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On the self-similar behavior of coagulation systems with injection

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Abstract. In this paper we prove the existence of a family of self-similar solutions for a class of coagulation equations with a constant flux of particles from the origin. These solutions are expected to describe the longtime asymptotics of Smoluchowski's coagulation equations with a time-independent source of clusters concentrated in small sizes. The self-similar profiles are shown to be smooth, provided the coagulation kernel is also smooth. Moreover, the self-similar profiles are estimated from above and from below by $x^{-(\gamma+3)/2}$ as $x \to 0$, where $\gamma < 1$ is the homogeneity of the kernel, and are proven to decay at least exponentially as $x \to \infty$.

1. Introduction

1.1. Aim of the paper

Smoluchowski's coagulation equation, introduced by the physicist Marian von Smoluchowski in 1916 (cf. [28]), is a mean field model describing a system of clusters evolving in time due to coagulation upon binary collision between clusters. The solution of Smoluchowski's equation, $f(t,x)$, represents the number density of clusters of size x at time t and is governed by the integro-differential equation

$$\partial_t f(t,x) = \mathbb{K}[f](t,x), \tag{1.1}$$

where $\mathbb{K}$ is the coagulation operator defined by

$$\begin{aligned} \mathbb{K}[f](t,x) := \frac{1}{2}\int_0^x K(x-y,y) f(t,x-y) f(t,y)\,dy \\ - \int_0^\infty K(x,y) f(t,x) f(t,y)\,dy. \end{aligned} \tag{1.2}$$

The kernel $K(x,y)$ is the coagulation rate of a cluster of size y with a cluster of size x, and it summarizes the microscopical mechanisms underlying coagulation. Different

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kernels may induce completely different dynamics. For an overview on Smoluchowski's coagulation equations with different kernels we refer to [1] and [2].

Equation (1.1), as well as its discrete counterpart, has been extensively used as a modeling tool. Polymerization [3], animal grouping [15], hemagglutination [25], planetesimal aggregation [17] and atmospheric aerosol formation [12], [21], [26] are just some examples of applications.

Several classes of coagulation kernels have been derived in the physical/chemical literature. The specific form of the kernels depends on mechanisms yielding the aggregation of the particles (cf. [12]). Many of the kernels relevant in applications satisfy

$$c_1(x^{\gamma+\lambda}y^{-\lambda}+y^{\gamma+\lambda}x^{-\lambda}) \leq K(x,y) \leq c_2(x^{\gamma+\lambda}y^{-\lambda}+y^{\gamma+\lambda}x^{-\lambda}), \tag{1.3}$$

where $0<c_1\leq c_2<\infty$ and $\gamma,\lambda\in\mathbb{R}$.

Two relevant kernels in aerosol science are the Brownian kernel and the free molecular coagulation kernel [12]. The exponents γ and λ in (1.3), associated to these two kernels are $(\gamma,\lambda)=(0,1/3)$ and $(\gamma,\lambda)=(1/6,1/2)$ respectively. Both kernels yield the aggregation rate for a set of molecules (monomers) immersed in the air. The difference between the two kernels is that, in the first case, the mean free path of the molecules is smaller than the cluster sizes, while it is much larger in the second case (for a more detailed discussion see [10] and [12]).

We focus on the coagulation equation with source

$$\partial_t f(t,x)=\mathbb{K}[f](t,x)+\eta(x), \tag{1.4}$$

with $\mathbb{K}$ defined by (1.2). The term η in (1.4) is a measure that represents a time-independent source of particles, which is the main difference of this equation compared to the classical pure coagulation equation (1.1).

Equation (1.4) has not been studied as much as the classical coagulation equation in the mathematical literature, despite its relevance in atmospheric physics (see for instance [21]). The existence of weak time-dependent solutions of (1.4) has been proven, under specific assumptions on the coagulation kernel, in [5] and [7]. The long-term asymptotic behavior of (1.4) has been studied in [4] with a combination of numerical simulations and matched asymptotics expansions for the kernels

$$K(x,y)=x^a y^b+y^a x^b,\quad a,b\geq 0. \tag{1.5}$$

This corresponds to $\gamma=a+b$ and $b=-\lambda$ in (1.3).

In the non-gelling regime, $a+b<1$ (and $a,b\geq 0$), the results of [4] suggest that the long-term behavior of a large class of solutions to (1.4) behave as the self-similar solution

$$f_s(t,x)=\frac{1}{t^{\frac{3+\gamma}{1-\gamma}}}\phi_s(y)\quad\text{with } y=\frac{x}{t^{\frac{2}{1-\gamma}}} \tag{1.6}$$

as time goes to infinity. According to [4] the self-similar profile ϕ_s behaves as the power law $y^{-\frac{\gamma+3}{2}}$ as y tends to zero and decays at least exponentially as y tends to infinity.

For x of order 1 we expect f to behave as a steady-state solution to (1.4) i.e. a solution to

$$\mathbb{K}[f](x) + \eta(x) = 0. \tag{1.7}$$

The solutions of (1.7) have been studied in [5] for bounded kernels, and in [16] for the discrete coagulation equation for kernels of the form (1.5) when a and b can take both positive and negative values. It is then proved that a solution to the discrete version of (1.7) exists for the range of exponents $\max\{\gamma + \lambda, -\lambda\} < 1$, $-1 \leq \gamma \leq 2$ and $|\gamma + 2\lambda| < 1$.

More recently, the existence of a solution to (1.7) has been studied in [10], for both the continuous and the discrete cases for kernels of the form (1.3). Specifically, it has been proven there that, if $|\gamma + 2\lambda| < 1$, then there exists at least a solution of (1.7) and, instead, if $|\gamma + 2\lambda| \geq 1$, then equation (1.7) does not have any solution. This implies that if the coagulation kernel is the Brownian kernel then there exists a solution of (1.7) and, if the coagulation kernel is the free molecular kernel, then (1.7) does not have any solution. We remark that for the kernels of the form (1.5) considered in [4] if $a + b < 1$, then, since $a, b \geq 0$, we have $|\gamma + 2\lambda| < 1$ and a steady state solving (1.7) always exists.

Results analogous to those in [10] have been obtained in [19] under different regularity assumptions on the coagulation kernels, the source η and the solutions, as well as an additional monotonicity assumption on the kernel.

In this paper we prove, under assumptions on the parameters γ and λ, the existence of a self-similar solution of the coagulation equation with constant flux coming from the origin, that can be formally written as

$$\partial_t(xf(x)) = x\mathbb{K}[f](x) + \delta_0, \tag{1.8}$$

where δ_0 is the Dirac mass at $\{0\}$. A precise definition of equation (1.8) will be presented in Definition 3.5. This result on the existence of self-similar solutions of equation (1.8) is the main novelty presented in this work. In agreement with the results obtained in [4], these self-similar solutions are expected to represent the longtime behavior of the solutions of (1.4), where we consider η to be a Radon measure with bounded first moment and decreasing fast enough for large sizes.

The self-similar profiles ϕ_s characterizing the self-similar solutions are constructed as the limit as $\varepsilon \to 0$ of a sequence of stationary solutions of certain coagulation equations with source η_ε, where we assume that $x\eta_\varepsilon(x)$ tends to the Dirac measure supported at $\{0\}$ as $\varepsilon \to 0$. This is the main technical novelty of this paper and it requires uniform estimates to be proved for the solutions. We also present some results on the regularity of the self-similar profiles and on their asymptotic behavior for small and large clusters.

We focus on non-gelling kernels (see [8] for a complete explanation of the gelation regimes), for which a stationary solution exists, i.e. we will consider

$$\begin{aligned} &\gamma < 1, \\ &|\gamma + 2\lambda| < 1. \end{aligned} \tag{1.9}$$

By [10] we know that, since $|\gamma + 2\lambda| < 1$, equation (1.4) admits a steady-state solution f. Hence, we expect the solutions of equation (1.4) to approach a steady-state solution when x is of order 1 and time goes to infinity. For every integrable function f we denote by J_f the flux associated with equation (1.7),

$$J_f(z) := \int_0^z \int_{z-x}^{\infty} xK(y,x) f(y) f(x)\, dy\, dx. \tag{1.10}$$

The analysis in [10] shows that the reason there exists at least a solution of (1.7) is that the contribution to J_f due to the interaction of particles of very different sizes, $x \ll y$ or $y \ll x$, is negligible compared to the contribution to J_f due to the interaction of particles of comparable sizes, $x \approx y$. On the contrary, when $|\gamma + 2\lambda| \geq 1$ (to be considered in another paper), the fact that a solution to (1.7) does not exist is due to the fact that the collisions between particles of very different sizes very quickly drive the mass towards infinity. Therefore, in the time-dependent problem we expect that if $|\gamma + 2\lambda| \geq 1$ one will need to take into account the interaction between particles of different sizes, and the behavior of the time-dependent solutions of (1.4) is expected to be different. In this case we expect that $f(t,x) \to 0$ for x of order 1 as time goes to infinity. For this range of parameters, the existence of self-similar solutions is not studied in [4] and might be the object of a future work.

1.2. Notation and plan of the paper

Before beginning with the technical content of the paper, hoping to help the reader, we clarify the notation that we adopt in this work.

First of all we employ the notation $\mathbb{R}_* := (0,\infty)$, $\mathbb{R}_+ := [0,\infty)$. Moreover, we denote by $\mathcal{L}$ the Lebesgue measure. For any interval $I \subset \mathbb{R}$, we denote by $C_c(I)$ the space of continuous functions with compact support endowed with the supremum norm, denoted by $\|\cdot\|_\infty$. We denote by $C_0(\mathbb{R}_*)$ the space of continuous functions vanishing at infinity, which is the completion of $C_c(\mathbb{R}_*)$. As before, we endow $C_0(\mathbb{R}_*)$ with the supremum norm.

We denote by $\mathcal{M}_+(I)$ the space of non-negative Radon measures on I. In the following, justified by the Riesz–Markov–Kakutani representation theorem, we frequently identify $\mathcal{M}_+(I)$ with the set of positive linear functionals on $C_c(\mathbb{R}_*)$. We adopt the notation

$$\mathcal{M}_{+,b}(I) := \{\mu \in \mathcal{M}_+(I) : \mu(I) < \infty\}$$

and endow this space with the total variation norm, which we denote by $\|\cdot\|$. Since we consider positive measures, we can easily compute the total variation norm of any measure $\mu \in \mathcal{M}_+(I)$, indeed $\|\mu\| = \mu(I)$. We will sometimes endow $\mathcal{M}_{+,b}(\mathbb{R}_*)$ with the weak-$*$ topology generated by the functionals $\langle \varphi, \mu\rangle = \int_I \varphi(x)\mu(dx)$.

We denote by $C(I, \mathcal{M}_{+,b}(\mathbb{R}_*))$ the space of continuous functions from the compact set $I \subset \mathbb{R}_+$ to the space of Radon bounded measures. We endow this space with the norm $\|f\|_I := \sup_{t\in I} \|f(t,\cdot)\|$.

Notice that $\|f\|_I < \infty$ because I is a compact set and $f(t,\cdot)$ is a Radon bounded measure. Assume Y is a normed space and $S \subset Y$. We use the notation $C^1([0,T];S;Y)$ for the collection of maps $f\colon [0,T] \to S$ such that f is continuous and there is $\dot{f} \in C([0,T];Y)$ for which the Fréchet derivative of f at any point $t \in [0,T]$ is given by $\dot{f}$. We also drop the normed space Y from the notation if it is clear from the context, in particular, if $S = \mathcal{M}_{+,b}(I)$ and $Y = \mathcal{M}_+(I)$ or $Y = S$. Clearly, if $f \in C^1([0,T];S;Y)$, the function $\dot{f}$ is unique and it can be found by requiring that for all $t \in (0,1)$,

$$\lim_{\varepsilon \to 0} \frac{\|f(t+\varepsilon) - f(t) - \varepsilon \dot{f}(t)\|_Y}{|\varepsilon|} = 0$$

and then taking the left and right limits to obtain the values $\dot{f}(0)$ and $\dot{f}(T)$. To keep the notation lighter, in some of the proofs we will denote by C or c a constant that may be different from line to line.

We denote by $\hat{f}$ the Fourier transform of $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{f}(y) := \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-ixy} f(x)\, dx.$$

We define the Sobolev spaces of fractional order s (negative or positive) as

$$H^s(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{H^s(\mathbb{R})} < \infty \right\},$$

where $\mathcal{S}(\mathbb{R})$ is the space of infinitely differentiable and rapidly decreasing functions and $\mathcal{S}'(\mathbb{R})$ is its dual (we refer to [6, Definition 14.6] for a precise definition), and the norm $\|\cdot\|_{H^s(\mathbb{R})}$ is given by

$$\|f\|^2_{H^s(\mathbb{R})} := \int_{\mathbb{R}} (1 + |x|^2)^s |\hat{f}(x)|^2\, dx.$$

Let Ω be an open set and let $s \in \mathbb{R}$. The space $H^s(\Omega)$ is the set of the restricted functions $f_{|\Omega}$ with $f\colon \mathbb{R} \to \mathbb{R}$ belonging to $H^s(\mathbb{R})$. It is a Banach space equipped with the norm

$$\|f\|_{H^s(\Omega)} := \inf\left\{ g \in H^s(\Omega) : g_{|\Omega} = f \right\}.$$

This definition is the same as in [22].

Finally, we will use the notation $f \sim g$ as $x \to x_0$ to indicate the asymptotic equivalence between the function f and the function g, i.e. $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$. Instead, we will use the notation $f \approx g$ to say that there exists a constant $M > 0$ such that $\frac{1}{M} \leq \frac{f}{g} \leq M$.

The organization of the paper is the following. In Section 2 we discuss the heuristic justification to study the self-similar solutions constructed in this paper. In Section 3 we explain in detail the setting in which we work, we present the definition of self-similar profile and state the main results of the paper. In Section 4 we prove the existence of a self-similar profile, while in Sections 5 and 6 we prove its properties. Finally, in Section 7, we prove that the self-similar solution f_s defined by (1.6), solves equation (1.8).

2. Heuristic argument

2.1. Scaling parameters

In this subsection, we explain how the exponents in (1.6) are computed. Since we are only considering non-gelling kernels, the change of mass in the system is only due to the contribution of the source. Indeed, multiplying equation (1.4) by x and integrating in x from 0 to ∞ we obtain formally

$$\frac{d}{dt}\int_0^\infty xf(t,x)\,dx = \int_0^\infty x\eta(dx) < \infty. \tag{2.1}$$

As a consequence, using the change of variables

$$f(t,x) = \frac{1}{t^\alpha}\phi\Big(\ln t, \frac{x}{t^\beta}\Big), \quad \xi = \frac{x}{t^\beta}, \quad \tau = \ln t \tag{2.2}$$

and considering the initial condition $f(0,x) = 0$, motivated by the fact that the total mass in the system is proportional to time, we conclude that

$$t^{2\beta-\alpha}\int_0^\infty \xi\phi(\tau,\xi)\,d\xi = \int_0^\infty \frac{x}{t^\alpha}\phi\Big(\ln t, \frac{x}{t^\beta}\Big)\,dx = \int_0^\infty xf(t,x)\,dx = t\int_0^\infty x\eta(dx),$$

which implies that $2\beta - \alpha = 1$. Hence the mass of the source is equal to the mass of ϕ.

Moreover, using (2.2) in the coagulation term (1.2) and using (1.3) yields the scaling

$$\mathbb{K}[f](t,x) \sim t^{\beta-2\alpha+\beta\gamma}\mathbb{K}[\phi](\tau,\xi).$$

On the other hand, the fact that

$$\partial_t\Big(t^{-\alpha}\phi\Big(\ln t, \frac{x}{t^\beta},\Big)\Big) = -t^{-\alpha-1}\Big(\alpha\phi\Big(\ln t, \frac{x}{t^\beta}\Big) + \beta\xi\partial_\xi\phi\Big(\ln t, \frac{x}{t^\beta}\Big) - \partial_\tau\phi\Big(\ln t, \frac{x}{t^\beta}\Big)\Big)$$

implies that $-1-\alpha = \beta - 2\alpha + \beta\gamma$. Recalling the condition $2\beta - \alpha = 1$ we conclude that

$$\beta = \frac{2}{1-\gamma}, \quad \alpha = \frac{3+\gamma}{1-\gamma}. \tag{2.3}$$

Notice that this scaling is in agreement with [4].

We now conclude this section by noticing that equation (1.4) has a scaling-invariance property. Indeed, if f solves (1.4) with a source of mass $\int_0^\infty x\eta(dx) = M_\eta$, then $\tilde{f}(t,y) = (M_\eta)^{\gamma/(1-\gamma)} f(t, M_\eta^{1/(1-\gamma)} y)$ solves (1.4) with a source of mass 1. Without loss of generality we therefore assume from now on that the source η has mass equal to 1.

2.2. Formal derivation of the equation for the self-similar profile

In this subsection we explain why we expect the solution of equation (1.4) to approach, as time tends to infinity, a self-similar solution for $x \gg 1$ and a steady state for $x \approx 1$, following an argument inspired by the one in [4].

When $|\gamma + 2\lambda| < 1$, holds we know, from [10], that a stationary solution $\bar{f}$ exists and we expect, in view of the numerical simulations in [4], that

$$f(t, x) \to \bar{f}(x) \text{ as } t \to \infty \quad \text{for } x \text{ of order } 1. \tag{2.4}$$

The fact that $J_{\bar{f}}(x) \to 1$ as $x \to \infty$ (proven in [10]) and that the simplest solution of $J_\phi(x) = 1$ is $\phi(x) = c_0 x^{-\frac{3+\gamma}{2}}$, with $c_0 = (\int_0^1 \int_1^\infty K(y, z) z^{-\frac{\gamma+3}{2}} y^{-\frac{\gamma+1}{2}} \, dy \, dz)^{-1/2}$, suggests that $\bar{f}(x) \sim c_0 x^{-\frac{3+\gamma}{2}}$ as $x \to \infty$. This yields the following matching condition:

$$f(t, x) \sim c_0 x^{-\frac{3+\gamma}{2}} \tag{2.5}$$

for $1 \ll x \ll t^{\frac{2}{1-\gamma}}$ or equivalently $1 \ll x \ll e^{\frac{2}{1-\gamma}\tau}$.

We now describe the asymptotic behavior of $f(t, x)$ in the self-similar region $x \approx t^{\frac{2}{1-\gamma}}$. Using the self-similar change of variables (2.2) in (1.4), we deduce that ϕ satisfies the following transport-coagulation equation with source,

$$\partial_\tau \phi(\tau, \xi) = \frac{3+\gamma}{1-\gamma} \phi(\tau, \xi) + \frac{2}{1-\gamma} \xi \partial_\xi \phi(\tau, \xi) + \mathbb{K}[\phi](\tau, \xi) + e^{\frac{4}{1-\gamma}\tau} \eta(\xi e^{\frac{2}{1-\gamma}\tau}) \tag{2.6}$$

for $\tau > 0$ and $\xi > 0$.

Since $\int_0^\infty x \eta(dx) = 1$, then in the region $\xi \approx 1$ the term $e^{\frac{4}{1-\gamma}\tau} \eta(\xi e^{\frac{2}{1-\gamma}\tau})$ tends to zero in the sense of measures as $\tau \to \infty$.

We make the self-similar ansatz, i.e. we assume that there exists a self-similar profile ϕ_s such that

$$\phi(\tau, \xi) \to \phi_s(\xi) \quad \text{as } \tau \to \infty \tag{2.7}$$

and conclude that ϕ_s solves

$$0 = \frac{3+\gamma}{1-\gamma} \phi_s(\xi) + \frac{2}{1-\gamma} \xi \partial_\xi \phi_s(\xi) + \mathbb{K}[\phi_s](\xi) \quad \text{for } \xi > 0. \tag{2.8}$$

By the matching condition (2.5), we know that

$$\phi_s \sim c_0 \xi^{-\frac{3+\gamma}{2}} \quad \text{as } \xi \to 0. \tag{2.9}$$

The fact that $\int_0^\infty \xi \phi_s(\xi) \, d\xi < \infty$ and the shape of the equation suggest that ϕ_s decays at least exponentially (see also the statement of Theorem 3.2); later we will justify the precise ansatz

$$\phi_s(\xi) \sim c e^{-L\xi} \xi^{-\gamma} \quad \text{as } \xi \to \infty, \tag{2.10}$$

for some $L > 0$. Rigorous upper estimates for ϕ_s supporting this asymptotic behavior will be also derived in Theorem 3.2. The matching condition (2.9), together with (2.10) then implies

$$\lim_{\xi \to 0} J_{\phi_s}(\xi) = 1. \tag{2.11}$$

We conclude that the self-similar profile satisfies equation (2.8) with the boundary condition (2.11).

We now derive a relation between the flux coming from the origin, (2.11), and $\int_0^\infty \xi\phi_s(d\xi)$. Multiplying (2.8) by ξ and noticing that $\xi\mathbb{K}[\phi_s](\xi) = -\partial_\xi J_{\phi_s}(\xi)$ we obtain

$$0 = -\xi\phi_s(\xi) + \partial_\xi\Big(\frac{2}{1-\gamma}\xi^2\phi_s(\xi) - J_{\phi_s}(\xi)\Big).$$

Integrating from 0 to infinity we deduce that

$$\int_0^\infty \xi\phi_s(\xi)\,d\xi = \lim_{\xi\to\infty}\Big(\frac{2}{1-\gamma}\xi^2\phi_s(\xi) - J_{\phi_s}(\xi)\Big) - \lim_{\xi\to 0}\Big(\frac{2}{1-\gamma}\xi^2\phi_s(\xi) - J_{\phi_s}(\xi)\Big). \tag{2.12}$$

Thanks to (2.9), and the assumption $\gamma < 1$ we deduce that $\lim_{\xi\to 0}\xi^2\phi_s(\xi) = 0$. Moreover, thanks to (2.10) we deduce that $\lim_{\xi\to\infty} J_{\phi_s}(\xi) = 0$ and $\lim_{\xi\to\infty}\xi^2\phi_s(\xi) = 0$. Therefore, combining (2.12) with (2.11) we obtain

$$\int_0^\infty \xi\phi_s(\xi)\,d\xi = 1.$$

Using the self-similar change of variables (2.2) we deduce that the self-similar solution f_s satisfies

$$\int_0^\infty x f_s(t,x)\,dx = t,$$

which is consistent with (2.1).

Finally, we justify (2.10). To this end we substitute in equation (2.8) the ansatz $\phi_s(\xi) \sim ce^{-L\xi}\xi^a$ as $\xi \to \infty$ and formally estimate the behavior at infinity of all the terms in (2.8), to deduce that $a = -\gamma$. We start by considering the coagulation term

$$\begin{aligned}
&\mathbb{K}[e^{-L\xi}\xi^a] \\
&\quad = \frac{c^2}{2}e^{-L\xi}\xi^a\int_0^\xi K(y,\xi-y)y^a(\xi-y)^a\,dy - c^2e^{-L\xi}\xi^a\int_0^\infty K(\xi,y)e^{-Ly}y^a\,dy \\
&\quad = c^2e^{-L\xi}\xi^{\gamma+1+2a}\Big(\frac{1}{2}\int_0^1 K(y,1-y)y^a(1-y)^a\,dy - \int_0^\infty K(1,y)e^{-L\xi y}y^a\,dy\Big) \\
&\quad \sim \frac{c^2}{2}e^{-L\xi}\xi^{\gamma+1+2a}\int_0^1 K(y,1-y)y^a(1-y)^a\,dy
\end{aligned}$$

and therefore

$$\mathbb{K}[e^{-L\xi}\xi^a] \sim c^2\Big(\int_0^1 K(y,1-y)y^a(1-y)^a\,dy\Big)e^{-L\xi}\xi^{\gamma+1+2a} \quad \text{as } \xi\to\infty. \tag{2.13}$$

On the other hand,

$$\frac{3+\gamma}{1-\gamma}\phi_s(\xi) + \frac{2}{1-\gamma}\xi\phi_s'(\xi) \sim -\frac{2cL}{1-\gamma}e^{-L\xi}\xi^{1+a} \quad \text{as } \xi \to \infty. \tag{2.14}$$

Combining (2.13) and (2.14) we deduce that $\phi_s(\xi) \sim ce^{-L\xi}\xi^{-\gamma}$ and

$$c = \frac{2L}{1-\gamma}\left(\int_0^1 K(y, 1-y)y^{-\gamma}(1-y)^{-\gamma}\,dy\right)^{-1}.$$

The power law behavior near the origin and the exponential behavior at infinity are supported by the numerical simulations in [4]. The estimates (3.3), (3.4) show that the mean of ϕ_s behaves as $x^{-(\gamma+3)/2}$ near the origin and the inequality (3.6) shows that ϕ_s decays at least exponentially for large sizes.

