf_{(x,\theta)\in\Omega\times I} n_\varepsilon(x,\theta). \tag{9}$$

(2) [Lipschitz bounds] *There exists* $C > 0$ *such that for all* ε *small enough,*

$$|\partial_x u_\varepsilon| \leq C\varepsilon \quad \textit{and} \quad |\partial_\theta u_\varepsilon| < C. \tag{10}$$

(3) [Bounds on ρ_ε] *For all* ε *small enough,* ρ_ε *is uniformly bounded in* $W^{2,p}(\Omega)$ *for all* $p \in [1, +\infty]$. *Moreover, there exist* $c, C > 0$ *(independent of the choice of* ε*) such that*

$$c \leq \rho_\varepsilon \leq C. \tag{11}$$

(4) [Bounds on u_ε] *The following holds true:*

$$\begin{aligned} &\lim_{\varepsilon \to 0} \sup_{(x,\theta) \in \Omega \times]-A,A[} u_\varepsilon \leq 0, \\ -a < &\lim_{\varepsilon \to 0} \inf_{(x,\theta) \in \Omega \times]-A,A[} u_\varepsilon, \end{aligned} \tag{12}$$

with $a > 0$.

Remark. In Theorem 3 (1), the interval I can be at the boundary of $]-A, A[$, i.e.

$$I =]-A, -A + \varepsilon[\quad \text{or} \quad I =]A - \varepsilon, A[.$$

The combination of the local and the non-local diffusion terms makes the establishment of such regularity estimates non-standard (see for instance [4] and [26] where such estimates were obtained for related models with a local diffusion term). Here, the Harnack inequality (9) is used to obtain the Lipschitz regularity estimate (10). However, the result is by itself interesting since it extends the classical Harnack inequality to elliptic operators with joint local and non-local diffusion terms.

Note finally that the constraint $\max_\theta u(\theta) = 0$ is a consequence of the Hopf–Cole transformation (5) and the fact that ρ_ε remains bounded away from 0, uniformly in ε. For a detailed proof of Theorem 2 we refer to Section 6.

1.5. Outline of the paper

In Section 2 we focus on the qualitative properties of the phenotypic density n and show which types of qualitative conclusions we can obtain thanks to our theoretical results presented above. In Section 3 we investigate, by some numerical simulations, the qualitative properties of n and confirm our theoretical results. In Section 4 we provide the existence of n_ε by proving Theorem 1. Next we prove the regularity results given by Theorem 3 in Section 5. Section 6 is devoted to the proof of Theorem 2.

As mentioned above, Appendix A is devoted to the existence of the eigenvalues $\lambda(\theta, \rho)$. We also prove some results in Appendix A that are stated and used in Section 2. Lemma 1 is proved in Appendix B.

The constants c, C are positive constants independent of the choice of ε and may change from line to line when there is no confusion possible.

2. Qualitative properties of the population density *n*

The objective of this section is to show how the asymptotic results provided in Theorem 2 imply qualitative results on the population density n, at the limit as $\varepsilon \to 0$. Recall from Theorem 2 that as $\varepsilon \to 0$ and along subsequences, n_ε tends weakly to a measure n and ρ_ε converges uniformly to a function ρ such that

$$\operatorname{supp} n \subset \Omega \times \big\{\theta \mid u(\theta) = 0\big\} \subset \Omega \times \big\{\theta \mid \lambda(\theta,\rho) = \min_{\theta'} \lambda(\theta',\rho) = 0\big\},$$

with $u\colon [-A, A] \to \mathbb{R}$ the viscosity to

$$\begin{cases} -|\partial_\theta u(\theta)|^2 = -\lambda(\theta,\rho), \quad \theta \in \,]-A, A[, \\ \partial_{\nu_\theta} u(\pm A) = 0, \\ \max u(\theta) = 0, \end{cases}$$

and $\lambda(\theta,\rho)$ the principal eigenvalue corresponding to the following problem:

$$\begin{cases} -\partial_{xx}\psi^\theta + L(\psi^\theta) - [R(\cdot,\theta) - \rho]\psi^\theta = \lambda(\theta,\rho)\psi^\theta & \text{in } \Omega, \\ \partial_{\nu_x}\psi^\theta = 0 & \text{in } \partial\Omega. \end{cases}$$

Moreover, from the first item of Theorem 3 we deduce that

$$\text{if } \theta_0 \in \operatorname{supp} n(x_0,\cdot) \text{ for some } x_0 \in \Omega, \text{ then } \theta_0 \in \operatorname{supp} n(x,\cdot) \text{ for all } x \in \Omega.$$

Consequently, we have

$$\operatorname{supp} n = \Omega \times \Gamma_\theta, \quad \Gamma_\theta \subset \big\{\theta \mid u(\theta) = 0\big\} \subset \big\{\theta \mid \lambda(\theta,\rho) = \min_{\theta'} \lambda(\theta',\rho) = 0\big\}.$$

Note that it may happen that $u(\theta) = 0$ for some $\theta \in [-A, A]$ but that θ does not belong to Γ_θ. Note also that Theorem 2 guarantees convergence of $(n_\varepsilon, \rho_\varepsilon)$ to (n, ρ), only along subsequences. It does not exclude the possibility of multiple limits (n, ρ) as $\varepsilon \to 0$.

We expect indeed that u would take its maximum at some distinct traits such that the phenotypic density n would have the form

$$n(x,\theta) = \sum_{i=1}^{d} \rho_i(x)\delta(\theta - \theta_i), \quad \rho(x) = \sum_{i=1}^{d} \rho_i(x), \quad \rho_i(x) > 0.$$

This expectation motivates the following definitions.

Definition 1.
- Any trait $\theta \in \Gamma_\theta$ is called an *emergent trait*.
- A population density is called *monomorphic* if the set of emergent traits, that is Γ_θ, is reduced to a single point.
- A population density is called *polymorphic* if it is not monomorphic.

With these definitions, any monomorphic population density is a Dirac mass with respect to θ, i.e.

$$n(x,\theta) = \rho(x)\delta(\theta - \bar{\theta}).$$

In Section 2.1 we will show that, for a particular choice of R, there is at most one possible monomorphic limit (n, ρ). In particular, under symmetry conditions on the set Ω, the only possible monomorphic outcome at the limit would be $n(x,\theta) = \rho(x)\delta(\theta)$. We next identify a situation in Section 2.2 where the phenotypic density n is indeed monomorphic. Finally, we show in Section 2.3 that a strong fragmentation of the environment may lead to polymorphic situations.

Before providing our qualitative results, we state the following technical result on the principal eigenvalue λ that in the next subsections will help us to obtain our qualitative results. This result is proved in Appendix A.

Proposition 1. *Under assumptions* (H1)–(H3), *the following identity holds true:*

$$\partial_\theta \lambda(\theta,\rho) = -\int_\Omega \partial_\theta R(x,\theta)\psi^\theta(x)^2\,dx. \tag{13}$$

Corollary 1. *Assume* (H1)–(H4) *and let* $\bar{\theta} \in \Gamma_\theta$ *be an emergent trait. We have*

$$\int_\Omega \partial_\theta R(x,\bar{\theta})\psi^{\bar{\theta}}(x)^2\,dx = 0.$$

Next we consider some examples with explicit expressions of R. We illustrate how Proposition 1 can be useful to characterize the emergent traits.

Example A. We fix $\theta_0 \in]-A, A[$, and we define

$$R(x,\theta) = r - g(\theta - \theta_0)^2.$$

We assume that r is large enough such that (H4) holds true for $\theta = \theta_0$. From Corollary 1, we deduce that at any emergent trait $\bar{\theta}$, we have

$$2g(\bar{\theta} - \theta_0)\int_\Omega \psi^{\bar{\theta}}(x)^2\,dx = 0.$$

We deduce that the unique emergent trait is $\bar{\theta} = \theta_0$. Therefore, the limit population is monomorphic. Of course, this example is a toy model and does not involve any spatial structure.

Example B. We define

$$R(x,\theta) = r - g(\theta - bx)^2. \tag{14}$$

From Corollary 1 we deduce that for any emergent trait $\bar{\theta}$, we have

$$\bar{\theta} = b\int_\Omega x\psi^{\bar{\theta}}(x)^2\,dx.$$

This is not enough to conclude that the number of emergent traits is finite. However, we can still remark that the emergent traits are fixed points of the application

$$\chi\colon \theta \mapsto b \int_\Omega x \psi^\theta(x)^2 \, dx. \tag{15}$$

2.1. At most one possible monomorphic outcome

In this subsection we restrict our study to monomorphic limits. We will prove that, when R is given by (14), the problem admits at most one monomorphic limit.

Proposition 2. *Assume* (H1)–(H4) *and let* $n_1(x,\theta) = \rho_1(x)\delta(\theta - \bar\theta_1)$ *and* $n_2(x,\theta) = \rho_2(x)\delta(\theta - \bar\theta_2)$ *be two monomorphic limits of the problem. Then* $\rho_1 = \rho_2$.

Additionally, if R *is given by* (14)*, then* $\theta_1 = \theta_2$.

Furthermore, if the domain Ω *is symmetric with respect to* $x = 0$*, then the only possible emergent trait corresponding to a monomorphic population is* $\bar\theta = 0$.

To prove Proposition 2, we will use the following Rayleigh quotient:

$$\mathcal{R}(\theta,\rho,\phi) = \frac{\int_\Omega |\partial_x \phi|^2 \, dx + \frac{\int_{\Omega\times\Omega} [\phi(x)-\phi(y)]^2 K(x-y) \, dx \, dy}{2} - \int_\Omega [R(x,\theta) - \rho(x)]\phi^2 \, dx}{\int_\Omega \phi(x)^2 \, dx}. \tag{16}$$

We recall the classical link between λ and $\mathcal{R}$:

$$\lambda(\theta,\rho) = \min_{\phi \in H^1(\Omega)} \mathcal{R}(\theta,\rho,\phi).$$

We will also use the following lemma.

Lemma 2. *Assume* (H1)–(H4) *and that* (n_ε) *converges in* $L^\infty(\mathrm{w}*(0,\infty); \mathcal{M}^1(\mathbb{R}^d))$ *to* $\rho(x)\delta(\theta - \theta_0)$*. Then* $\rho(\cdot)$ *is the principal eigenfunction corresponding to the operator* $-\Delta_x + L - [R(\cdot,\theta_0) - \rho]$.

Proof. By passing weakly to the limit in equation (E), we obtain

$$-\partial_{xx}\rho + L\rho = \rho(R(x,\theta_0) - \rho),$$

which implies that ρ is the principal eigenfunction corresponding to the operator $-\Delta_x + L - [R(\cdot,\theta_0) - \rho]$. ∎

Proof of Proposition 2. Let $n_1(x,\theta) = \rho_1(x)\delta(\theta - \theta_1)$ and $n_2(x,\theta) = \rho_2(x)\delta(\theta - \theta_2)$ be two possible limits. Our objective is to prove that

$$\rho_1 = \rho_2 \quad \text{for generic } R \text{ satisfying (H2)},$$

and

$$\theta_1 = \theta_2 \quad \text{if, furthermore, } R \text{ is of the form of (14)}.$$

From Theorem 2 we obtain

$$\min_{\theta} \lambda(\theta, \rho_i) = \lambda(\theta_i, \rho_i) = 0.$$

Let ψ_i be the positive eigenfunction associated to the operator $-\Delta_x + L - [R(\cdot, \theta_i) - \rho_i]$ by Proposition 6 with $\|\psi_i\|_{L^2} = 1$ (i.e. $\psi_i = \psi^{\theta_i}$). Notice that since the limit is monomorphic and thanks to Lemma 2, we have for each case $\rho_i = c_i \psi_i$ with $c_i > 0$. Using the Rayleigh quotient introduced in (16), we deduce that

$$\begin{aligned} 0 &= \int_\Omega c_1^2 \psi_1(x)^2\, dx \times \mathcal{R}(\theta_1, c_1\psi_1, c_1\psi_1) \\ &= c_1^2 \int_\Omega |\partial_x \psi_1|^2\, dx + c_1^2 \frac{\int_{\Omega\times\Omega} [\psi_1(x) - \psi_1(y)]^2 K(x-y)\, dx\, dy}{2} \\ &\quad - c_1^2 \int_\Omega [R(x, \theta_1) - c_1\psi_1]\psi_1^2\, dx \end{aligned} \tag{17}$$

and

$$\begin{aligned} 0 &\le \int_\Omega c_1^2 \psi_1(x)^2\, dx \times \mathcal{R}(\theta_1, c_2\psi_2, c_1\psi_1) \\ &= c_1^2 \int_\Omega |\partial_x \psi_1|^2\, dx + c_1^2 \frac{\int_{\Omega\times\Omega} [\psi_1(x) - \psi_1(y)]^2 K(x-y)\, dx\, dy}{2} \\ &\quad - c_1^2 \int_\Omega [R(x, \theta_1) - c_2\psi_2]\psi_1^2\, dx. \end{aligned} \tag{18}$$

The last inequality holds since thanks to Theorem 2 we have

$$0 \le \lambda(\theta_1, c_2\psi_2) \le \mathcal{R}(\theta_1, c_2\psi_2, c_1\psi_1).$$

By subtracting (17) from (18), we deduce that

$$\int_\Omega (c_1\psi_1(x))^3\, dx \le \int_\Omega (c_1\psi_1(x))^2 c_2\psi_2(x)\, dx. \tag{19}$$

Following similar computations, we obtain

$$\int_\Omega (c_2\psi_2(x))^3\, dx \le \int_\Omega (c_2\psi_2(x))^2 c_1\psi_1(x)\, dx. \tag{20}$$

By combining (19) and (20), we deduce that

$$\int_\Omega [c_1\psi_1(x) - c_2\psi_2(x)]^2 (c_1\psi_1(x) + c_2\psi_2(x))\, dx \le 0.$$

Since $c_{1,2}\psi_{1,2} > 0$, we deduce that $c_1\psi_1 = c_2\psi_2$, and hence $\rho_1 = \rho_2$.

