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Green function for linearized Navier–Stokes around a boundary shear layer profile for long wavelengths

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Abstract. This paper is the continuation of a program, initiated in Grenier and Nguyen [SIAM J. Math. Anal. 51 (2019); J. Differential Equations 269 (2020)], to derive pointwise estimates on the Green function of Orr–Sommerfeld equations. In this paper we focus on long wavelength perturbations, more precisely horizontal wave numbers α of order $\nu^{1/4}$, which correspond to the lower boundary of the instability area for monotonic profiles.

1. Introduction

We are interested in the study of linearized Navier–Stokes around a given fixed profile $U_s = (U(z), 0)$ in the inviscid limit $\nu \to 0$. Namely, we consider the set of equations

$$\partial_t v + U_s \cdot \nabla v + v \cdot \nabla U_s + \nabla p - \nu \Delta v = 0, \tag{1.1}$$

$$\nabla \cdot v = 0, \tag{1.2}$$

where $0 < \nu \ll 1$, posed on the half-plane $x \in \mathbb{R}$, $z > 0$, with the no-slip boundary conditions

$$v = 0 \quad \text{on } z = 0. \tag{1.3}$$

The linear problem (1.1)–(1.3) is a very classical problem that has led to a huge physical and mathematical literature, focussing in particular on the linear stability, on the dispersion relation, on the study of eigenvalues and eigenmodes, and on the onset of nonlinear instabilities and turbulence [1, 15]. We also mention several efforts in proving linear to nonlinear stability and instability around shear flows in the small viscosity limit [2–5, 9].

Throughout this paper, we will assume that $U(z)$ is holomorphic near $z = 0$, that $U(0) = 0$, that $U'(0) > 0$, that $U(z) > 0$ for any $z > 0$, and that U converges exponentially fast at ∞, to some positive constant U_+,

$$0 < U_+ = \lim_{z \to \infty} U(z) < \infty,$$

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as well as all its derivatives (which converge to 0). Note in particular that this class of profiles includes for instance the exponential profile

$$U(z) = U_+(1 - e^{-\beta z}),$$

where $\beta > 0$. As such a profile has no inflection point, according to Rayleigh's inflection criterion, it is stable with respect to linearized Euler equations. However, strikingly, a small viscosity has a destabilizing effect. That is, *all such shear profiles are unstable for large enough Reynolds numbers* ν^{-1} [6, 7].

More precisely, for such shear flows there exist lower and upper marginal stability branches $\alpha_{\text{low}}(\nu) \sim \nu^{1/4}$ and $\alpha_{\text{up}}(\nu) \sim \nu^{1/6}$, so that whenever the horizontal wave number α belongs to $[\alpha_{\text{low}}(\nu), \alpha_{\text{up}}(\nu)]$, the linearized Navier–Stokes equations about this shear profile have an eigenfunction and a corresponding eigenvalue λ_ν with

$$\Re\lambda_\nu \sim \nu^{1/2}. \tag{1.4}$$

Heisenberg [11, 12], then Tollmien and Lin [13, 14] were among the first physicists to use asymptotic expansions to study this spectral instability. We refer to Drazin and Reid [1] and Schlichting [15] for a complete account of the physical literature on the subject, and to [6, 7] for a complete mathematical proof of this instability.

To study the linear stability of U_s we first introduce the vorticity of the perturbation

$$\omega = \nabla \times v = \partial_z v_1 - \partial_x v_2,$$

which leads to

$$(\partial_t + U\partial_x)\omega + v_2 U'' - \nu\Delta\omega = 0, \tag{1.5}$$

together with $v = \nabla^\perp \phi$ and $\Delta\phi = \omega$, where ϕ is the related stream function. The no-slip boundary condition (1.3) becomes $\phi = \partial_z\phi = 0$ on $\{z = 0\}$.