2.3. Longtime asymptotics

In this section we present a different argument justifying the self-similar behavior of solutions of (1.4). We show that, if the self-similar profile can be uniquely identified as the solution of a coagulation equation with constant flux coming from the origin (equation (1.8)), then the self-similar solution describes the longtime behavior of the solutions to the coagulation equation with source (1.4). Since the investigation of the uniqueness of the coagulation equation with constant flux coming from the origin is still an open problem, the following argument represents only a formal heuristic motivation for the study of self-similar solutions of equation (1.4).

Nevertheless, the self-similar ansatz (1.6) is corroborated, at least for some of the kernels considered here, by the numerical simulations and heuristic explanations in [4] and by [20] from the point of view of physics.

Let us consider a solution f to (1.4) with initial condition f_0 such that $f_0(y) < cy^{\omega}$ with $\omega < -\frac{\gamma+3}{2}$ and $c > 0$, and a positive constant R. Since we are interested in the longtime behavior and we are assuming $\gamma < 1$, we consider the following change of variables:

$$x = \xi R, \quad t = R^{\frac{1-\gamma}{2}}s, \quad s > 0.$$

The scaling in size balances the scaling in time in such a way that the function F_R defined by

$$F_R(s,\xi) := R^{(\gamma+3)/2}f(R\xi, R^{\frac{1-\gamma}{2}}s) \tag{2.15}$$

satisfies the coagulation equation

$$\partial_s F_R(s,\xi) = \mathbb{K}[F_R](s,\xi) + \eta_R(\xi) \tag{2.16}$$

with the source

$$\eta_R(\xi) := R^2\eta(R\xi)$$

and the initial condition $F_R(0,\xi) = R^{\frac{\gamma+3}{2}}f_0(R\xi)$.

Integrating equation (2.16) against a test function $\varphi \in C([0,T]\times\mathbb{R}_+)$ we obtain

$$\begin{aligned}\int_{\mathbb{R}_*} \xi\varphi(t,\xi)F_R(t,\xi)\,d\xi &= \int_{\mathbb{R}_*} \xi\varphi(0,\xi)F_R(0,\xi)\,d\xi + \int_0^t\int_{\mathbb{R}_*} \xi\partial_s\varphi(s,\xi)F_R(s,\xi)\,d\xi\,ds \\ &+ \int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \frac{K(y,\xi)}{2}\big[\varphi(s,\xi+y)(\xi+y)-\xi\varphi(s,\xi)-y\varphi(s,y)\big]\cdot \\ &\qquad \times F_R(s,\xi)F_R(s,y)\,d\xi\,dy\,ds \\ &+ \int_0^t\int_{\mathbb{R}_*} \xi\varphi(s,\xi)\eta_R(\xi)\,d\xi\,ds. \end{aligned} \tag{2.17}$$

Since the source η_R decays fast enough and

$$\int_0^\infty \xi\eta_R(\xi)\,d\xi = 1,$$

we infer that

$$\lim_{R\to\infty}\int_{\mathbb{R}_*} \xi\varphi(s,\xi)\eta_R(\xi)\,d\xi = \lim_{R\to\infty}\int_{\mathbb{R}_*} \varphi\Big(s,\frac{y}{R}\Big)y\eta(y)\,dy = \varphi(s,0)$$

and $y\eta_R(y)\to\delta_0(y)$ as $R\to\infty$.

Assuming that the solution F_R is unique and that the limit of F_R when $R\to\infty$ exists, passing to the limit as $R\to\infty$ in equation (2.16), we deduce that F, the limit of F_R, solves the coagulation equation with constant flux coming from the origin in the sense of Definition 3.5.

We will prove in Theorem 3.6 that the function $s^{-\frac{\gamma+3}{1-\gamma}}\phi_s(\xi s^{-\frac{2}{1-\gamma}})$, with ϕ_s solving (2.8) with the boundary condition (2.11), satisfies equation (3.10) for every test function $\varphi\in C^1([0,T],C(\mathbb{R}_+))$. Assuming that (3.10) has a unique solution we conclude that

$$F(s,\xi) = \lim_{R\to\infty} F_R(s,\xi) = s^{-\frac{\gamma+3}{1-\gamma}}\phi_s(\xi s^{-\frac{2}{1-\gamma}}). \tag{2.18}$$

Choosing $R = e^{\frac{2}{1-\gamma}\tau}$ and $s=1$ in (2.15), we conclude that

$$F_{e^{\frac{2\tau}{1-\gamma}}}(1,\xi) = e^{\frac{3+\gamma}{1-\gamma}\tau} f(e^\tau,\xi e^{\frac{2\tau}{1-\gamma}}),$$

where f is a solution of equation (1.4).

From (2.18) and the fact that $F(1,\xi)=\phi_s(\xi)$, we deduce that, as τ tends to infinity,

$$e^{\frac{3+\gamma}{1-\gamma}\tau} f(e^\tau,\xi e^{\frac{2}{1-\gamma}\tau}) \to \phi_s(\xi).$$

This implies the self-similar ansatz (1.6).

3. Setting and main results

Given $\gamma, \lambda \in \mathbb{R}$ and $c_1, c_2 > 0$, a continuous symmetric function $K: \mathbb{R}_* \times \mathbb{R}_* \to \mathbb{R}_+$ is a coagulation kernel of parameters γ and λ if it satisfies the inequalities

$$c_1 w(x,y) \leq K(x,y) \leq c_2 w(x,y) \quad \forall x, y \in \mathbb{R}_*, \tag{3.1}$$

where

$$w(x,y) = \frac{x^{\gamma+\lambda}}{y^{\lambda}} + \frac{y^{\gamma+\lambda}}{x^{\lambda}}.$$

We assume that the coagulation kernel is homogeneous, with homogeneity γ, that is, for any $b > 0$ it satisfies

$$K(bx, by) = b^{\gamma} K(x,y) \quad \forall x, y \in \mathbb{R}_*.$$

We now give the definition of a self-similar profile for equation (1.8).

Definition 3.1. Let K be a homogeneous symmetric coagulation kernel $K \in C(\mathbb{R}_* \times \mathbb{R}_*)$ satisfying (3.1) with homogeneity $\gamma < 1$. A self-similar profile of equation (1.8) with respect to the kernel K is a measurable function $\phi \geq 0$ with

$$\int_{\mathbb{R}_*} x\phi(x)\,dx = 1 \quad \text{and} \quad J_{\phi} \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_*),$$

where J_{ϕ} is defined by (1.10), and it satisfies

$$J_{\phi}(z) = 1 - \int_0^z x\phi(x)\,dx + \frac{2}{1-\gamma} z^2 \phi(z) \quad \text{a.e. } z > 0. \tag{3.2}$$

Theorem 3.2 (Existence). *Let K be a homogeneous symmetric coagulation kernel $K \in C(\mathbb{R}_* \times \mathbb{R}_*)$ satisfying (3.1), with homogeneity $\gamma < 1$ and $|\gamma + 2\lambda| < 1$. Then there exists a self-similar profile ϕ as in Definition 3.1. Moreover, there exist positive constants C and b_1 with $b_1 < 1$ such that*

$$\frac{1}{z}\int_{b_1 z}^{z} \phi(x)\,dx \leq \frac{C}{z^{(3+\gamma)/2}} \quad \text{for any } z > 0 \tag{3.3}$$

and there exist two constants $b_2 \in (0,1)$ and $c > 0$, depending on the parameters of the kernel γ, λ as well as on c_1, c_2 in (3.1), such that

$$\frac{1}{z}\int_{b_2 z}^{z} \phi(x)\,dx \geq \frac{c}{z^{(3+\gamma)/2}}, \quad z \in (0,1]. \tag{3.4}$$

There exists a positive constant L such that

$$\int_1^{\infty} e^{Lx}\phi(x)\,dx < \infty \tag{3.5}$$

and a positive constant ρ such that

$$\limsup_{z\to\infty} \phi(z) e^{\rho z} < \infty. \tag{3.6}$$

Theorem 3.3 (Regularity). *Assume K, λ and γ are as in the assumptions of Theorem 3.2 and assume ϕ to be the self-similar profile whose existence is proven in Theorem 3.2. Assume that for every $0 < y \leq 1$ we have $K(\cdot, y) \in H^l(\mathbb{R}_*)$ with*

$$\sup_{0<y\leq 1} y^{-\min\{\gamma+\lambda,-\lambda\}} \|K(\cdot, y)\|_{H^l(1/2,2)} < \infty. \tag{3.7}$$

Then the self-similar profile ϕ is such that $x^2\phi(x) \in H^l(\mathbb{R}_*)$.

If $l \geq 3/2$ then $\phi \in C^1(\mathbb{R}_*)$ and it satisfies

$$\int_0^{x/2} \big[K(x-y, y)\phi(x-y) - K(x, y)\phi(x)\big]\phi(y)\, dy + \int_{x/2}^{\infty} K(x, y)\phi(x)\phi(y)\, dy \\ + \frac{3+\gamma}{1-\gamma}\phi(x) + \frac{2}{1-\gamma} x\partial_x\phi(x) = 0 \quad \forall x \in \mathbb{R}_*, \tag{3.8}$$

with the boundary condition (2.11).

Remark 3.4. Due to the homogeneity and the symmetry of the kernel, inequality (3.7) implies that for every $(a, b) \subset \mathbb{R}_+$ with $0 < a < b$, we have

$$\sup_{0<y\leq 1} y^{-\min\{\gamma+\lambda,-\lambda\}} \|K(\cdot, y)\|_{H^l((a,b))} < \infty.$$

Definition 3.5 (Coagulation equation with constant flux coming from the origin). Let K be a homogeneous symmetric coagulation kernel $K \in C(\mathbb{R}_* \times \mathbb{R}_*)$ satisfying (3.1) with homogeneity $\gamma < 1$, and let $T > 0$. We say that $F \in C([0, T], \mathcal{M}_+(\mathbb{R}_+))$ is a solution of the coagulation equation with constant flux coming from the origin with initial condition $F(0, \cdot) = 0$, if

$$\sup_{s\in[0,T]} \int_{(0,1]} \xi^q F(s, d\xi) < \infty \qquad \sup_{s\in[0,T]} \int_{[1,\infty)} \xi^p F(s, d\xi) < \infty \tag{3.9}$$

for $q = \min\{1 + \gamma + \lambda, 1 - \lambda, 1\}$ and $p = \max\{\gamma + \lambda, -\lambda\}$ and if it solves the equation

$$\int_{\mathbb{R}_*} \xi\varphi(t, \xi)F(t, d\xi) \\ = \int_0^t \int_{\mathbb{R}_*} \xi\partial_s\varphi(s, \xi)F(s, d\xi)\, ds + \int_0^t \varphi(s, 0)\, ds \\ + \frac{1}{2}\int_0^t \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(\eta, \xi)\big[(\xi+\eta)\varphi(s, \xi+\eta) - \xi\varphi(s, \xi) - \eta\varphi(s, \eta)\big] \\ \times F(s, d\xi)F(s, d\eta)\, ds \tag{3.10}$$

for every test function $\varphi \in C^1([0, T], C_c^1(\mathbb{R}_+))$ and every $0 \leq t < T$.

Theorem 3.6. *Assume K, λ and γ are as in the assumptions of Theorem 3.2. Let ϕ be a self-similar profile as in Definition 3.1. Then $F(t, d\xi) := t^{-\frac{\gamma+3}{1-\gamma}}\phi(\xi t^{-\frac{2}{1-\gamma}})\, d\xi$ solves the coagulation equation with flux coming from the origin in the sense of Definition 3.5.*

Remark 3.7. We underline that the moment bounds (3.3), (3.4), (3.5), estimate (3.6) and the regularity properties in Theorem 3.3 are proven only for the self-similar profile whose existence is stated in Theorem 3.2. We do not know whether these properties are more general, i.e. whether they hold for all the self-similar profiles as in Definition 3.1. (Since we do not prove uniqueness of the self-similar profile, the existence of many self-similar profiles is not excluded).

4. Existence of a self-similar profile

We briefly explain the technique we adopt to prove the existence of a solution of equation (2.8) with the boundary condition (2.11).

The main idea is to approximate a solution of (2.8) with a sequence of solutions ϕ_ε of equation

$$0 = \frac{3+\gamma}{1-\gamma}\phi_\varepsilon(\xi) + \frac{2}{1-\gamma}\xi\partial_\xi\phi_\varepsilon(\xi) + \mathbb{K}[\phi_\varepsilon](\xi) + \eta_\varepsilon(\xi), \tag{4.1}$$

where η_ε is a smooth function with support $[\varepsilon, 2\varepsilon]$ and such that

$$\int_{\mathbb{R}_*} x\eta_\varepsilon(x)\,dx = \int_{\mathbb{R}_*} y\eta(dy) = 1.$$

To prove the existence of a solution of (4.1) we follow an approach which is extensively used in the analysis of kinetic equations; see for instance [9] and [13]. We first prove the existence of solutions of (4.1) by considering the corresponding truncated time-dependent evolution problem:

$$\partial_t\phi_\varepsilon(\tau,\xi) = -\phi_\varepsilon(\tau,\xi) + \frac{2}{1-\gamma}\frac{1}{\xi}\partial_\xi(\xi^2\Xi_\varepsilon(\xi)\phi_\varepsilon(\tau,\xi)) + \mathbb{K}_{a,R}[\phi_\varepsilon](\tau,\xi) + \eta_\varepsilon(\xi), \tag{4.2}$$

where Ξ_ε is a smooth monotone function $\Xi_\varepsilon\colon\mathbb{R}_* \to \mathbb{R}_+$ such that $\Xi_\varepsilon(x) = 1$ if $x \geq 2\varepsilon$ and $\Xi_\varepsilon(x) = 0$ if $x \leq \varepsilon$, while $\mathbb{K}_{R,a}$ is the truncated coagulation operator defined by

$$\begin{aligned}\mathbb{K}_{R,a}[f](\tau,\xi) := \frac{\zeta_R(\xi)}{2}\int_0^\xi K_a(\xi-z,z)f(t,\xi-z)f(t,z)\,dz \\ - \int_0^\infty K_a(\xi,z)f(t,\xi)f(t,z)\,dz,\end{aligned}$$

where K_a is a kernel bounded from above by a and from below by $1/a$ and ζ_R is a truncation function of parameter $R > 0$, i.e. it is a smooth function $\zeta_R\colon\mathbb{R}_* \to \mathbb{R}_+$ such that $\zeta_R(x) = 1$ if $x \leq R$ while $\zeta_R(x) = 0$ if $x \geq 2R$. The specific truncation in the growth term of (4.2) has been chosen in order to ensure that the set $\{\phi : \int_0^\infty \xi\phi(\xi)\,d\xi \leq 1\}$ is invariant under the evolution equation corresponding to (4.2).

In Section 4.1 we prove the existence of a weak solution to (4.2). In Section 4.2 we prove, using the Tychonoff fixed point theorem, the existence of a stationary weak solution of equation (4.2),

$$0 = -\phi_\varepsilon(\xi) + \frac{2}{1-\gamma}\frac{1}{\xi}\partial_\xi(\xi^2\Xi_\varepsilon(\xi)\phi_\varepsilon(\xi)) + \mathbb{K}_{a,R}[\phi_\varepsilon](\xi) + \eta_\varepsilon(\xi), \tag{4.3}$$

and in Section 4.3 we study the properties of ϕ_ε. In Section 4.4 we show that the solutions of (4.3) approximate a self-similar profile in the sense of Definition 3.1 as $R \to \infty$, $a \to \infty$, $\varepsilon \to 0$.

4.1. Truncated time-dependent coagulation equation with source written in self-similar variables

We introduce now some terminology that will enable us to define the truncated equation. As we will see in the proof of Proposition 4.4, to prove the existence of a solution of equation (4.2), we prove the existence of a solution of an auxiliary equation, obtained via a time-dependent change of variables.

The aim of the rest of this section is to prove the existence of a solution, for every $\varphi \in C^1([0,T], C_c^1(\mathbb{R}_*))$, of the following truncated equation:

$$\begin{aligned}
&\int_{\mathbb{R}_*}\Phi(t,d\xi)\varphi(t,\xi) - \int_{\mathbb{R}_*}\Phi_0(d\xi)\varphi(0,\xi) - \int_0^t\int_{\mathbb{R}_*}\partial_s\varphi(s,\xi)\Phi(s,d\xi)\,ds \\
&\quad = \frac{1}{2}\int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}K_a(\xi,z)\big[\varphi(s,\xi+z)\zeta_R(\xi+z) - \varphi(s,\xi) - \varphi(s,z)\big] \\
&\qquad\qquad \times \Phi(s,d\xi)\Phi(s,dz)\,ds \\
&\qquad - \int_0^t\int_{\mathbb{R}_*}\varphi(s,\xi)\Phi(s,d\xi)\,ds - \frac{2}{1-\gamma}\int_0^t\int_{\mathbb{R}_*}\Xi_\varepsilon(\xi)\partial_\xi\varphi(s,\xi)\xi\Phi(s,d\xi)\,ds \\
&\qquad + \frac{2}{1-\gamma}\int_0^t\int_{\mathbb{R}_*}\Xi_\varepsilon(\xi)\varphi(s,\xi)\Phi(s,d\xi) + \int_0^t\int_{\mathbb{R}_*}\varphi(s,\xi)\eta_\varepsilon(\xi)\,d\xi\,ds,
\end{aligned} \tag{4.4}$$

where we are assuming that $\gamma < 1$ and $\Phi_0 \in \mathcal{M}_{+,b}(\mathbb{R}_*)$ with $\Phi_0((0,\varepsilon]\cup[2R,\infty)) = 0$.

The proof of the existence of a solution of equation (4.4) is standard, in the sense that it is based on the Banach fixed point theorem. More precisely, we will use a change of variables to obtain an equation which is easier to analyze (cf. (4.12)). This equation looks complicated, but it does not contain any transport terms, and, therefore, it is suitable for a fixed point argument.

To prepare the change of variables, we introduce the following notation. We denote by $X(t,x)$ the solution of the characteristic ODE,

$$\frac{\partial X(t,x)}{\partial t} = -\beta X(t,x)\Xi_\varepsilon(X(t,x)), \quad X(0,x) = x. \tag{4.5}$$

We also introduce the function $\ell\colon [0,T]\times\mathbb{R}_*\times\mathbb{R}_* \to \mathbb{R}_+$ which is the function that satisfies

$$X(t,\ell(t,x,y)) = X(t,x) + X(t,y). \tag{4.6}$$

The function ℓ is well defined because the map $x \mapsto X(t,x)$ is a diffeomorphism for every time t.

A time-dependent truncation of parameter R is a function $\theta_R \in C^\infty([1,T]\times\mathbb{R}_*\times\mathbb{R}_*)$ defined by

$$\theta_R(t,x,y) := \zeta_R(X(\ln t,x)+X(\ln t,y)), \tag{4.7}$$

where ζ_R is a truncation function of parameter R.

We also define a truncated time-dependent kernel. A time-dependent coagulation kernel of parameter $a>0$ is a continuous map $K_{a,T}\colon [1,T]\times\mathbb{R}_*\times\mathbb{R}_*\to\mathbb{R}_+$ defined by

$$K_{a,T}(t,x,y) := t^{-2}K_a(X(\ln t,x),X(\ln t,y)), \tag{4.8}$$

where K_a is a bounded coagulation kernel of bound a. We also introduce a new auxiliary source $\widetilde{\eta_\varepsilon}$ defined by

$$\widetilde{\eta_\varepsilon}(t,x) = e^{-g(\ln t,x)}\frac{\partial X(\ln t,x)}{\partial x}\eta_\varepsilon(X(\ln t,x)), \quad t>1,\ x>0, \tag{4.9}$$

where g is defined by

$$g(\tau,x) := \beta\int_0^\tau \Xi_\varepsilon(X(s,x))\,ds \quad \text{for every } \tau>0, x>0. \tag{4.10}$$

Notice that, by the change of variables formula, this implies that for every test function φ and every time $t>0$,

$$\int_{\mathbb{R}_*}\varphi(\xi)\eta_\varepsilon(\xi)\,d\xi = \int_{\mathbb{R}_*}\varphi(X(t,x))e^{g(t,x)}\widetilde{\eta_\varepsilon}(e^t,x)\,dx. \tag{4.11}$$

Proposition 4.1. *Let $T>1$, $\gamma<1$, $\beta=2/(1-\gamma)$, and consider a source $\widetilde{\eta_\varepsilon}$, a kernel $K_{a,T}$, a truncation θ_R and a truncation Ξ_ε. Let ℓ be the function defined by (4.6). Consider an initial condition $f_1\in\mathcal{M}_{+,b}(\mathbb{R}_*)$ with $f_1((0,\varepsilon]\cup(2R,\infty))=0$. Then there exists a unique solution to the equation*

$$\begin{aligned}\int_{\mathbb{R}_*}&\varphi(x)\dot f(t,dx)\\ &= \int_{\mathbb{R}_*}\varphi(x)\widetilde{\eta_\varepsilon}(t,x)\,dx\\ &\quad+\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}\frac{K_{a,T}(t,x,y)}{2}\big(\Lambda[\varphi](t,x,y)-\varphi(x)e^{g(\ln t,y)}-\varphi(y)e^{g(\ln t,x)}\big)\\ &\qquad\times f(t,dx)f(t,dy)\end{aligned} \tag{4.12}$$

for any $\varphi\in C_c(\mathbb{R}_*)$ and $t\in[1,T]$, with $f(1,\cdot)=f_1(\cdot)$ and where

$$\Lambda[\varphi](t,x,y) := \varphi(\ell(\ln t,x,y))e^{-g(\ln t,\ell(\ln t,x,y))+g(\ln t,x)+g(\ln t,y)}\theta_R(t,x,y). \tag{4.13}$$

The solution $f \in C^1([1,T],\mathcal{M}_{+,b}(\mathbb{R}_*))$ has the following properties for every $t \in [1,T]$:

$$\int_{\mathbb{R}_*} f(t,dx) \le T\|\eta_\varepsilon\| + \|f_1\| \tag{4.14}$$

and

$$f(t,(0,\varepsilon] \cup (2Rt^\beta,\infty)) = 0. \tag{4.15}$$

Before starting with the proof of this proposition we provide some definitions and two auxiliary lemmas that help in its proof. Let us define the operator $\mathcal{F}$, which is a contraction, as will be shown in the proof of Proposition 4.1, whose fixed point is the solution of (4.12).

Consider $f \in C([1,T],\mathcal{M}_{+,b}(\mathbb{R}_*))$ with $f(1,\cdot) = f_1(\cdot)$. We denote by b and h_R the following expressions:

$$b[f](t,x) := \int_{\mathbb{R}_*} K_{a,T}(t,x,y) f(t,dy) e^{g(\ln t,x)},$$
$$h_R(t,s,x,y) := \theta_R(s,x,y) e^{-\int_s^t b[f](\xi,x)\,d\xi}.$$

The operator $\mathcal{F}[f](t)\colon C_0(\mathbb{R}_*) \mapsto \mathbb{R}_*$ is defined by

$$\langle \mathcal{F}[f](t),\varphi\rangle := \langle \mathcal{F}_1[f](t),\varphi\rangle + \langle \mathcal{F}_2[f](t),\varphi\rangle + \langle \mathcal{F}_3[f](t),\varphi\rangle \tag{4.16}$$

for $t \in [1,T]$, where the operators $\mathcal{F}_i\colon C_0(\mathbb{R}_*) \mapsto \mathbb{R}_*$ are defined by

$$\langle \mathcal{F}_1[f](t),\varphi\rangle := \int_{\mathbb{R}_*} \varphi(x) e^{-\int_1^t b[f](s,x)\,ds} f_1(dx),$$
$$\langle \mathcal{F}_2[f](t),\varphi\rangle := \int_1^t \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \Lambda[\varphi](s,x,y) e^{-\int_s^t b[f](\xi,x)\,d\xi} \frac{K_{a,T}(s,x,y)}{2} \times f(s,dx) f(s,dy)\,ds,$$
$$\langle \mathcal{F}_3[f](t),\varphi\rangle := \int_{\mathbb{R}_*} \varphi(x) \int_1^t e^{-\int_s^t b[f](v,x)dv} \tilde{\eta}_\varepsilon(s,x)\,ds\,dx.$$

We define the set $\mathcal{X}_\varepsilon$ as

$$\mathcal{X}_\varepsilon := \big\{ f \in \mathcal{M}_+(\mathbb{R}_*) : f((0,\varepsilon]) = 0 \big\}. \tag{4.17}$$

The set $\mathcal{X}_\varepsilon$ is a closed set both with respect to the weak-$*$ topology and the norm topology on $\mathcal{M}_b(\mathbb{R}_*)$; thus it is a Banach space with respect to the total variation norm.