We next assume that R is given by (14) and conclude thanks to Corollary 1. Indeed, since $\partial_\theta R(x, \theta_i) = -2g(\theta_i - bx)$, we deduce that

$$\theta_1 = \frac{-b}{2g} \int_\Omega x c_1 \psi_1^2(x)\, dx = \frac{-b}{2g} \int_\Omega x c_2 \psi_2^2(x)\, dx = \theta_2.$$

Finally, note that under symmetry conditions, if $n(x,\theta) = \rho(x)\delta(\theta)$ is a possible monomorphic limit, then $\tilde{n}(x,\theta) = n(-x,-\theta)$ is also a possible monomorphic outcome. We hence deduce in this case that $\theta = 0$. ■

2.2. A small selection pressure leads to a monomorphic population

First, we state a technical result about the dependence of $\lambda(\theta,\rho)$ with respect to g. This proposition is proved at the end of Appendix A.

Proposition 3. *Assume that R is given by (14). Under hypotheses (H1)–(H3), there holds*

$$\lambda(\theta,\rho) \text{ is non-decreasing with respect to } g, \tag{21}$$

and

$$\int_\Omega |\partial_\theta \psi^\theta(x)|^2\, dx \xrightarrow[g\to 0]{} 0. \tag{22}$$

Moreover, the above limit is uniform with respect to θ.

Next we use Proposition 3 to provide a condition that ensures the existence of a set of parameters such that the limit is monomorphic.

Proposition 4. *Assume that R is given by (14). Under hypotheses (H1)–(H4), there exists $g_0 > 0$ such that if $g \in]0, g_0[$ then there exists a unique emergent trait.*

Proof. First, we differentiate χ (defined by (15)) with respect to θ, to obtain

$$\chi'(\theta) = 2b \int_\Omega x\psi^\theta(x)\partial_\theta\psi^\theta(x)\, dx.$$

Thanks to the Cauchy–Schwarz inequality and Proposition 3, we deduce that

$$\begin{aligned} |\chi'(\theta)| &\le 2b \sup_{x\in\Omega}|x| \int_\Omega \psi^\theta(x)^2\, dx \int_\Omega \partial_\theta\psi^\theta(x)^2\, dx \\ &= 2b \sup_{x\in\Omega}|x| \int_\Omega \partial_\theta\psi^\theta(x)^2\, dx \xrightarrow[g\to 0]{} 0. \end{aligned}$$

Since the last inequality does not depend on the choice of θ, we deduce the existence of a uniform $g_0 > 0$ such that for all $g \in]0, g_0[$ we have

$$|\chi'(\theta)| < 1.$$

Thanks to Theorem 2 there exists at least one fixed point $\bar{\theta}$ to χ. Moreover, since χ is a contraction mapping, we recover that the fixed point $\bar{\theta}$ is unique. ■

2.3. A strong fragmentation of the environment leads to polymorphism

In this subsection we consider the growth rate given by (14) and spatial domains of the type

$$\Omega_d = (-d-a, -d) \cup (d, d+a).$$

Proposition 5. *Under assumptions* (H1)–(H4)*, for $d \geq d_0$, with d_0 a large enough constant, the trait* 0 *is not included in the support of the phenotypic density n. As a consequence, the population density is not monomorphic.*

Proof. We first note that, for fixed d, there always exists $r(d)$ such that for all $r \geq r(d)$, (H4) is satisfied so that thanks to Theorem 1 the population persists. We thus can assume that, up to adjusting the constant r, we are in a situation where the population persists.

We prove that $\theta = 0$ is not an emergent trait. Let us suppose by contradiction that $\theta = 0$ is included in the support of $n(x, \cdot)$. Then the Rayleigh quotient (16) implies that

$$\lambda(0, \rho) = \inf_{\phi \in H^1(\Omega)} \mathcal{R}(0, \rho, \phi) = 0 \leq \lambda(\theta, \rho) \quad \text{for all } \theta \in [-A, A].$$

Let $\phi_0 \in H^1(\Omega)$ be such that

$$\inf_{\phi \in H^1(\Omega)} \mathcal{R}(0, \rho, \phi) = \mathcal{R}(0, \rho, \phi_0) = 0, \quad \|\phi_0\|_{L^2(\Omega)} = 1, \quad \phi_0 > 0.$$

We choose

$$\theta_1 = -d - a/2, \quad \theta_2 = d + a/2.$$

We also define

$$\Omega_1 = [-d - a, -d], \quad \Omega_2 = [d, d + a], \quad \phi_0^{(1)} = \phi_0 \mathbb{1}_{\Omega_1}, \quad \phi_0^{(2)} = \phi_0 \mathbb{1}_{\Omega_2}.$$

We will prove that when d is large enough,

$$\|\phi_0^{(1)}\|_{L^2} \lambda(\theta_1, \rho) + \|\phi_0^{(2)}\|_{L^2} \lambda(\theta_2, \rho) < \lambda(0, \rho) = 0. \tag{23}$$

Since $\|\phi_0^{(1)}\|_{L^2} + \|\phi_0^{(2)}\|_{L^2} = 1$, inequality (23) would imply

$$\min(\lambda(\theta_1, \rho), \lambda(\theta_2, \rho)) < 0,$$

which is in contradiction with the positiveness of the eigenvalues $\lambda(\theta, \rho)$ established in Theorem 2 (2).

We consider the positive function ϕ_i, for $i = 1, 2$, which minimizes $\mathcal{R}(0, \rho, \cdot)$ restricted to the set Ω_i, that is, the operator

$$\mathcal{R}(0, \rho, \Omega_i, \phi) = \frac{\int_{\Omega_i} |\partial_x \phi|^2 \, dx + \frac{\int_{\Omega_i \times \Omega_i} [\phi(x) - \phi(y)]^2 K(x-y) \, dx \, dy}{2} - \int_{\Omega_i} [R(x, 0) - \rho(x)] \phi^2 \, dx}{\|\phi\|^2_{L^2(\Omega_i)}},$$

and such that

$$\|\phi_i\|_{L^2(\Omega_i)} = 1.$$

Note that here ϕ_i has its support in Ω_i, while ϕ_0 has its support in Ω_d. To prove (23), it is enough to prove that

$$\|\phi_0^{(1)}\|^2_{L^2} \mathcal{R}(\theta_1, \rho, \phi_1) + \|\phi_0^{(2)}\|^2_{L^2} \mathcal{R}(\theta_2, \rho, \phi_2) < \mathcal{R}(0, \rho, \phi_0).$$

We compute

$$\begin{aligned}
&\|\phi_0^{(i)}\|_{L^2}^2 \mathcal{R}(\theta_i,\rho,\phi_i)\\
&\quad = \|\phi_0^{(i)}\|_{L^2}^2\left[\mathcal{R}(\theta_i,\rho,\Omega_i,\phi_i) + \frac{1}{2}\int_{\substack{\Omega_i\times\Omega_j\\ i\neq j}} \phi_i^2(x)K(x-y)\,dx\,dy\right]\\
&\quad \le \|\phi_0^{(i)}\|_{L^2}^2\left[\mathcal{R}(\theta_i,\rho,\Omega_i,\phi_0^{(i)}) + \frac{1}{2}\int_{\substack{\Omega_i\times\Omega_j\\ i\neq j}} \phi_i^2(x)K(x-y)\,dx\,dy\right]\\
&\quad = \int_{\Omega_i} |\partial_x\phi_0^{(i)}|^2\,dx + \frac{1}{2}\int_{\Omega_i\times\Omega_i} [\phi_0^{(i)}(x)-\phi_0^{(i)}(y)]^2K(x-y)\,dx\,dy\\
&\qquad - \int_{\Omega_i}[R(x,\theta_i)-\rho(x)]\phi_0^{(i)\,2}(x)\,dx + \frac{1}{2}\|\phi_0^{(i)}\|_{L^2}^2\int_{\substack{\Omega_i\times\Omega_j\\ i\neq j}} \phi_i^2(x)K(x-y)\,dx\,dy.
\end{aligned}$$

Combining the inequality above, for $i=1$ and $i=2$, we obtain

$$\begin{aligned}
&\|\phi_0^{(1)}\|_{L^2}^2\mathcal{R}(\theta_1,\rho,\phi_1) + \|\phi_0^{(2)}\|_{L^2}^2\mathcal{R}(\theta_2,\rho,\phi_2)\\
&\qquad \le \mathcal{R}(0,\rho,\phi_0)\\
&\qquad\quad + \int_{\Omega_1}[R(x,0)-R(x,\theta_1)]\phi_0^2(x)\,dx + \int_{\Omega_2}[R(x,0)-R(x,\theta_2)]\phi_0^2(x)\,dx\\
&\qquad\quad + \frac{1}{2}\|\phi_0^{(1)}\|_{L^2}^2\int_{\Omega_1\times\Omega_2}\phi_1^2(x)K(x-y)\,dx\,dy\\
&\qquad\quad + \frac{1}{2}\|\phi_0^{(2)}\|_{L^2}^2\int_{\Omega_2\times\Omega_1}\phi_2^2(x)K(x-y)\,dx\,dy.
\end{aligned}$$

We next note that

$$\begin{aligned}
\int_{\Omega_i}[R(x,0)-R(x,\theta_i)]\phi_0^2(x)\,dx &= -g\theta_i\int_{\Omega_i}(2x-\theta_i)\phi_0^2(x)\,dx\\
&\le -g(d^2-a^2/4)\int_{\Omega_i}\phi_0^2(x)\,dx.
\end{aligned}$$

We also have

$$\int_{\Omega_i\times\Omega_j}\phi_i^2(x)K(x-y)\,dx\,dy \le a\|K\|_{L^\infty(\mathbb{R})}\int_{\Omega_i}\phi_i^2(x)\,dx = a\|K\|_{L^\infty(\mathbb{R})}.$$

We deduce that

$$\begin{aligned}
&\|\phi_0^{(1)}\|_{L^2}^2\mathcal{R}(\theta_1,\rho,\phi_1) + \|\phi_0^{(2)}\|_{L^2}^2\mathcal{R}(\theta_2,\rho,\phi_2)\\
&\qquad \le \mathcal{R}(0,\rho,\phi_0) - g(d^2-a^2/4) + \frac{a}{2}\|K\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

Therefore, for d large enough, we obtain

$$\|\phi_0^{(1)}\|_{L^2}^2\mathcal{R}(\theta_1,\rho,\phi_1) + \|\phi_0^{(2)}\|_{L^2}^2\mathcal{R}(\theta_2,\rho,\phi_2) < \mathcal{R}(0,\rho,\phi_0).$$

We conclude that $\theta=0$ is not an emergent trait.

It remains to show that the population density cannot be monomorphic. Indeed, if we assume that n is monomorphic with $\bar{\theta}$ as the only emergent trait, then since the domain Ω is symmetric and according to Proposition 2, it follows that $\bar{\theta} = 0$. This is in contradiction with the fact that $\theta = 0$ is not an emergent trait. ■

3. Some numerical illustrations

In this section we illustrate the numerical solutions of (E), for some particular examples, considering the following type of growth rate:

$$R(x, \theta) = r - g(bx - \theta)^2.$$

We recall from Example 1 that in this case, r is a maximal growth rate and $-g(bx - \theta)^2$ models the selection. The parameter g is the selection pressure whereas b is the gradient of the environment. We provide numerical examples where we vary this set of parameters.

To find a numerical solution of (E), we solve numerically the parabolic equation

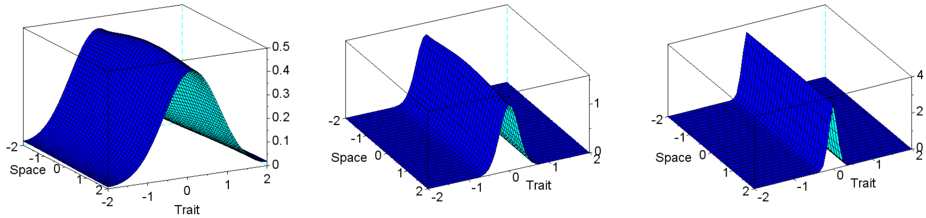
$$\begin{cases} \partial_t n_\varepsilon - \partial_{xx} n_\varepsilon - \varepsilon^2 \partial_{\theta\theta} n_\varepsilon + L(n_\varepsilon) = [R - \rho_\varepsilon] n_\varepsilon & \text{in } \mathbb{R}^+ \times \Omega \times]-A, A[, \\ \rho_\varepsilon(t, x) = \displaystyle\int_{-A}^{A} n_\varepsilon(t, x, \theta)\, d\theta & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_{\nu_x} n_\varepsilon = \partial_{\nu_\theta} n_\varepsilon = 0, & \\ n(t = 0, x, \theta) = n_0(x, \theta). & \end{cases} \tag{E_t}$$

We implement equation (E_t) by a semi-implicit finite difference method. We stop the algorithm when we find a numerical steady state of (E_t): a numerical solution of (E).

First, we underline that in all the numerical resolutions, the density of the population concentrates around one or several distinct trait(s). Moreover, these emergent traits are present everywhere in space thanks to the local and the non-local migration. However, the density of the population at the position x with a emergent trait θ_m depends on whether this trait θ_m is adapted or not to the position x.