We then take the Fourier transform in the tangential variables with Fourier variable α and the Laplace transform in time with dual variable $-i\alpha c$, following the traditional notation. In other words, we study solutions of linearized Navier–Stokes equations which are of the form

$$v = \nabla^\perp(e^{i\alpha(x-ct)}\phi_\alpha(z)).$$

This leads to the classical Orr–Sommerfeld equation,

$$\text{Orr}_{\alpha,c}(\phi_\alpha) := -\varepsilon\Delta_\alpha^2\phi_\alpha + (U - c)\Delta_\alpha\phi_\alpha - U''\phi_\alpha = 0, \tag{1.6}$$

where

$$\varepsilon = \frac{\nu}{i\alpha},$$

together with the boundary conditions

$$\phi_{\alpha|_{z=0}} = \partial_z\phi_{\alpha|_{z=0}} = 0, \qquad \lim_{z\to\infty}\phi_\alpha(z) = 0, \tag{1.7}$$

and where

$$\Delta_\alpha = \partial_z^2 - \alpha^2.$$

The aim of this paper is to give bounds on the Green function of the Orr–Sommerfeld equation when α is of order $\nu^{1/4}$ and c is of the same order, which correspond to one of the boundaries of the instability area. This restricted study appears to be sufficient to construct linear and nonlinear instabilities for the full nonlinear Navier–Stokes equations [8–10].

We first observe that since $U''(z)$ decays exponentially fast to zero as $z \to +\infty$, equation (1.6) "converges" to the constant-coefficient equation

$$\mathrm{Orr}_+(\phi) = -\varepsilon\Delta_\alpha^2\phi_\alpha + (U_+ - c)\Delta_\alpha\phi_\alpha = 0, \tag{1.8}$$

which has four independent solutions, with two slow modes $e^{\pm\mu_s z}$ and two fast modes $e^{\pm\mu_f^+ z}$, where

$$\mu_s := |\alpha|, \quad \mu_f(z) := \sqrt{\frac{U - c + \varepsilon\alpha^2}{\varepsilon}}, \quad \mu_f^+ = \lim_{z\to\infty}\mu_f(z). \tag{1.9}$$

Here, we take the positive real part of the square root. Note in particular that

$$|\mu_f(z)| \geq \left|\sqrt{\frac{\Im c + \nu\alpha}{\varepsilon}}\right| = \left|\sqrt{\frac{\alpha\,\Im c + \nu\alpha^2}{\nu}}\right| = \mathcal{O}(\nu^{-1/4}) \tag{1.10}$$

for $\alpha = \mathcal{O}(\nu^{1/4})$, and let $c = \mathcal{O}(\nu^{1/4})$, with $|\Im c| \geq \sigma_0\nu^{1/4}$. That is, slow and fast modes have distinct behavior at $z = \infty$.

In order to construct the Green function of the Orr–Sommerfeld equations, we need to construct all four independent solutions. In previous joint work with Guo [6], we were able to construct two exact slow and fast decaying solutions using an exact Rayleigh–Airy iterative scheme. The scheme is rather delicate to construct the remaining two growing solutions. In this paper, we provide a much simplified iterative scheme to construct both decaying and growing modes to (1.6). The simplification is due to the fact that we only need to construct approximate solutions and approximate Green functions. The exact Green function follows by the standard iteration.

The slow approximate solutions will be solutions of the Rayleigh equation

$$(U - c)\Delta_\alpha\phi - U''\phi = 0 \tag{1.11}$$

with boundary condition $\phi(0) = 0$. They will be constructed by perturbation of the case $\alpha = 0$ where the Rayleigh equation degenerates in

$$\mathrm{Ray}_0(\phi) = (U - c)\partial_z^2\phi - U''\phi. \tag{1.12}$$

The main observation is that $\phi_{1,0} = U - c$ is a particular solution of (1.12). Let $\phi_{2,0}$ be the other solution of this equation such that the Wronskian $W[\phi_{1,0}, \phi_{2,0}]$ equals 1. We will construct approximate solutions to the Orr–Sommerfeld equation which satisfy

$$\phi_{s,-}^{\rm app}(0) = U(0) - c + \alpha U_+^2 \phi_{2,0}(0) + \mathcal{O}(\alpha^2), \tag{1.13}$$

$$\partial_z \phi_{s,-}^{\rm app}(0) = U'(0) + \mathcal{O}(\alpha). \tag{1.14}$$

The "fast approximate solutions" will emerge in the balance between $-\varepsilon \Delta_\alpha^2 \phi$ and $(U - c)\Delta_\alpha \phi$. Keeping in mind that α is small, they will be constructed starting from solutions of the simplified equation