Lemma 4.2. *Assume γ, β, η_ε, $K_{a,T}$, Ξ_ε, ℓ and θ_R are as in Proposition 4.1. The operator $\mathcal{F}$ defined by (4.16), for any initial condition $f_1 \in \mathcal{X}_\varepsilon$, maps $C([1,T],\mathcal{X}_\varepsilon)$ into itself.*

Consider an initial condition f_1 for equation (4.12); we denote by X_T the set defined by

$$X_T := \big\{ f \in C([1,T],\mathcal{X}_\varepsilon) : \|f - f_1\|_{[1,T]} \le 1 + \|f_1\| \big\}. \tag{4.18}$$

Lemma 4.3. *Under the assumptions of Lemma 4.2, we deduce that if*

$$T - 1 \leq \frac{C(\eta_\varepsilon, a)}{1 + \|f_1\|} \tag{4.19}$$

for a suitable constant $C(\eta_\varepsilon, a) > 0$, then, for every $f, g \in X_T$, it holds that

$$\|\mathcal{F}[f](\cdot) - \mathcal{F}[g](\cdot)\|_{[1,T]} \leq C_T \|f - g\|_{[1,T]} \tag{4.20}$$

with $0 < C_T < \frac{1}{2}$ and

$$\|\mathcal{F}[f_1] - f_1\|_{[1,T]} \leq D_T(1 + \|f_1\|_{[1,T]}) \tag{4.21}$$

with $0 < D_T < \frac{1}{2}$.

The proofs of these lemmas are postponed to the appendix, as they are based on elementary sequences of inequalities.

Proof of Proposition 4.1. Thanks to Lemma 4.2 we already know that $\mathcal{F}$ maps $C([1,T], \mathcal{X}_\varepsilon)$ into itself.

By Lemma 4.3 we also know that if T satisfies (4.19), then $\mathcal{F}$ is a contraction and for every $f \in X_T$,

$$\|\mathcal{F}[f] - f_1\|_{[1,T]} \leq (C_T + D_T)(1 + \|f_1\|_{[1,T]})$$

with $C_T + D_T < 1$.

By the Banach fixed point theorem we deduce that if T satisfies (4.19), then the operator $\mathcal{F}$ has a unique fixed point f in X_T. Notice that if $f \in C([1,T], \mathcal{X}_\varepsilon)$, then $\mathcal{F}[f] \in C^1([1,T], \mathcal{X}_\varepsilon)$, therefore the map $f : [1,T] \to \mathcal{X}_\varepsilon$ is Fréchet differentiable. Differentiating $\mathcal{F}[f] = f$ we deduce that f satisfies equation (4.12). (For the details of this computation we refer to [29, proof of Lemma 5.6].)

For the moment we only know that the solution of equation (4.12) exists if T is small enough. Let us show, following the strategy of [10] and [29], that we can extend this solution to the whole line $[1,\infty)$. To this end we first prove inequality (4.14). Considering in (4.12) a test function $\varphi \in C_c(\mathbb{R}_+)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $[\varepsilon, 2T^\beta R]$ we obtain the following a priori estimate, for every $t > 1$:

$$\begin{aligned}
&\int_{\mathbb{R}_*} \varphi(x) \dot{f}(t, dx) \\
&\quad = \int_{\mathbb{R}_*} \varphi(x) \tilde{\eta}_\varepsilon(t, x)\, dx \\
&\qquad + \int_{(\varepsilon,\infty)} \int_{(\varepsilon,\infty)} \frac{K_{a,T}(t,x,y)}{2} \left(\Lambda[\varphi](t,x,y) - \varphi(x) e^{g(\ln t, y)} - \varphi(y) e^{g(\ln t, x)}\right) \\
&\qquad\qquad \times f(t, dx) f(t, dy) \\
&\quad \leq \|\eta_\varepsilon\|,
\end{aligned}$$

where the last inequality is a consequence of (4.11) and of the fact that, due to the definition of ℓ and the monotonicity of Ξ_ε, $\max\{g(\ln t, x), g(\ln t, y)\} - g(\ln t, \ell(\ln t, x, y)) \leq 0$, hence

$$\Lambda[\varphi](t, x, y) - \varphi(x)e^{g(\ln t, y)} - \varphi(y)e^{g(\ln t, x)} \leq 0 \quad t > 0,\ x, y > \varepsilon \tag{4.22}$$

for our choice of test function.

Therefore, for every $t \in [1, T]$, inequality (4.14) holds and a unique solution of (4.12) exists in the interval $[1, T_1]$ with

$$T_1 - 1 := \frac{C(\eta_\varepsilon, a)}{1 + \|f_1\|}.$$

To extend the solution, preserving uniqueness and differentiability, we update the initial condition to $f_2(\cdot) := f(\frac{T_1}{2}, \cdot)$ and by (4.19) and (4.14) we deduce that there exists a unique solution on $[1, \frac{T_1}{2} + T_2]$ with

$$T_2 - 1 := \frac{C(\eta_\varepsilon, a)}{1 + \|f_1\| + \frac{C(\eta_\varepsilon, a)\|\eta_\varepsilon\|}{1 + \|f_1\|}} \leq \frac{C(\eta_\varepsilon, a)}{1 + \|f_2\|}.$$

Iterating this argument, thanks to (4.14) and (4.19) we deduce that a unique solution exists on the whole real line. We refer to [29, proof of Proposition 5.8] for the details.

We show that $f(t, (2Rt^\beta, \infty)) = 0$. Considering a test function φ_n which approaches $\chi_{(2Rt^\beta, \infty)}$ in equation (4.12), we deduce that $\int_{\mathbb{R}_*} \varphi_n(x)\dot{f}(t, dx) \leq 0$ for every $n \geq 0$ and the desired conclusion follows by the Lebesgue dominated convergence theorem. ■

Proposition 4.4. *Let γ, β, $K_{a,T}$, η_ε, Ξ_ε and θ_R be as in Proposition 4.1 and let f denote the solution of equation (4.12) with respect to $K_{a,T}$, η_ε, Ξ_ε and θ_R. Then $\Phi \in C^1([0, T], \mathcal{M}_{+,b}(\mathbb{R}_*))$, defined, by duality, as the function with measure values that satisfies, for every $\varphi \in C_b(\mathbb{R}_*)$ and every $\tau \in [0, T]$, the equality*

$$\int_{\mathbb{R}_*} \varphi(\xi)\Phi(\tau, d\xi) = \int_{\mathbb{R}_*} \varphi(X(\tau, x))e^{-\tau + g(\tau, x)} f(e^\tau, dx), \tag{4.23}$$

where g is given by (4.10), is a solution of equation (4.4) with respect to K_a, η_ε, ζ_R and Ξ_ε and the initial condition $\Phi_0 \in \mathcal{M}_+(\mathbb{R}_*)$ with $\Phi_0((0, \varepsilon] \cup (2R, \infty)) = 0$. The solution Φ has the following properties for every $\tau \in [0, T]$:

$$\Phi(\tau, (0, \varepsilon] \cup (2R, \infty)) = 0,$$
$$\int_{\mathbb{R}_*} \Phi(\tau, d\xi) \leq e^{\frac{1+\gamma}{1-\gamma}T}(e^T\|\eta_\varepsilon\| + \|\Phi_0\|). \tag{4.24}$$

Proof. By the change of variables (4.23), for every $\varphi \in C^1([0, T], C_c^1(\mathbb{R}_*))$,

$$\frac{d}{d\tau}\left(\int_{\mathbb{R}_*} \varphi(\tau, \xi)\Phi(\tau, d\xi)\right) = \frac{d}{d\tau}\left(\int_{\mathbb{R}_*} \varphi(\tau, X(\tau, x))e^{-\tau + g(\tau, x)} f(e^\tau, dx)\right)$$

$$= \int_{\mathbb{R}_*} \frac{d}{d\tau}\big(\varphi(\tau, X(\tau,x))e^{-\tau+g(\tau,x)}\big) f(e^\tau, dx)$$
$$+ \int_{\mathbb{R}_*} \varphi(\tau, X(\tau,x))e^{g(\tau,x)} \frac{d}{de^\tau} f(e^\tau, dx).$$

Expanding the first term and using the definition of g, (4.10), we obtain

$$\int_{\mathbb{R}_*} \frac{d}{d\tau}\big(\varphi(\tau, X(\tau,x))e^{-\tau+\beta\int_0^\tau \Xi_\varepsilon(X(s,x))\,ds}\big) f(e^\tau, dx)$$
$$= \int_{\mathbb{R}_*} \big[\partial_1\varphi(\tau, X(\tau,x)) - \beta X(\tau,x)\Xi_\varepsilon(X(\tau,x))\partial_2\varphi(\tau, X(\tau,x)) - \varphi(\tau, X(\tau,x))$$
$$+ \beta\Xi_\varepsilon(X(\tau,x))\varphi(\tau, X(\tau,x))\big]e^{-\tau+\beta\int_0^\tau \Xi_\varepsilon(X(s,x))\,ds} f(e^\tau, dx)$$
$$= \int_{\mathbb{R}_*} \big[\partial_1\varphi(\tau,z) - \beta z\Xi_\varepsilon(z)\partial_2\varphi(\tau,z) - \varphi(\tau,z) + \beta\Xi_\varepsilon(z)\varphi(\tau,z)\big]\Phi(\tau, dz).$$

On the other hand, since f is the fixed point of $\mathcal{F}$, choosing the test function

$$\psi(\tau,x) := \varphi(\tau, X(\tau,x))e^{g(\tau,x)} \tag{4.25}$$

in (4.12) we deduce that

$$\int_{\mathbb{R}_*} \psi(\tau,x)\frac{d}{de^\tau} f(e^\tau, dx)$$
$$= \int_{\mathbb{R}_*} \psi(\tau,x)\tilde{\eta}_\varepsilon(e^\tau,x)\,dx$$
$$+ \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \frac{K_{a,T}(e^\tau,x,y)}{2}\big(\Lambda[\psi(\tau,\cdot)](e^\tau,x,y) - \psi(\tau,x)e^{g(\tau,y)}$$
$$- \psi(\tau,y)e^{g(\tau,x)}\big) f(e^\tau,dx) f(e^\tau,dy),$$

which together with (4.8), (4.11) and (4.25) implies that

$$\int_{\mathbb{R}_*} \varphi(\tau, X(\tau,x))e^{g(\tau,x)} \frac{d}{de^\tau} f(e^\tau, dx)$$
$$= \int_{\mathbb{R}_*} \varphi(\tau, X(\tau,x))e^{g(\tau,x)}\tilde{\eta}_\varepsilon(e^\tau,x)\,dx$$
$$+ \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \frac{K_{a,T}(e^\tau,x,y)}{2} e^{g(\tau,y)+g(\tau,x)}\big[\varphi(\tau, X(\tau,x)+X(\tau,y))\theta_R(e^\tau,x,y)$$
$$- \varphi(\tau, X(\tau,x)) - \varphi(\tau, X(\tau,y))\big]$$
$$\times f(e^\tau,dx) f(e^\tau,dy)$$
$$= \int_{\mathbb{R}_*} \varphi(\tau,\xi)\eta_\varepsilon(\xi)\,d\xi$$
$$+ \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \frac{K_a(X(\tau,x), X(\tau,y))}{2}\big[\varphi(\tau, X(\tau,x)+X(\tau,y))\zeta_R(X(\tau,x)+X(\tau,y))$$
$$- \varphi(\tau, X(\tau,x)) - \varphi(\tau, X(\tau,y))\big]$$
$$\times e^{-\tau+g(\tau,x)} f(e^\tau,dx)e^{-\tau+g(\tau,y)} f(e^\tau,dy).$$

By the change of variables (4.23), we deduce that

$$\int_{\mathbb{R}_*} \varphi(\tau, X(\tau,x))e^{g(\tau,x)} \frac{d}{de^\tau} f(e^\tau, dx)$$
$$= \int_{\mathbb{R}_*} \varphi(\tau,\xi)\eta_\varepsilon(\xi)\, d\xi$$
$$= \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \frac{K_a(\xi,z)}{2}\big[\zeta_R(z+\xi)\varphi(\tau,z+\xi) - \varphi(\tau,\xi) - \varphi(\tau,z)\big]\Phi(\tau,d\xi)\Phi(\tau,dz).$$

Summarizing, we have proven that Φ satisfies (4.4) for all $\varphi \in C^1([0,T], C_c^1(\mathbb{R}_*))$.

Since f is a solution of equation (4.12), then (4.15) holds. Considering a test function $\varphi = 1$ in $(0,\varepsilon] \cup (2Rt^\beta, \infty)$ in (4.23) and recalling that, if $x > 2\varepsilon$, then $X(t,x) = t^{-\beta}x$ and for every $\varepsilon \geq x > 0$ and every $t > 1$ we have $X(t,x) = x$, we conclude that for every $\tau \in [0,T]$,

$$\Phi(\tau, (0,\varepsilon] \cup (2R,\infty)) = 0.$$

By inequality (4.14) and by the fact that $f_1 = \Phi_0$, we deduce that

$$\sup_{\tau\in[0,T]} \int_{\mathbb{R}_*} \Phi(\tau,d\xi) = \sup_{\tau\in[0,T]} \int_{\mathbb{R}_*} e^{-\tau+\beta\int_0^\tau \Xi_\varepsilon(X(s,x))\,ds} f(e^\tau, dx)$$
$$\leq \sup_{t\in[1,e^T]} \int_{\mathbb{R}_*} t^{-1+\beta} f(t,dx) \leq e^{\frac{1+\gamma}{1-\gamma}T}(e^T\|\eta_\varepsilon\| + \|\Phi_0\|).$$

The bound (4.24) follows. ∎

4.2. Existence of steady-state solutions for the truncated coagulation equation with source written in self-similar variables

In this subsection we prove the existence of steady-state solutions of equation (4.4) (see Proposition 4.6).

Definition 4.5. Let $\gamma < 1$. We say that a measure $\Phi \in \mathcal{M}_{+,b}(\mathbb{R}_*)$ is a steady-state solution of equation (4.4) with respect to K_a, ζ_R, Ξ_ε and η_ε if it solves the following equation for every $\varphi \in C_c^1(\mathbb{R}_*)$:

$$\int_{\mathbb{R}_*} \varphi(x)\Phi(dx)$$
$$= \int_{\mathbb{R}_*} \varphi(x)\eta_\varepsilon(x)\,dx + \frac{2}{1-\gamma}\int_{\mathbb{R}_*} \Xi_\varepsilon(x)(\varphi(x) - x\varphi'(x))\Phi(dx)$$
$$+ \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \frac{K_a(x,y)}{2}\big[\zeta_R(x+y)\varphi(x+y) - \varphi(x) - \varphi(y)\big]\Phi(dx)\Phi(dy). \quad (4.26)$$

In the following we will denote by $\chi_{z,n}$ the mollification of the characteristic function $\chi_{(0,z]}$,

$$\chi_{z,n}(x) = \int_{\mathbb{R}_*} \chi_{(0,z]}(x-y)\rho_n(y)\,dy, \quad (4.27)$$

where ρ_n are the mollifiers considered in [14]. In the following, we use the notation

$$g_{z,n}(x) := \int_{x-z}^{\frac{1}{n}} c_n e^{\frac{1}{(ny)^2-1}} \, dy. \tag{4.28}$$

Classical results for mollifiers yield $\chi_{z,n} \to \chi_{[0,z]}$ in $L^1(\mathbb{R}_*)$.

Proposition 4.6. *Let $\gamma < 1$. There exists a steady-state solution for equation (4.4) corresponding to K_a, η_ε, Ξ_ε and ζ_R, $\Phi \neq 0$, as in (4.5), satisfying*

$$\Phi((0, \varepsilon] \cup (2R, \infty)) = 0, \tag{4.29}$$

$$0 < \int_{\mathbb{R}_*} x \Phi(dx) \leq 1. \tag{4.30}$$

Before proceeding with the proof of Proposition 4.6 we present and prove two auxiliary results.

Lemma 4.7. *Under the assumptions of Proposition 4.4, we have that each solution $\Phi \in C^1([1, T], \mathcal{M}_{+,b}(\mathbb{R}_*))$ of (4.4) corresponding to an initial condition $\Phi_0 \in \mathcal{M}_{+,b}(\mathbb{R}_*)$ such that $\int_{\mathbb{R}_*} \xi \Phi_0(d\xi) \leq 1$, and to K_a, η_ε, ζ_R, Ξ_ε satisfies $\int_{\mathbb{R}_*} \xi \Phi(t, d\xi) \leq 1$.*

Proof. We consider the test function $\varphi_n^M(\xi) := \xi \chi_{M,n}(\xi)$, with $\chi_{M,n}$ defined by (4.27) in equation (4.4) and we pass to the limit as M tends to infinity in all the terms of (4.4). First, we notice that

$$\int_{(M,M+1/n]} \xi g_{M,n}(\xi) \Phi(t, d\xi) \to 0 \quad \text{as } M \to \infty;$$

indeed, for every $\xi \in \mathbb{R}_*$, we have $\chi_{(M,M+1/n]}(\xi) g_{M,n}(\xi) \leq 1$ and

$$\int_{\mathbb{R}_*} \xi \Phi(t, d\xi) \leq 2R \Phi(t, \mathbb{R}_*) < \infty.$$

Therefore, by the Lebesgue dominated convergence theorem,

$$\lim_{M \to \infty} \int_{\mathbb{R}_*} \varphi_n^M(\xi) \Phi(t, d\xi) = \int_{\mathbb{R}_*} \xi \Phi(t, d\xi).$$

A similar argument can be used to prove that

$$\int_{\mathbb{R}_*} \varphi_n^M(\xi) \eta_\varepsilon(\xi) \, d\xi \, ds \to \int_{\mathbb{R}_*} \xi \eta_\varepsilon(\xi) \, d\xi \quad \text{as } M \to \infty.$$

If $M > 2R$, then

$$\varphi_n^M(\xi + z) \zeta_R(\xi + z) - \varphi_n^M(\xi) - \varphi_n^M(z) = -\xi \chi_{M,n}(\xi) - z \chi_{M,n}(z).$$

This implies that

$$\lim_{M\to\infty}\int_0^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}K_a(\xi,z)\big[\varphi_n^M(\xi+z)\zeta_R(\xi+z)-\varphi_n^M(\xi)-\varphi_n^M(z)\big]$$
$$\times\,\Phi(s,d\xi)\Phi(s,dz)\,ds\le 0.$$

Let us now consider the term

$$\int_{[0,M+1/n]}\xi^2\Xi_\varepsilon(\xi)\partial_\xi(\chi_{M,n}(\xi))\Phi(t,d\xi)=\int_{(M,M+1/n]}\Xi_\varepsilon(\xi)\xi^2\partial_\xi(g_{M,n}(\xi))\Phi(t,d\xi).$$

Since $g'_{M,n}(x)=-c_n e^{\frac{1}{(n(x-M))^2-1}}$, by the Lebesgue dominated convergence theorem we can conclude that

$$\lim_{M\to\infty}\int_{\mathbb{R}_*}\Xi_\varepsilon(\xi)\chi_{(M,M+1/n]}(\xi)\xi^2\partial_\xi(g_{M,n}(\xi))\Phi(t,d\xi)=0.$$

Passing to the limit as $M\to\infty$ in equation (4.4), thanks to the specific form of the truncation for small particles in (4.2), we deduce that

$$\partial_t\int_0^\infty x\Phi(t,dx)\le 1-\int_0^\infty x\Phi(t,dx).$$

Therefore, since by assumption $\int_0^\infty x\Phi_0(dx)\le 1$, we deduce that $\int_0^\infty x\Phi(t,dx)\le 1$ for every $t>0$. ∎

Lemma 4.8. *Under the assumptions of Proposition 4.6, for every $T>0$ there exists a unique solution $\varphi\in C^1([0,T],C_c^1(\mathbb{R}_*))$, with $\varphi(T,\cdot):=\psi\in C_c^1(\mathbb{R}_*)$, of the equation*

$$\partial_s\varphi(s,x)-\varphi(s,x)+\frac{2}{1-\gamma}\Xi_\varepsilon(x)(\varphi(s,x)-x\partial_x\varphi(s,x))+\mathbb{L}[\varphi](s,x)=0, \tag{4.31}$$

where $\mathbb{L}$ is defined by

$$\mathbb{L}[\varphi](s,x):=\frac{1}{2}\int_{\mathbb{R}_*}K_a(x,y)\big(\varphi(s,x+y)\zeta_R(x+y)-\varphi(s,x)-\varphi(s,y)\big)\mu_s(dy), \tag{4.32}$$

where $\mu_s=\Phi(s,\cdot)+\Psi(s,\cdot)$ and Φ and Ψ are two solutions of (4.4) with initial conditions Φ_0 and Ψ_0 respectively.

Proof. Equation (4.31) includes a transport term. Therefore, we proceed by integrating along the characteristics. Since the associated ODE is (4.5), we can rewrite equation (4.31) as

$$\frac{d}{ds}\varphi(s,X(s,y))=\varphi(s,X(s,y))-\mathbb{L}[\varphi](s,X(s,y))$$
$$-\frac{2}{1-\gamma}\Xi_\varepsilon(X(s,y))\varphi(s,X(s,y)).$$

We aim now to apply the Banach fixed point theorem. We therefore rewrite the equation in a fixed point form $\varphi(s, X(s, y)) = \mathcal{T}[\varphi](s, X(s, y))$, where

$$\mathcal{T}[\varphi](s, y) := \psi(X(t, y)) + \int_s^t \Big(\frac{2}{1-\gamma}\,\Xi_\varepsilon(X(r, y))\varphi(r, X(r, y)) - \varphi(r, X(r, y))\Big)\, dr + \int_s^t \mathbb{L}[\varphi](r, X(r, y))\, dr.$$

Recall that $\varphi \in Y := C([0, T], C_c(\mathbb{R}_*))$ and Y is a Banach space with the norm $\|\cdot\|_Y := \sup_{[0,T]} \sup_{\mathbb{R}_*} |\cdot|$. We show that the operator $\mathcal{T} : Y \to Y$ is a contraction.

Let us consider $\varphi_1, \varphi_2 \in Y$; then

$$\begin{aligned}\mathcal{T}[\varphi_1](s, y) - \mathcal{T}[\varphi_2](s, y) = &\int_s^t (\mathbb{L}[\varphi_1] - \mathbb{L}[\varphi_2])(r, X(r, y))\, dr \\ &+ \int_s^t \Big(\frac{2\,\Xi_\varepsilon(X(r, y))}{1-\gamma}\big(\varphi_1(r, X(r, y)) - \varphi_2(r, X(r, y))\big) \\ &\qquad - \big(\varphi_1(r, X(r, y)) - \varphi_2(r, X(r, y))\big)\Big)\, dr.\end{aligned}$$

We start by estimating the last term,

$$\begin{aligned}&\Big|\int_s^t \Big(\frac{2\,\Xi_\varepsilon(X(r, y))}{1-\gamma}\big(\varphi_1(r, X(r, y)) - \varphi_2(r, X(r, y))\big) \\ &\qquad - \big(\varphi_1(r, X(r, y)) - \varphi_2(r, X(r, y))\big)\Big)\, dr\Big| \\ &\leq T\Big|\frac{3-\gamma}{1-\gamma}\Big|\|\varphi_1 - \varphi_2\|_Y, \end{aligned} \tag{4.33}$$

and then we analyze the first term,

$$\begin{aligned}&\int_s^t (\mathbb{L}[\varphi_1] - \mathbb{L}[\varphi_2])(r, X(r, y))\, dr \\ &\quad\leq \frac{a}{2}\int_s^t \int_0^\infty \big[(\varphi_1 - \varphi_2)(r, x + X(r, y))\zeta_R(x + y) - (\varphi_1 - \varphi_2)(r, x) \\ &\qquad\qquad - (\varphi_1 - \varphi_2)(r, X(r, y))\big]\mu_r(dx) \\ &\quad\leq \frac{3a}{2}\|\varphi_1 - \varphi_2\|_Y \int_s^t \mu_r(\mathbb{R}_*)\, dr = \frac{3a}{2}\|\varphi_1 - \varphi_2\|_Y \int_s^t (\Phi(r, \mathbb{R}_*) + \Psi(r, \mathbb{R}_*))\, dr.\end{aligned}$$

To prove that $\mathcal{T}$ is a contraction, we need to control the quantity

$$\int_s^t (\Phi(r, \mathbb{R}_*) + \Psi(r, \mathbb{R}_*))\, dr.$$

Thanks to (4.24) and to the fact that $\gamma < 1$, we know that for every $r > 0$,

$$\Psi(r, \mathbb{R}_*) \leq e^{r\frac{1+\gamma}{1-\gamma}}(\|\Psi_0\| + e^r\|\eta_\varepsilon\|) \leq e^{r\frac{2}{1-\gamma}}(\|\Psi_0\| + \|\eta_\varepsilon\|)$$

and

$$\Phi(r, \mathbb{R}_*) \leq e^{r\frac{1+\gamma}{1-\gamma}}(\|\Phi_0\| + e^r\|\eta_\varepsilon\|) \leq e^{r\frac{2}{1-\gamma}}(\|\Phi_0\| + \|\eta_\varepsilon\|).$$

Therefore,

$$\int_s^t (\Phi(r,\mathbb{R}_*) + \Psi(r,\mathbb{R}_*))\,dr \le 2(\max\{\|\Psi_0\|,\|\Phi_0\|\} + \|\eta_\varepsilon\|)\int_s^t e^{r\frac{2}{1-\gamma}}\,dr$$
$$= 2(\max\{\|\Psi_0\|,\|\Phi_0\|\} + \|\eta_\varepsilon\|)\frac{1-\gamma}{3-\gamma}(e^{t\frac{3-\gamma}{1-\gamma}} - e^{s\frac{3-\gamma}{1-\gamma}}).$$

Since for any $\alpha > 0$ and $1 > t > s > 0$ then $e^{\alpha t} - e^{\alpha s} \le e^{\alpha t} - 1 \le t(e^{\alpha} - 1)$, we conclude, recalling (4.33), that for every $T \ge 0$,

$$\|\mathcal{T}[\varphi_1] - \mathcal{T}[\varphi_2]\|_Y \le \|\varphi_1 - \varphi_2\|_Y\left(\left|\frac{3-\gamma}{1-\gamma}\right|T + \frac{3a}{2}\int_s^t (\Phi(\mathbb{R}_*,r) + \Psi(\mathbb{R}_*,r))\,dr\right)$$
$$\le cT\|\varphi_1 - \varphi_2\|_Y,$$

where

$$c := \left|\frac{3-\gamma}{1-\gamma}\right| + 3a\frac{1-\gamma}{3-\gamma}(e^{\frac{3-\gamma}{1-\gamma}} - 1)(\max\{\|\Psi_0\|,\|\Phi_0\|\} + \|\eta_\varepsilon\|).$$

If $T < \min\{\frac{1}{c}, 1\}$, then the operator $\mathcal{T}$ is a contraction. By the Banach fixed point theorem we conclude there exists a unique solution to the equation $\varphi(t,\cdot) = \mathcal{T}\varphi(\cdot,t)$ for $t \in [0,T]$.