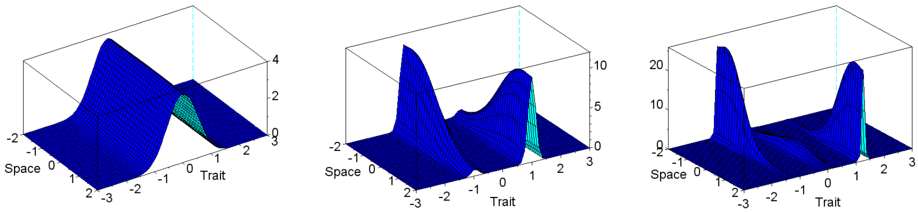
Figure 1 illustrates the convergence of n_ε to a Dirac mass as ε goes to 0. The only variation is with respect to the parameter $\varepsilon = 0.1, 0.01$, and 0.001.

Next we focus on the qualitative properties established in Section 2; Figures 2 and 3 are numerical illustrations of Proposition 2. We fix Ω as a single connected component and we investigate the dependence on the parameter g. We recover numerically that as $g \to 0$ the limit density is monomorphic with an emergent trait at $\theta = 0$. For larger values of g, the phenotypic density concentrates around several distinct traits. For each simulation, we also provide the numerical distributions of ρ_ε (Figure 3); this density seems to be centered around the point which maximizes $R(\cdot, \bar{\theta})$ (where $\bar{\theta}$ is any emergent trait). Therefore, when the emergent trait is unique, ρ_ε is increasing on $]-2, 0[$ and then decreasing on $]0, 2[$



(a) $\varepsilon = 0.1$ **(b)** $\varepsilon = 0.01$ **(c)** $\varepsilon = 0.001$

Figure 1. Variation of the numerical solutions of (E) with respect to ε. The others parameters are fixed as follows: $r = 1$, $b = 1$, $g = 0.1$, $\Omega =]-2, 2[$, and $A = 2$. We observe that the distribution of the population concentrates around the emergent trait $\theta = 0$.



(a) $g = 0.01$ **(b)** $g = 1$ **(c)** $g = 5$

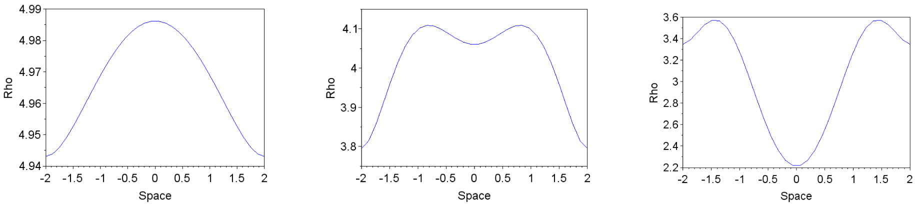
Figure 2. Variation of the numerical solutions n_ε of (E) with respect to g. The other parameters are fixed as follows: $r = 5$, $b = 1$, $\varepsilon = 0.01$, $\Omega =]-2, 2[$, and $A = 3$. We recover that if g is small then the population is monomorphic. For large values of g, there exist several distinct emergent traits.

whereas the spatial distribution can be more involved whenever there exist several distinct emergent traits.

To conclude, we present in Figure 4 a numerical illustration of Proposition 5. Here, the free parameter is the distance between the two connected components of Ω. We recover that increasing the distance between the two connected components may lead to multiple emergent traits.

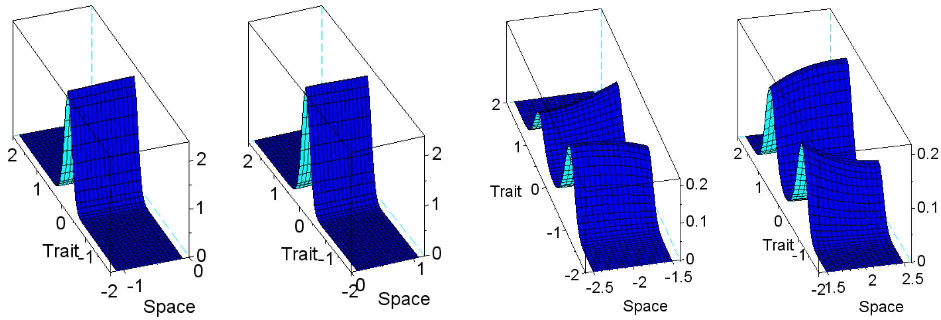
4. Existence of a non-trivial solution of (E)

As mentioned in the introduction, we recall that the proof of existence of a non-trivial solution is an adaptation of Lam and Lou [23, proof of Theorem 2.1]. The major difference is the presence of the integral operator L. Therefore, we only provide the main elements dealing with the integral operator L. We also skip the proof of non-existence of a non-trivial solution, when (H4) does not hold, which follows from classical arguments.



(a) $g = 0.01$ **(b)** $g = 1$ **(c)** $g = 5$

Figure 3. Variation of the numerical density ρ_ε of (E) with respect to g. The other parameters are fixed as follows: $r = 5$, $b = 1$, $\varepsilon = 0.01$, $\Omega =]-2, 2[$, and $A = 3$.



(a) $\Omega =]-1.1, -0.1[\cup]0.1, 1.1[$ **(b)** $\Omega =]-2.5, -1.5[\cup]1.5, 2.5[$

Figure 4. Variation of the numerical solutions n_ε of (E) with respect to the distance between the two connected components of Ω. The other parameters are fixed as follows: $r = 1$, $b = 1$, $g = 1$, $\varepsilon = 0.01$, and $A = 2$. We observe that for this set of parameters, increasing the distance induces a polymorphic density population.

Proof of Theorem 1. We fix $\varepsilon \in]0, \varepsilon_0[$ (where ε_0 is given by (4)). Let $\tau \in [0, 1]$ and n_τ be a solution of

$$\begin{cases} -\partial_{xx} n_\tau - \varepsilon^2 \partial_{\theta\theta} n_\tau + L n_\tau = n_\tau (R - \tau \rho_\tau - (1 - \tau) n_\tau) & \text{in } \Omega \times]-A, A[, \\ \rho_\tau(x) = \displaystyle\int_{]-A,A[} n_\tau(x, \theta)\, d\theta & \text{in } \Omega, \\ \partial_{\nu_x} n_\tau(x, \theta) = 0 \text{ on } \partial\Omega \times]-A, A[, \quad \partial_{\nu_\theta} n_\tau(x, \pm A) = 0 \text{ on } \Omega \times \{\pm A\}. \end{cases} \tag{E_τ}$$

It is well known that for $\tau = 0$, according to (4), there exists a non-trivial steady solution n_0. As in [23], we prove that there exists a constant $C_\varepsilon > 1$ (which may depend on ε) such that we have for any $\tau \in [0, 1]$,

$$C_\varepsilon^{-1} \leq \int_{-A}^{A} \int_\Omega n_\tau \, dx \, d\theta \leq C_\varepsilon.$$

Then one can conclude using a topological degree argument.

The lower bound. Let v_τ be such that $n_\tau = \xi_\varepsilon v_\tau$ (where ξ_ε is provided by (3)). First, we remark that

$$L(v_\tau \xi_\varepsilon) = v_\tau L(\xi_\varepsilon) + \xi_\varepsilon L(v_\tau) + \Lambda(v_\tau, \xi_\varepsilon)$$

and

$$\Lambda(v_\tau, \xi_\varepsilon)(x) = \int_\Omega [(v_\tau(x) - v_\tau(y))(\xi_\varepsilon(y) - \xi_\varepsilon(x))] K(x-y)\, dy.$$

Then v_τ is a solution of

$$\begin{aligned} &-\xi_\varepsilon \partial_{xx} v_\tau - 2\partial_x \xi_\varepsilon \partial_x v_\tau - \varepsilon^2 \xi_\varepsilon \partial_{\theta\theta} v_\tau - 2\varepsilon^2 \partial_\theta \xi_\varepsilon \partial_\theta v_\tau + \xi_\varepsilon L(v_\tau) + \Lambda(v_\tau, \xi_\varepsilon) + \mu_\varepsilon \xi_\varepsilon v_\tau \\ &\qquad = -v_\tau \xi_\varepsilon [\tau \rho_\tau + (1-\tau) n_\tau]. \end{aligned}$$

If we multiply it by $\frac{\xi_\varepsilon}{v_\tau}$, we obtain

$$\frac{-\partial_x(\xi_\varepsilon^2 \partial_x v_\tau) - \varepsilon^2 \partial_\theta(\xi_\varepsilon^2 \partial_\theta v_\tau) + \xi_\varepsilon^2 L(v_\tau) + \xi_\varepsilon \Lambda(v_\tau, \xi_\varepsilon)}{v_\tau} = \xi_\varepsilon^2(-\mu_\varepsilon - \tau\rho_\tau - (1-\tau) n_\tau).$$

Next we integrate over all the domain:

$$\begin{aligned} &\int_{-A}^{A} \int_\Omega \frac{-\partial_x(\xi_\varepsilon^2 \partial_x v_\tau) - \varepsilon^2 \partial_\theta(\xi_\varepsilon^2 \partial_\theta v_\tau) + \xi_\varepsilon^2 L(v_\tau) + \xi_\varepsilon \Lambda(v_\tau, \xi_\varepsilon)}{v_\tau}\, dx\, d\theta \\ &\quad = \int_{-A}^{A} \int_\Omega \frac{-\partial_x(\xi_\varepsilon^2 \partial_x v_\tau) - \varepsilon^2 \partial_\theta(\xi_\varepsilon^2 \partial_\theta v_\tau)}{v_\tau}\, dx\, d\theta \\ &\qquad + \int_{-A}^{A} \int_\Omega \frac{\xi_\varepsilon^2 L(v_\tau) + \xi_\varepsilon \Lambda(v_\tau, \xi_\varepsilon)}{v_\tau}\, dx\, d\theta \\ &\quad = I_1 + I_2. \end{aligned}$$

We next prove that I_1 and I_2 are negative. For I_1, by an integration by parts, we have

$$\begin{aligned} I_1 &= \int_{-A}^{A} \int_\Omega \frac{-\partial_x(\xi_\varepsilon^2 \partial_x v_\tau) - \varepsilon^2 \partial_\theta(\xi_\varepsilon^2 \partial_\theta v_\tau)}{v_\tau}\, dx\, d\theta \\ &= -\int_{-A}^{A} \int_\Omega \frac{\xi_\varepsilon^2}{v_\tau^2}(|\partial_x v_\tau|^2 + \varepsilon^2 |\partial_\theta v_\tau|^2)\, dx\, d\theta \le 0. \end{aligned}$$

For I_2, using that K is even and the Fubini theorem, we obtain

$$\begin{aligned} I_2 &= \int_{-A}^{A} \int_\Omega \frac{\xi_\varepsilon^2(x) L(v_\tau)(x) + \xi_\varepsilon(x) \Lambda(v_\tau, \xi_\varepsilon)(x)}{v_\tau(x)}\, dx\, d\theta \\ &= \int_{-A}^{A} \int_\Omega \frac{\xi_\varepsilon(x)}{v_\tau(x)} \int_\Omega [(v_\tau(x) - v_\tau(y))\xi_\varepsilon(y)] K(x-y)\, dy\, dx\, d\theta \\ &= -\int_{-A}^{A} \int_\Omega \int_\Omega \Big[\frac{\xi_\varepsilon(x)\xi_\varepsilon(y)}{v_\tau(x) v_\tau(y)}(v_\tau(y) - v_\tau(x))^2\Big] K(x-y)\, dy\, dx\, d\theta - I_2. \end{aligned}$$

We deduce that

$$I_2 = -\frac{1}{2}\int_{-A}^{A}\int_{\Omega}\int_{\Omega}\Big[\frac{\xi_\varepsilon(x)\xi_\varepsilon(y)}{v_\tau(x)v_\tau(y)}(v_\tau(y)-v_\tau(x))^2\Big]K(x-y)\,dy\,dx\,d\theta \le 0.$$

Therefore, we have

$$\int_{-A}^{A}\int_{\Omega}\xi_\varepsilon^2[-\mu_\varepsilon - \tau\rho_\tau - (1-\tau)n_\tau]\,dx\,d\theta \le 0.$$

Thanks to (4), we conclude that for ε small enough,

$$\frac{|\lambda(\theta_0,0)|}{2} \le -\mu_\varepsilon = -\mu_\varepsilon\int_{-A}^{A}\int_{\Omega}\xi_\varepsilon^2\,dx\,d\theta \le \sup(\xi_\varepsilon^2)[\tau+(1-\tau)]\int_{-A}^{A}\int_{\Omega}n_\tau\,dx\,d\theta.$$

The upper bound. First, we remark that thanks to the Neumann boundary conditions and the parity of K, we have

$$\int_{-A}^{A}\int_{\Omega}-\partial_{xx}(n_\tau - \partial_{\theta\theta}n_\tau + L(n_\tau))\,dx\,d\theta = 0.$$

Therefore, if we integrate (E_τ) with respect to x and θ, we obtain

$$\begin{aligned}\Big(\frac{(1-\tau)}{2A|\Omega|}+\frac{\tau}{|\Omega|}\Big)\|n_\tau\|_{L^1}^2 &= \frac{\tau}{|\Omega|}\Big(\int_{\Omega}\rho_\tau\,dx\Big)^2 + \frac{(1-\tau)}{2A|\Omega|}\Big(\int_{-A}^{A}\int_{\Omega}n_\tau\,dx\,d\theta\Big)^2\\ &\le \tau\int_{\Omega}\rho_\tau^2\,dx + (1-\tau)\int_{-A}^{A}\int_{\Omega}n_\tau^2\,dx\,d\theta\\ &= \int_{-A}^{A}\int_{\Omega}Rn_\tau\,dx\,d\theta \le C_R\int_{-A}^{A}\int_{\Omega}n_\tau\,dx\,d\theta = C_R\|n_\tau\|_{L^1}.\end{aligned}$$

Conclusion. It follows that there exists a bounded non-trivial solution n_ε of (E). Moreover, we have indeed proved that there exist constants $c, C > 0$ such that

$$\frac{c}{\sup\xi_\varepsilon^2} \le \int_{-A}^{A}\int_{\Omega}n_\varepsilon\,dx\,d\theta \le C. \qquad \blacksquare$$

5. Regularity results

In this section we prove Theorem 3. The subsections correspond respectively to the proofs of items (1), (2), (3), and (4) of Theorem 3. But, we need an intermediate result: ρ_ε is uniformly bounded.