$$-\varepsilon \partial_z^4 \phi + (U - c)\partial_z^2 \phi = 0.$$

As c is small, and as $U'(0) \neq 0$, there exists a unique $z_c \in \mathbb{C}$ near 0 such that

$$U(z_c) = c. \tag{1.15}$$

Such a z_c is called a "critical layer" in the physics literature. It turns out that all the instability is driven by what happens near this critical layer. Near z_c, equation (1.15) is a perturbation of the Airy equation

$$-\varepsilon \partial_z^2 \psi + U'(0)(z - z_c)\psi = 0 \tag{1.16}$$

posed on $\psi = \partial_z^2 \phi$. The fast approximate solutions are thus constructed as perturbations of second primitives of classical Airy functions. This construction will be detailed in Section 2, where we will construct two approximate solutions $\phi_{f,\pm}^{\rm app}$ to the Orr–Sommerfeld equation, with fast behavior and with

$$\phi_{f,-}^{\rm app}(0) = {\rm Ai}(2, -\gamma z_c) + \mathcal{O}(\nu^{1/4}), \tag{1.17}$$

$$\partial_z \phi_{f,-}^{\rm app}(0) = \gamma\, {\rm Ai}(1, -\gamma z_c) + \mathcal{O}(1), \tag{1.18}$$

where

$$\gamma = \left(\frac{i\alpha U'(z_c)}{\nu}\right)^{1/3} = \mathcal{O}(\nu^{-1/4}), \tag{1.19}$$

and where ${\rm Ai}(1, .)$ and ${\rm Ai}(2, .)$ are the first and the second primitives of the classical Airy function Ai. We now introduce the Tietjens function, defined by

$${\rm Ti}(z) = \frac{{\rm Ai}(1, z)}{{\rm Ai}(2, z)}.$$

The Tietjens function is a classical special function in physics, precisely known and tabulated. Then

$$\frac{\partial_z \phi_{f,-}^{\rm app}(0)}{\phi_{f,-}^{\rm app}(0)} = \gamma\, {\rm Ti}(-\gamma z_c) + \mathcal{O}(1). \tag{1.20}$$

In this paper we will bound the Green function of the Orr–Sommerfeld equations. More precisely, for each fixed $\alpha \in \mathbb{R}_+$ and $c \in \mathbb{C}$, we let $G_{\alpha,c}(x, z)$ be the corresponding

Green kernel of the Orr–Sommerfeld problem. By definition, for each $x \in \mathbb{R}$ and $c \in \mathbb{C}$, $G_{\alpha,c}(x,z)$ solves

$$\mathrm{Orr}_{\alpha,c}(G_{\alpha,c}(x,\cdot)) = \delta_x(\cdot)$$

on $z \geq 0$, together with the boundary conditions

$$G_{\alpha,c}(x,0) = \partial_z G_{\alpha,c}(x,0) = 0, \qquad \lim_{z\to\infty} G_{\alpha,c}(x,z) = 0.$$

That is, for $z \neq x$, the Green function $G_{\alpha,c}(x,z)$ solves the homogenous Orr–Sommerfeld equations, together with the following jump conditions across $z = x$:

$$[\partial_z^k G_{\alpha,c}(x,z)]_{|z=x} = 0, \quad [\varepsilon \partial_z^3 G_{\alpha,c}(x,z)]_{|z=x} = -1$$

for $k = 0, 1, 2$. Here, the jump $[f(z)]_{|z=x}$ across $z = x$ is defined to be the value of the right limit subtracted from that of the left limit as $z \to x$.

The main result in this paper is as follows.