Since $\mathcal{T}\varphi(\cdot,x)$ defines a differentiable function for every $x \in \mathbb{R}_*$ we conclude that $\varphi \in C^1([0,T], C_c(\mathbb{R}_*))$. We can now extend the existence of a fixed point for the operator $\mathcal{T}$ on $C^1([0,T], C_c(\mathbb{R}_*))$ for an arbitrary $T > 0$. For this it is enough to notice that, since c and T do not depend on the initial condition ψ, we can iterate the argument and deduce that there exists a unique solution of (4.31). ∎

Proof of Proposition 4.6. We introduce the semigroup $\{S(t)\}_{t\ge 0}$ with values in $\mathcal{M}_{+,b}(\mathbb{R}_*)$ and defined by $S(0)\Phi_0(\cdot) = \Phi_0(\cdot)$ and $S(t)\Phi_0(\cdot) = \Phi(t,\cdot)$, where Φ is the solution of (4.4) with respect to K_a, Ξ_ε, η_ε, ζ_R and the initial condition Φ_0, defined by (4.23). Namely, Φ is the measure defined by equality (4.23) for every $\varphi \in C_c(\mathbb{R}_*)$ with f being the unique solution of the fixed point equation (4.12)

We split the proof into steps: first of all we show the existence of a weak-$*$ compact invariant region. Then we prove the weak-$*$ continuity of the operator $\Phi_0 \mapsto S(t)\Phi_0$. By the Tychonoff fixed point theorem we conclude that for every $t > 0$ the operator $S(t)$ has a fixed point $\widehat{\Phi}_t$. As a last step we show that a steady state of (4.4), as in Definition 4.5, can be obtained from $\widehat{\Phi}_t$ by passing to the limit as t goes to zero.

Step 1*: Existence of an invariant region.* Let us consider the set $P \subset \mathcal{M}_{+,b}(\mathbb{R}_*)$ defined by

$$P := \left\{H \in \mathcal{M}_{+,b}(\mathbb{R}_*) : H((0,\varepsilon] \cup (2R,\infty)) = 0,\ \int_0^\infty xH(dx) \le 1\right\}.$$

Notice that $P \subset B(0,\frac{1}{\varepsilon}) := \{H \in \mathcal{M}_+(\mathbb{R}_*) : \|H\| \le \frac{1}{\varepsilon}\}$ where $\|\cdot\|$ is the total variation norm. By the Banach–Alaoglu theorem we conclude that P is compact in the weak-$*$ topology, since it is a closed subset of the set $B(0,\frac{1}{\varepsilon})$, which is compact in the weak-$*$ topology.

Proposition 4.4 implies that if $\Phi_0((0,\varepsilon] \cup (2R,\infty)) = 0$, then $S(t)\Phi_0((0,\varepsilon] \cup (2R,\infty)) = 0$. By Lemma 4.7 we conclude that P is an invariant region.

Step 2*: Weak-$*$ continuity.* To be able to apply the Tychonoff fixed point theorem, we need to check that the map $\Phi_0 \mapsto S(t)\Phi_0$ is continuous in the weak-$*$ topology.

To this end, it is enough to show that, for every test function $\psi \in C_c(\mathbb{R}_*)$,

$$\int_{\mathbb{R}_*} \psi(x)(\Phi - \Psi)(t, dx) = \int_{\mathbb{R}_*} \psi(x)(\Phi_0 - \Psi_0)(dx), \tag{4.34}$$

where Φ and Ψ are two solutions of (4.4) corresponding to the two initial conditions Φ_0 and Ψ_0, respectively.

With this aim, we notice that the measure $(\Phi - \Psi)(t, \cdot)$ is a solution of

$$\begin{aligned}
&\int_{\mathbb{R}_*} \varphi(t, x)(\Phi - \Psi)(t, dx) \\
&\quad = \int_{\mathbb{R}_*} \varphi(t, x)(\Phi_0 - \Psi_0)(dx) \\
&\qquad + \int_0^t \int_{\mathbb{R}_*} \partial_s \varphi(s, x)(\Phi - \Psi)(s, dx)\, ds + \int_0^t \int_{\mathbb{R}_*} \mathcal{L}[\varphi](x, s)(\Phi - \Psi)(s, dx)\, ds \\
&\qquad + \int_0^t \int_{\mathbb{R}_*} \Big[\frac{2\Xi_\varepsilon(x)}{1-\gamma}(\varphi(s, x) - \partial_x \varphi(s, x)x) - \varphi(x)\Big](\Phi - \Psi)(s, dx)\, ds,
\end{aligned}$$

where $\mathcal{L}$ is given by (4.32). By Lemma 4.8 we conclude that (4.34) holds and, hence, we can prove that the map $\Phi_0 \mapsto S(t)\Phi_0$ is weak-$*$ continuous as in [23, proof of Proposition 2.8].

Step 3*: Time continuity.* The function $S(t)$ has a fixed point $\widehat{\Phi}_t$ for every time $t \geq 0$. We now show that the map

$$t \to S(t)\Phi_0 \tag{4.35}$$

is weak-$*$ continuous for any Φ_0 in $\mathcal{M}_+(\mathbb{R}_*)$.

Since Φ solves (4.4), for every $\varphi \in C_c^1(\mathbb{R}_*)$ it holds that

$$\begin{aligned}
&\int_0^\infty \varphi(x)\big[\Phi(\tau_1, dx) - \Phi(\tau_2, dx)\big] \\
&\quad = \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}_*} \varphi(x)\Phi(\tau, dx) \\
&\qquad - \frac{2}{1-\gamma} \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}_*} \Xi_\varepsilon(x)\partial_x\varphi(x)x\Phi(\tau, dx)d\tau + \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}_*} \varphi(x)\eta_\varepsilon(dx) \\
&\qquad + \frac{1}{2} \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} K_a(x, y)\big[\varphi(x + y) - \varphi(x) - \varphi(y)\big]\Phi(\tau, dx)\Phi(\tau, dy)d\tau \\
&\qquad + \frac{2}{1-\gamma} \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}_*} \Xi_\varepsilon(x)\varphi(x)x\Phi(\tau, dx)d\tau \\
&\quad \leq (\tau_2 - \tau_1)C(\eta_\varepsilon, \varphi, \gamma) + (\tau_2 - \tau_1)^2 C(a, \varphi, \gamma),
\end{aligned}$$

where $C(\eta_\varepsilon, \varphi, \gamma)$ and $C(a, \varphi, \gamma)$ are positive constants. Therefore, the function (4.35) is continuous in the weak-$*$ topology.

Since the set P is compact and metrizable and the map (4.35) is continuous, we conclude by [9, Theorem 1.2] that there exists a measure $\widehat{\Phi}$ such that $S(t)\widehat{\Phi} = \widehat{\Phi}$. The measure $\widehat{\Phi}$ is a solution of equation (4.26).

Finally, $\Phi \neq 0$ because 0 does not solve (4.26), whence $\int_0^\infty x\Phi(dx) > 0$. ∎

4.3. Properties of the steady-state solutions

We state now some important properties of the solutions of equation (4.26).

Lemma 4.9 (Regularity). *If $\Phi \in \mathcal{M}_+(\mathbb{R}_*)$ is a solution of equation (4.26) with respect to K_a, η_ε, Ξ_ε and ζ_R, as in (4.5), then $\Phi \ll \mathcal{L}$.*

Proof. We follow a similar strategy to the one used in [18]. If we consider the test function

$$\varphi(x) := -\int_{(x,\infty)} \frac{1}{z}\chi(z)\,dz, \tag{4.36}$$

with $\chi \in C_c(\mathbb{R}_*)$ in (4.26), then we obtain, for any $p, q \in \mathbb{R}_*$ such that $1/p + 1/q = 1$,

$$\frac{2}{1-\gamma}\left|\int_{\mathbb{R}_*} \Xi_\varepsilon(x)\chi(x)\Phi(dx)\right| \leq \left(\eta_\varepsilon(\mathbb{R}_*) + \frac{|1+\gamma|}{1-\gamma}\Phi(\mathbb{R}_*) + \frac{3a}{2}\Phi(\mathbb{R}_*)^2\right) \times \left\|\frac{1}{z}\right\|_{L^p(\mathcal{K})}\|\chi\|_{L^q(\mathcal{K})},$$

where $\mathcal{K}$ is the support of χ. By the density of $C_c(\mathbb{R}_*)$ in $L^q(\mathbb{R}_*)$, we conclude that for any $q < \infty$ and any compact set $\mathcal{K}$,

$$\left|\int_{\mathbb{R}_*} \chi(x)\Xi_\varepsilon(x)\Phi(dx)\right| \leq C(\mathcal{K}, \gamma, \varepsilon, \Phi)\|\chi\|_{L^q(\mathcal{K})} \quad \forall \chi \in L^q(\mathcal{K}).$$

This implies that the measure $\Xi_\varepsilon(x)\Phi(dx)$ is absolutely continuous with respect to the Lebesgue measure. Thus Φ is absolutely continuous with respect to the Lebesgue measure on $(0, \varepsilon)$ and since $\Phi((0, \varepsilon]) = 0$ we deduce that hence $\Phi \ll \mathcal{L}$. ∎

Lemma 4.10. *Every steady-state solution Φ of (4.4), defined as in Definition 4.5, with density ϕ, corresponding to K_a, η_ε, Ξ_ε and ζ_R, satisfies the inequality*

$$\int_0^z \int_{z-x}^\infty K_a(x,y)x\phi(x)\phi(y)\,dy\,dx \leq \int_0^z x\eta_\varepsilon(dx) + \frac{2}{1-\gamma}z^2\phi(z) \tag{4.37}$$

for almost every $z > 0$.

Proof. If we consider the test function $\varphi_z^n(x) = x\chi_{z,n}(x)$, with $\chi_{z,n}$ given by formula (4.27), in equation (4.26), we obtain

$$\int_0^{z+\frac{1}{n}} \varphi_z^n(x)\eta_\varepsilon(x)\,dx - \int_0^{z+\frac{1}{n}} \varphi_z^n(x)\phi(x)\,dx - \frac{2}{1-\gamma}\int_0^{z+\frac{1}{n}} \Xi_\varepsilon(x)x\varphi_z^{n\prime}(x)\phi(x)\,dx$$
$$+ \frac{2}{1-\gamma}\int_0^{z+\frac{1}{n}} \Xi_\varepsilon(x)\varphi_z^n(x)\phi(x)\,dx$$

$$= -\frac{1}{2}\int_0^{z+\frac{1}{n}}\int_0^{z+\frac{1}{n}} K_a(x,y)\big[\zeta_R(x+y)\varphi_z^n(x+y)-\varphi_z^n(x)-\varphi_z^n(y)\big] \times \phi(x)\,dx\,\phi(y)\,dy.$$

Applying the Lebesgue dominated convergence theorem we prove that

$$\lim_{n\to\infty}\int_0^\infty \varphi_z^n(x)\eta_\varepsilon(dx) = \int_0^z x\eta_\varepsilon(dx), \quad \lim_{n\to\infty}\int_0^\infty \varphi_z^n(x)\phi(x)\,dx = \int_0^z x\phi(x)\,dx$$

and

$$\lim_{n\to\infty}\int_0^\infty \Xi_\varepsilon(x)\varphi_z^n(x)\phi(x)\,dx = \int_0^z \Xi_\varepsilon(x)x\phi(x)\,dx.$$

We now aim to show that

$$\int_0^{z+1/n} \Xi_\varepsilon(x)x(x\chi_{z,n}(x))'\phi(x)\,dx \to \int_0^z \Xi_\varepsilon(x)x\phi(x)\,dx - \Xi_\varepsilon(z)z^2\phi(z)$$

as $n\to\infty$. Notice that $\chi'_{z,n}(x)=\rho_n(x-z)$, where ρ_n are the mollifiers introduced in Section 4.2. As a consequence of the properties of the mollifiers, we know that for every $f\in L^1(\mathbb{R}_*)$,

$$\|\rho_n * f - f\|_1 \to 0 \quad \text{as } n\to\infty,$$

where with $\|\cdot\|_1$ we denote the L^1 norm and with $*$ the classical convolution product. This implies that, up to a subsequence,

$$\int_{\mathbb{R}_*} \Xi_\varepsilon(x)x^2\phi(x)\chi'_{z,n}(x)\,dx = \int_{\mathbb{R}_*} \Xi_\varepsilon(x)x^2\phi(x)\rho_n(x-z)\,dx \to \Xi_\varepsilon(z)z^2\phi(z) \text{ a.e.}$$

as n goes to infinity.

On the other hand, it is possible to prove as in [10, proof of Lemma 2.7] that

$$-\lim_{n\to\infty}\int_0^{z+\frac{1}{n}}\int_0^{z+\frac{1}{n}} K_a(x,y)\big[\varphi_z^n(x+y)-\varphi_z^n(x)-\varphi_z^n(y)\big]\phi(x)\,dx\,\phi(y)\,dy = \int_0^z\int_{z-x}^\infty K_a(x,y)x\phi(x)\,dx\,\phi(y)\,dy$$

as n goes to infinity. Since, by definition of the truncation we have $\zeta_R\le 1$,

$$-\int_0^{z+\frac{1}{n}}\int_0^{z+\frac{1}{n}} K_a(x,y)\big[\zeta_R(x+y)\varphi_z^n(x+y)-\varphi_z^n(x)-\varphi_z^n(y)\big]\phi(x)\,dx\,\phi(y)\,dy \geq -\int_0^{z+\frac{1}{n}}\int_0^{z+\frac{1}{n}} K_a(x,y)\big[\varphi_z^n(x+y)-\varphi_z^n(x)-\varphi_z^n(y)\big]\cdot\phi(x)\,dx\,\phi(y)\,dy,$$

the statement of the lemma follows. ■

To prove the estimates for the solutions of (4.26), [10, Lemma 2.10] will be useful. We recall the statement here.

Lemma 4.11. *Suppose $d > 0$ and $b \in (0,1)$ and assume that $L \in (0,\infty]$ is such that $L \geq d$. Consider some $\mu \in \mathcal{M}_+(\mathbb{R}_*)$ and $\varphi \in C(\mathbb{R}_*)$, with $\varphi \geq 0$.*

(1) *Suppose $L < \infty$, and assume that there is $g \in L^1([d,L])$ such that $g \geq 0$ and*

$$\frac{1}{z}\int_{[bz,z]} \varphi(x)\mu(dx) \leq g(z) \quad \text{for } z \in [d,L]. \tag{4.38}$$

Then

$$\int_{[d,L]} \varphi(x)\mu(dx) \leq \frac{\int_{[d,L]} g(z)\,dz}{|\ln b|} + Lg(L).$$

(2) *If $L = \infty$ and there is $g \in L^1([d,\infty))$ such that $g \geq 0$ and*

$$\frac{1}{z}\int_{[bz,z]} \varphi(x)\mu(dx) \leq g(z) \quad \text{for } z \geq d,$$

then

$$\int_{[d,\infty)} \varphi(x)\mu(dx) \leq \frac{\int_{[d,\infty)} g(z)\,dz}{|\ln b|}.$$

Lemma 4.12. *The density ϕ of every solution of equation (4.26) with respect to K_a, Ξ_ε, η_ε and ζ_R satisfying (4.30) and (4.29) is such that*

$$\frac{1}{z}\int_{8z/9}^{z} \phi(x)\,dx \leq C\left(\frac{a}{z^3}\right)^{1/2}, \quad z \in [0,2R], \tag{4.39}$$

for some $C > 0$ independent of ε, a, R, and such that

$$\int_y^\infty \phi(x)\,dx \leq C_{a,\varepsilon} y^{-1/2}, \quad y \in [1,2R], \tag{4.40}$$

for a positive constant $C_{a,\varepsilon} > 0$ independent of R.

Proof. Since ϕ satisfies (4.37), it follows that for almost every $z > 0$,

$$J_\phi(z) \leq 1 + \frac{2}{1-\gamma} z^2 \phi(z). \tag{4.41}$$

Noting that

$$\left[2z/3, z\right]^2 \subset \left\{(x,y) \in \mathbb{R}_*^2 \mid 0 < x \leq z,\ y > z - x\right\} =: \Omega_z, \tag{4.42}$$

as well as the lower bound for the kernel

$$J_\phi(z) \geq \int_{\frac{2z}{3}}^{z}\int_{\frac{2z}{3}}^{z} K_a(x,y) x\phi(x)\phi(y)\,dx\,dy \geq \frac{cz}{a}\left(\int_{\frac{2z}{3}}^{z}\phi(x)\,dx\right)^2 \quad \text{for } z \leq 2R, \tag{4.43}$$

for some constant $c > 0$ independent of a, R and ε.

Combining (4.41) and (4.43) we conclude that

$$\int_{[2z/3,z]} \phi(x)\,dx \le \left(\frac{a}{c}\right)^{1/2}\left(\frac{1+\frac{2}{1-\gamma}z^2\phi(z)}{z}\right)^{\frac{1}{2}}, \quad \text{a.e. } z \le 2R. \tag{4.44}$$

Since $\frac{2}{1-\gamma}z^2\phi(z) \ge 0$, integrating (4.44) over $[w, 2w]$, with $w \in [0, R]$, we obtain

$$\int_w^{2w}\int_{\frac{2z}{3}}^{z} \phi(x)\,dx\,dz \le \left(\frac{a}{c}\right)^{1/2}\left(\int_w^{2w}\left(\frac{1}{z}\right)^{\frac{1}{2}} dz + \int_w^{2w}\left(\frac{2}{1-\gamma}z\phi(z)\right)^{\frac{1}{2}} dz\right).$$

By the Cauchy–Schwarz inequality we conclude that

$$\int_w^{2w} (z\phi(z))^{\frac{1}{2}}\,dz \le \left(\int_w^{2w} dz\right)^{1/2}\left(\int_w^{2w} z\phi(z)\,dz\right)^{1/2} \le w^{1/2}.$$

Notice that, for the last inequality, we have used (4.30).

Combining all the above inequalities we conclude that

$$\int_w^{2w}\int_{\frac{2z}{3}}^{z} \phi(x)\,dx\,dz \le \left(\frac{a}{c}\left(1+\frac{2}{(1-\gamma)}\right)\right)^{1/2} w^{1/2}.$$

Moreover, by observing that

$$\left[8w/9, w\right] \times \left[w, 4w/3\right] \subset \left\{(x,z) \in \mathbb{R}_*^2 : 2z/3 < x < z,\ z \in [w, 2w]\right\},$$

we deduce that

$$\int_w^{2w}\int_{\frac{2z}{3}}^{z} \phi(x)\,dx\,dz \ge \int_w^{4w/3}\int_{8w/9}^{w} \phi(x)\,dx\,dz = w/3\int_{8w/9}^{w} \phi(x)\,dx.$$

Consequently, adopting the notation $\widetilde{C} = 3c^{-1/2}(\frac{3-\gamma}{1-\gamma})^{1/2}$, we conclude that for any $w \in [0, 2R]$,

$$w\int_{8w/9}^{w} \phi(x)\,dx \le \widetilde{C}a^{1/2}w^{1/2}.$$

Let us prove (4.40). Thanks to inequality (4.39), hypothesis (4.38) of Lemma 4.11 holds with $d = y$, $b = 8/9$, $L = 2R$, $g(z) = Cz^{-3/2}a^{1/2}$, and implies

$$\int_y^{2R} \phi(x)\,dx \le \left(C\frac{a}{R}\right)^{\frac{1}{2}} + C\frac{\int_y^{2R}(\frac{a}{z^3})^{\frac{1}{2}}\,dz}{\ln(3/2)}.$$

Since

$$\int_y^{2R}\left(\frac{a}{z^3}\right)^{\frac{1}{2}} dz \le 2\left(\frac{a}{y}\right)^{\frac{1}{2}}, \quad \left(\frac{a}{R}\right)^{\frac{1}{2}} \le 2^{1/2}\left(\frac{a}{y}\right)^{1/2} \text{ and } \int_{2R}^{\infty}\Phi(dx) = 0$$

and the result follows. ■

4.4. Proof of the existence of a self-similar solution

The aim of this section is to prove that, as $R \to \infty$, $a \to \infty$ and $\varepsilon \to 0$, each solution $\Phi_{\varepsilon,a,R}$ of (4.26) with respect to K_a, ζ_R, Ξ_ε and η_ε, converges, in the weak-$*$ topology, to a measure Φ whose density is a self-similar profile as in Definition 3.1.

Lemmas 4.13 and 4.16 describe the bounds and the properties that the limiting measures $\Phi_{\varepsilon,a}$ and Φ_ε respectively satisfy. In the proof of Theorem 3.2 we will use these bounds and properties to prove the existence of a self-similar profile.

Lemma 4.13. *Consider a sequence* $\{R_n\} \subset \mathbb{R}_*$ *such that* $\lim_{n\to\infty} R_n = \infty$. *Let* Φ_{ε,a,R_n} *be a solution of* (4.26) *with respect to* K_a, Ξ_ε, η_ε *and* ζ_{R_n}. *There exists a measure* $\Phi_{\varepsilon,a} \in \mathcal{M}_+(\mathbb{R}_*)$ *such that*

$$\Phi_{\varepsilon,a,R_n} \rightharpoonup \Phi_{\varepsilon,a} \quad \text{as } n \to \infty, \text{ in the weak-}* \text{ topology.} \tag{4.45}$$

The measure $\Phi_{\varepsilon,a}$ *is absolutely continuous with respect to the Lebesgue measure and satisfies the equation*

$$\begin{aligned}\int_{\mathbb{R}_*} &\Phi_{\varepsilon,a}(dx)\varphi(x)\\ &= \int_{\mathbb{R}_*} \varphi(x)\eta_\varepsilon(dx) + \frac{2}{1-\gamma}\int_{\mathbb{R}_*} \Xi_\varepsilon(x)(\varphi(x) - x\varphi'(x))\Phi_{\varepsilon,a}(dx)\\ &\quad + \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_a(x,y)[\varphi(x+y) - \varphi(x) - \varphi(y)]\Phi_{\varepsilon,a}(dx)\Phi_{\varepsilon,a}(dy)\end{aligned} \tag{4.46}$$

for every $\varphi \in C_c^1(\mathbb{R}_*)$. *Moreover,* $\Phi_{\varepsilon,a}$ *satisfies the growth bound*

$$\frac{1}{z}\int_{[8z/9,z]} \Phi_{\varepsilon,a}(dx) \le C\left(\frac{1}{z^3 \min\{a, z^\gamma\}}\right)^{1/2}, \quad z > 0, \tag{4.47}$$

for some positive C.