Lemma 3. *Under assumptions (H1)–(H4), we have that for all $\varepsilon < \varepsilon_0$,*

$$0 \le \rho_\varepsilon \le C_R$$

(where C_R is introduced in (H2)). Moreover, there exists $C > 0$ such that for all ε small enough,

$$\|\rho_\varepsilon\|_{W^{2,p}(\Omega)} \le C.$$

Proof. *The L^∞-bounds.* It is obvious that $\rho_\varepsilon > 0$. If we integrate (E) with respect to θ, we obtain

$$\begin{cases} -\partial_{xx}\rho_\varepsilon + L\rho_\varepsilon = \displaystyle\int_{-A}^{A} R(\cdot,\theta)n_\varepsilon(\cdot,\theta)\,d\theta - \rho_\varepsilon^2 & \text{in } \Omega, \\ \partial_{\nu_x}\rho_\varepsilon = 0 & \text{in } \partial\Omega. \end{cases} \tag{E_ρ}$$

Recalling the L^∞ bounds on R (H2), it follows that

$$-\partial_{xx}\rho_\varepsilon + L\rho_\varepsilon \le C_R\rho_\varepsilon - \rho_\varepsilon^2.$$

We conclude thanks to the maximum principle that $\rho_\varepsilon \le C_R$.

The $W^{2,p}(\Omega)$ bounds. Thanks to the L^∞ bounds on R, K, ρ_ε (assumptions (H2), (H3) and the previous inequality), we may write (E_ρ) in the form

$$-\partial_{xx}\rho_\varepsilon = f_\varepsilon,$$

with $f_\varepsilon \in L^\infty(\Omega)$ uniformly bounded. The result follows from the standard elliptic estimates. ∎

Corollary 2. *There exists a constant $C > 0$ such that for all ε small enough,*

$$|\partial_x\rho_\varepsilon| \le C. \tag{24}$$

5.1. A Harnack inequality

The first step to prove the first item of Theorem 3 is to prove the result in the interior of $\Omega \times]-A, A[$.

Theorem 4. *For all $(x_0, \theta_0) \in \Omega \times]-A, A[$, and $R_0 > 0$ such that*

$$B_{3R_0}(x_0) \times B_{3\varepsilon R_0}(\theta_0) \subset \Omega \times]-A, A[,$$

there exists $C(R_0) > 0$ such that

$$\sup_{(x,\theta)\in B_{R_0}(x_0)\times B_{\varepsilon R_0}(\theta_0)} n_\varepsilon(x,\theta) \le C(R_0) \inf_{(x,\theta)\in B_{R_0}(x_0)\times B_{\varepsilon R_0}(\theta_0)} n_\varepsilon(x,\theta). \tag{25}$$

Next we prove that we can extend the solution thanks to a reflective argument (see [10, Remark 9, p. 275]).

We perform the change of variable $\tilde n(x,\theta) = n_\varepsilon(x, \varepsilon\theta)$. Therefore, we consider the following scaled equation:

$$\begin{cases} -\partial_{xx}\tilde n - \partial_{\theta\theta}\tilde n + L\tilde n = \tilde n[\widetilde R - \rho] & \text{in } \Omega \times]-\varepsilon^{-1}A, \varepsilon A[, \\ \partial_{\nu_x}\tilde n = \partial_{\nu_\theta}\tilde n = 0 & \text{in } \partial(\Omega \times]-\varepsilon^{-1}A, \varepsilon A[). \end{cases} \tag{E'}$$

We have denoted by $\widetilde R$ the function $\widetilde R(x,\theta) = R(x, \varepsilon\theta)$. Note that $\widetilde R$ still verifies (H2).

Proof of Theorem 4. Let $(x_0, \theta_0) \in \Omega \times]-\varepsilon^{-1}A, \varepsilon^{-1}A[$ and a radius $R_0 > 0$ be such that $B_{3R_0}(x_0, \theta_0) \subset \Omega \times]-\varepsilon^{-1}A, \varepsilon^{-1}A[$. If we denote $f(x, \theta) = \int_\Omega \tilde{n}(y, \theta)K(y - x)\, dy$, according to (H3) it follows that $f \in L^\infty(B_{2R_0}(x_0, \theta_0))$. From the classical Harnack inequality, [19, Theorems 9.20 and 9.22, pp. 244–246], and using (H2), we deduce the existence of $C_1 > 0$ (depending on R_0) such that

$$\begin{aligned}
&\sup_{(x,\theta)\in B_{R_0}(x_0,\theta_0)} \tilde{n}(x, \theta) \\
&\quad \le C_1 \inf_{(x,\theta')\in B_{R_0}(x_0,\theta_0)} \tilde{n}(x, \theta') + C_1 \sup_{(x,\theta'')\in B_{2R_0}(x_0,\theta_0)} |f(x, \theta'')| \\
&\quad \le C_1 \inf_{(x,\theta')\in B_{R_0}(x_0,\theta_0)} \tilde{n}(x, \theta') + C_1 C_K \sup_{\theta''\in B_{2R_0}(\theta_0)} \int_\Omega \tilde{n}(x, \theta'')\, dx. \qquad (26)
\end{aligned}$$

The main element of the proof is to prove the following claim:

$$\exists\, C > 0 \text{ such that } \sup_{\theta\in B_{2R_0}(\theta_0)} \int_\Omega \tilde{n}(x, \theta)\, dx \le C \inf_{(x,\theta)\in B_{R_0}(x_0,\theta_0)} \tilde{n}(x, \theta). \qquad (27)$$

It is clear that if (27) holds true, the conclusion follows.

First, we integrate (E') with respect to x. It follows, thanks to the Neumann boundary conditions, that for all $\theta \in B_{3R_0}(\theta_0)$ we have

$$\begin{aligned}
-\partial_{\theta\theta} \int_\Omega \tilde{n}(x, \theta)\, dx = {}& \frac{\int_\Omega \tilde{n}(x, \theta)\big(\widetilde{R}(x, \theta) - \rho(x) - \int_\Omega K(x - y)\, dy\big)\, dx}{\int_\Omega \tilde{n}(x, \theta)\, dx} \int_\Omega \tilde{n}(x, \theta)\, dx \\
&+ \frac{\int_\Omega \int_\Omega \tilde{n}(y, \theta)K(x - y)\, dy\, dx}{\int_\Omega \tilde{n}(x, \theta)\, dx} \int_\Omega \tilde{n}(x, \theta)\, dx.
\end{aligned}$$

Thanks to the L^∞-bounds on K, $\widetilde{R}$, ρ (assumptions (H2), (H3) and Lemma 3), and the Fubini theorem, we have

$$-C \le \frac{\int_\Omega \tilde{n}(x, \theta)\big(\widetilde{R}(x, \theta) - \rho(x) - \int_\Omega K(x - y)\, dy\big)\, dx}{\int_\Omega \tilde{n}(x, \theta)\, dx} < C$$

and

$$\frac{\int_\Omega \int_\Omega \tilde{n}(y, \theta)K(x - y)\, dy\, dx}{\int_\Omega \tilde{n}(x, \theta)\, dx} \le C_K|\Omega|.$$

It follows that

$$-C \int_\Omega \tilde{n}(x, \theta)\, dx \le -\partial_{\theta\theta} \int_\Omega \tilde{n}(x, \theta)\, dx \le C \int_\Omega \tilde{n}(x, \theta)\, dx.$$

Hence, we apply the Harnack inequality to $\theta \in B_{3R_0}(\theta_0) \mapsto \int_\Omega \tilde{n}(x, \theta)\, dx$ into the ball $B_{2R_0}(\theta_0)$ and we deduce the existence of a constant $C_2 > 0$ such that

$$\sup_{\theta\in B_{2R_0}(\theta_0)} \int_\Omega \tilde{n}(x, \theta)\, dx \le C_2 \inf_{\theta\in B_{2R_0}(\theta_0)} \int_\Omega \tilde{n}(x, \theta)\, dx. \qquad (28)$$

Next, thanks to the L^∞-bounds on K, $\widetilde{R}$, ρ (assumptions (H2), (H3) and Lemma 3), it follows that in $\Omega \times B_{2R_0}(\theta_0)$,

$$c_K \inf_{\theta \in B_{2R_0}(\theta_0)} \int_\Omega \tilde{n}(y,\theta)\,dy \le c_K \int_\Omega \tilde{n}(y,\theta)\,dy \le \int_\Omega \tilde{n}(y,\theta) K(x-y)\,dy \le (-\partial_{xx} - \partial_{\theta\theta})\tilde{n} + C\tilde{n}.$$

From an inequality developed by Krylov (we refer to [11, Theorem 7.1, p. 565] and the reference therein), we deduce the existence of a constant $C_3 > 0$ such that

$$\inf_{B_{2R_0}(\theta_0)} \int_\Omega \tilde{n}(x,\theta)\,dx \le C_3 \inf_{B_{R_0}(x_0,\theta_0)} \tilde{n}(x,\theta). \tag{29}$$

Combining the previous inequality with (28) and (29) yields

$$\sup_{\theta \in B_{2R_0}(\theta_0)} \int_\Omega \tilde{n}(x,\theta)\,dx \le C_2 \inf_{\theta \in B_{2R_0}(\theta_0)} \int_\Omega \tilde{n}(x,\theta)\,dx \le C_2 C_3 \inf_{(x,\theta) \in B_{R_0}(x_0,\theta_0)} \tilde{n}(x,\theta).$$

This concludes the proof. ■

5.2. Lipschitz estimates

We prove Theorem 3 (2) by the Bernstein method.

Proof of Theorem 3 (2). We recall the main equation satisfied by u_ε:

$$\frac{-\partial_{xx} u_\varepsilon}{\varepsilon} - \frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2} - \varepsilon \partial_{\theta\theta} u_\varepsilon - |\partial_\theta u_\varepsilon|^2 + \int_\Omega [1 - e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}}] K(x-y)\,dy = R(x,\theta) - \rho_\varepsilon \tag{30}$$

with Neumann boundary conditions. The first step is to differentiate (30) with respect to x and multiply it by $\frac{\partial_x u_\varepsilon}{\varepsilon^2}$:

$$-\frac{\partial_{xxx} u_\varepsilon \partial_x u_\varepsilon}{\varepsilon^3} - \frac{\partial_x(\frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2}) \partial_x u_\varepsilon}{\varepsilon^2} + \int_\Omega e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}} K(x-y)\,dy \frac{\partial_x u_\varepsilon^2}{\varepsilon^3} - \frac{\partial_x |\partial_\theta u_\varepsilon|^2 \partial_x u_\varepsilon}{\varepsilon^2} - \frac{\partial_{x\theta\theta} u_\varepsilon \partial_x u_\varepsilon}{\varepsilon} = \frac{\big(\int_\Omega [e^{\frac{u_\varepsilon(y) - u_\varepsilon(x)}{\varepsilon}} - 1] \partial_x K(x-y)\,dy + \partial_x R - \partial_x \rho_\varepsilon\big) \partial_x u_\varepsilon}{\varepsilon^2}.$$

Remarking that

$$\partial_{xxx} u_\varepsilon \partial_x u_\varepsilon = \frac{\partial_{xx}(|\partial_x u_\varepsilon|^2)}{2} - (\partial_{xx} u_\varepsilon)^2$$

and

$$\partial_{x\theta\theta} u_\varepsilon \partial_x u_\varepsilon = \frac{\partial_{\theta\theta}(|\partial_x u_\varepsilon|^2)}{2} - (\partial_{\theta x} u_\varepsilon)^2$$

yields

$$-\frac{\partial_{xx}(\frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2})}{2\varepsilon}+\frac{(\partial_{xx}u_\varepsilon)^2}{\varepsilon^3}-\frac{\partial_x(\frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2})\partial_x u_\varepsilon}{\varepsilon^2}+\int_\Omega e^{\frac{u_\varepsilon(y)-u_\varepsilon(x)}{\varepsilon}}K(x-y)\,dy\frac{(\partial_x u_\varepsilon)^2}{\varepsilon^3}$$
$$-\frac{\partial_x|\partial_\theta u_\varepsilon|^2\partial_x u_\varepsilon}{\varepsilon^2}+\frac{(\partial_{\theta x}u_\varepsilon)^2}{\varepsilon}-\frac{\varepsilon\partial_{\theta\theta}(\frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2})}{2}$$
$$=\frac{\left(\int_\Omega[e^{\frac{u_\varepsilon(y)-u_\varepsilon(x)}{\varepsilon}}-1]\partial_x K(x-y)\,dy+\partial_x R-\partial_x\rho_\varepsilon\right)\partial_x u_\varepsilon}{\varepsilon^2}. \tag{31}$$