Theorem 1.1. *Let $U(z)$ be a smooth monotone shear profile so that $U(0) = 0$, $U'(0) > 0$, and $U(z)$ converges exponentially fast to a nonzero constant at $z = \infty$. Let σ_0 be arbitrarily small and positive, $\alpha = \mathcal{O}(\nu^{1/4})$, and $c = \mathcal{O}(\nu^{1/4})$, with $|\Im c| \geq \sigma_0 \nu^{1/4}$, such that*

$$|W[\phi_{s,-}^{\mathrm{app}}, \phi_{f,-}^{\mathrm{app}}]| \geq \sigma_0. \tag{1.21}$$

Let $G_{\alpha,c}(x,z)$ be the Green function of the Orr–Sommerfeld problem. Then there exists a smooth function $P(x)$ and there are universal positive constants θ_0, C_0 so that

$$\begin{aligned} &\Big| G_{\alpha,c}(x,z) - \frac{P(x)\phi_{s,-}(z)}{\nu^{1/4}} \Big| \\ &\quad \leq \frac{C_0}{\varepsilon \mu_f^2(x)} \Big(e^{-\theta_0 |\alpha|\,|x-z|} + \frac{1}{|\mu_f(x)|} e^{-\theta_0 |\int_x^z \Re \mu_f(y)\,dy|} \Big) \end{aligned} \tag{1.22}$$

uniformly for all $x, z \geq 0$. Similarly,

$$\begin{aligned} \Big| \partial_z G_{\alpha,c}(x,z) - \frac{P(x)\partial_z \phi_{s,-}(z)}{\nu^{1/4}} \Big| &\leq \frac{C_0}{\varepsilon \mu_f^2(x)} \Big(e^{-\theta_0 |\alpha|\,|x-z|} \\ &\qquad + \frac{|\mu_f(z)|}{|\mu_f(x)|} e^{-\theta_0 |\int_x^z \Re \mu_f(y)\,dy|} \Big), \\ \Big| \partial_z^2 G_{\alpha,c}(x,z) - \frac{P(x)\phi_{s,-}(z)}{\nu^{1/4}(U-c)} \Big| &\leq \frac{C_0}{\varepsilon \mu_f^2(x)} \Big(\frac{1}{|U-c|} e^{-\theta_0 |\alpha|\,|x-z|} \\ &\qquad + \frac{|\mu_f(z)|^2}{|\mu_f(x)|} e^{-\theta_0 |\int_x^z \Re \mu_f(y)\,dy|} \Big). \end{aligned} \tag{1.23}$$

Let us comment (1.21). We have

$$\begin{aligned} W[\phi_{s,-}^{\mathrm{app}}, \phi_{f,-}^{\mathrm{app}}] &= \gamma \psi_{s,0}^{\mathrm{app}}(0)\,\mathrm{Ti}(-\gamma z_c) \phi_{f,-}^{\mathrm{app}}(0) - \partial_z \phi_{s,-}^{\mathrm{app}}(0) \phi_{f,-}^{\mathrm{app}}(0) \\ &= -(\gamma c\,\mathrm{Ti}(-\gamma z_c) + U'(0))\,\mathrm{Ai}(2, -\gamma z_c) + \mathcal{O}(\nu^{1/4}). \end{aligned}$$

Note that both terms under the brackets are of order $\mathcal{O}(1)$, since γc is of order $\mathcal{O}(1)$. The Wronskian vanishes if there exists a linear combination of $\phi_{s,-}^{\rm app}$ and $\phi_{f,-}^{\rm app}$ which satisfies the boundary conditions, namely if there exists an approximate eigenmode of $\mathrm{Orr}_{\alpha,c}$ (recalling that $\phi_{s,-}^{\rm app}$ and $\phi_{f,-}^{\rm app}$ are only approximate solutions of $\mathrm{Orr}_{\alpha,c}$). We have to remain away from such approximate modes, since nearby there exist true eigenmodes where $\mathrm{Orr}_{\alpha,c}$ is no longer invertible. Note that σ_1 may be taken arbitrarily small.

Note that in this theorem we are at a distance $\mathcal{O}(\nu^{1/4})$ from a simple eigenmode ψ_0. It is therefore expected that $\mathrm{Orr}_{\alpha,c}$ is of order $\mathcal{O}(\nu^{-1/4})$ and that

$$\mathrm{Orr}_{\alpha,c}^{-1}(\psi) = \nu^{-1/4}\left(\int_0^{+\infty} P(z)\psi(z)\,dz\right)\psi_0 + \mathcal{O}(1). \tag{1.24}$$

As $\psi_0 = \phi_{s,-} + \mathcal{O}(\nu^{1/4})$, $G_{\alpha,c}$ is only bounded by $\mathcal{O}(\nu^{-1/4})$, and its main component is $\nu^{-1/4}P\phi_{s,-}$.