Proof. We use inequality (4.39), proven in Lemma 4.12, and apply Lemma 4.11 with $d = \varepsilon$, $b = 8/9$, $L = 2R_n$, $g(z) = Cz^{-3/2}a^{1/2}$ to conclude that

$$\int_{[\varepsilon,2R_n]} \Phi_{\varepsilon,a,R_n}(dx) \le 2a\varepsilon^{-1/2}.$$

From (4.29) we deduce that the sequence $\{\Phi_{\varepsilon,a,R_n}\}$ is bounded in the weak-$*$ topology.

By the Banach–Alaoglu theorem we deduce that the sequence $\{\Phi_{\varepsilon,a,R_n}\}$ admits a subsequence, $\{\Phi_{\varepsilon,a,R_{n_k}}\}$, which converges in the weak-$*$ topology, namely, there exists a measure $\Phi_{\varepsilon,a}$ such that

$$\Phi_{\varepsilon,a,R_{n_k}} \rightharpoonup \Phi_{\varepsilon,a} \quad \text{as } k \to \infty, \text{ in the weak-}* \text{ topology.}$$

Since for every $n > 0$,

$$\int_{\mathbb{R}_*} \Phi_{\varepsilon,a,R_n}(dx) \le C_{\varepsilon,a},$$

we conclude, by passing to the limit as n tends to infinity, that

$$\int_{\mathbb{R}_*} \Phi_{\varepsilon,a}(dx) \leq C_{\varepsilon,a}. \tag{4.48}$$

We would like to show that $\Phi_{\varepsilon,a}$ satisfies equation (4.46). Since $\Phi_{\varepsilon,a,R_n} \rightharpoonup \Phi_{\varepsilon,a}$ as $n \to \infty$ in the weak-$*$ topology, we immediately conclude that, for every $\varphi \in C_c^1(\mathbb{R}_*)$,

$$\begin{aligned}
\int_{\mathbb{R}_*} \varphi(x)\Phi_{\varepsilon,a,R_n}(dx) &\to \int_{\mathbb{R}_*} \varphi(x)\Phi_{\varepsilon,a}(dx) && \text{as } n \to \infty,\\
\int_{\mathbb{R}_*} \Xi_\varepsilon(x)\varphi(x)\Phi_{\varepsilon,a,R_n}(dx) &\to \int_{\mathbb{R}_*} \Xi_\varepsilon(x)\varphi(x)\Phi_{\varepsilon,a}(dx) && \text{as } n \to \infty,\\
\int_{\mathbb{R}_*} \Xi_\varepsilon(x)x\varphi'(x)\Phi_{\varepsilon,a,R_n}(dx) &\to \int_{\mathbb{R}_*} \Xi_\varepsilon(x)x\varphi'(x)\Phi_{\varepsilon,a}(dx) && \text{as } n \to \infty.
\end{aligned}$$

It is not straightforward to conclude that

$$\begin{aligned}
&\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_a(x,y)\big[\zeta_{R_n}(x+y)\varphi(x+y) - \varphi(x) - \varphi(y)\big]\Phi_{\varepsilon,a,R_n}(dx)\Phi_{\varepsilon,a,R_n}(dy)\\
&\quad\to \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_a(x,y)\big[\varphi(x+y) - \varphi(x) - \varphi(y)\big]\Phi_{\varepsilon,a}(dx)\Phi_{\varepsilon,a}(dy) \quad \text{as } n \to \infty.
\end{aligned}$$

The main difficulty lies in the fact that the function

$$(x,y) \mapsto K_a(x,y)\big[\zeta_{R_n}(x+y)\varphi(x+y) - \varphi(x) - \varphi(y)\big]$$

has not, in general, a compact support. Here, the estimate (4.40) can be used. The details of the proof are shown in [10, proof of Theorem 2.3] and we omit them here. We conclude that $\Phi_{\varepsilon,a}$ is a solution of equation (4.46).

An adaptation of the proof of Lemma 4.9 allows us to conclude that $\Phi_{\varepsilon,a} \ll \mathcal{L}$. Indeed, if we consider the test function defined by (4.36), then for any $p, q \in \mathbb{R}_*$ such that $1/p + 1/q = 1$, we deduce that for some $C(\varepsilon, a, \gamma) > 0$ we have

$$\frac{2}{1-\gamma}\left|\int_{\mathbb{R}_*} \chi(x)\Phi_{\varepsilon,a}(dx)\right| \leq C(\varepsilon,a,\gamma)\left\|\frac{1}{z}\right\|_{L^p(\text{supp}(\chi))}\|\chi\|_{L^q(\text{supp}(\chi))}.$$

By the density of $C_c(\mathbb{R}_*)$ in $L^q(\mathbb{R}_*)$ we conclude that for any q and any compact set $\mathcal{K}$,

$$\left|\int_{\mathbb{R}_*} \chi(x)\Phi_{\varepsilon,a}(dx)\right| \leq C(\mathcal{K},\gamma,\varepsilon,a)\|\chi\|_{L^q(\mathcal{K})} \quad \forall \chi \in L^q(\mathcal{K}).$$

This implies that the measure $\Phi_{\varepsilon,a}$ is absolutely continuous with respect to the Lebesgue measure.

Let us prove estimate (4.47). First of all, as in the proof of Lemma 4.10, we can conclude that, if $\Phi_{\varepsilon,a}$ satisfies (4.46), then the density $\phi_{a,\varepsilon}$ satisfies

$$\tilde{J}_{\phi_{\varepsilon,a}}(z) = \int_0^z x\eta_\varepsilon(dx) - \int_0^z x\phi_{\varepsilon,a}(x)\,dx + \frac{2}{1-\gamma}\Xi_\varepsilon(z)z^2\phi_{\varepsilon,a}(z) \tag{4.49}$$

for almost every $z > 0$ and where

$$\tilde{J}_{\phi_{\varepsilon,a}}(z) := \int_0^z \int_{z-x}^{\infty} K_a(x,y) x \phi_{\varepsilon,a}(x)\, dx\, \phi_{\varepsilon,a}(y)\, dy. \tag{4.50}$$

From (4.49), it follows that for almost every $z > 0$,

$$\tilde{J}_{\phi_{\varepsilon,a}}(z) \leq 1 + \frac{2}{1-\gamma} z^2 \phi_{\varepsilon,a}(z). \tag{4.51}$$

Noting that $[2z/3, z]^2 \subset \Omega_z$, where Ω_z is defined by (4.42), as well as the condition on the kernel (3.1), we write

$$\tilde{J}_{\phi_{\varepsilon,a}}(z) \geq cz \min\{z^\gamma, a\} \left(\int_{\frac{2z}{3}}^{z} \phi_{\varepsilon,a}(x)\, dx \right)^2 \tag{4.52}$$

for some constant $c > 0$ independent of a and ε.

Combining (4.51) and (4.52) we conclude that

$$\int_{\frac{2z}{3}}^{z} \phi_{\varepsilon,a}(x)\, dx \leq \left(\frac{1}{c}\right)^{1/2} \left(\frac{1 + \frac{2}{1-\gamma} z^2 \phi_{\varepsilon,a}(z)}{z \min\{z^\gamma, a\}} \right)^{\frac{1}{2}}, \quad \text{a.e. } z > 0. \tag{4.53}$$

Since $z^2 \phi_{\varepsilon,a}(z) \geq 0$, by integrating (4.53) over $[w, 2w]$, with $w \geq 0$, we obtain that there exists a constant $\tilde{c}(\gamma) > 0$ such that

$$\int_w^{2w} \int_{\frac{2z}{3}}^{z} \phi_{\varepsilon,a}(x)\, dx\, dz \leq \tilde{c}(\gamma) \left(\int_w^{2w} \left(\frac{1}{z \min\{z^\gamma, a\}} \right)^{\frac{1}{2}} dz + \int_w^{2w} \left(\frac{z^2 \phi_{\varepsilon,a}(z)}{z \min\{z^\gamma, a\}} \right)^{\frac{1}{2}} dz \right).$$

By the Cauchy–Schwarz inequality we conclude that

$$\int_w^{2w} \left(\frac{1}{z \min\{z^\gamma, a\}} \right)^{\frac{1}{2}} dz \leq (\ln 2)^{1/2} \left(\int_w^{2w} \frac{1}{\min\{z^\gamma, a\}}\, dz \right)^{1/2}$$

and

$$\int_w^{2w} \left(\frac{z \phi_{\varepsilon,a}(z)}{\min\{z^\gamma, a\}} \right)^{\frac{1}{2}} dz \leq \left(\int_w^{2w} \frac{1}{\min\{z^\gamma, a\}}\, dz \right)^{1/2}.$$

Combining all the above inequalities we conclude that there exists a constant $c(\gamma) > 0$ such that

$$\int_w^{2w} \int_{\frac{2z}{3}}^{z} \phi_{\varepsilon,a}(x)\, dx\, dz \leq c(\gamma) \left(\int_w^{2w} \frac{1}{\min\{z^\gamma, a\}}\, dz \right)^{1/2}.$$

Moreover, by observing that $[\frac{8w}{9}, w] \times [w, \frac{4w}{3}] \subset \{(x,z) \in \mathbb{R}_*^2 : \frac{2z}{3} < x < z,\ z \in [w, 2w]\}$ we deduce that

$$\int_w^{4w/3} \int_{8w/9}^{w} \phi_{\varepsilon,a}(x)\, dx\, dz \geq \int_w^{4w/3} \int_{8w/9}^{w} \phi_{\varepsilon,a}(x)\, dx\, dz = w/3 \int_{8w/9}^{w} \phi_{\varepsilon,a}(x)\, dx$$

and, consequently, adopting the notation $\widetilde{C} = 3c(\gamma)$ we conclude that for any $w > 0$,

$$w \int_{8w/9}^{w} \phi_{\varepsilon,a}(x)\, dx \le \widetilde{C} \Big(\int_{w}^{2w} \frac{1}{\min\{z^{\gamma}, a\}}\, dz \Big)^{1/2}.$$

Notice that $\widetilde{C}$ is independent of a, ε.

If $a^{1/\gamma} \notin [w, 2w]$, then

$$w \int_{8w/9}^{w} \phi_{\varepsilon,a}(x)\, dx \le 2^{1/2} \widetilde{C} \max\Big\{1, \Big(\frac{1}{1-\gamma}\Big)^{1/2}\Big\} \Big(\frac{w}{\min\{a, w^{\gamma}\}}\Big)^{1/2},$$

whereas if $a^{1/\gamma} \in [w, 2w]$, then

$$\begin{aligned} w \int_{8w/9}^{w} \phi_{\varepsilon,a}(x)\, dx &\le \widetilde{C} \Big(\int_{w}^{a^{1/\gamma}} \frac{1}{z^{\gamma}}\, dz + \int_{a^{1/\gamma}}^{2w} \frac{1}{a}\, dz \Big)^{1/2} \\ &\le \widetilde{C} \Big(\frac{a^{1/\gamma - 1}}{1-\gamma} + \frac{2w}{a} \Big)^{1/2} \\ &\le 2^{1/2} \widetilde{C} \Big(\frac{w}{a(1-\gamma)} + \frac{w}{a} \Big)^{1/2} \\ &\le 2^{1/2} \widetilde{C} \max\Big\{1, \Big(\frac{1}{1-\gamma}\Big)^{1/2}\Big\} \Big(\frac{w}{\min\{a, w^{\gamma}\}}\Big)^{1/2}. \end{aligned}$$

The statement of the lemma follows by selecting $C = 2^{1/2} \widetilde{C} \max\{1, (\frac{1}{1-\gamma})^{1/2}\}$. ∎

In the following definition we explain how we truncate the coagulation kernel K to obtain a bounded coagulation kernel. Let us adopt the notation

$$p := \max\big\{\lambda, -(\gamma + \lambda)\big\}. \tag{4.54}$$

Each homogeneous coagulation kernel K of parameters γ, λ and with homogeneity γ can be written as

$$K(x, y) = (x + y)^{\gamma} F\Big(\frac{x}{x+y}\Big), \tag{4.55}$$

with $F\colon (0,1) \to \mathbb{R}_+$ being a smooth function such that

$$F(s) = F(1-s) \quad \text{and} \quad \frac{C_1}{s^p (1-s)^p} \le F(s) \le \frac{C_2}{s^p (1-s)^p} \tag{4.56}$$

for any $s \in (0,1)$ and for some constants C_1, C_2 satisfying $0 < C_1 \le C_2 < \infty$.

Definition 4.14. Assume K is a homogeneous coagulation kernel of parameters $\gamma, \lambda \in \mathbb{R}$ and homogeneity γ. We say that K_a is the bounded coagulation kernel corresponding to K and of bound $a > 0$ if

$$K_a(x, y) := 1/a + \min\big\{(x+y)^{\gamma}, a\big\} F_a\Big(\frac{x}{x+y}\Big), \quad x, y \in \mathbb{R}_*,$$

where F_a is a smooth non-negative function such that

$$F_a(s) := \begin{cases} F(s) & \text{if } F(s) \leq Aa^\sigma, \\ 0 & \text{if } F(s) \geq Aa^\sigma, \end{cases}$$

where $A > 0$ is a constant independent of a, $\sigma = 0$ if $p \leq 0$ while $\sigma > 0$ if $p > 0$ and $\gamma \leq 0$ and, finally, $0 < \sigma < \frac{p}{\gamma}$ if $p > 0$ and $\gamma > 0$.

This definition is taken from [10] and even if it might seem odd, it allows us to pass to the limit as a goes to infinity in (4.46). The main properties of this truncation of the kernel, which motivated us to introduce Definition 4.14, are exposed in the following lemma.

Lemma 4.15. *Let $\{a_n\} \subset \mathbb{R}_*$ such that $a_n \to \infty$ as $n \to \infty$. For every n, let K_{a_n} be a bounded coagulation kernel of bound a_n, corresponding to a homogeneous coagulation kernel K of parameters γ and λ, with homogeneity $\gamma < 1$ and $|\gamma + 2\lambda| < 1$. Let $\mathcal{C}$ be a compact subset of $\mathbb{R}_*$ and $M > 0$. For each $x \in \mathcal{C}$ and $y > M$ it holds that*

(1) *if $\gamma, p \leq 0$, then*

$$\min\{(x+y)^\gamma, a_n\} F_{a_n}\left(\frac{x}{x+y}\right) \leq c_3, \quad c_3 > 0;$$

(2) *if $p \leq 0$ and $\gamma > 0$, then*

$$\begin{aligned} &\min\{(x+y)^\gamma, a_n\} F_{a_n}\left(\frac{x}{x+y}\right) \\ &\quad \leq c_4(y^{-\lambda} + y^{\gamma+\lambda})\chi_{\{y \leq a_n{}^{1/\gamma}\}}(y) \\ &\qquad + c_4 a_n (y^\lambda + y^{-\gamma-\lambda})\chi_{\{y > a_n{}^{1/\gamma}\}}(y), \quad c_4 > 0; \end{aligned}$$

(3) *if $p > 0$ then*

$$\begin{aligned} &\min\{(x+y)^\gamma, a_n\} F_{a_n}\left(\frac{x}{x+y}\right) \\ &\quad \leq c_5(y^{-\lambda} + y^{\gamma+\lambda})\chi_{\{y \leq c_* a_n{}^{-\sigma/p}\}}(y), \quad c_5,\ c_* > 0. \end{aligned}$$

Let Φ_{ε,a_n} be a solution of (4.46), with respect to K_{a_n} and η_ε. Then, for every $\varphi \in C_c(\mathbb{R}_*)$, we have

$$\int_{\mathbb{R}_*}\int_{(M,\infty)} \min\{(x+y)^\gamma, a_n\} F_{a_n}\left(\frac{x}{x+y}\right)\varphi(y)\Phi_{\varepsilon,a_n}(dx)\Phi_{\varepsilon,a_n}(dy) \to 0 \tag{4.57}$$

as $M \to \infty$.

For the proof of Lemma 4.15 we refer to [10, proof of Theorem 2.3]. The main idea is that Definition 4.14 allows us to prove the inequalities presented in the lemma, which imply (4.57).

Lemma 4.16. *Assume K is a homogeneous symmetric coagulation kernel $K \in C(\mathbb{R}_* \times \mathbb{R}_*)$ satisfying (3.1) with homogeneity $\gamma < 1$ and $|\gamma + 2\lambda| < 1$. Consider a sequence $\{a_n\} \subset \mathbb{R}_*$ such that $\lim_{n\to\infty} a_n = \infty$ and the sequence of bounded coagulation kernels $\{K_{a_n}\}$ corresponding to K. Let Φ_{ε,a_n} be a solution of (4.46) with respect to η_ε, Ξ_ε and with respect to K_{a_n}. There exists a measure Φ_ε such that*

$$\Phi_{\varepsilon,a_n} \rightharpoonup \Phi_\varepsilon \quad \text{as } n \to \infty \text{, in the weak-}* \text{ topology.} \tag{4.58}$$

The measure Φ_ε is absolutely continuous with respect to the Lebesgue measure, with density ϕ_ε. It satisfies the bounds

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi_\varepsilon(dx) < \infty, \quad \int_{\mathbb{R}_*} x^{-\lambda}\Phi_\varepsilon(dx) < \infty \tag{4.59}$$

and

$$\frac{1}{z}\int_{[8z/9,z]} \Phi_\varepsilon(dx) \le \left(\frac{C}{z^{3+\gamma}}\right)^{\frac{1}{2}}, \quad z > 0. \tag{4.60}$$

Moreover, it solves for every $\varphi \in C_c(\mathbb{R}_*)$ the equation

$$\begin{aligned}\int_0^\infty \Phi_\varepsilon(dx)\varphi(x) &= \int_0^\infty \varphi(x)\eta_\varepsilon(dx) + \frac{2}{1-\gamma}\int_0^\infty \Xi_\varepsilon(x)(\varphi(x) - x\varphi'(x))\Phi_\varepsilon(dx)\,dx \\ &\quad + \frac{1}{2}\int_0^\infty\int_0^\infty K(x,y)\big[\varphi(x+y) - \varphi(x) - \varphi(y)\big]\Phi_\varepsilon(dx)\Phi_\varepsilon(dy).\end{aligned} \tag{4.61}$$

Proof. We know that $\Phi_{\varepsilon,a}((0,\varepsilon]) = 0$. Therefore inequality (4.47) is non-trivial when $z > \varepsilon$. Let us consider the case $\gamma \le 0$. In this case we have $z^\gamma \le \varepsilon^\gamma$. Since ε is fixed and $a_n \to \infty$ as $n \to \infty$, there exists an $\bar{n}$ such that $z^\gamma < a_n$ for every $n \ge \bar{n}$.

Consequently, for every $n \ge \bar{n}$, we conclude that

$$\frac{1}{z}\int_{[8z/9,z]} \Phi_{\varepsilon,a_n}(dx) \le \frac{C^{1/2}}{z^{(\gamma+3)/2}}.$$

Applying Lemma 4.11 to the rescaled measure $x^{\gamma+\lambda}\Phi_{\varepsilon,a_n}(dx)$ we conclude that, if $n \ge \bar{n}$, then

$$\int_{[\varepsilon,\infty)} x^{\gamma+\lambda}\Phi_{\varepsilon,a_n}(dx) \le C^{1/2}\int_{[\varepsilon,\infty)} z^{-(\gamma+3)/2} z^{\gamma+\lambda}\,dz < C_\varepsilon,$$

where C_ε is a constant which depends only on ε and where the last inequality comes from the bound $|\gamma + 2\lambda| < 1$.

Since $\Phi_{\varepsilon,a}((0,\varepsilon]) = 0$, the sequence of rescaled measures $\{x^{\gamma+\lambda}\Phi_{\varepsilon,a_n}(dx)\}_{n\ge\bar{n}}$ belongs to a compact set, and we conclude by the Banach–Alaoglu theorem that there exists a subsequence of $\{x^{\gamma+\lambda}\Phi_{\varepsilon,a_n}(dx)\}$ which converges, in the weak-$*$ topology, to a measure μ. This implies that, if we denote by Φ_ε the measure defined by $\Phi_\varepsilon(dx) := x^{-(\gamma+\lambda)}\mu(dx)$, we have, up to a subsequence, that

$$\Phi_{\varepsilon,a_n} \rightharpoonup \Phi_\varepsilon$$

as $n \to \infty$, in the weak-$*$ topology.

Let us now consider the case $0 < \gamma < 1$. If $z^\gamma > a$ it holds that

$$\frac{1}{z}\int_{[8z/9,z]} \Phi_{\varepsilon,a}(dx) \leq \frac{C^{1/2}}{az^{3/2}}.$$

If, instead, $z^\gamma \leq a$, then

$$\frac{1}{z}\int_{[8z/9,z]} \Phi_{\varepsilon,a}(dx) \leq \frac{C^{1/2}}{z^{(3+\gamma)/2}}.$$

By applying Lemma 4.11 to the scaled measure Φ_{ε,a_n} with $a_n \geq 1$, we conclude that

$$\int_{[\varepsilon,\infty)} \Phi_{\varepsilon,a_n}(dx) \leq C^{1/2}(\varepsilon^{-1/2} + \varepsilon^{(-1+\gamma)/2}).$$

Therefore, the sequence $\{\Phi_{\varepsilon,a_n}\}$ belongs to a compact set, and $\Phi_{\varepsilon,a_n} \rightharpoonup \Phi_\varepsilon$ as $n \to \infty$, in the weak-$*$ topology.

Passing to the limit as $a \to \infty$ in inequality (4.47) we obtain

$$\frac{1}{z}\int_{[8z/9,z]} \Phi_\varepsilon(dx) \leq \left(\frac{C}{z^{3+\gamma}}\right)^{\frac{1}{2}}, \quad z > 0.$$

By applying Lemma 4.11 with $g(z) = z^{-\frac{3+\gamma}{2}}$ and using the assumption $|\gamma + 2\lambda| < 1$, we deduce that

$$\int_{\mathbb{R}_*} x^{-\lambda}\Phi_\varepsilon(dx) < \infty \quad \text{and} \quad \int_{\mathbb{R}_*} x^{\lambda+\gamma}\Phi_\varepsilon(dx) < \infty.$$

Let us pass to the limit as n tends to infinity in all the terms of equation (4.46).