In the second step, we differentiate (30) with respect to θ and multiply by $\partial_\theta u_\varepsilon$. With computations similar to those presented above, we find

$$-\frac{\partial_{xx}(|\partial_\theta u_\varepsilon|^2)}{2\varepsilon}+\frac{(\partial_{\theta x}u_\varepsilon)^2}{\varepsilon}-\partial_\theta\frac{|\partial_x u_\varepsilon|^2}{\varepsilon^2}\partial_\theta u_\varepsilon-\frac{\varepsilon}{2}\partial_{\theta\theta}(|\partial_\theta u_\varepsilon|^2)+\varepsilon(\partial_{\theta\theta}u_\varepsilon)^2$$
$$-\partial_\theta|\partial_\theta u_\varepsilon|^2\partial_\theta u_\varepsilon+\int_\Omega\frac{\left(\partial_\theta u_\varepsilon(x)^2-\partial_\theta u_\varepsilon(x)\partial_\theta u_\varepsilon(y)\right)}{\varepsilon}e^{\frac{u_\varepsilon(y)-u_\varepsilon(x)}{\varepsilon}}K(x-y)\,dy$$
$$=\partial_\theta R\partial_\theta u_\varepsilon. \tag{32}$$

Next we introduce

$$p_\varepsilon(x,\theta)=\frac{|\partial_x u_\varepsilon(x,\theta)|^2}{\varepsilon^2}+|\partial_\theta u_\varepsilon(x,\theta)|^2. \tag{33}$$

If we combine (31) and (32) and rewrite in terms of p_ε, it follows that

$$-\frac{\partial_{xx}p_\varepsilon}{2\varepsilon}-\frac{\varepsilon\partial_{\theta\theta}p_\varepsilon}{2}$$
$$+\frac{1}{\varepsilon}\int_\Omega[p_\varepsilon(x,\theta)-\partial_\theta u_\varepsilon(x,\theta)\partial_\theta u_\varepsilon(y,\theta)]e^{\frac{u_\varepsilon(y,\theta)-u_\varepsilon(x,\theta)}{\varepsilon}}K(x-y)\,dy$$
$$-\frac{\partial_x p_\varepsilon\partial_x u_\varepsilon}{\varepsilon^2}-\partial_\theta p_\varepsilon\partial_\theta u_\varepsilon+\frac{2(\partial_{x\theta}u_\varepsilon)^2}{\varepsilon}+\frac{(\partial_{xx}u_\varepsilon)^2}{\varepsilon^3}+\varepsilon(\partial_{\theta\theta}u_\varepsilon)^2$$
$$=\left(\int_\Omega[e^{\frac{u_\varepsilon(y)-u_\varepsilon(x)}{\varepsilon}}-1]\partial_x K(x-y)\,dy+\partial_x R-\partial_x\rho_\varepsilon\right)\frac{\partial_x u_\varepsilon}{\varepsilon^2}+\partial_\theta R\partial_\theta u_\varepsilon. \tag{34}$$

Let $(x_\varepsilon,\theta_\varepsilon)$ be such that

$$\sup_{(x,\theta)\in\Omega\times]-A,A[}p_\varepsilon(x,\theta)=p_\varepsilon(x_\varepsilon,\theta_\varepsilon).$$

Thanks to the Neumann boundaries conditions, we deduce that $(x_\varepsilon,\theta_\varepsilon)\notin\partial\Omega\times\partial(]-A,A[)$. Therefore, we distinguish three cases: either $(x_\varepsilon,\theta_\varepsilon)\in\Omega\times]-A,A[$ or $(x_\varepsilon,\theta_\varepsilon)\in\partial\Omega\times]-A,A[$ or $(x_\varepsilon,\theta_\varepsilon)\in\Omega\times\{\pm A\}$.

Case 1: $(x_\varepsilon,\theta_\varepsilon)\in\Omega\times]-A,A[$. First, we bound the right-hand-side of (34). Indeed, thanks to the Harnack inequality (Theorem 3 (1)) and the L^∞-bounds on the derivative of K, R, and ρ_ε (assumptions (H2), (H3) and (24) in Corollary 2), it follows that

$$\left(\int_\Omega[e^{\frac{u_\varepsilon(y)-u_\varepsilon(x)}{\varepsilon}}-1]\partial_x K(x-y)\,dy+\partial_x R-\partial_x\rho_\varepsilon\right)\frac{\partial_x u_\varepsilon}{\varepsilon^2}+\partial_\theta R\partial_\theta u_\varepsilon\le\frac{C\sqrt{p}}{\varepsilon}. \tag{35}$$

Next we evaluate (34) at $(x_\varepsilon, \theta_\varepsilon)$. We claim that

$$\begin{aligned} &-\partial_{xx} p_\varepsilon(x_\varepsilon,\theta_\varepsilon) \geq 0, \quad -\partial_{\theta\theta} p_\varepsilon(x_\varepsilon,\theta_\varepsilon) \geq 0, \quad \partial_x p_\varepsilon(x_\varepsilon,\theta_\varepsilon) = \partial_\theta p_\varepsilon(x_\varepsilon,\theta_\varepsilon) = 0, \\ &\frac{1}{\varepsilon}\int_\Omega [p_\varepsilon(x_\varepsilon,\theta_\varepsilon) - \partial_\theta u_\varepsilon(x_\varepsilon,\theta_\varepsilon)\partial_\theta u_\varepsilon(y,\theta_\varepsilon)] e^{\frac{u_\varepsilon(y,\theta_\varepsilon)-u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y)\,dy \geq 0. \end{aligned} \tag{36}$$

Indeed, the first inequalities follow easily since $p(x_\varepsilon,\theta_\varepsilon) = \max p_\varepsilon$ and the last inequality holds true thanks to the following computations:

$$\begin{aligned} &\frac{1}{\varepsilon}\int_\Omega [p_\varepsilon(x_\varepsilon,\theta_\varepsilon) - \partial_\theta u_\varepsilon(x_\varepsilon,\theta_\varepsilon)\partial_\theta u_\varepsilon(y,\theta_\varepsilon)] e^{\frac{u_\varepsilon(y,\theta_\varepsilon)-u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y)\,dy \\ &\geq \frac{1}{\varepsilon}\Big[\int_\Omega p_\varepsilon(x_\varepsilon,\theta_\varepsilon) e^{\frac{u_\varepsilon(y,\theta_\varepsilon)-u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y)\,dy \\ &\qquad - \frac{1}{2}\int_\Omega \partial_\theta u_\varepsilon^2(x_\varepsilon,\theta_\varepsilon) e^{\frac{u_\varepsilon(y,\theta_\varepsilon)-u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y)\,dy \\ &\qquad - \frac{1}{2}\int_\Omega \partial_\theta u_\varepsilon^2(y,\theta_\varepsilon) e^{\frac{u_\varepsilon(y,\theta_\varepsilon)-u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y)\,dy\Big] \\ &\geq \frac{1}{2\varepsilon}\Big[\int_\Omega p_\varepsilon(x_\varepsilon,\theta_\varepsilon) e^{\frac{u_\varepsilon(y,\theta_\varepsilon)-u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y)\,dy \\ &\qquad - \int_\Omega p_\varepsilon(y,\theta_\varepsilon) e^{\frac{u_\varepsilon(y,\theta_\varepsilon)-u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon}} K(x_\varepsilon - y)\,dy\Big] \\ &\geq 0. \end{aligned}$$

We deduce thanks to (35) and (36) (and noticing $\frac{2(\partial_{x\theta}u_\varepsilon)^2(x_\varepsilon,\theta_\varepsilon)}{\varepsilon} \geq 0$) that

$$\begin{aligned} \frac{1}{2\varepsilon}\Big[\frac{\partial_{xx}u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon} + \varepsilon\partial_{\theta\theta}u_\varepsilon(x_\varepsilon,\theta_\varepsilon)\Big]^2 &\leq \frac{1}{\varepsilon}\Big[\Big(\frac{\partial_{xx}u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon}\Big)^2 + (\varepsilon\partial_{\theta\theta}u_\varepsilon(x_\varepsilon,\theta_\varepsilon))^2\Big] \\ &\leq \frac{C\sqrt{p_\varepsilon(x_\varepsilon,\theta_\varepsilon)}}{\varepsilon}. \end{aligned}$$

Hence, using the original equation (30), we deduce that

$$\begin{aligned} &\Big[-p_\varepsilon(x_\varepsilon,\theta_\varepsilon) + \int_\Omega (1 - e^{\frac{u_\varepsilon(y,\theta_\varepsilon)-u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon}}) K(x_\varepsilon - y)\,dy - R(x_\varepsilon,\theta_\varepsilon) + \rho_\varepsilon(x_\varepsilon)\Big]^2 \\ &\quad \leq C\sqrt{p_\varepsilon(x_\varepsilon,\theta_\varepsilon)}. \end{aligned} \tag{37}$$

Thanks to the L^∞-bounds on K, R, and ρ_ε (assumptions (H2), (H3) and Lemma 3), it follows that $p_\varepsilon(x_\varepsilon,\theta_\varepsilon)$ is uniformly bounded with respect to ε. The conclusion follows.

Case 2: $(x_\varepsilon,\theta_\varepsilon) \in \partial\Omega \times]-A, A[$. First, note that $p_\varepsilon(x_\varepsilon,\theta_\varepsilon) = |\partial_\theta u_\varepsilon(x_\varepsilon,\theta_\varepsilon)|^2$ in this case. We claim also that p_ε verifies the Neumann boundary conditions at $(x_\varepsilon,\theta_\varepsilon)$. Indeed, according to the Neumann boundary conditions satisfied by u_ε, we can use a reflective argument and differentiate p_ε on the boundary. We obtain

$$\partial_x p_\varepsilon(x_\varepsilon,\theta_\varepsilon) = \frac{2\partial_x u_\varepsilon(x_\varepsilon,\theta_\varepsilon)\partial_{xx}u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon^2} + 2\partial_\theta u_\varepsilon(x_\varepsilon,\theta_\varepsilon)\partial_{x\theta}u_\varepsilon(x_\varepsilon,\theta_\varepsilon) = 0$$

because

$$\partial_x u_\varepsilon(x_\varepsilon, \theta_\varepsilon) = 0 \quad \text{and} \quad \partial_{x\theta} u_\varepsilon(x_\varepsilon, \theta_\varepsilon) = 0.$$

Since $p(x_\varepsilon, \theta_\varepsilon) = \max p_\varepsilon$, we deduce that

$$-\partial_{xx} p_\varepsilon(x_\varepsilon, \theta_\varepsilon) \geq 0.$$

We conclude that (36) and (35) also hold true in this case and the conclusion follows from the same computations as in the previous case.

Case 3: $(x_\varepsilon, \theta_\varepsilon) \in \Omega \times \{\pm A\}$. This case is treated in the same manner as the previous case. ∎

5.3. The bounds on ρ_ε

We recall the equation (E_ρ) satisfied by ρ_ε:

$$\begin{cases} -\partial_{xx}\rho_\varepsilon + L\rho_\varepsilon = \int_{-A}^{A} R(x,\theta) n_\varepsilon(x,\theta)\, d\theta - \rho_\varepsilon^2 & \text{in } \Omega, \\ \partial_{\nu_x}\rho_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{E_ρ}$$

Proof of Theorem 3 (3). The uniform bound from above and the $W^{2,p}$ bounds on ρ_ε are already provided in Lemma 3. Here we prove the uniform lower bound. We start by proving that $0 < c \leq \sup \rho_\varepsilon$. Next we prove that $c < \rho_\varepsilon$ holds true in the whole domain Ω.