2. The Airy operator

In this section we construct two approximate solutions of the Orr–Sommerfeld equation, called $\phi_{f,\pm} = \phi_{f,\pm}^{\rm app}$, with fast increasing or decreasing behaviors. For these approximate solutions, it turns out that the zeroth-order term $U''\phi_{f,\pm}$ may be neglected. Moreover, as α is small, α^2 terms may also be neglected. This simplifies the Orr–Sommerfeld operator in the so-called modified Airy operator defined by

$$\mathrm{Airy} = \mathcal{A}\partial_z^2, \tag{2.1}$$

where

$$\mathcal{A} := -\varepsilon\partial_z^2 + (U-c). \tag{2.2}$$

Note that

$$\mathrm{Orr}_{\alpha,c} = \mathrm{Airy} + \mathrm{OrrAiry}, \tag{2.3}$$

where

$$\mathrm{OrrAiry} = 2\varepsilon\alpha^2\partial_z^2 - \varepsilon\alpha^4 - \alpha^2(U-c) - U''.$$

Note also that $U-c$ behaves like $U'(z_c)(z-z_c)$ for z near z_c, hence $\mathcal{A}$ is very similar to the classical Airy operator $\partial_z^2 - z$ when z is close to z_c. The main difficulty lies in the fact that the "phase" $U(z)-c$ almost vanishes when z is close to $\Re z_c$, hence we have to distinguish between two cases: $z \le \sigma_1$ and $z \ge \sigma_1$ for some small σ_1. The first case is handled through a Langer transformation, which reduces (2.1) to the classical Airy equation. The second case may be treated using a classical WKB expansion.

We will prove the following proposition.

Proposition 2.1. *Let N be an arbitrarily large number. There exist two smooth functions $\phi_\pm^{\rm app}(z)$, depending on N, to the Orr–Sommerfeld equations such that*

$$|\mathcal{A}\partial_z^2\phi_\pm^{\rm app}| \le C\nu^N|\phi_\pm^{\rm app}|, \tag{2.4}$$

$$|\mathrm{Orr}_{\alpha,c}(\phi_\pm^{\rm app})| \le C|\phi_\pm^{\rm app}|. \tag{2.5}$$

Moreover, for $z \gg \nu^{1/4}$ *and for* $k = 1, 2, 3$, *as* $\nu \to 0$,

$$\frac{\partial_z^k \phi_-^{\mathrm{app}}(z)}{\phi_-^{\mathrm{app}}}(z) \sim (-1)^k \mu_f^k(z), \qquad \frac{\partial_z^k \phi_+^{\mathrm{app}}(z)}{\phi_+^{\mathrm{app}}}(z) \sim \mu_f^k(z), \tag{2.6}$$

and for any $x_1 < x_2$, *there holds*

$$\left|\frac{\phi_\pm^{\mathrm{app}}(x_2)}{\phi_\pm^{\mathrm{app}}(x_1)}\right| \le C \exp\left(\pm \int_{x_1}^{x_2} \Re \mu_f(y)\, dy\right). \tag{2.7}$$

To prove this proposition we construct $\psi_\pm^{\mathrm{app}} = \partial_z^2 \phi_\pm^{\mathrm{app}}$ for $z < z_c$ in Section 2.2 using Langer's transformation introduced in (2.1) and for $z > z_c$ in Section 2.3 using the classical WKB method. We then match these two constructions in Section 2.4, integrate them twice in Section 2.5, and detail the Green function of the Airy operator in Section 2.7.