From the weak-$*$ convergence of Φ_{ε,a_n} to Φ_ε, we conclude that, for every $\varphi \in C_c^1(\mathbb{R}_*)$,

$$\begin{aligned}
\int_{\mathbb{R}_*} \varphi(x)\Phi_{\varepsilon,a_n}(dx) &\to \int_{\mathbb{R}_*} \varphi(x)\Phi_\varepsilon(dx), && n \to \infty, \\
\int_{\mathbb{R}_*} \Xi_\varepsilon(x)x\varphi'(x)\Phi_{\varepsilon,a_n}(dx) &\to \int_{\mathbb{R}_*} \Xi_\varepsilon(x)x\varphi'(x)\Phi_\varepsilon(dx), && n \to \infty, \\
\int_{\mathbb{R}_*} \Xi_\varepsilon(x)\varphi(x)\Phi_{\varepsilon,a_n}(dx) &\to \int_{\mathbb{R}_*} \Xi_\varepsilon(x)\varphi(x)\Phi_\varepsilon(dx), && n \to \infty.
\end{aligned}$$

For the proof of the fact that, for every $\varphi \in C_c(\mathbb{R}_*)$,

$$\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_{a_n}(x,y)\big[\varphi(x+y) - \varphi(y) - \varphi(x)\big]\Phi_{\varepsilon,a_n}(dx)\Phi_{\varepsilon,a_n}(dy)$$

converges, as n tends to infinity, to

$$\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\big[\varphi(x+y)-\varphi(y)-\varphi(x)\big]\Phi_\varepsilon(dx)\Phi_\varepsilon(dy),$$

we refer to [10, proof of Theorem 2.3]. Nevertheless, we find it instructive to provide the main steps of the proof here. First of all, since K_a and K are continuous functions we have, for any compact subset of ${\mathbb{R}_*}^2$, $(xy)^q K_a(x,y) \to (xy)^q K(x,y)$, uniformly in x, y as $a \to \infty$ for any $q \in \mathbb{R}$. Consequently, for any $\varphi \in C_c(\mathbb{R}_*)$,

$$\begin{aligned}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} & K_{a_n}(x,y)\varphi(x+y)\Phi_{\varepsilon,a_n}(dx)\Phi_{\varepsilon,a_n}(dy)\\ &\to \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\varphi(x+y)\Phi_\varepsilon(dx)\Phi_\varepsilon(dy)\end{aligned}$$

as $n\to\infty$. For the same reason we know that

$$\begin{aligned}\int_{\mathbb{R}_*}\int_{(0,M]} & K_{a_n}(x,y)\varphi(x)\Phi_{\varepsilon,a_n}(dy)\Phi_{\varepsilon,a_n}(dx)\\ &\to \int_{\mathbb{R}_*}\int_{(0,M]} K(x,y)\varphi(x)\Phi_\varepsilon(dy)\Phi_\varepsilon(dx)\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}_*}\int_{(0,M]} & K_{a_n}(x,y)\varphi(y)\Phi_{\varepsilon,a_n}(dx)\Phi_{\varepsilon,a_n}(dy)\\ &\to \int_{\mathbb{R}_*}\int_{(0,M]} K(x,y)\varphi(y)\Phi_\varepsilon(dx)\Phi_\varepsilon(dy).\end{aligned}$$

To conclude we just need now to show that, as $M\to\infty$,

$$\int_{\mathbb{R}_*}\int_{(M,\infty)} K_{a_n}(x,y)\varphi(y)\Phi_{\varepsilon,a_n}(dx)\Phi_{\varepsilon,a_n}(dy)\to 0$$

and

$$\int_{\mathbb{R}_*}\int_{(M,\infty)} K_{a_n}(x,y)\varphi(x)\Phi_{\varepsilon,a_n}(dy)\Phi_{\varepsilon,a_n}(dx)\to 0.$$

Notice that, by the definition of K_{a_n},

$$\begin{aligned}&\int_{\mathbb{R}_*}\int_{(M,\infty)} K_{a_n}(x,y)\varphi(x)\Phi_{\varepsilon,a_n}(dy)\Phi_{\varepsilon,a_n}(dx)\\ &\quad\le \frac{1}{a_n}\int_{\mathbb{R}_*}\int_{(M,\infty)} \varphi(x)\Phi_{\varepsilon,a_n}(dy)\Phi_{\varepsilon,a_n}(dx)\\ &\qquad+\int_{\mathbb{R}_*}\int_{(M,\infty)} \min\{(x+y)^\gamma,a\}F_{a_n}\Big(\frac{x}{x+y}\Big)\varphi(x)\Phi_{\varepsilon,a_n}(dy)\Phi_{\varepsilon,a_n}(dx).\end{aligned}\tag{4.62}$$

Thanks to (4.40), we have

$$\int_{\mathbb{R}_*}\int_{(M,\infty)} \varphi(x)\Phi_{\varepsilon,a_n}(dy)\Phi_{\varepsilon,a_n}(dx)\le cM^{-1/2},$$

where c is just a positive constant.

By Lemma 4.15 we know that

$$\int_{\mathbb{R}_*}\int_{(M,\infty)} \min\{(x+y)^\gamma, a\} F_{a_n}\Big(\frac{x}{x+y}\Big)\varphi(x)\Phi_{\varepsilon,a_n}(dy)\Phi_{\varepsilon,a_n}(dx) \to 0$$

as $M \to \infty$. The same holds also for the term

$$\int_{\mathbb{R}_*}\int_{(M,\infty)} \min\{(x+y)^\gamma, a\} F_{a_n}\Big(\frac{x}{x+y}\Big)\varphi(y)\Phi_{\varepsilon,a_n}(dx)\Phi_{\varepsilon,a_n}(dy).$$

We conclude that Φ_ε satisfies the following equation for every $\varphi \in C^1(\mathbb{R}_*)$:

$$\int_0^\infty \Phi_\varepsilon(dx)\varphi(x) = \int_0^\infty \varphi(x)\eta_\varepsilon(dx) + \frac{2}{1-\gamma}\int_0^\infty \Xi_\varepsilon(x)(\varphi(x) - x\varphi'(x))\Phi_\varepsilon(dx)$$
$$+\frac{1}{2}\int_0^\infty\int_0^\infty K(x,y)[\varphi(x+y)-\varphi(x)-\varphi(y)]\Phi_\varepsilon(dx)\Phi_\varepsilon(dy).$$

As in the proof of Lemma 4.9, we can choose the test function φ given by expression (4.36) to conclude that, for any $p, q \in \mathbb{R}_*$ such that $1/p + 1/q = 1$, we have

$$\frac{2}{1-\gamma}\left|\int_{\mathbb{R}_*} \chi(x)\Xi_\varepsilon(x)\Phi_\varepsilon(dx)\right|$$
$$\leq \frac{3}{2}\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi_\varepsilon(dx)\int_{\mathbb{R}_*} x^{-\lambda}\Phi_\varepsilon(dx)\left\|\frac{1}{z}\right\|_{L^p(U)}\|\chi\|_{L^q(U)}$$
$$+\eta_\varepsilon(\mathbb{R}_*)\left\|\frac{1}{z}\right\|_{L^p(U)}\|\chi\|_{L^q(U)} + \frac{|1+\gamma|}{1-\gamma}\Phi_\varepsilon(\mathbb{R}_*)\left\|\frac{1}{z}\right\|_{L^p(U)}\|\chi\|_{L^q(U)},$$

where we are using the notation $\operatorname{supp}(\chi) := U$. By the density of $C_c(\mathbb{R}_*)$ in $L^q(\mathbb{R}_*)$ we conclude that for any q and any compact set $\mathcal{K}$,

$$\left|\int_{\mathbb{R}_*} \Xi_\varepsilon(x)\chi(x)\Phi_\varepsilon(dx)\right| \leq C(\mathcal{K},\gamma,\varepsilon,\Phi_\varepsilon)\|\chi\|_{L^q(\mathcal{K})} \quad \forall \chi \in L^q(\mathcal{K}).$$

Therefore, $\Phi_\varepsilon \ll \mathcal{L}$. ∎

We now introduce some notation, which will be employed in the following, and a lemma, taken from [10], which will be important for the proof of Theorem 3.2.

For a given $\delta > 0$, we consider the partition $\mathbb{R}_+^2 = \Sigma_1(\delta) \cup \Sigma_2(\delta) \cup \Sigma_3(\delta)$ with

$$\Sigma_1(\delta) = \{(x,y) \mid y > x/\delta\},$$
$$\Sigma_2(\delta) = \{(x,y) \mid \delta x \leq y \leq x/\delta\},$$
$$\Sigma_3(\delta) = \{(x,y) \mid y < \delta x\},$$

and, if $\mu \in \mathcal{M}_+(\mathbb{R}_*)$ is such that for every $z > 0$,

$$\int_{\Omega_z} K(x,y)x\mu(dx)\mu(dy) < \infty,$$

where Ω_z is defined by (4.42), then we define

$$J_j(z,\delta,\mu) := \int_{\Omega_z \cap \Sigma_j(\delta)} K(x,y)x\mu(dx)\mu(dy) \quad \text{for } z>0,$$

for $j = 1, 2, 3$.

Notice that

$$\sum_{j=1}^{3} J_j(z,\delta,\mu) = \int_{\Omega_z} K(x,y)x\mu(dx)\mu(dy).$$

Lemma 4.17. *Let K be a homogeneous symmetric coagulation kernel $K \in C(\mathbb{R}_* \times \mathbb{R}_*)$ satisfying (3.1) with $|\gamma + 2\lambda| < 1$. Suppose that the measure $\mu \in \mathcal{M}_+(\mathbb{R}_*)$ satisfies*

$$\frac{1}{z}\int_{[z/2,z]} \mu(dx) \leq \frac{A}{z^{(\gamma+3)/2}} \qquad \forall z>0. \tag{4.63}$$

Then, for every $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ depending on ε, as well as on γ and λ and on the constants c_1 and c_2 of inequality (3.1), but independent of A, such that for any $\delta \leq \delta_\varepsilon$ we have

$$\sup_{z>0} J_1(z,\delta,\mu) \leq \varepsilon A^2, \tag{4.64}$$

$$\sup_{R>0} \frac{1}{R}\int_{[R,2R]} J_3(z,\delta,\mu)\,dz \leq \varepsilon A^2. \tag{4.65}$$

Proposition 4.18. *Assume K, η, λ and γ are as in the assumptions of Theorem 3.2. There exists a $\Phi \in \mathcal{M}_+(\mathbb{R}_*)$ with*

$$J_\Phi \in L^\infty_{\mathrm{loc}}(\mathbb{R}_*)$$

solving

$$\int_{\mathbb{R}_*} \varphi(z)\left(J_\Phi(z)\,dz - dz + \int_{(0,z]} x\Phi(dx)\,dz - \frac{2}{1-\gamma}z^2\Phi(dz)\right) = 0 \tag{4.66}$$

for every test function $\varphi \in C_c(\mathbb{R}_*)$ and satisfying (3.3) and the inequalities

$$\int_{\mathbb{R}_*} x\Phi(dx) \leq 1, \tag{4.67}$$

$$\int_{(1,\infty)} x^p\Phi(dx) < \infty, \quad \int_{(0,1]} x^q\Phi(dx) < \infty, \tag{4.68}$$

with $q = \min\{\gamma+\lambda+1, 1-\lambda\}$ and $p = \max\{\gamma+\lambda, -\lambda\}$.

Proof. Consider the sequences $\{\varepsilon_n\}$, $\{a_m\}$ and $\{R_k\}$, with $\lim_{n\to\infty}\varepsilon_n = 0$, $\lim_{m\to\infty} a_m = \infty$ and $\lim_{k\to\infty} R_n = \infty$. Consider a sequence of measures $\{\Phi_{\varepsilon_n,a_m,R_k}\}$ that solve (4.26) with respect to ζ_{R_n}, K_{a_n} and η_{ε_n}. By Lemmas 4.13 and 4.16 we know that $\Phi_{\varepsilon_n,a_m,R_k}$ converges in the weak-$*$ topology to the absolutely continuous measure Φ_{ε_n} as m and k

go to infinity, solves (4.61) and satisfies the bound (4.60). We would like to prove that there exists a measure Φ such that $\Phi_{\varepsilon_n} \rightharpoonup \Phi$ as $n \to \infty$, in the weak-$*$ topology, and that Φ solves (4.66), (4.68) and (4.67).

Consequently, we use the following diagonal argument, which is similar to the one used in [10, Section 7]. We notice that if $I_k := [2^{-k}, 2^k]$, then $\mathbb{R}_* = \bigcup_{k=1}^{\infty} I_k$. The restricted sequence $\{\Phi_{\varepsilon_n | I_k}\}$ on I_k has a convergent subsequence $\{\Phi_{\varepsilon_{n_l} | I_k}\}$. Since if $k > m$, then $I_m \subset I_k$, we can use a diagonal argument to conclude that, up to a subsequence, there exists a measure $\Phi \in \mathcal{M}_+(\mathbb{R}_*)$ such that $\Phi_{\varepsilon_n} \rightharpoonup \Phi$ as $n \to \infty$, in the weak-$*$ topology.

Since Φ is the weak-$*$ limit of Φ_{ε_n} as $n \to \infty$ thanks to (4.30), we know that

$$\int_{\mathbb{R}_*} x \Phi_{\varepsilon_n}(dx) \leq 1.$$

Passing to the limit for $n \to \infty$ we deduce (4.67). Similarly, passing to the limit in (4.59) we deduce (3.3).

Thanks to inequality (3.3), Lemma 4.11 and the assumption $|\gamma + 2\lambda| < 1$ we can prove (4.68).

We aim now to show that $J_\Phi \in L^\infty_{\text{loc}}(\mathbb{R}_*)$. Notice that

$$J_\Phi(z) = \int_{(z,\infty)} \int_{(0,z]} x K(x,y) \Phi(dx)\Phi(dy) + \int_{(0,z]} \int_{(z-y,z]} x K(x,y) \Phi(dx)\Phi(dy),$$

and using (4.68) and (3.3) we deduce that for every $z > 0$,

$$\begin{aligned}\int_{(z,\infty)} \int_{(0,z]} x K(x,y) \Phi(dx)\Phi(dy) &\leq c_2 \int_{(z,\infty)} y^{-\lambda} \Phi(dy) \int_{(0,z]} x^{1+\gamma+\lambda} \Phi(dx) \\ &\quad + c_2 \int_{(z,\infty)} y^{\gamma+\lambda} \Phi(dy) \int_{(0,z]} x^{1-\lambda} \Phi(dx) \\ &< c_5 < \infty,\end{aligned}$$

where c_5 is a positive constant depending on γ and λ and independent of z.

On the other hand,

$$\begin{aligned}\int_{(0,z]} \int_{(z-y,z]} x K(x,y) \Phi(dx)\Phi(dy) &\leq c_2 \int_{(0,z]} \int_{(z-y,z]} x^{1-\lambda} \Phi(dx) y^{\gamma+\lambda} \Phi(dy) \\ &\quad + c_2 \int_{(0,z]} \int_{(z-y,z]} x^{1+\lambda+\gamma} \Phi(dx) y^{-\lambda} \Phi(dy).\end{aligned}$$

Again, from (3.3) we conclude, applying Lemma 4.11, that

$$\int_{(z-y,z]} x^{1-\lambda} \Phi(dx) \leq c \int_{z-y}^{z} x^{1-\lambda-(3+\gamma)/2} \, dx = z^{(1-2\lambda-\gamma)/2} - (z-y)^{(1-2\lambda-\gamma)/2}.$$

Notice that for every $p > 0$, if $x \geq y$, then

$$(x+y)^p - x^p \leq C(p) x^{p-1} y, \tag{4.69}$$

while if $x \leq y$,

$$(x+y)^p - x^p \leq c(p) y^p, \tag{4.70}$$

where $c(p)$ and $C(p)$ are two positive constants. We conclude that if $y \leq \frac{z}{2}$, we have

$$\int_{(z-y,z]} x^{1-\lambda} \Phi(dx) \leq z^{(-1-2\lambda-\gamma)/2} y.$$

If instead $y > \frac{z}{2}$, we obtain

$$\int_{(z-y,z]} x^{1-\lambda} \Phi(dx) \leq y^{(1-2\lambda-\gamma)/2}.$$

Consequently, for every $z > 0$ we have

$$\int_{(0,z/2]} \int_{(z-y,z]} x^{1-\lambda} \Phi(dx) y^{\gamma+\lambda} \Phi(dy) \leq z^{(-1-2\lambda-\gamma)/2} \int_{(0,z]} y^{1+\gamma+\lambda} \Phi(dy) < c_6 < \infty,$$

where $c_6 > 0$ is independent of z, and

$$\int_{(z/2,z]} \int_{(z-y,z]} x^{1-\lambda} \Phi(dx) y^{\gamma+\lambda} \Phi(dy) < \int_{(z/2,z]} y^{(\gamma+1)/2} \Phi(dy) < c z^{\frac{\gamma+3}{2}} < \infty,$$

with $c(z) > 0$ for every z.

We conclude that, if $\mathcal{K}$ is a compact subset of $\mathbb{R}_*$, then there exists a constant $c > 0$ such that

$$\sup_{z \in \mathcal{K}} \int_{(0,z]} \int_{(z-y,z]} x^{1-\lambda} \Phi(dx) y^{\gamma+\lambda} \Phi(dy) < c < \infty.$$

Similarly, it is possible to prove that there exists another constant $c > 0$ such that

$$\int_{(0,z]} \int_{(z-y,z]} x^{1+\lambda+\gamma} \Phi(dx) y^{-\lambda} \Phi(dy) < c < \infty$$

and we conclude that $J_\Phi \in L^\infty_{\text{loc}}(\mathbb{R}_*)$.

We now prove that Φ solves (4.66) using a similar argument to [10, Section 7]. First of all we prove that the measure Φ_{ε_n} satisfies, for every test function $\varphi \in C_c(\mathbb{R}_*)$, the equality

$$\begin{aligned} \int_{\mathbb{R}_*} \varphi(z) J_{\Phi_{\varepsilon_n}}(z)\, dz = \int_{\mathbb{R}_*} \varphi(z) \int_{(0,z]} x \eta_{\varepsilon_n}(dx)\, dz - \int_{\mathbb{R}_*} \varphi(z) \int_{(0,z]} x \Phi_{\varepsilon_n}(dx)\, dz \\ + \frac{2}{1-\gamma} \int_{\mathbb{R}_*} \Xi_\varepsilon(z) z^2 \varphi(z) \Phi_{\varepsilon_n}(dz). \end{aligned} \tag{4.71}$$

Since the measure Φ_ε satisfies (4.61), we conclude, as in Lemma 4.10, that the density ϕ_{ε_n} of Φ_{ε_n} satisfies

$$J_{\phi_{\varepsilon_n}}(z) = \int_0^z x \eta_{\varepsilon_n}(x)\, dx - \int_0^z x \phi_{\varepsilon_n}(x)\, dx + \frac{2}{1-\gamma} \Xi_\varepsilon(z) z^2 \phi_{\varepsilon_n}(z) \quad \text{a.e. } z > 0.$$

Integrating against a test function $\varphi \in C_c(\mathbb{R}_*)$ we conclude that equation (4.71) holds. We aim now to pass to the limit as n goes to infinity in equation (4.71).

We plan to show that estimate (4.60) implies that Φ satisfies the hypothesis of Lemma 4.17. To this end, we only need to ensure that

$$\frac{1}{z}\int_{z/2}^{z} \phi_{\varepsilon_n}(x)\,dx \le \left(\frac{C}{z^{3+\gamma}}\right)^{\frac{1}{2}}, \quad z>0, \tag{4.72}$$

for some constant $C>0$.

Notice that there exists $m>0$ such that $[z/2,z]\subset \bigcup_{i=1}^{m}[(8/9)^i z,(8/9)^{i-1}z]$. Since for every i,

$$\frac{1}{z}\int_{(8/9)^i z}^{(8/9)^{i-1}z} \phi_{\varepsilon_n}(x)\,dx = \frac{1}{z}\left(\frac{8}{9}\right)^{i-1}\int_{z}^{8z/9} \phi_{\varepsilon_n}(y)\,dy \le \left(\frac{C}{z^{3+\gamma}}\right)^{\frac{1}{2}}, \quad z>0,$$

then (4.72) holds and we can apply Lemma 4.17 to conclude that for any $\bar{\varepsilon}>0$ there is a $\delta_0>0$ depending on $\bar{\varepsilon}$ and γ, such that for any $\delta\le\delta_0$ and any $\varphi\in C_c(\mathbb{R}_*)$,

$$\left|\sum_{j\in\{1,3\}}\int_{(0,\infty)} J_j(z,\delta,\Phi_{\varepsilon_n})\varphi(z)\,dz\right| \le C\bar{\varepsilon}\|\varphi\|_{L^\infty(0,\infty)}. \tag{4.73}$$

Since φ is compactly supported and for every compact set K the set $\bigcup_{z\in K}\Sigma_2\cap\Omega_z$ is bounded, using the fact that Φ_{ε_n} converges to Φ in the weak-$*$ topology, we have the following limits as $n\to\infty$:

$$\begin{aligned}
\int_0^\infty J_2(z,\delta,\Phi_{\varepsilon_n})\varphi(z)\,dz &\to \int_0^\infty J_2(z,\delta,\Phi)\varphi(z)\,dz,\\
\int_0^\infty\int_{(0,z)} x\Phi_{\varepsilon_n}(dx)\varphi(z)\,dz &\to \int_0^\infty\int_{(0,z)} x\Phi(dx)\varphi(z)\,dz,\\
\int_0^\infty\int_{(0,z)} \Xi_{\varepsilon_n}(x)x\Phi_{\varepsilon_n}(dx)\varphi(z)\,dz &\to \int_0^\infty\int_{(0,z)} x\Phi(dx)\varphi(z)\,dz,\\
\int_0^\infty \Xi_{\varepsilon_n}(z)z^2\varphi(z)\Phi_{\varepsilon_n}(z)\,dz &\to \int_0^\infty z^2\varphi(z)\Phi(z)\,dz.
\end{aligned}$$

These limits, together with (4.73), imply an upper estimate for the difference between the right-hand side of (4.71) and $\int_{(0,\infty)} J_2(z,\delta,\Phi)\varphi(z)\,dz$,

$$\begin{aligned}
&\Bigg|\int_0^\infty J_2(z,\delta,\Phi)\varphi(z)\,dz - \int_0^\infty \varphi(z)\,dz + \int_0^\infty\int_{(0,z]} x\Phi(dx)\varphi(z)\,dz\\
&\qquad - \frac{2}{1-\gamma}\int_0^\infty z^2\varphi(z)\Phi(z)\,dz\Bigg|\\
&\le C\bar{\varepsilon}\|\varphi\|_\infty.
\end{aligned} \tag{4.74}$$

Using (3.3), we can again apply Lemma 4.17 to Φ to conclude that

$$\left|\sum_{j\in\{1,3\}}\int_0^\infty J_j(z,\delta,\Phi)\varphi(z)\,dz\right| \le C\bar{\varepsilon}\|\varphi\|_\infty. \tag{4.75}$$

Finally, estimates (4.74) and (4.75) imply

$$\begin{aligned}\Big|\int_0^\infty \int_{\Omega_z} xK(x,y)\Phi(dx)\Phi(dy)\varphi(z)\,dz - \int_0^\infty \varphi(z)\,dz \\ + \int_0^\infty \int_{(0,z]} x\Phi(dx)\varphi(z)\,dz + \frac{2}{1-\gamma}\int_{\mathbb{R}_*} z^2\varphi(z)\Phi(dz)\Big| \\ \le C\bar{\varepsilon}\|\varphi\|_\infty\end{aligned}$$

for any $\bar{\varepsilon} > 0$ and any $\varphi \in C_c(0,\infty)$, which implies that Φ satisfies (4.66). ■

5. Moment bounds satisfied by the self-similar profile

Proposition 5.1 (Moment bounds). *Assume K, η, λ and γ are as in the assumptions of Theorem 3.2. The solution of (4.66) with respect to K, η, constructed in Proposition (4.18), satisfies for every $\mu \in \mathbb{R}$,*

$$\int_1^\infty x^\mu \Phi(dx) < \infty. \tag{5.1}$$

Proof. The bound (5.1) for $\mu < 1$ follows directly from (4.67).

We now focus on proving the bound for $\mu > 1$. Integrating (3.2) against a positive test function $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and denoting $\psi(z) := \int_0^z \varphi(x)\,dx$, we obtain

$$\begin{aligned}\frac{2}{1-\gamma}\int_{\mathbb{R}_*} z^2\varphi(z)\Phi(dz) \\ \le \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \big[\psi(x+y)-\psi(x)\big] xK(x,y)\Phi(dy)\Phi(dx).\end{aligned} \tag{5.2}$$

Let δ be a small positive constant satisfying the two conditions

$$\max\{\gamma, \gamma+\lambda, -\lambda\} + \delta \le 1 \quad \text{and} \quad 1-\lambda+\delta \le 1 \text{ if } \lambda > 0, \tag{5.3}$$

and choose $\varphi(x) = x^{\delta-1}\chi_{(1,\infty)}(x)$, where $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. Plugging φ into (5.2), we obtain

$$\begin{aligned}\frac{2}{1-\gamma}\int_{[1,\infty)} x^{1+\delta}\Phi(dx) \le \frac{1}{\delta}\int_{\mathbb{R}_*}\int_{[1,\infty)} \big[(x+y)^\delta - x^\delta\big] xK(x,y)\Phi(dx)\Phi(dy) \\ + \frac{1}{\delta}\int_{\mathbb{R}_*}\int_{(1-y,1]} (x+y)^\delta xK(x,y)\Phi(dx)\Phi(dy).\end{aligned} \tag{5.4}$$

Next we derive estimates for each term of the right-hand side. We start by dividing the domain of integration of the first term into two regions defined by $y < 1$, $x \ge 1$ and by $x, y \ge 1$, respectively.

In the region $y < 1$, $x \geq 1$, using the upper bound for the kernel (3.1) and the estimates (4.69)–(4.70), we have

$$\begin{aligned}\int_{(0,1)}\int_{[1,\infty)}\big[(x+y)^{\delta}-x^{\delta}\big]xK(x,y)\Phi(dx)\Phi(dy)\\ \leq C(\delta)\int_{(0,1)}\int_{[1,\infty)}x^{\delta}yK(x,y)\Phi(dx)\Phi(dy)\\ \leq C(\delta)c_2\int_{(0,1)}\int_{[1,\infty)}(x^{\delta+\gamma+\lambda}y^{1-\lambda}+y^{1+\gamma+\lambda}x^{\delta-\lambda})\Phi(dx)\Phi(dy),\end{aligned} \tag{5.5}$$

which is finite by (4.67)–(4.68).