A lower bound on $\sup \rho_\varepsilon$. Assume by contradiction that there exists a sequence ε_k such that

$$\varepsilon_k \xrightarrow[k \to +\infty]{} 0 \quad \text{and} \quad \sup \rho_{\varepsilon_k} \xrightarrow[k \to +\infty]{} 0.$$

Next, if we multiply (E) by ξ_{ε_k} (introduced in (3)) and we integrate by parts, we obtain

$$\mu_\varepsilon \int_{-A}^{A} \int_\Omega n_{\varepsilon_k} \xi_{\varepsilon_k}\, dx\, d\theta = -\int_{-A}^{A} \int_\Omega \rho_{\varepsilon_k} n_{\varepsilon_k} \xi_{\varepsilon_k}\, dx\, d\theta.$$

We deduce, thanks to (4), that for k large enough,

$$\frac{|\lambda(\theta_0, 0)|}{2} \leq -\mu_{\varepsilon_k} \leq \sup \rho_{\varepsilon_k} \frac{\int_{-A}^{A} \int_\Omega n_{\varepsilon_k} \xi_{\varepsilon_k}\, dx\, d\theta}{\int_{-A}^{A} \int_\Omega n_{\varepsilon_k} \xi_{\varepsilon_k}\, dx\, d\theta}.$$

It is in contradiction with the hypothesis $\sup \rho_{\varepsilon_k} \xrightarrow[k \to +\infty]{} 0$. Therefore, there exists a constant $c > 0$ such that

$$\forall \varepsilon \in]0, \varepsilon_0[, \quad c \leq \sup \rho_\varepsilon. \tag{38}$$

The lower bound on ρ_ε *in the whole domain* Ω. Let $\varepsilon < \varepsilon_0$ and $x_0 \in \Omega$ be such that

$$\rho_\varepsilon(x_0) = \sup \rho_\varepsilon.$$

We conclude, thanks to (38) and the Lipschitz estimates obtained in Theorem 3 (2), that for all $x \in \Omega$,

$$\rho_\varepsilon(x) = \int_{-A}^{A} e^{\frac{u_\varepsilon(x,\theta)}{\varepsilon}}\, d\theta = \int_{-A}^{A} e^{\frac{u_\varepsilon(x,\theta) - u_\varepsilon(x_0,\theta) + u_\varepsilon(x_0,\theta)}{\varepsilon}}\, d\theta \geq \rho_\varepsilon(x_0) e^{-C} \geq c e^{-C}.$$ ∎

5.4. The bounds on u_ε

Proof of Theorem 3 (4). First, we prove that there exists $a > 0$ such that $-a < u_\varepsilon$. Thanks to Theorem 3 (3), we know that there exists $c > 0$ such that for all ε small enough we have

$$c < \int_{-A}^{A} n_\varepsilon(x,\theta)\, d\theta.$$

We deduce the existence of $(x_0, \theta_0) \in \Omega \times]-A, A[$ such that

$$\frac{c}{2A} \le n_\varepsilon(x_0, \theta_0).$$

Hence, it follows that

$$\varepsilon \log\Big(\frac{c}{2A}\Big) \le u_\varepsilon(x_0, \theta_0).$$

We conclude thanks to the Lipschitz estimates established in Theorem 3 (2) that

$$\forall (x,\theta) \in \Omega \times]-A, A[, \quad -a \le -2CA + \varepsilon\Big[\log\Big(\frac{c}{2A}\Big) - C|\Omega|\Big] \le u_\varepsilon(x,\theta). \tag{39}$$

Next, we prove that $\lim_{\varepsilon\to 0} \sup_{(x,\theta)\in\Omega\times]-A,A[} u_\varepsilon(x,\theta) \le 0$. We prove it by contradiction. Assume that there exists $a > 0$ and sequences ε_k, (x_k, θ_k) such that

$$\varepsilon_k \xrightarrow[k\to+\infty]{} 0 \quad \text{and} \quad u_{\varepsilon_k}(x_k, \theta_k) > a.$$

Using the Lipschitz estimates provided by Theorem 3 (2), it follows for all $\theta \in (B_{\frac{a}{4C}}(\theta_k) \cap]-A, A[)$,

$$u_{\varepsilon_k}(x_k,\theta) = u_{\varepsilon_k}(x_k,\theta) - u_{\varepsilon_k}(x_k,\theta_k) + u_{\varepsilon_k}(x_k,\theta_k) \ge -C|\theta-\theta_k| + a \ge \frac{a}{2},$$

where C corresponds to the Lipschitz estimate given by (10). We deduce that

$$\rho_{\varepsilon_k}(x_k) \ge \min\Big(2A, \frac{a}{4C}\Big) e^{\frac{a}{2\varepsilon_k}}.$$

We conclude that $\liminf_{k\to+\infty} \rho_{\varepsilon_k}(x_k) = +\infty$. This is in contradiction with the L^∞ bounds on ρ_ε established in Theorem 3 (3). ∎

6. Convergence to the Hamilton–Jacobi equation

Proof of Theorem 2. We prove here the three items of Theorem 2.

Proof of (1): Convergence of ρ_ε. Thanks to Theorem 3 (3), it follows that for ε small enough, $0 < c \le \rho_\varepsilon \le C$ and $\|\rho_\varepsilon\|_{W^{2,p}(\Omega)} \le C$. We deduce from the classical Sobolev injection (see [10]) that ρ_ε converges, along subsequences, strongly in $W^{1,p}(\Omega)$ and in particular uniformly to ρ and ρ verifies

$$0 < c \le \rho \le C.$$

Proof of (2): (i) Convergence to the Hamilton–Jacobi equation. The convergence of u_ε to a viscosity solution of the Hamilton–Jacobi equation (6) can be obtained thanks to the regularity results given in Theorem 3 and a perturbed test function argument, following the heuristic argument provided in Section 1.4 .

From the Lipschitz estimates and the bounds established in Theorem 3 (2),(4), we deduce thanks to the Arzelà–Ascoli theorem that up to a subsequence, $(u_\varepsilon)_{\varepsilon>0}$ converges locally uniformly to some continuous function u. Moreover, the limit function u does not depend on x.

We prove that u is a viscosity solution of

$$\begin{cases} -|\partial_\theta u|^2 = -\lambda(\theta,\rho), \\ \partial_{\nu_\theta} u(\pm A) = 0, \end{cases}$$

with $\lambda(\theta,\rho)$ the principal eigenvalue of (2). First, we focus on the equation in the interior of the domain and then we treat the boundary conditions.

The interior equation. We recall that for a fixed value θ, Proposition 6 provides the existence of a sequence of principal eigenvalues $\lambda(\theta,\rho_\varepsilon)$ associated with a sequence of positive eigenfunctions $(\psi_\varepsilon^\theta)_{\varepsilon>0}$ of the operator $-\partial_{xx} + L - (R(x,\theta)-\rho_\varepsilon)$ with Neumann boundary conditions, i.e.

$$\begin{cases} -\partial_{xx}\psi_\varepsilon^\theta + L(\psi_\varepsilon^\theta) - (R(x,\theta)-\rho_\varepsilon)\psi_\varepsilon^\theta = \lambda(\theta,\rho_\varepsilon)\psi_\varepsilon^\theta & \text{in } \Omega, \\ \partial_{\nu_x}\psi_\varepsilon^\theta = 0 & \text{on } \partial\Omega. \end{cases} \tag{40}$$

Since $\psi_\varepsilon^\theta > 0$, we introduce

$$\Psi_\varepsilon^\theta = \ln(\psi_\varepsilon^\theta).$$

Let ϕ be a test function such that $u-\phi$ has a strict maximum at $\theta \in]-A,A[$. Then there exists $(x_\varepsilon,\theta_\varepsilon) \in \bar{\Omega} \times]-A,A[$ such that $\theta_\varepsilon \xrightarrow[\varepsilon\to 0]{} \theta$ and

$$\max_{(x,\theta)\in\bar{\Omega}\times]-A,A[} u_\varepsilon(x,\theta) - \phi(\theta) - \varepsilon\Psi_\varepsilon^{\theta_\varepsilon}(x) = u_\varepsilon(x_\varepsilon,\theta_\varepsilon) - \phi(\theta_\varepsilon) - \varepsilon\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon).$$

We distinguish two cases: either $x_\varepsilon \in \Omega$ or $x_\varepsilon \in \partial\Omega$.

Case 1: $x_\varepsilon \in \Omega$. Since u_ε is a classical solution of (E_{HC}), we deduce that it is also a viscosity solution, and therefore

$$\begin{aligned} &-\frac{\partial_{xx}(\phi(\theta_\varepsilon)+\varepsilon\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon))}{\varepsilon} - \frac{[\partial_x(\phi(\theta_\varepsilon)+\varepsilon\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon))]^2}{\varepsilon^2} \\ &\quad + \int_\Omega [1-e^{\Psi_\varepsilon^{\theta_\varepsilon}(y)-\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)}]K(x_\varepsilon-y)\,dy - \varepsilon\partial_{\theta\theta}(\phi(\theta_\varepsilon)+\varepsilon\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)) \\ &\quad - [\partial_\theta(\phi(\theta_\varepsilon)+\varepsilon\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon))]^2 - R(x_\varepsilon,\theta_\varepsilon) + \rho_\varepsilon(x_\varepsilon) \le 0. \end{aligned}$$

Remarking that ϕ does not depend on x and the θ value is fixed in $\Psi_\varepsilon^{\theta_\varepsilon}$, we deduce that

$$- \partial_{xx}\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon) - [\partial_x\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)]^2 + \int_\Omega [1 - e^{\Psi_\varepsilon^{\theta_\varepsilon}(y) - \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)}]K(x_\varepsilon - y)\,dy$$
$$- R(x_\varepsilon, \theta_\varepsilon) + \rho_\varepsilon(x_\varepsilon) - \varepsilon\partial_{\theta\theta}\phi(\theta_\varepsilon) - [\partial_\theta\phi(\theta_\varepsilon)]^2 \le 0. \tag{41}$$

Next we observe that (40) implies

$$- \partial_{xx}\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon) - [\partial_x\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)]^2$$
$$+ \int_\Omega [1 - e^{\Psi_\varepsilon^{\theta_\varepsilon}(y) - \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)}]K(x_\varepsilon - y)\,dy - R(x_\varepsilon, \theta_\varepsilon) + \rho_\varepsilon(x_\varepsilon)$$
$$= \lambda(\theta_\varepsilon, \rho_\varepsilon).$$

Therefore, passing to the limit $\varepsilon \to 0$, thanks to the continuity of $\lambda(\theta, \rho)$ with respect to θ and ρ (Proposition 6), it follows that

$$-[\partial_\theta\phi(\theta)]^2 \le -\lambda(\theta, \rho).$$

Case 2: $x_\varepsilon \in \partial\Omega$. First, we remark that in this case,

$$-\partial_x u_\varepsilon(x_\varepsilon, \theta_\varepsilon) = -\partial_x\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon) = 0.$$

Therefore, we deduce that

$$-\partial_x[u_\varepsilon(x_\varepsilon, \theta_\varepsilon) - \phi(\theta_\varepsilon) - \varepsilon\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)] = 0.$$

Moreover, since

$$(u_\varepsilon - \phi - \varepsilon\Psi_\varepsilon^{\theta_\varepsilon})(x_\varepsilon, \theta_\varepsilon) = \max(u_\varepsilon - \phi - \varepsilon\Psi_\varepsilon^{\theta_\varepsilon}),$$

we have firstly by a reflective argument that

$$- \partial_{xx}(u_\varepsilon - \phi - \varepsilon\Psi_\varepsilon^{\theta_\varepsilon})(x_\varepsilon, \theta_\varepsilon) \ge 0, \tag{42}$$

and secondly we have

$$u_\varepsilon(y, \theta_\varepsilon) - u_\varepsilon(x_\varepsilon, \theta_\varepsilon) \le \varepsilon[\Psi_\varepsilon(y) - \Psi_\varepsilon(x_\varepsilon)]. \tag{43}$$

Inequalities (42) and (43) lead to

$$-\partial_{xx}\varepsilon\Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon) \le -\partial_{xx}u_\varepsilon(x_\varepsilon, \theta_\varepsilon)$$

and

$$\int_\Omega [1 - e^{\Psi_\varepsilon^{\theta_\varepsilon}(y) - \Psi_\varepsilon^{\theta_\varepsilon}(x_\varepsilon)}]K(x_\varepsilon - y)\,dy \le \int_\Omega [1 - e^{\frac{u_\varepsilon(y,\theta_\varepsilon) - u_\varepsilon(x_\varepsilon,\theta_\varepsilon)}{\varepsilon}}]K(x_\varepsilon - y)\,dy.$$

Therefore, the conclusion follows from a similar computation to above.

The boundary conditions. Let ϕ be a test function such that $u - \phi$ has a strict maximum at A (the proof works the same for $-A$). Then there exists $(x_\varepsilon, \theta_\varepsilon) \in \bar{\Omega} \times [-A, A]$ such that

$$\theta_\varepsilon \xrightarrow[\varepsilon \to 0]{} A \quad \text{and} \quad \max_{(x,\theta)\in\bar{\Omega}\times[-A,A]} u_\varepsilon - \phi = (u_\varepsilon - \phi)(x_\varepsilon, \theta_\varepsilon).$$

We distinguish two cases: $(x_\varepsilon, \theta_\varepsilon) \in \bar{\Omega} \times]-A, A[$ or $(x_\varepsilon, \theta_\varepsilon) \in \bar{\Omega} \times \{A\}$.

Case 1: $(x_\varepsilon, \theta_\varepsilon) \in \bar{\Omega} \times]-A, A[$. In this case, by a similar analysis to above, we deduce that

$$-[\partial_\theta \phi(\theta_\varepsilon)]^2 \le -\lambda(\theta_\varepsilon, \rho) + o_\varepsilon(1).$$

Case 2: $(x_\varepsilon, \theta_\varepsilon) \in \bar{\Omega} \times \{A\}$. In this case, since the maximum is reached on the boundary, we deduce thanks to the boundary conditions of u_ε,

$$-\partial_{\nu_\theta} \phi(\theta_\varepsilon) = \partial_{\nu_\theta}(u_\varepsilon - \phi)(x_\varepsilon, \theta_\varepsilon) \ge 0.$$

Taking the inferior limit, we conclude that

$$\min(-[\partial_\theta \phi(A)]^2 + \lambda(A, \rho), \partial_{\nu_\theta} \phi(\pm A)) \le 0,$$

which corresponds to the boundary condition in the viscosity sense.

Finally, u is a subsolution of (6) in a viscosity sense. With similar arguments, u is also a supersolution. We conclude that u is a viscosity solution of (6).

(ii) The constraint. The constraint $\max_\theta u(\theta) = 0$ in (6) is a consequence of the Hopf–Cole transformation (5) and the fact that ρ_ε remains bounded away from 0, uniformly in ε.

If the constraint does not hold true, it follows, thanks to (12), that $\sup_{\theta\in[-A,A]} u(\theta) < -a < 0$. Hence for ε small enough, and thanks to the uniform convergence of u_ε to u, we deduce that $\max_{(x,\theta)\in\Omega\times[-A,A]} u_\varepsilon(x, \theta) < -\frac{a}{2}$, which implies that $\rho_\varepsilon < c$ for ε sufficiently small. This is in contradiction with Theorem 3 (3). We conclude that $\max_{\theta\in[-A,A]} u(\theta) = 0$.