2.1. A primer on Langer's transformation

The first step is to construct approximate solutions to $\mathcal{A}\psi = 0$, starting from solutions of the genuine Airy equation $\varepsilon\psi'' = y\psi$, thanks to the so-called Langer transformation that we will now detail. Let $B(x)$ and $C(x)$ be two smooth functions. In 1931, Langer introduced the following method to build approximate solutions to the varying coefficient Airy-type equation

$$-\varepsilon\phi'' + C(x)\phi = 0, \tag{2.8}$$

starting from solutions to the similar Airy-type equation

$$-\varepsilon\psi'' + B(x)\psi = 0. \tag{2.9}$$

We assume that both B and C vanish at some point x_0, and that their derivatives at x_0 do not vanish. Let ψ be any solution to (2.9). Let f and g be two smooth functions, to be chosen later. Then

$$\phi(x) = f(x)\psi(g(x))$$

satisfies

$$-\varepsilon\phi'' + C(x)\phi = -\varepsilon f''\psi - 2\varepsilon f'\psi' g' - B(g(x))(g')^2 f\psi - \varepsilon f\psi' g'' + C(x) f\psi.$$

Note that f may be seen as a modulation of amplitude and g as a change of phase. If we choose g such that

$$B(g(x))(g')^2 = C(x) \tag{2.10}$$

and f such that

$$2f'g' + fg'' = 0, \tag{2.11}$$

we then have

$$-\varepsilon\phi'' + C(x)\phi = -\varepsilon f''\psi.$$

Hence ϕ may be considered as an approximate solution to $-\varepsilon\phi'' + C(x)\phi = 0$.

Note that (2.11) may be solved, yielding

$$f(x) = \frac{1}{\sqrt{g'(x)}}. \tag{2.12}$$

As a consequence, the link between solutions to (2.8) and (2.9) is given by

$$\phi(x) = \frac{1}{\sqrt{g'(x)}}\psi(g(x)). \tag{2.13}$$

Now, as for the choice of the function $g(x)$, let B_1 be the primitive of $\sqrt{B}$ which vanishes at x_0 and let C_1 be the primitive of $\sqrt{C}$ which vanishes at x_0. Then the square root of (2.10) may be rewritten as

$$B_1(g(x)) = C_1(x). \tag{2.14}$$

Note that both B_1 and C_1 behave like $C_0(x-x_0)^{3/2}$ near x_0. Hence (2.14) may be solved for x near x_0. This defines a smooth function g which satisfies $g(x_0) = x_0$. Moreover, if $B'(x_0) = C'(x_0)$ then $g'(x_0) = 1$.

2.2. Airy critical points

In this section we use Langer's transformation to construct approximate solutions to $\mathcal{A}\psi = 0$ starting from solutions of the genuine Airy equation.

Let c be of order $\nu^{1/4}$. Then there exists a unique $z_c \in \mathbb{C}$ near 0 such that $U(z_c) = c$. Note that z_c is also of order $\nu^{1/4}$ since $U'(0) \neq 0$. Expanding U near z_c to first order we get the approximate equation

$$-\varepsilon\partial_z^2\psi + U'(z_c)(z-z_c)\psi = 0, \tag{2.15}$$

which is the classical Airy equation. Let us assume that $\Re U'(z_c) > 0$, the opposite case being similar. A first solution to (2.15) is given by

$$A(z) := \mathrm{Ai}(\gamma(z-z_c)), \tag{2.16}$$

where Ai is the classical Airy function, solution of $\mathrm{Ai}'' = x\,\mathrm{Ai}$, and where $\varepsilon\gamma^3 = U'(z_c)$, namely

$$\gamma = \Big(\frac{i\alpha U'(z_c)}{\nu}\Big)^{1/3}.$$

Note that since α is of order $\nu^{1/4}$, γ is of order $\nu^{-1/4}$, and that

$$\arg(\gamma) = \frac{\pi}{6} + \mathcal{O}(\nu^{-1/4}).$$

Moreover, as x goes to $\pm\infty$, with argument $i\pi/6$,

$$\mathrm{Ai}(x) \sim \frac{1}{2\sqrt{\pi}}\frac{e^{-2x^{3/2}/3}}{x^{1/4}}(1+\mathcal{O}(|x|^{-3/2})).$$

In particular, $\mathrm{Ai}'(x)/\mathrm{Ai}(x) \sim -x^{1/2}$ for large x. Hence, as $\gamma(z-z_c)$ goes to infinity, $A(z)$ goes to 0 and

$$\frac{A'(z)}{A(z)} \sim -\gamma^{3/2}(z-z_c)^{1/2} = -\Big(\frac{i\alpha U'(z_c)}{\nu}\Big)^{1/2}(z-z_c)^{1/2} \sim -\sqrt{B(z)}, \tag{2.17}$$

with

$$B(z) = \varepsilon^{-1}U'(z_c)(z-z_c).$$

More precisely, using the next order expansion for $A(z)$, we get

$$\frac{A'(z)}{A(z)} = -\sqrt{B(z)}(1+\mathcal{O}(\nu^{3/8}|z-z_c|^{-3/2})) \tag{2.18}$$

for $|\gamma(z-z_c)| \gg 1$. Here, we have used the fact that γ is of order $\nu^{-1/4}$.