In the second region $x, y \geq 1$ we use the symmetry of the kernel, as well as (4.69) and (4.70), to obtain

$$\begin{aligned}\iint_{[1,\infty)^2}\big[(x+y)^{\delta}-x^{\delta}\big]xK(x,y)\Phi(dy)\Phi(dx)\\ \leq C(\delta)\iint_{\{1\leq y\leq x\}}x^{\delta}yK(x,y)\Phi(dy)\Phi(dx)\\ +c(\delta)\iint_{\{1\leq x\leq y\}}y^{\delta}xK(x,y)\Phi(dy)\Phi(dx) \qquad (5.6)\\ =(C(\delta)+c(\delta))\iint_{\{1\leq y\leq x\}}x^{\delta}yK(x,y)\Phi(dy)\Phi(dx). \qquad (5.7)\end{aligned}$$

The upper bound for the kernel (3.1) implies

$$x^{\delta}yK(x,y)\leq c_2(x^{\delta+\gamma+\lambda}y^{1-\lambda}+y^{1+\gamma+\lambda}x^{\delta-\lambda}),$$

which yields the following estimates, for $1 \leq y \leq x$:

$$\begin{aligned}&x^{\delta}yK(x,y)\leq c_2(x^{\delta+\gamma+\lambda}y+x^{1+\delta-\lambda}y^{\gamma+\lambda}), && \lambda>0,\\ &x^{\delta}yK(x,y)\leq c_2\Big(x^{\delta+\gamma}y\Big(\frac{x}{y}\Big)^{\lambda}+x^{\delta-\lambda+\gamma+\lambda}y\Big), && \lambda\leq 0,\ \gamma+\lambda>0,\\ &x^{\delta}yK(x,y)\leq c_2\Big(x^{\delta+\gamma}y\Big(\frac{x}{y}\Big)^{\lambda}+x^{\delta-\lambda}y\Big), && \lambda\leq 0,\ \gamma+\lambda\leq 0.\end{aligned}$$

The bounds (4.67)–(4.68), together with the condition on δ (5.3), then ensure that (5.7) is finite.

We now focus on the second term of (5.4). We divide the domain of integration into two regions, defined by $y > 1$, $x \leq 1$ and by $1 - y < x \leq 1$, $y \leq 1$. In the first region $y > 1$, $x \leq 1$, it holds that

$$\int_{(1,\infty)}\int_{(0,1]}(x+y)^{\delta}xK(x,y)\Phi(dx)\Phi(dy)\leq 2^{\delta}\int_{(1,\infty)}\int_{(0,1]}y^{\delta}xK(x,y)\Phi(dx)\Phi(dy).$$

Similarly to the region $y < 1, x \geq 1$ (cf. (5.5)), this integral is finite by (3.1) and (4.67)–(4.68). In the second region $1 - y < x \leq 1$, $y \leq 1$, we notice that $(x+y)^\delta \leq 2^\delta$. Therefore,

$$\int_{(0,1]} \int_{(1-y,1]} (x+y)^\delta x K(x,y) \Phi(dx)\Phi(dy) \leq 2^\delta \int_{(0,1]} \int_{(1-x,1]} x K(x,y) \Phi(dx)\Phi(dy),$$

which is bounded by $J_\Phi(1)$, which is finite by Proposition 4.18.

Thus, we conclude that

$$\frac{2}{1-\gamma} \int_{[1,\infty)} z^{1+\delta} \Phi(dz) < \infty. \tag{5.8}$$

Now let us select $\varphi(x) = x^{n\delta - 1} \chi_{(1,\infty)}(x)$ in (5.2):

$$\begin{aligned} \frac{2}{1-\gamma} \int_{\mathbb{R}_*} x^{1+n\delta} \phi(x)\, dx \leq \frac{1}{n\delta} \int_{\mathbb{R}_*} \int_{[1,\infty)} [(x+y)^{n\delta} - x^{n\delta}] x K(x,y) \Phi(dx)\Phi(dy) \\ + \frac{1}{n\delta} \int_{\mathbb{R}_*} \int_{(1-y,1]} (x+y)^{n\delta} x K(x,y) \Phi(dx)\Phi(dy). \end{aligned}$$

We divide the domains of integration as before and conclude that, for some constant $c_{\delta,n}$ that depends on δ and n, the following estimate holds:

$$\begin{aligned} \frac{2}{1-\gamma} \int_{\mathbb{R}_*} x^{1+n\delta} \phi(x)\, dx \leq c_{\delta,n} \Bigg(\int_{(0,1)} \int_{[1,\infty)} x^{n\delta} y K(x,y) \phi(y) \phi(x)\, dx\, dy \\ + \iint_{\{x \geq y \geq 1\}} x^{n\delta} y K(x,y) \Phi(dx)\Phi(dy) \\ + \int_{(1,\infty)} \int_{(0,1]} y^{n\delta} x K(x,y) \Phi(dx)\Phi(dy) \\ + \int_{(0,1]} \int_{(1-x,1]} x K(x,y) \Phi(dx)\Phi(dy) \Bigg). \end{aligned}$$

The last term is bounded by $c_{\delta,n} J_\Phi(1) < \infty$. The third term may be estimated exactly as the first term. The first and second terms may be estimated as before using the upper bound for the kernel (3.1):

$$x^{n\delta} y K(x,y) \leq c_2 (x^{(n-1)\delta + \delta + \gamma + \lambda} y^{1-\lambda} + y^{1+\gamma+\lambda} x^{(n-1)\delta + \delta - \lambda}).$$

It then follows by the choice of δ that, for some constant $\tilde{c}_{\delta,n}$ depending on δ and n, as well as on the parameters of the kernel, we have

$$\int_{(1,\infty)} x^{1+n\delta} \Phi(dx) \leq 3\tilde{c}_{\delta,n} \int_{(1,\infty)} x^{1+(n-1)\delta} \Phi(dx) + c_{\delta,n} J_\Phi(1). \tag{5.9}$$

The bound (5.1) follows by induction combining (5.8) and (5.9). ∎

Proposition 5.2. *Assume K, η, λ and γ are as in the assumptions of Theorem 3.2. The solution Φ of (4.66), constructed in Proposition 4.18, satisfies (3.5) for some positive constant L.*

Proof. We now prove the exponential bound following the approach of [11]. Let us introduce the notation $\Psi_a(L) := \int_{\mathbb{R}_*} \frac{x^2}{\min\{x,a\}} e^{L\min\{x,a\}}\Phi(dx)$. Notice that

$$\Psi_a(0) = \int_{(0,a)} \frac{x^2}{\min\{x,a\}}\Phi(dx) + \int_{[a,\infty)} \frac{x^2}{a}\Phi(dx) \le 1 + \frac{M_2}{a} \tag{5.10}$$

and

$$\Psi_a'(L) = \int_{\mathbb{R}_*} x^2 e^{L\min\{x,a\}}\Phi(dx).$$

Considering the test function $\varphi'(x) = e^{L\min\{x,a\}}$ in (5.2), we deduce that

$$\begin{aligned}\Psi_a'(L) &\le c(\gamma)\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} xK(x,y)\int_x^{x+y} e^{L\min\{z,a\}}\,dz\,\Phi(dy)\Phi(dx)\\ &\le c(\gamma)\int_{\mathbb{R}_*} x^{\gamma+\lambda}\mu_{a,L}(dx)\int_{\mathbb{R}_*} y^{-\lambda}\mu_{a,L}(dy),\end{aligned}$$

where we are using the notation $\mu_{a,L}(dx) := xe^{L\min\{x,a\}}\Phi(dx)$.

Since $\gamma+\lambda < 1$ and $-\lambda < 1$, the maps $x \mapsto x^{-\frac{1}{\lambda}}$ and $x \mapsto x^{\frac{1}{\lambda+\gamma}}$ are convex functions. By the Jensen inequality we obtain

$$\begin{aligned}\|\mu_{L,a}\|_1 \int_{\mathbb{R}_*} x^{\gamma+\lambda}\frac{\mu_{L,a}(dx)}{\|\mu_{L,a}\|_1} &\le \|\mu_{L,a}\|_1^{1-\gamma-\lambda}\left(\int_{\mathbb{R}_*} x\mu_{L,a}(dx)\right)^{\gamma+\lambda}\\ &\le \Psi_a(L)^{1-\gamma-\lambda}\Psi_a'(L)^{\gamma+\lambda}\end{aligned}$$

and similarly

$$\|\mu_{L,a}\|_1\int_{\mathbb{R}_*} x^{-\lambda}\frac{\mu_{L,a}(dx)}{\|\mu_{L,a}\|_1} \le \Psi_a(L)^{1+\lambda}\Psi_a'(L)^{-\lambda}.$$

As a consequence $\Psi_a'(L) \le c(\gamma)\Psi_a(L)^{2-\gamma}\Psi_a'(L)^{\gamma}$, or equivalently,

$$\Psi_a'(L) \le \Psi_a(L)^{\frac{2-\gamma}{1-\gamma}} c(\gamma)^{\frac{1}{1-\gamma}}.$$

This implies that if $L \le \Psi_a(0)^{-\frac{1}{1-\gamma}} c(\gamma)^{-\frac{1}{1-\gamma}}$, then

$$\Psi_a(L) \le (\Psi_a(0)^{-\frac{1}{1-\gamma}} - c(\gamma)^{\frac{1}{1-\gamma}} L)^{\gamma-1}.$$

Since by (5.10) it follows that $\Psi_a(0) \to 1$ as $a \to \infty$, we deduce that if $L \le c(\gamma)^{-\frac{1}{1-\gamma}}$ then

$$\int_{(1,\infty)} e^{Lx}\Phi(dx) \le \limsup_{a\to\infty}\Psi_a(L) < \infty. \qquad \blacksquare$$

Proof of Theorem 3.2. Let Φ be a solution of (4.66) constructed in Proposition 4.18. We know that Φ satisfies (3.5) and (3.3). We now prove that Φ is absolutely continuous. Since Φ is a solution of (4.66), by selecting $\varphi(x) = \frac{1}{z^2}\chi(x)$ with $\chi \in C_c(\mathbb{R}_*)$, we conclude that

$$\begin{aligned}\frac{2}{1-\gamma}\left|\int_{\mathbb{R}_*}\chi(z)\Phi(dz)\right| &\le \left|\int_{\mathbb{R}_*} J_\Phi(z)\frac{1}{z^2}\chi(z)\varphi(z)\,dz\right| + \left|\int_{\mathbb{R}_*}\int_{(z,\infty)} x\Phi(dx)\frac{1}{z^2}\chi(z)\,dz\right|\\ &\le (1+\|J_{|\operatorname{supp}\chi}\|_\infty)\|1/z^2\|_{L^p(\operatorname{supp}\chi)}\|\chi\|_{L^q(\operatorname{supp}\chi)} < \infty.\end{aligned}$$

Therefore, by density we know that for every $q > 1$ and any compact set $\mathcal{K} \subset \mathbb{R}_*$,

$$\left| \int_{\mathbb{R}_*} \chi(z)\Phi(dz) \right| \leq C(\eta, \lambda, \gamma, \Phi) \|\chi\|_{L^q(\mathcal{K})}, \quad \chi \in L^q(\mathcal{K}).$$

This implies that Φ is absolutely continuous and that its density ϕ satisfies (3.2). We now show (3.6). First of all we notice that

$$J_\phi(z) = \int_z^\infty x\phi \, dx + \frac{2}{1-\gamma} z^2 \phi(z) \quad \text{for a.e. } z > 0.$$

As a consequence we deduce that

$$\phi(z) \leq \frac{1-\gamma}{2z^2} J_\phi(z) \quad \text{for a.e. } z > 0. \tag{5.11}$$

Assume $z > 1$, hence we can rewrite J_ϕ in the following way:

$$\begin{aligned} J_\phi(z) &= \int_0^z \int_{z-x}^\infty K(x, y) x \phi(y) \phi(x) \, dy \, dx \\ &= \int_0^1 \int_{z-x}^\infty K(x, y) x \phi(y) \phi(x) \, dy \, dx \\ &\quad + \int_1^z \int_{z-x}^\infty K(x, y) x \phi(y) \phi(x) \, dy \, dx. \end{aligned} \tag{5.12}$$

Using (4.68), we have that, if z is large enough, then there exists a constant $\widetilde{L} > 0$ such that

$$\begin{aligned} &\int_0^1 \int_{z-x}^\infty K(x, y) x \phi(y) \phi(x) \, dy \, dx \\ &\quad \leq c_2 \int_0^1 x^{\gamma+\lambda+1} \phi(x) \int_{z-1}^\infty y^{-\lambda} \phi(y) \, dy \, dx \\ &\qquad + c_2 \int_0^1 x^{1-\lambda} \phi(x) \int_{z-1}^\infty y^{\gamma+\lambda} \phi(y) \, dy \, dx \\ &\quad \leq e^{-z\widetilde{L}}. \end{aligned}$$

We now focus on the second term of (5.12), which for large enough z can be written as the sum of two terms in the following way:

$$\begin{aligned} &\int_1^z \int_{z-x}^\infty K(x, y) x \phi(y) \phi(x) \, dy \, dx \\ &\quad = \int_{z-1}^z \int_{z-x}^\infty K(x, y) x \phi(y) \phi(x) \, dy \, dx \\ &\qquad + \int_1^{z-1} \int_{z-x}^\infty K(x, y) x \phi(y) \phi(x) \, dy \, dx. \end{aligned} \tag{5.13}$$

Moreover, since

$$\begin{aligned}\int_1^{z-1}\int_{z-x}^\infty K(x,y)x\phi(y)\phi(x)\,dy\,dx &\le c_2\int_{z/2}^\infty x^{\gamma+\lambda+1}\phi(x)\,dx\int_1^\infty y^{-\lambda}\phi(y)\,dy\\ &+c_2\int_{z/2}^\infty x^{1-\lambda}\phi(x)\,dx\int_1^\infty y^{\gamma+\lambda}\phi(y)\,dy\\ &+c_2\int_1^\infty x^{\gamma+\lambda+1}\phi(x)\,dx\int_{z/2}^\infty y^{-\lambda}\phi(y)\,dy\\ &+c_2\int_1^\infty x^{1-\lambda}\phi(x)\,dx\int_{z/2}^\infty y^{\gamma+\lambda}\phi(y)\,dy,\end{aligned}$$

from this we deduce, thanks to (3.5), that

$$\int_1^{z-1}\int_{z-x}^\infty K(x,y)x\phi(y)\phi(x)\,dy\,dx\le ce^{-zL_1}$$

for suitable constants $L_1>0$ and $c>0$. We focus now on the first term of (5.13). Notice that there exists a positive constant $L_4>0$ such that

$$\int_{z-1}^z\int_1^\infty xK(x,y)\phi(y)\phi(x)\,dy\,dx\le ce^{-L_4z}.$$

Using (5.11) we deduce that

$$\int_{z-1}^z\int_{z-x}^1 K(x,y)x\phi(y)\phi(x)\,dy\,dx\le c(\gamma)\int_{z-1}^z\int_{z-x}^1\frac{K(x,y)}{x}\phi(y)J_\phi(x)\,dy\,dx.$$

Combining all the above inequalities we have that there exists a $\rho>0$ such that

$$J_\phi(x)\le ce^{-z\rho}+c(\gamma)\int_{z-1}^z\int_{z-x}^1\frac{K(x,y)}{x}\phi(y)J_\phi(x)\,dy\,dx.$$

On the other hand, using Lemma 4.11 we deduce that

$$\begin{aligned}&\int_{z-1}^z\int_{z-x}^1\frac{K(x,y)}{x}\phi(y)J_\phi(x)\,dy\,dx\\ &\le\sup_{x\in[z-1,z]}J_\phi(x)\int_{z-1}^z\int_{z-x}^1\frac{K(x,y)}{x}\phi(y)\,dy\,dx\\ &\le\sup_{x\in[z-1,z]}J_\phi(x)\left[\int_{z-1}^z\frac{x^{\gamma+\lambda}}{x}\int_{z-x}^1 y^{\frac{-2\lambda-\gamma-3}{2}}\,dy\,dx+\int_{z-1}^z\frac{x^{-\lambda}}{x}\int_{z-x}^1 y^{\frac{2\lambda+\gamma-3}{2}}\,dy\,dx\right]\\ &\le c\sup_{x\in[z-1,z]}J_\phi(x)\left[\int_{z-1}^z x^{\gamma+\lambda-1}(z-x)^{\frac{-2\lambda-\gamma-1}{2}}\,dx+\int_{z-1}^z x^{-\lambda-1}(z-x)^{\frac{2\lambda+\gamma-1}{2}}\,dx\right]\\ &\le c\sup_{x\in[z-1,z]}J_\phi(x)\left[(z-1)^{\gamma+\lambda-1}\int_0^1 y^{\frac{-2\lambda-\gamma-1}{2}}\,dy+(z-1)^{-\lambda-1}\int_0^1 y^{\frac{2\lambda+\gamma-1}{2}}\,dy\right].\end{aligned}$$

This allows us to deduce that there exists an N_ε such that for every $z > N_\varepsilon$,

$$J_\phi(z) \le ce^{-z\rho} + \varepsilon \sup_{x\in[z-1,z]} J_\phi(x), \tag{5.14}$$

with $e^{-\rho} > \frac{\varepsilon}{1-\varepsilon} > 0$. Let

$$\tilde{J}_n := \sup_{z\in[n-1,n]} J_\phi(z);$$

then by (5.14) we deduce that

$$\tilde{J}_n \le \varepsilon \tilde{J}_{n-1} + \varepsilon \tilde{J}_n + ce^{-n\rho}, \tag{5.15}$$

and by the local boundedness of J_ϕ we have that there exists a $c_0 > 0$ such that $\tilde{J}_{N_\varepsilon} \le c_0$. Formula (5.15) implies that for every $k > 0$,

$$\tilde{J}_{N_\varepsilon+k} \le \frac{c}{1-\varepsilon} e^{-\rho(N_\varepsilon+k)} \sum_{i=0}^{k} e^{\rho i} \left(\frac{\varepsilon}{1-\varepsilon}\right)^i + c_0 e^{-\rho k}.$$

As a consequence, for large z we have $J_\phi(z) \le ce^{-\rho z}$, where c is a positive constant.

The inequality $\phi(z) \le ce^{-\rho z}$ as $z \to \infty$ follows by the fact that $J_\phi(z) \le ce^{-\rho z}$ and by equation (3.2).

We now prove the lower power-law bound (3.4). Let C be any positive constant. For small enough positive constants z_0 and ε such that $1 - C\varepsilon - \int_0^{z_0} x\phi(x)\,dx \ge \frac{1}{2}$, it follows by Lemma 4.17 that there is a $\delta_0 > 0$ depending on ε and γ such that for any $\delta \le \delta_0$ and any $\varphi \in C_c(\mathbb{R}_*)$,

$$J_2(z,\delta,\phi) \ge 1 + \frac{2}{1-\gamma} z^2\phi(z) - C\varepsilon - \int_0^z x\phi(x)x\,dx \ge \frac{1}{2}, \quad \text{a.e. } z \in (0, z_0]. \tag{5.16}$$

Moreover, using a geometrical argument, we deduce that there exists a constant $b \in (0,1)$ depending on δ such that, for all $z \in [R, 2R]$,

$$\Omega_z \cap \Sigma_2(\delta) \subset \left(\frac{\delta}{\delta+1} z, z\right] \times \left(\frac{\delta}{\delta+1} z, \frac{z}{\delta}\right] \subset \left(\sqrt{b}R, R/\sqrt{b}\right]^2,$$

with Ω_z defined by (4.42), which together with the upper bound for the kernel (3.1), $xK(x,y) \le cR^{\gamma+1}$, yields

$$\int_{[R,2R]} J_2(z,\delta,\phi)\,dz \le cR^{\gamma+2} \left(\int_{(\sqrt{b}R, R/\sqrt{b}]} \phi(x)\,dx\right)^2, \quad R > 0,$$

for a constant $c > 0$ depending only on γ, λ, c_1 and c_2. This together with (5.16) implies

$$\frac{R}{2} \le cR^{\gamma+2} \left(\int_{(\sqrt{b}R, R/\sqrt{b}]} \phi(x)\,dx\right)^2, \quad R \in (0, z_0].$$

Hence, the result follows after substituting $R/\sqrt{b}$ by z.

Finally, $\int_{\mathbb{R}_*} x\phi(x)\,dx = 1$ follows by the fact that for every sequence $\{z_n\}$ such that $\lim_{n\to\infty} z_n = \infty$ we have $\lim_{n\to\infty} J_\phi(z_n) = 0$ and, thanks to (3.5), up to a subsequence $\lim_{n\to\infty} z_n^2\phi(z_n) = 0$. ∎

6. Regularity of the self-similar profiles

Proof of Theorem 3.3. We divide the proof into steps.

Step 1. For every $\psi \in C_c^\infty(\mathbb{R})$, every $\beta > 0$ and $0 < s < 1$ it holds that

$$\|\psi(\cdot + y) - \psi(\cdot)\|_{H^{-\beta}(\mathbb{R})} \leq |y|^s \|\psi\|_{H^{s-\beta}}(\mathbb{R}), \quad y \in \mathbb{R}. \tag{6.1}$$

This follows by the fact that

$$\begin{aligned}
\|\Delta_y \psi(\cdot)\|^2_{H^{-\beta}(\mathbb{R})} &= \int_{\mathbb{R}} (1 + |x|^2)^{-\beta} |\Delta_y \hat{\psi}(x)|^2 \, dx \\
&= \int_{\mathbb{R}} (1 + |x|^2)^{-\beta} |\hat{\psi}(x)|^2 |e^{ixy} - 1|^2 \, dx \\
&\leq c \int_{\mathbb{R}} (1 + |x|^2)^{-\beta} |\hat{\psi}(x)|^2 |x|^{2s} |y|^{2s} \, dx \\
&\leq c|y|^{2s} \int_{\mathbb{R}} |\hat{\psi}(x)|^2 (1 + |x|^2)^{s-\beta} \, dx \\
&= c|y|^{2s} \|\psi(\cdot)\|^2_{H^{s-\beta}(\mathbb{R})},
\end{aligned}$$

where we are using the notation $\Delta_y \psi(\cdot) := \psi(\cdot + y) - \psi(\cdot)$.

Step 2. Let U be an open set and let us define the operator $T : C_c^\infty(\mathbb{R}) \to C^1(U)$ as

$$T[\psi](x) := \int_0^1 K(x, y)\phi(y)\Delta_y \psi(x) \, dy, \quad x \in U.$$

Then

$$\|T[\psi]\|_{L^2(U)} \leq C \|\psi\|_{H^{\bar{s}}(\mathbb{R})} \tag{6.2}$$

for some positive constant $C > 0$, and

$$1 > \bar{s} > \frac{\gamma + 1}{2} - \min\{\gamma + \lambda, -\lambda\} \geq \frac{1}{2}. \tag{6.3}$$

To prove (6.2) we notice that

$$\begin{aligned}
\|T[\psi]\|_{L^2(U)} &\leq \int_0^1 \|K(x, y)\phi(y)\Delta_y \psi(x)\|_{L^2(U)} \, dy \\
&\leq C \int_0^1 y^{\min\{\gamma+\lambda,-\lambda\}} \phi(y) \|\Delta_y \psi(x)\|_{L^2(\mathbb{R})} \, dy \\
&\leq C \int_0^1 y^{\min\{\gamma+\lambda,-\lambda\}+\bar{s}} \phi(y) \, dy \|\psi\|_{H^{\bar{s}}(\mathbb{R})},
\end{aligned}$$

where, for the last inequality, we use the fact that $\|\Delta_y \psi(x)\|_{L^2(\mathbb{R})} \leq C|y|^{\bar{s}} \|\psi\|_{H^{\bar{s}}(\mathbb{R})}$; see [27].

Inequalities (3.3) and (6.3) imply that $\int_0^1 y^{\min\{\gamma+\lambda,-\lambda\}+\bar{s}} \phi(y) \, dy < \infty$ and, therefore, (6.2) follows.