Proof of (3). The convergence of n_ε and the inclusion property. The first inclusion property in (7) can be obtained thanks to the Hopf–Cole transformation (5) and the uniform convergence of u_ε to u. The second inclusion property in (7) is a consequence of the Hamilton–Jacobi equation (6) and the fact that the zero level set of u is also the set of the maximum points of u. We detail these arguments below.

Thanks to the L^∞ bounds on ρ_ε (Theorem 3 (3)), we deduce that

$$c \le \|n_\varepsilon\|_{L^1(\Omega\times]-A,A[)} \le C.$$

It follows that n_ε converges up to a subsequence and in the sense of measures to a measure n. The measure n is non-negative and not trivial. We next prove that

$$\operatorname{supp} n_\varepsilon \subset \Omega \times \{\theta \in]-A, A[\mid u(\theta) = 0\}.$$

Indeed, let $\phi \in C_c^\infty(\Omega \times]-A, A[)$ be any positive test function such that

$$\operatorname{supp}\phi \subset \Omega \times \left\{\theta \in]-A, A[\mid u(\theta) = 0\right\}^c. \tag{44}$$

We prove that $\int_\Omega \int_{-A}^{A} \phi(x,\theta) n(x,\theta)\, dx\, d\theta = 0$.

To this end, we introduce $-a = \sup_{\operatorname{supp}\phi} u$. According to (44), it follows that $a > 0$. We deduce that for all ε small enough and all $(x, \theta) \in \Omega \times \operatorname{supp}\phi$, we have

$$u_\varepsilon(x,\theta) \le -\frac{a}{2}. \tag{45}$$

We conclude that

$$\begin{aligned}
\int_\Omega \int_{-A}^{A} \phi(x,\theta) n(x,\theta)\, d\theta\, dx &= \int_\Omega \int_{\operatorname{supp}\phi(x,\cdot)} \phi(x,\theta) n(x,\theta)\, d\theta\, dx \\
&= \lim_{\varepsilon\to 0} \int_\Omega \int_{\operatorname{supp}\phi(x,\cdot)} \phi(x,\theta) n_\varepsilon(x,\theta)\, d\theta\, dx \\
&= \lim_{\varepsilon\to 0} \int_\Omega \int_{\operatorname{supp}\phi(x,\cdot)} \phi(x,\theta) e^{\frac{u_\varepsilon(x,\theta)}{\varepsilon}}\, d\theta\, dx \\
&\le \lim_{\varepsilon\to 0} \int_\Omega \int_{\operatorname{supp}\phi(x,\cdot)} \phi(x,\theta) e^{\frac{-a}{2\varepsilon}}\, d\theta\, dx \\
&= 0.
\end{aligned}$$

We finally prove that

$$\left\{u(\theta) = 0\right\} \subset \left\{\lambda(\theta, \rho) = 0\right\}. \tag{46}$$

To this end, note first that since u is a Lipschitz continuous function, it is a.e. differentiable. Therefore, (6) implies that

$$\lambda(\theta, \rho) \ge 0 \quad \text{for a.e. } \theta.$$

Moreover, since λ is continuous with respect to θ, the above inequality holds indeed for all θ. To prove (46), it is therefore enough to prove that for any θ_0 such that $u(\theta_0) = 0$, we have

$$\lambda(\theta_0, \rho) \le 0.$$

This property can be derived by testing the equation in (6) against the test function $\varphi(\theta) \equiv 0$ at the point θ_0 for a viscosity subsolution criterion.

This concludes the proof of (3). ∎

A. Existence and properties of $\lambda(\theta, \rho)$

In this section we first establish a Hopf lemma. It is obtained by a classical argument, but for the sake of completeness and because of the presence of the less classical non-local operator L, we provide the proof. Next we verify the existence of $\lambda(\theta, \rho)$. To finish we provide the proof of some properties of λ already stated in the article (namely Propositions 1 and 3).

A.1. A Hopf lemma

In this section we prove the following Hopf lemma:

Lemma 4 (Hopf lemma). *Let u be a smooth function defined on Ω such that*

$$- \partial_{xx} u + L(u) + c(x)u \geq 0, \tag{47}$$

with c a non-negative bounded smooth function. If there exists $x_0 \in \partial\Omega$ such that $\min_{x\in\Omega} u(x) = u(x_0) < 0$ then either u is constant or

$$\partial_{\nu_x} u(x) < 0. \tag{48}$$

The proof is in the spirit of the classical proof of the Hopf lemma (see [16, p. 250]).

Proof. Up to a scaling, there is no loss of generality if we assume that $B(0,1) \subset \Omega$ and $x_0 = 1$. Next we define

$$v(x) = [e^{-\frac{3}{4}\lambda} - e^{-\lambda \max(0,|x|^2-\frac{1}{4})}]\mathbb{1}_{B(0,1)}(x)$$

for λ a positive constant. We underline that $v(x) = e^{-\frac{3}{4}\lambda} - 1$ in $B(0, \frac{1}{2})$. Next we claim that by taking λ large enough, for all $x \in B(0,1)\backslash B(0, \frac{3}{4})$ there holds

$$\begin{aligned} -\partial_{xx} v(x) &= 2\lambda e^{-\lambda \max(0,|x|^2-\frac{1}{4})}(2\lambda|x|^2 - 1) > 0, \\ Lv(x) &\geq \int_{B(0,1)} e^{-\lambda \max(0,|y|^2-\frac{1}{4})} K(x-y)\,dy \\ &\quad - \int_{\Omega} e^{-\lambda \max(0,|x|^2-\frac{1}{4})} K(x-y)\,dy > 0. \end{aligned} \tag{49}$$

The first inequality of (49) follows from a straightforward computation. For the second inequality, according to assumption (H3) we have

$$\begin{aligned} \liminf_{\lambda\to+\infty} \int_{B(0,1)} e^{-\lambda \max(0,|y|^2-\frac{1}{4})} K(x-y)\,dy - \int_{\Omega} e^{-\lambda \max(0,|x|^2-\frac{1}{4})} K(x-y)\,dy \\ \geq c_K |B(0, \tfrac{1}{2})| > 0. \end{aligned}$$

Therefore, if λ is large enough, (49) holds true.

Next we claim that if u is not constant, the minimum cannot be reached in the interior of Ω. Otherwise, we deduce the existence of $x_1 \in \Omega$ such that $u(x_1) = -\min_{x\in\Omega} u < 0$. Since c is non-negative, we have

$$-\partial_{xx} u(x_1) \leq 0, \quad Lu(x_1) < 0, \quad \text{and} \quad c(x_1)u(x_1) \leq 0.$$

Therefore, we deduce that

$$-\partial_{xx} u(x_1) + Lu(x_1) + c(x_1)u(x_1) < 0.$$

This is in contradiction with assumption (47).

We deduce that $\min_{x\in\partial B(0,\frac{3}{4})} u(x_0) - u(x) < 0$. Next, taking ε small enough, there holds that

$$\forall x \in \partial B(0, \tfrac{3}{4}), \quad u(x_0) - u(x) - \varepsilon v(x) < 0.$$

Since $v = 0$ on $\partial B(0, 1)$ and by definition of x_0, it follows that

$$\forall x \in \partial B(0, 1), \quad u(x_0) - u(x) - \varepsilon v(x) \leq 0.$$

Moreover, according to (47), (49) and remarking that $u(x_0) - \varepsilon v(x) \geq 0$ for ε small enough, we deduce that for all $x \in B(0, 1)\backslash B(0, \frac{3}{4})$,

$$\begin{aligned} - \partial_{xx}(u(x_0) - u(x) - \varepsilon v(x)) + L(u(x_0) - u - \varepsilon v)(x) \\ + c(x)(u(x_0) - u(x) - \varepsilon v(x)) \leq 0. \end{aligned}$$

We deduce thanks to the maximum principle that $u(x_0) - u(x) - \varepsilon v(x) \leq 0$ for all $x \in B(0, 1)\backslash B(0, \frac{3}{4})$. We conclude that

$$\partial_{\nu_x} u(x_0) \leq -\partial_{\nu_x} \varepsilon v(x_0) = -\varepsilon 2\lambda e^{-\frac{3\lambda}{4}} < 0. \qquad \blacksquare$$

A.2. Existence of a principal eigenpair

Proposition 6. *Under hypotheses* (H1)–(H3)*, for a fixed bounded smooth function ρ and a fixed value $\theta \in]-A, A[$, there exists a principal eigenvalue $\lambda(\theta, \rho)$ of the operator $-\partial_{xx}\psi + L(\psi) - (R(\cdot, \theta) - \rho)\psi$ with Neumann boundary conditions, i.e.*

$$\begin{cases} -\partial_{xx}\psi^\theta + L(\psi^\theta) - (R(\cdot, \theta) - \rho)\psi^\theta = \lambda(\theta, \rho)\psi^\theta & \text{in } \Omega, \\ \partial_{\nu_x}\psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{50}$$

The associated eigenfunction ψ^θ has a constant sign and is unique up to multiplication by a constant. Moreover, the function $\lambda(\theta, \rho)$ and ψ^θ are C^1 with respect to θ and $\rho \in H^1(\Omega)$.

In the following, we will consider that ψ^θ is positive and of L^2 norm equal to 1.

Proof. First, we prove the existence of the principal eigenpair by verifying that we can apply the Krein–Rutman theorem (see [32, p. 122]). Since it is classical, we do not provide all the details. The cone of functions where we apply the Krein–Rutman theorem is

$$K = \overline{\{u \in C^{1+\alpha}(\Omega) \mid u > 0 \text{ and } \partial_{\nu_x} u = 0\}}.$$

We define $\mathcal{L}(v)$ as the unique solution of

$$\begin{cases} -\partial_{xx}\mathcal{L}(v) + \int_\Omega [\mathcal{L}(v)(x) - \mathcal{L}(v)(y)]K(x-y)\,dy \\ \qquad -(R(\cdot, \theta) - \rho - C)\mathcal{L}(v) = v & \text{in } \Omega, \\ \partial_{\nu_x}\mathcal{L}(v) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $C > \sup_{x\in\Omega}(R(x,\theta) - \rho(x))$ and $v \in K$. The operator $\mathcal{L}$ is linear, compact thanks to the elliptic estimates. We have to prove that

$$\forall v \in K\backslash\{0\}, \quad \mathcal{L}(v) \in \text{int}(K).$$

Let v be in K with v not trivial. By elliptic regularity, it follows that $\mathcal{L}(v) \in C^{1+\alpha}$ and $\partial_{\nu_x}\mathcal{L}(v) = 0$. It remains to prove that $\mathcal{L}(v) > 0$.

First we prove that if $\mathcal{L}(v)$ is constant then it is necessarily a positive constant. Next we prove that if $\mathcal{L}(v)$ varies then $\mathcal{L}(v) > 0$.

Assume that $\mathcal{L}(v) = c$. Let $\bar{x} \in \Omega$ be such that $v(\bar{x}) > 0$. Moreover, the choice of C gives $-(R(\bar{x},\theta) - \rho(\bar{x}) - C) > 0$ and since $-\partial_{xx}\mathcal{L}(v) = L(\mathcal{L}(v)) = 0$, we deduce that

$$\mathcal{L}(v)(\bar{x}) = c = \frac{v(\bar{x})}{-(R(\bar{x},\theta) - \rho(\bar{x}) - C)} > 0.$$

Next we suppose that $\mathcal{L}(v)$ is not constant. Assume by contradiction that there exists x such that $\mathcal{L}(v)(x) \leq 0$. Let $\bar{x}' \in \overline{\Omega}$ be such that

$$\inf_{x\in\Omega} \mathcal{L}(v)(x) = \mathcal{L}(v)(\bar{x}').$$

Then either $\bar{x}' \in \Omega$ or $\bar{x}' \in \partial\Omega$. In the first case, we deduce that

$$-\partial_{xx}\mathcal{L}(v)(\bar{x}') \leq 0 \quad \text{and} \quad L(\mathcal{L}(v))(\bar{x}') < 0,$$

which leads to the contradiction

$$0 \leq v(\bar{x}') = -\partial_{xx}\mathcal{L}(v)(\bar{x}') + L(\mathcal{L}(v))(\bar{x}') - (R(\bar{x}',\theta) - \rho(\bar{x}') - C)\mathcal{L}(v)(\bar{x}') < 0.$$

If $\bar{x}' \in \partial\Omega$, since $\mathcal{L}(v)$ is not constant, we deduce from Lemma 4 that $\partial_{\nu_x}\mathcal{L}(v)(\bar{x}') < 0$. It is in contradiction with the Neumann boundary condition.

We conclude that we can apply the Krein–Rutman theorem and the conclusion follows.

Next we focus on the regularity of λ and ψ^θ with respect to θ and ρ. The result follows directly from the implicit function theorem applied to

$$\begin{aligned} G\colon (\phi,\lambda,\theta,\rho) &\in H^1(\Omega)\times\mathbb{R}\times\mathbb{R}\times H^1(\Omega) \\ &\mapsto \left(-\partial_{xx}\phi + L\phi - [R(\cdot,\theta) - \rho + \lambda]\phi, \int_\Omega \phi(x)^2\,dx - 1\right). \end{aligned}$$

The interested reader may refer to [16, Theorem 2, Chapter 11] for technical details in a finite-dimensional setting. ∎

The existence of the solution of (3) is also due to the Krein–Rutman theorem, and therefore we do not provide the proof of existence.

A.3. Some properties of λ

We prove in this subsection two propositions that are stated in Section 2: Propositions 1 and 3.