Another independent solution to (2.15) is given by $\mathrm{Ci}(\gamma(z-z_c))$, where

$$\mathrm{Ci} = -i\pi(\mathrm{Ai} + i\,\mathrm{Bi}),$$

with $\mathrm{Bi}(\cdot)$ being the other classical Airy function. In this case $|\mathrm{Ci}(\gamma(z-z_c))|$ goes to $+\infty$ as $z-z_c$ goes to $+\infty$, with a plus instead of the minus in the corresponding formula (2.17). Precisely,

$$\frac{\gamma\,\mathrm{Ci}'(\gamma(z-z_c))}{\mathrm{Ci}(\gamma(z-z_c))} = \sqrt{B(z)}(1+\mathcal{O}(\nu^{3/8}|z-z_c|^{-3/2})). \tag{2.19}$$

We now use Langer's transformation introduced in the previous section. As $U(z)$ and $U'(z_c)(z-z_c)$ vanish at the same point with the same derivative at that point, we use Langer's transformation with

$$C(z) = \varepsilon^{-1}(U(z)-c)$$

and

$$B(z) = \varepsilon^{-1}U'(z_c)(z-z_c).$$

Then, introducing $g(z)$ in accordance with (2.10), we have that $g(z)$ is locally well defined for z in a neighborhood of z_c, independent of small ε. Since z_c is of order $\nu^{1/4}$, $g(z)$ is thus defined for $0 \le z \le \sigma_1$ for some positive σ_1, independent of ν. Moreover, $g(z_c) = z_c$ and $g'(z_c) = 1$. Now we use the two independent solutions $\mathrm{Ai}(\gamma(z-z_c))$ and $\mathrm{Ci}(\gamma(z-z_c))$ to (2.15) to construct the two approximate solutions to $\mathcal{A}\phi = 0$, which reads

$$-\varepsilon\partial_z^2\psi + (U(z)-c)\psi = 0. \tag{2.20}$$

Indeed, through Langer's transformation (see (2.13)) we set

$$\widetilde{\mathrm{Ai}}(z) := \frac{1}{\sqrt{g'(z)}}\,\mathrm{Ai}\big(\gamma(g(z)-z_c)\big)$$

and

$$\widetilde{\mathrm{Ci}}(z) := \frac{1}{\sqrt{g'(z)}}\,\mathrm{Ci}\big(\gamma(g(z)-z_c)\big).$$

It follows that $\widetilde{\mathrm{Ai}}(z)$ and $\widetilde{\mathrm{Ci}}(z)$ are two approximate solutions of $\mathcal{A}\phi = 0$ in the sense that

$$\mathcal{A}\,\widetilde{\mathrm{Ai}} = -\varepsilon f''\,\mathrm{Ai}\big(\gamma(g(z)-z_c)\big), \quad \mathcal{A}\,\widetilde{\mathrm{Ci}} = -\varepsilon f''\,\mathrm{Ci}\big(\gamma(g(z)-z_c)\big),$$

recalling that $f(z) = 1/\sqrt{g'(z)}$. Note that the error term is of order $\varepsilon \sim \nu^{3/4}$. Note also that to first order, for z of order $\nu^{1/4}$, $\widetilde{\mathrm{Ai}}(z)$ equals $\mathrm{Ai}(\gamma(z-z_c))$ since $g'(z_c) = 1$.