Step 3. If $l \geq \beta > 1/2$, then

$$\|T[\psi]\|_{H^{-\beta}(U)} \leq C\|\psi\|_{H^{\bar{s}-\beta}(\mathbb{R})}. \tag{6.4}$$

First of all we notice that if $f \in H^l(U)$ and $g \in H^{-\beta}(\mathbb{R})$, then $fg \in H^{-\beta}(U)$. This is due to the fact that, if $\beta > 1/2$,

$$\begin{aligned}\|fg\|_{H^{-\beta}(U)} &= \inf_{\{E\in H^{-\beta}(\mathbb{R}):E_{|U}=fg\}} \|E\|_{H^{-\beta}(\mathbb{R})}\\ &\leq \inf_{\{F\in H^{-\beta}(\mathbb{R}):F_{|U}=f\}} \|gF\|_{H^{-\beta}(\mathbb{R})}\\ &\leq \inf_{\{F\in H^{-\beta}(\mathbb{R}):F_{|U}=f\}} \|F\|_{H^{-\beta}(\mathbb{R})}\|g\|_{H^{-\beta}(\mathbb{R})}\\ &= \|f\|_{H^{-\beta}(U)}\|g\|_{H^{-\beta}(\mathbb{R})} < \infty.\end{aligned}$$

Since $K \in H^l(\mathbb{R}_*)$ as well as the fact that it satisfies (3.7), we deduce that

$$\begin{aligned}\|T[\psi]\|_{H^{-\beta}(U)} &\leq \left\| \int_0^1 K(x,y)\phi(y)\Delta_y\psi(x)\,dy \right\|_{H^{-\beta}(U)}\\ &\leq \int_0^1 \|K(x,y)\Delta_y\psi(x)\|_{H^{-\beta}(U)}\phi(y)\,dy\\ &\leq C\int_0^1 y^{\min\{\gamma+\lambda,-\lambda\}}\phi(y)\|\Delta_y\psi(x)\|_{H^{-\beta}(\mathbb{R})}\,dy\\ &\leq C\|\psi\|_{H^{\bar{s}-\beta}(\mathbb{R})}\int_0^1 y^{\min\{\gamma+\lambda,-\lambda\}+\bar{s}}\phi(y)\,dy\\ &\leq C\|\psi\|_{H^{\bar{s}-\beta}(\mathbb{R})}\end{aligned}$$

for some positive constant C. To deduce the second last inequality, we applied (6.1).

Step 4. In this last step we show how to combine the results of the previous steps to prove the regularity of the self-similar profile. Considering a test function $\varphi(z) = \psi'(z)$ with $\psi \in C_c^\infty(\mathbb{R}_*)$ and $\mathrm{supp}(\psi) = [a,b]$ for some $a > b > 0$ in (4.66), we deduce that

$$\frac{2}{1-\gamma}\int_{\mathbb{R}_*} z^2\psi'(z)\phi(z)\,dz = \int_{\mathbb{R}_*} x\phi(x)\Big(\psi(x) + \int_{\mathbb{R}_*}\Delta_y\psi(x)K(x,y)\phi(y)\Big)\,dx\,dy.$$

Therefore,

$$\begin{aligned}\frac{2}{1-\gamma}\int_{\mathbb{R}_*}|z^2\psi'(z)\phi(z)|\,dz \leq{}& \|\psi\|_{L^2([a,b])}\|x\phi(x)\|_{L^2([a,b])}\\ &+\int_a^b\int_1^\infty \Delta_y\psi(x)xK(x,y)\phi(x)\phi(y)\,dy\,dx\\ &+\int_a^b xT[\psi](x)\phi(x)\,dx\\ &+\int_0^a\int_{a-x}^\infty \psi(x+y)xK(x,y)\phi(x)\phi(y)\,dx\,dy.\end{aligned}$$

We notice that, for some $C > 0$,

$$\int_a^b \int_1^\infty \Delta_y \psi(x) x K(x,y)\phi(x)\phi(y)\,dy\,dx \\ \leq C\|\psi\|_{L^2([a,b])} \int_a^b \int_1^\infty x K(x,y)\phi(x)\phi(y)\,dy\,dx \leq C\|\psi\|_{L^2([a,b])}$$

and

$$\int_0^a \int_{a-x}^\infty \psi(x+y) x K(x,y)\phi(x)\phi(y)\,dy\,dx \leq \|\psi\|_{L^2([a,b])} J_\phi(a) \leq C\|\psi\|_{L^2([a,b])}.$$

Thanks to Step 2 we deduce that

$$\begin{aligned}\int_a^b x T[\psi](x)\phi(x)\,dx &\leq \|x\phi(x)\|_{L^2([a,b])}\|T[\psi]\|_{L^2([a,b])} \\ &\leq \|x\phi(x)\|_{L^2([a,b])}\|T[\psi]\|_{L^2(\mathbb{R})} \\ &\leq \|x\phi(x)\|_{L^2([a,b])}\|\psi\|_{H^{\bar{s}}(\mathbb{R})}.\end{aligned}$$

By denoting with Θ the function $z \mapsto z^2\phi(z)$ we conclude that

$$\int_0^\infty \psi'(z)\Theta(z)\,dz \leq C\|\psi\|_{H^{\bar{s}}(\mathbb{R})}.$$

This inequality implies that $\Theta'(z) \in (H^{\bar{s}})'(\mathbb{R}) = H^{-\bar{s}}(\mathbb{R})$ and therefore $\Theta \in H^{1-\bar{s}}(\mathbb{R})$. For every test function $\zeta \in C_c^\infty(\mathbb{R})$, we have $\Theta\zeta \in H^{1-\bar{s}}(\mathbb{R})$.

Assume now that $\zeta\Theta \in H^{n(1-\bar{s})}(\mathbb{R})$ for any test function ζ and for some $n \geq 1$. Then if $l \geq n(\bar{s}-1)$, considering a test function which is equal to $\frac{1}{x}$ on $[a,b]$ we deduce that

$$\begin{aligned}\int_a^b x T[\psi](x)\phi(x)\,dx &\leq C\int_a^b \zeta(x) x^2 T[\psi](x)\phi(x)\,dx \\ &\leq \|\zeta\Theta\|_{H^{n(1-\bar{s})}(\mathbb{R})}\|T[\psi](x)\|_{H^{n(\bar{s}-1)}(a,b)} \\ &\leq \|\zeta\Theta\|_{H^{n(1-\bar{s})}(\mathbb{R})}\|\psi\|_{H^{(n+1)\bar{s}-n}(\mathbb{R})}.\end{aligned}$$

This implies that

$$\int_0^\infty z^2\psi'(z)\phi(z)\,dz \leq c\|\psi\|_{H^{(n+1)\bar{s}-n}(\mathbb{R})}$$

and, therefore, $\Theta \in H^{(n+1)(1-\bar{s})}(\mathbb{R})$ and the desired result follows.

Recalling the Sobolev embeddings ([6]) and differentiating (3.2), we deduce that ϕ satisfies (2.8). ■

7. Self-similar solutions for the coagulation with constant flux coming from the origin

Proof of Theorem 3.6. The fact that F satisfies (3.9) follows by (4.68).

We notice that for every $\varepsilon > 0$ and every $\varphi \in C^1([0,T], C_c^1(\mathbb{R}_*))$,

$$\begin{aligned}\int_{\mathbb{R}_*} \xi\varphi(\varepsilon,\xi) F(\varepsilon, d\xi) &= \int_{\mathbb{R}_*} \xi\varphi(\varepsilon,\xi)\varepsilon^{-\frac{\gamma+3}{1-\gamma}}\phi(\xi\varepsilon^{-\frac{2}{1-\gamma}})\, d\xi \\ &\leq \int_{\mathbb{R}_*} \varphi(\varepsilon, y\varepsilon^{\frac{2}{1-\gamma}})\varepsilon\phi(y)\, dy \\ &\leq \varepsilon\|\varphi\|_\infty,\end{aligned}$$

and therefore $\int_{\mathbb{R}_*} \xi\varphi(t,\xi)F(t,\xi)\, d\xi \to 0$ as $t \to 0$. Consequently, via a change of variables and integral manipulations, we deduce that

$$\begin{aligned}&\int_{\mathbb{R}_*} \xi\varphi(t,\xi)F(t,\xi)\, d\xi - \int_0^t \int_{\mathbb{R}_*} \xi\partial_s\varphi(s,\xi)F(s,\xi)\, d\xi \\ &\quad= \int_0^t \int_{\mathbb{R}_*} x\varphi(s, xs^{\frac{2}{1-\gamma}})\phi(x)\, dx\, ds \\ &\qquad+ \frac{2}{1-\gamma}\int_0^t \int_{\mathbb{R}_*} x^2\partial_x\varphi(s, xs^{\frac{2}{1-\gamma}})\phi(x)\, dx\, ds \\ &\quad= \int_0^t \int_{\mathbb{R}_*} \int_{(0,x]} \partial_y\varphi(s, ys^{\frac{2}{1-\gamma}})\, dy\, x\phi(x)\, dx\, ds \\ &\qquad+ \frac{2}{1-\gamma}\int_0^t \int_{\mathbb{R}_*} x^2\partial_x\varphi(s, xs^{\frac{2}{1-\gamma}})\phi(x)\, dx\, ds \\ &\quad= \int_0^t \int_{\mathbb{R}_*} \partial_z\varphi(s, zs^{\frac{2}{1-\gamma}})\left(\int_{(z,\infty)} x\phi(x)\, dx + \frac{2}{1-\gamma}z^2\phi(z)\, dz\right) ds.\end{aligned}$$

Since Φ solves (4.66), by considering the test function $\psi(x) = \partial_x\varphi(s, xs^{\frac{2}{1-\gamma}})$ for a fixed $s > 0$, we deduce that

$$\int_{\mathbb{R}_*} \xi\varphi(t,\xi)F(t,\xi)\, d\xi - \int_0^t \int_{\mathbb{R}_*} \xi\partial_s\varphi(s,\xi)F(s,\xi)\, d\xi = \int_0^t \int_{\mathbb{R}_*} \partial_z\varphi(s, zs^{\frac{2}{1-\gamma}})J_\phi(z)\, dz.$$

Since

$$\begin{aligned}&\int_{\mathbb{R}_*} \partial_z\varphi(s, zs^{\frac{2}{1-\gamma}})J_\phi(z)\, dz \\ &\quad= \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \left(\varphi(s, (z+\xi)s^{\frac{2}{1-\gamma}}) - \varphi(s, zs^{\frac{2}{1-\gamma}})\right) zK(z,\xi)\phi(z)\, dz\, \phi(\xi)\, d\xi \\ &\quad= \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} \frac{K(y,x)}{2}\left((y+x)\varphi(s,x+y) - x\varphi(s,x) - y\varphi(s,y)\right) F(s,dy)F(s,dx),\end{aligned}$$

thus F solves (3.10) for every $\varphi \in C^1([0,T], C_c^1(\mathbb{R}_*))$.

Every test function $\varphi \in C^1([0,T], C_c^1(\mathbb{R}_+))$ can be approximated by a sequence of functions $\{\varphi_n\} \subset C^1([0,T], C_c(\mathbb{R}_*))$ defined by $\varphi_n(s,x) = \zeta(xn)\varphi(s,x)$, with $\zeta \in$

$C^\infty(\mathbb{R}_+)$ such that $\zeta(x) = 1$ if $x \geq 1$ and $\zeta(x) = 0$ if $x \leq 1/2$. For every $n \in \mathbb{N}$ it holds that

$$\int_{\mathbb{R}_*} \xi \varphi_n(t,\xi) F(t,\xi)\, d\xi = \int_0^t \int_{\mathbb{R}_+} \xi \partial_s \varphi_n(s,\xi) F(s,\xi)\, d\xi\, ds + \int_0^t \int_{\mathbb{R}_+} \partial_\xi \varphi_n(s,\xi) J_{F(s,\cdot)}(\xi)\, d\xi\, ds. \tag{7.1}$$

Since $\partial_\xi \varphi_n(s,\xi) = \varphi(s,\xi) n\zeta'(\xi n) + \zeta(\xi n)\partial_\xi \varphi(s,\xi)$, $J_{F(s,\xi)} = J_\phi(\xi s^{-\frac{2}{1-\gamma}}) < \infty$ and, moreover, $\int_0^\infty \xi F(t,\xi)\, d\xi \leq t$, by the Lebesgue dominated convergence theorem we deduce that

$$\int_0^t \int_{\mathbb{R}_+} \zeta(\xi n)\partial_\xi \varphi(s,\xi) J_{F(s,\cdot)}(\xi)\, d\xi\, ds \to \int_0^t \int_{\mathbb{R}_+} \partial_\xi \varphi(s,\xi) J_{F(s,\cdot)}(\xi)\, d\xi\, ds \quad \text{as } n \to \infty,$$

$$\int_{\mathbb{R}_*} \xi \varphi_n(t,\xi) F(t,\xi)\, d\xi \to \int_{\mathbb{R}_*} \xi \varphi(t,\xi) F(t,\xi)\, d\xi \quad \text{as } n \to \infty,$$

$$\int_0^t \int_{\mathbb{R}_+} \xi \partial_s \varphi_n(s,\xi) F(s,\xi)\, d\xi\, ds \to \int_0^t \int_{\mathbb{R}_+} \xi \partial_s \varphi(s,\xi) F(s,\xi)\, d\xi\, ds \quad \text{as } n \to \infty.$$

For any $n > 0$,

$$\int_0^\varepsilon \int_{\mathbb{R}_+} n\zeta'(\xi n)\varphi(s,\xi) J_{F(s,\cdot)}(\xi)\, d\xi\, ds \to 0 \quad \text{as } \varepsilon \to 0.$$

This, together with the fact that for any $n \in \mathbb{N}$ it holds that $\int_0^\infty n\zeta'(\xi n)\, d\xi = 1$, that $\operatorname{supp}(n\zeta'(\cdot n)) \subset [\frac{1}{2n}, \frac{1}{n}]$ and that $\int_0^\infty x\phi(x)\, dx = 1$ implies

$$\begin{aligned}
&\int_\varepsilon^t \int_{\mathbb{R}_+} n\zeta'(\xi n)\varphi(s,\xi) J_{F(s,\cdot)}(\xi)\, d\xi\, ds \\
&\quad = \int_\varepsilon^t \int_{\mathbb{R}_+} n\zeta'(\xi n)\varphi(s,\xi) \\
&\qquad\qquad \times \left[1 - \int_0^{\xi s^{-\frac{2}{1-\gamma}}} x\phi(x)\, dx + \frac{2}{1-\gamma}(\xi s^{-\frac{2}{1-\gamma}})^2 \phi(\xi s^{-\frac{2}{1-\gamma}})\right] d\xi\, ds \\
&\quad \to \int_0^t \varphi(s,0)\, ds \quad \text{as } n \to \infty \text{ and } \varepsilon \to 0.
\end{aligned}$$

Notice that we have used the fact that

$$\begin{aligned}
&\int_\varepsilon^t \int_{\mathbb{R}_+} n\zeta'(\xi n)\varphi(s,\xi) \int_0^{\xi s^{-\frac{2}{1-\gamma}}} x\phi(x)\, dx\, d\xi\, ds \\
&\qquad \leq \int_\varepsilon^t \int_{\mathbb{R}_+} n\zeta'(\xi n)\varphi(s,\xi) \int_0^{\frac{s^{-\frac{2}{1-\gamma}}}{n}} x\phi(x)\, dx\, d\xi\, ds \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

and that, by Lemma 4.17,

$$\begin{aligned}\int_\varepsilon^t \int_{\mathbb{R}_+} n\zeta'(\xi n)\varphi(s,\xi)(\xi s^{-\frac{2}{1-\gamma}})^2\phi(\xi s^{-\frac{2}{1-\gamma}})\,d\xi\,ds \\ \leq \int_\varepsilon^t \int_{1/2n}^{1/n} \varphi(s,\xi)(\xi s^{-\frac{2}{1-\gamma}})^2\phi(\xi s^{-\frac{2}{1-\gamma}})\,d\xi\,ds \\ \leq c\Big(\frac{1}{n}\Big)^{\frac{3-\gamma}{2}} \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Passing to the limit as $n \to \infty$ in equation (7.1), we deduce that F is a solution of the coagulation with constant flux coming from the origin in the sense of Definition 3.5. ∎

A. Proofs of Lemmas 4.2 and 4.3

In this section we write the proof of some auxiliary lemmas.

Proof of Lemma 4.2. First of all we show that, for each $f \in C([1,T],\mathcal{M}_{+,b}(\mathbb{R}_*))$ and for every $t \in [1,T]$, the functional $\mathcal{F}[f](t)$ is a linear and continuous functional on $C_0(\mathbb{R}_*)$ and, consequently, defines a measure in $\mathcal{M}_{+,b}(\mathbb{R}_*)$. The linearity follows directly from the definition.

To check the continuity we notice that for every $\varphi_1,\varphi_2 \in C_c(\mathbb{R}_*)$,

$$|\langle \mathcal{F}_1[f](t),\varphi_1-\varphi_2\rangle| \leq \|\varphi_1-\varphi_2\|_\infty \|\Phi_0\|$$

and

$$\begin{aligned}\langle \mathcal{F}_2[f](t),\varphi_1-\varphi_2\rangle &\leq \frac{a}{2}\|\varphi_1-\varphi_2\|_\infty \max\Big\{\frac{1}{|\beta\gamma|}t^{\beta\gamma+1},t\Big\}\|f\|^2_{[1,T]},\\ \langle \mathcal{F}_3[f](t),\varphi_1-\varphi_2\rangle &\leq \|\varphi_1-\varphi_2\|_\infty \|\eta_\varepsilon\|\widetilde{T},\end{aligned}$$

where $\widetilde{T} := T-1$. Combining all the above inequalities we conclude that

$$\langle \mathcal{F}[f](t),\varphi_1-\varphi_1\rangle \leq \|\varphi_1-\varphi_2\|\Big(\|\Phi_0\| + \frac{a}{2}\max\Big\{\frac{1}{|\beta\gamma|}t^{\beta\gamma+1},t\Big\}\|f\|^2_{[1,T]} + \|\eta_\varepsilon\|\widetilde{T}\Big).$$

Given the fact that $C_0^*(\mathbb{R}_*)$ is isomorphic with the space $\mathcal{M}_b(\mathbb{R}_*)$ (see for instance [24]), we conclude that the operator $\mathcal{F}[f](t)$ defines a measure.

We now check that $\mathcal{F}$ maps $C([1,T],\mathcal{X}_\varepsilon)$ into itself. We have already shown that $\mathcal{F}[f](t) \in \mathcal{M}_{+,b}(\mathbb{R}_*)$. The fact that $\mathcal{F}[f](t)((0,\varepsilon]) = 0$ follows easily by the fact that $\mathcal{F}_i(t)((0,\varepsilon]) = 0$ for $i = 1,2$. Indeed, $\mathcal{F}_i(t)$ are defined as integrals over measures that are equal to zero in the set $(0,\varepsilon]$. Moreover, since for any $t > 0$ we have $\ell(t,x,y) \geq \max\{x,y\}$, we deduce that for every test function with support contained in $(0,\varepsilon]$ we have that $\Lambda[\varphi] = 0$ if $x, y > \varepsilon$. Thus, for this choice of test function,

$$\langle \mathcal{F}_2[f](t), \varphi \rangle = \int_1^t \int_{(\varepsilon,\infty)} \int_{(\varepsilon,\infty)} \Lambda[\varphi](s,x,y) e^{-\int_s^t b[f](\xi,x)\,d\xi} \frac{K_{a,T}(s,x,y)}{2} \times f(s,dx) f(s,dy)\,ds = 0,$$

hence $\mathcal{F}_2(t)((0,\varepsilon]) = 0$.

Let us first show that $\mathcal{F}[f]$ is a continuous map from $[1,T]$ to $\mathcal{M}_{+,b}(\mathbb{R}_*)$. To this end we notice that for every $t_2 \geq t_1 > 0$, for every $\varphi \in C_0(\mathbb{R}_*)$ with $\|\varphi\| \leq 1$, we have

$$|\langle \varphi, \mathcal{F}_1[f](t_2) - \mathcal{F}_1[f](t_1) \rangle| \leq a \max\left\{ \frac{1}{|\gamma\beta|} \left({t_2}^{\gamma\beta+1} - {t_1}^{\gamma\beta+1} \right), t_2 - t_1 \right\} \|\Phi_0\| \|f\|_{[1,T]}.$$

On the other side, we have

$$|\langle \varphi, \mathcal{F}_2[f](t_2) - \mathcal{F}_2[f](t_1) \rangle| \leq \frac{a}{2} \|f\|_{[1,T]}^2 \max\left\{ \frac{1}{|\gamma\beta|} (t_1^{\gamma\beta+1} - t_2^{\gamma\beta+1}), t_1 - t_2 \right\} \times \left(1 + a \max\left\{ \frac{1}{\gamma\beta} \tilde{T}^{\gamma\beta+1}, \tilde{T} \right\} \|f\|_{[1,T]} \right).$$

Moreover,

$$|\langle \varphi, \mathcal{F}_3[f](t_2) - \mathcal{F}_3[f](t_1) \rangle| \leq (t_2 - t_1) \big(a \max\{ \tilde{T}^{\gamma\beta}, \tilde{T} \} \|f\|_{[1,T]} + 1 \big) \|\eta_\varepsilon\|.$$

If $\gamma < 0$, then

$$\max\left\{ \frac{1}{|\gamma\beta|} (t_1^{\gamma\beta+1} - t_2^{\gamma\beta+1}), t_1 - t_2 \right\} \leq \max\left\{ 1, \frac{1}{|\gamma\beta|} \right\} (t_1 - t_2)$$

and, if $\gamma \geq 0$, then

$$\max\left\{ \frac{1}{|\gamma\beta|} (t_1^{\gamma\beta+1} - t_2^{\gamma\beta+1}), t_1 - t_2 \right\} \leq \max\left\{ 1, \frac{1}{|\gamma\beta|} \right\} (t_1^{\gamma\beta+1} - t_2^{\gamma\beta+1}).$$

Therefore, the continuity of $\mathcal{F}[f]$ follows by the above inequalities. ■

Proof of Lemma 4.3. For every $\varphi \in C_c(\mathbb{R}_*)$ with $\|\varphi\| \leq 1$ we have

$$|\langle \varphi, \mathcal{F}_1[f](t) - \mathcal{F}_1[g](t) \rangle| \leq a \max\left\{ \frac{1}{\gamma\beta} \tilde{T}^{\gamma\beta+1}, \tilde{T} \right\} \|f - g\|_{[1,T]} \|f_1\|,$$

$$|\langle \varphi, \mathcal{F}_3[f](t) - \mathcal{F}_3[g](t) \rangle| \leq a \|f - g\|_{[1,T]} \max\left\{ \frac{1}{\gamma\beta} \tilde{T}^{\gamma\beta+1}, \tilde{T} \right\} \|\eta_\varepsilon\|,$$

$$\langle \varphi, \mathcal{F}_2[f](t) - \mathcal{F}_2[g](t) \rangle \leq 2a(1 + 2\|f_1\|) \|f - g\|_{[1,T]} \max\left\{ \frac{1}{\gamma\beta} \tilde{T}^{\gamma\beta+1}, \tilde{T} \right\} + a^2 \|f - g\|_{[1,T]} (1 + 2\|f_1\|)^2 \max\left\{ \frac{1}{\gamma^2\beta^2} \tilde{T}^{2(\gamma\beta+1)}, \tilde{T}^2 \right\},$$

where $\tilde{T} := T - 1$. To obtain the above inequalities we have used the fact that $|e^{-x_1} - e^{-x_2}| \leq |x_1 - x_2|$, whenever $x_1 > 0$ and $x_2 > 0$.

Summarizing,

$$\sup_{t\in[1,T]} \langle \varphi, \mathcal{F}[f](t) - \mathcal{F}[g](t)\rangle \leq C_T \|f - g\|_{[1,T]},$$

where

$$\begin{aligned} C_T &:= a \max\Big\{\frac{1}{|\gamma\beta|}\tilde{T}^{\gamma\beta+1}, \tilde{T}\Big\} \\ &\quad \times \Big(\|\eta_\varepsilon\| + (1 + 2\|f_1\|)^2 a \max\Big\{\frac{1}{|\gamma\beta|}\tilde{T}^{\gamma\beta+1}, \tilde{T}\Big\} + 3(1 + 2\|f_1\|)\Big). \end{aligned} \tag{A.1}$$

It is possible to verify that if

$$\tilde{T} < \frac{1}{\min\{1, \frac{1}{|\gamma\beta|}\}}\Big(\frac{1}{10a}\min\Big\{\frac{1}{\|\eta_\varepsilon\|}, \frac{1}{1 + 2\|f_1\|}\Big\}\Big), \tag{A.2}$$

then $C_T < \frac{1}{2}$.

The inequalities

$$\begin{aligned} \|\mathcal{F}_1[f] - f\|_{[1,T]} &\leq a\|f\|^2_{[1,T]} \max\Big\{\frac{1}{|\gamma\beta|}\tilde{T}^{\gamma\beta+1}, \tilde{T}\Big\}, \\ \|\mathcal{F}_2[f]\|_{[1,T]} &\leq a\|f\|^2_{[1,T]} \max\Big\{\frac{1}{|\gamma\beta|}\tilde{T}^{\gamma\beta+1}, \tilde{T}\Big\}, \\ \|\mathcal{F}_3[f]\|_{[1,T]} &\leq \|\eta_\varepsilon\| T \leq \|\eta_\varepsilon\| \max\Big\{\frac{1}{|\gamma\beta|}\tilde{T}^{\gamma\beta+1}, \tilde{T}\Big\} \end{aligned}$$

imply that if T satisfies (A.2), then (4.21) holds for $D_T < \frac{1}{2}$. ■

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