Proof of Proposition 1. First recall from Proposition 6 that λ and ψ^θ are C^1 functions with respect to θ. Differentiating (2) with respect to θ leads to

$$\begin{aligned} &\partial_\theta\lambda(\theta,\rho)\psi^\theta + \lambda(\theta,\rho)\partial_\theta\psi^\theta \\ &\quad = -\partial_{xx}\partial_\theta\psi^\theta + L(\partial_\theta\psi^\theta) - [R(\cdot,\theta)-\rho]\partial_\theta\psi^\theta - \partial_\theta R(\cdot,\theta)\psi^\theta. \end{aligned} \tag{51}$$

We multiply (51) by ψ^θ and integrate by parts, recalling $\int_\Omega \psi^\theta(x)^2\,dx = 1$, to obtain

$$\begin{aligned} \partial_\theta\lambda(\theta,\rho) = &-\int_\Omega \partial_\theta R(x,\theta)\psi^\theta(x)^2\,dx \\ &+\int_\Omega \partial_x\partial_\theta\psi^\theta\partial_x\psi^\theta + L(\psi^\theta)\partial_\theta\psi^\theta \\ &\qquad - [R(x,\theta)-\rho-\lambda(\theta,\rho)]\partial_\theta\psi^\theta\psi^\theta\,dx. \end{aligned} \tag{52}$$

We remark that multiplying (2) by $\partial_\theta\psi^\theta$ and integrating by parts leads to

$$\int_\Omega \partial_x\psi^\theta\partial_x\partial_\theta\psi^\theta + L(\psi^\theta)\partial_\theta\psi^\theta - [R(x,\theta)-\rho-\lambda(\theta,\rho)]\psi^\theta\partial_\theta\psi^\theta\,dx = 0.$$

The conclusion follows. ■

Proof of Proposition 3. We focus on the proof of (22), since the proof of (21) follows by straightforward computations.

Since the function R is C^2 with respect to θ and g, according to the implicit function theorem, the eigenfunction ψ^θ is C^2 with respect to its parameters g and θ. In this proof, we will take this dependence with respect to the parameters into account in the notation: we will denote ψ^θ by $\psi^{\theta,g}$, $\lambda(\theta)$ by , and R by R_g.

First, we establish that

$$\int_\Omega (\partial_\theta\psi^{\theta,g}(x))^2\,dx \xrightarrow[g\to 0]{} 0$$

for a fixed value $\theta \in]-A,A[$. Since $\|\psi^{\theta,g}\|^2_{L^2(\Omega)} = 1$, we deduce that

$$\int_\Omega \psi^{\theta,g}(x)\partial_\theta\psi^{\theta,g}(x)\,dx = 0.$$

If we differentiate (50) with respect to θ, we have that $\partial_\theta\psi^{\theta,g}$ is a solution in Ω of

$$\begin{aligned} &-\partial_{xx}\partial_\theta\psi^{\theta,g} + L(\partial_\theta\psi^{\theta,g}) - [R_g(\cdot,\theta)-\rho]\partial_\theta\psi^{\theta,g} - \partial_\theta R_g(\cdot,\theta)\psi^{\theta,g} \\ &\quad = \lambda(\theta,g)\partial_\theta\psi^{\theta,g} + \partial_\theta\lambda(\theta,g)\psi^{\theta,g}. \end{aligned}$$

We then multiply the above equation by $\psi^{\theta,g}$ and integrate over Ω to obtain

$$\int_\Omega (\partial_x(\partial_\theta \psi^{\theta,g}))^2\,dx - \frac{\int_{\Omega\times\Omega}[\partial_\theta\psi^{\theta,g}(x) - \partial_\theta\psi^{\theta,g}(y)]^2 K(x-y)\,dy\,dx}{2}$$

$$- \int_\Omega [R_g(x,\theta) - \rho(x)]\partial_\theta\psi^{\theta,g}(x)^2\,dx$$

$$= \lambda(\theta,g)\int_\Omega (\partial_\theta\psi^{\theta,g})^2\,dx + \int_\Omega \partial_\theta R_g(x,\theta)\psi^{\theta,g}(x)\partial_\theta\psi^{\theta,g}(x)\,dx$$

$$+ \partial_\theta\lambda(\theta,g)\int_\Omega \psi^{\theta,g}(x)\partial_\theta\psi^{\theta,g}(x)\,dx.$$

Remarking that $\partial_\theta R_0 = 0$ and recalling that $\int_\Omega \psi^{\theta,0}(x)\partial_\theta\psi^{\theta,0}(x)\,dx = 0$, we deduce that $\partial_\theta\psi^{\theta,0}$ belongs to the eigenspace associated to the principal eigenvalue $\lambda(\theta,0)$ of the operator $-\partial_{xx} + L - [R_0 - \rho]$. Since this space is one-dimensional, engendered by $\psi^{\theta,0}$ and using again that $\psi^{\theta,0}$ is orthogonal to $\partial_\theta\psi^{\theta,0}$, we conclude that

$$\int_\Omega (\partial_\theta\psi^{\theta,g}(x))^2\,dx \xrightarrow[g\to 0]{} 0.$$

Next, we prove that this convergence is uniform with respect to θ. By compactness of $[-A,A]\times[0,\tilde g]$ (for some $\tilde g > 0$), we deduce that there exists a uniform constant $C > 0$ such that for all $(\theta,g)\in[-A,A]\times[0,\tilde g]$ we have

$$|\partial_{\theta\theta}\psi^{\theta,g}| < C.$$

It follows that for any $\theta_1,\theta_2 \in]-A,A[$, we have

$$\int_\Omega (\partial_\theta\psi^{\theta_1,g}(x))^2\,dx \le \int_\Omega (\partial_\theta\psi^{\theta_2,g}(x))^2\,dx + 2C^2\,\mathrm{mes}(\Omega)|\theta_1-\theta_2|^2$$

(where $\mathrm{mes}(\Omega)$ stands for the Lebesgue measure of Ω). Next we fix $\mu > 0$ and we prove that for $g < g_0$ (for some $g_0 > 0$) we have (independently of the choice of θ)

$$\int_\Omega (\partial_\theta\psi^{\theta,g}(x))^2\,dx < \mu.$$

By compactness of $[-A,A]$, there exists an integer $i_0 > 0$ and $\theta_i \in]-A,A[$ with $i \in \{1,\ldots,i_0\}$ such that

$$[-A,A] \subset \bigcup_{i=1}^{i_0} B\Big(\theta_i, \sqrt{\frac{\varepsilon}{4C^2\,\mathrm{mes}(\Omega)}}\Big).$$

Next we define $g_i = \sup\{g' \in [0,\tilde g] : \forall g < g',\ \int_\Omega(\partial_\theta\psi^{\theta_i,g}(x))^2\,dx < \frac{\mu}{2}\}$ (notice that $g_i > 0$). By setting $g_0 = \min_{i\in\{1,\ldots,i_0\}} g_i$, for any $\theta \in]-A,A[$, we conclude that for all $g < g_0$ we have

$$\int_\Omega (\partial_\theta\psi^{\theta,g}(x))^2\,dx \le \int_\Omega (\partial_\theta\psi^{\theta_i,g}(x))^2\,dx + 2C^2\,\mathrm{mes}(\Omega)|\theta-\theta_i|^2 < \mu$$

with $i \in \{1,\ldots,i_0\}$ such that $\theta \in B(\theta_i, \sqrt{\frac{\varepsilon}{4C^2\,\mathrm{mes}(\Omega)}})$. This concludes the proof of the uniform convergence. ■

B. Proof of Lemma 1

The proof of Lemma 1 follows essentially the steps of the proof of the convergence of u_ε (i.e. Theorem 2 (2)). Therefore, we will only emphasize the differences between the two proofs. We made the choice to provide the proof of the convergence of u_ε rather than the convergence of μ_ε because it is the result that motivated the current study.

Proof of Lemma 1. We recall the equation satisfied by μ_ε and ξ_ε:

$$\begin{cases} -\partial_{xx}\xi_\varepsilon - \varepsilon^2\partial_{\theta\theta}\xi_\varepsilon + L\xi_\varepsilon - R\xi_\varepsilon = \mu_\varepsilon\xi_\varepsilon & \text{in } \Omega\times]-A,A[, \\ \partial_{\nu_x}\xi_\varepsilon = \partial_{\nu_\theta}\xi_\varepsilon = 0 & \text{on } \partial(\Omega\times]-A,A[). \end{cases} \tag{3}$$

The existence of ξ_ε is ensured by the Krein–Rutman theorem. Moreover, according to the Krein–Rutman theorem, the sign of ξ_ε is constant. Therefore, we consider that $\xi_\varepsilon > 0$, $\|\xi_\varepsilon\|_{L^2} = 1$ and we define

$$v_\varepsilon = \varepsilon \ln(\xi_\varepsilon).$$

Next we prove that μ_ε is bounded from below and above respectively by $-\sup R$ and $-\inf R$.

First, we focus on the upper bound. Let $(\bar{x},\bar{\theta}) \in \bar{\Omega}\times[-A,A]$ be such that

$$\sup_{(x,\theta)\in\bar{\Omega}\times[-A,A]} \xi_\varepsilon(x,\theta) = \xi_\varepsilon(\bar{x},\bar{\theta}).$$

If $(\bar{x},\bar{\theta}) \in \Omega\times]-A,A[$, it follows that

$$(-\partial_{xx}\xi_\varepsilon - \varepsilon^2\partial_{\theta\theta}\xi_\varepsilon + L(\xi_\varepsilon))(\bar{x},\bar{\theta}) \leq 0.$$

From (3), we deduce that

$$\mu_\varepsilon \leq -R(\bar{x},\bar{\theta}) \leq -\inf R.$$

If $(\bar{x},\bar{\theta})$ belongs to $\partial(\Omega\times]-A,A[)$, we conclude with a reflective argument and the same computations as in the previous case. In any case, for all $\varepsilon > 0$ we have

$$\mu_\varepsilon < -\inf R.$$

Next we focus on the lower bound. Let $(\underline{x},\underline{\theta}) \in \bar{\Omega}\times[-A,A]$ be such that

$$\inf_{(x,\theta)\in\bar{\Omega}\times[-A,A]} \xi_\varepsilon = \xi_\varepsilon(\underline{x},\underline{\theta}).$$

With similar arguments to the upper bound, we deduce that

$$-\sup R \leq -R(\underline{x},\underline{\theta}) \leq \mu_\varepsilon.$$

Therefore, μ_ε is uniformly bounded from below and above, thus μ_ε converges along subsequences to μ.

Next, as we have established Lipschitz and uniform bounds on u_ε, we can prove that there exists a constant $C > 0$ such that

$$|\partial_x v_\varepsilon| < C\varepsilon, \quad |\partial_\theta v_\varepsilon| < C, \quad -C < \lim_{\varepsilon\to 0} \inf_{\Omega\times]-A,A[} v_\varepsilon, \quad \text{and} \quad \lim_{\varepsilon\to 0} \sup_{\Omega\times]-A,A[} v_\varepsilon \le 0.$$

Therefore, we deduce that v_ε converges along subsequences to v. Moreover, with similar computations to the proof of Theorem 2 (2), we deduce that v is a viscosity solution of

$$\begin{cases} -[\partial_\theta v(\theta)]^2 = -\lambda(\theta, -\mu), \\ \max\limits_{\theta\in[-A,A]} v(\theta) = 0. \end{cases} \tag{53}$$

Next we claim that

$$\lambda(\theta, -\mu) = \lambda(\theta, 0) - \mu. \tag{54}$$

We postpone the proof of this claim to the end of this paragraph. Thanks to (53) and (54) we deduce that

$$\begin{cases} -[\partial_\theta v(\theta)]^2 = -\lambda(\theta, 0) + \mu, \\ \max\limits_{\theta\in[-A,A]} v(\theta) = 0. \end{cases}$$

Note that $-\lambda(\theta, 0) + \mu \le 0$ for all $\theta \in [-A, A]$. Next we introduce $\theta_m \in [-A, A]$ such that

$$v(\theta_m) = \max_{\theta\in[-A,A]} v(\theta).$$

It follows that $\partial_\theta v(\theta_m) = 0$ and $-\lambda(\theta_m, 0) + \mu = 0 = \max(-\lambda(\theta, 0) + \mu)$. We deduce thanks to (H4) that

$$0 = \max(-\lambda(\theta, 0) + \mu) = -\min(\lambda(\theta, 0)) + \mu = -\lambda(\theta_0, 0) + \mu.$$

We conclude that

$$\lambda(\theta_0, 0) = \mu.$$

We finish the proof by remarking that the previous convergence result holds for any subsequence of μ_ε. Therefore, we conclude that

$$\lim_{\varepsilon\to 0} \mu_\varepsilon = \lambda(\theta_0, 0).$$

It remains to prove (54). Let ψ_μ^θ be the principal eigenfunction associated to the principal eigenvalue of $\lambda(\theta, -\mu)$ with μ a constant,

$$\begin{cases} -\partial_{xx}\psi_\mu^\theta + L\psi_\mu^\theta - [R(\cdot, \theta) + \mu]\psi_\mu^\theta = \lambda(\theta, -\mu)\psi_\mu^\theta & \text{in } \Omega, \\ \partial_{\nu_x}\psi_\mu^\theta = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows that

$$-\partial_{xx}\psi_\mu^\theta + L\psi_\mu^\theta - R(\cdot, \theta)\psi_\mu^\theta = (\lambda(\theta, -\mu) + \mu)\psi_\mu^\theta.$$

Since μ is constant, $\psi_\mu^\theta > 0$ and by the uniqueness of the positive eigenfunction of $-\partial_{xx} + L - R(\cdot, \theta)$ (up to multiplication by a scalar), we deduce that $\lambda(\theta, -\mu) + \mu = \lambda(\theta, 0)$. ∎

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