Moreover, for $\gamma(z-z_c) \gg 1$, or equivalently, $|z-z_c| \gg \nu^{1/4}$, using (2.18), we get

$$\frac{\partial_z \widetilde{\mathrm{Ai}}(z)}{\widetilde{\mathrm{Ai}}(z)} \sim g'(z)\frac{A'(g(z))}{A(g(z))} \sim -g'(z)\sqrt{B(g(z))} \sim -\sqrt{C(z)} \sim -\mu_f(z), \tag{2.21}$$

and more precisely,

$$\frac{\partial_z \widetilde{\mathrm{Ai}}(z)}{\widetilde{\mathrm{Ai}}(z)} \sim -\mu_f(z)(1+\mathcal{O}(\nu^{3/8}|z-z_c|^{-3/2})). \tag{2.22}$$

Note in particular that when $|z-z_c| \gtrsim 1$, the above error of approximation is of order $\nu^{3/8}$. Similarly, for higher derivatives in $|z-z_c| \gg \nu^{1/4}$, we get

$$\frac{\partial_z^k \widetilde{\mathrm{Ai}}(z)}{\widetilde{\mathrm{Ai}}(z)} \sim (-1)^k \mu_f^k(z). \tag{2.23}$$

Similarly, using (2.19), we have

$$\frac{\partial_z \widetilde{\mathrm{Ci}}(z)}{\widetilde{\mathrm{Ci}}(z)} \sim \mu_f(z)(1+\mathcal{O}(\nu^{3/8}|z-z_c|^{-3/2})). \tag{2.24}$$

The higher derivatives also satisfy similar bounds to (2.23).

2.3. Away from the critical layer

If $z-z_c$ is small then g is well defined, precisely on $[0,\sigma_1]$ for some small σ_1 as in the previous section. However, if $z > \sigma_1$, then Langer's transformation is no longer useful, and we may directly use a WKB expansion. We look for solutions ψ of the form

$$\psi(z) = e^{\theta(z)/\varepsilon^{1/2}} \tag{2.25}$$

to the equation $\mathcal{A}\psi = \varepsilon\partial_z^2\psi - (U-c)\psi = 0$. Note that

$$\varepsilon\partial_z^2\psi = (\theta'^2 + \varepsilon^{1/2}\theta'')\psi.$$

Hence we look for θ such that

$$\theta'^2 + \varepsilon^{1/2}\theta'' = (U-c). \tag{2.26}$$

Note that as z is away from the critical layer z_c, $U(z)-c$ is of order 1 and never vanishes.

We shall solve (2.26) in an approximate way by looking for θ of the form

$$\theta = \sum_{i=0}^{M} \varepsilon^{i/2} \theta_i$$

for some arbitrarily large M. The profiles θ_i may be constructed by iteration, starting from

$$\theta_0' = \pm\sqrt{U(z) - c}. \tag{2.27}$$

Indeed, plugging the ansatz for θ into (2.26) and matching the order in ε, we are led to define θ_i inductively through the relation

$$\theta_0' \theta_i' = -\theta_{i-1}'' - \sum_{j+k=i-1} \theta_j' \theta_k'$$

for $i \geq 1$, noting that θ_0' never vanishes on $z > \sigma_1$ (since c is of order $\nu^{1/4}$). In (2.27), we take the positive real part of the square root (of the complex number). The $-$ choice in (2.27) leads to an approximate solution $\psi_{f,-}^{\mathrm{app}}$ of (2.25) that tends to 0 at $z = +\infty$ and the $+$ choice gives an approximate solution $\psi_{f,+}^{\mathrm{app}}$ of (2.25) that tends to $+\infty$ at $z = +\infty$.

In addition, by construction, we have

$$\theta'^2 + \varepsilon^{1/2} \theta'' = (U - c) + \mathcal{O}(\nu^N),$$

where N can be chosen arbitrarily large, provided M is sufficiently large. Therefore, the approximate solutions $\psi_{f,\pm}^{\mathrm{app}}$ satisfy

$$|\mathcal{A} \psi_{f,\pm}^{\mathrm{app}}| \leq \nu^N |\psi_{f,\pm}^{\mathrm{app}}|.$$

Note that

$$\partial_z \psi_{f,\pm}^{\mathrm{app}}(\sigma_1) = \pm \mu_f(\sigma_1)(1 + \mathcal{O}(\nu^{1/4})) \psi_{f,\pm}^{\mathrm{app}}(\sigma_1). \tag{2.28}$$

More generally,

$$\partial_z^k \psi_{f,\pm}^{\mathrm{app}}(z) = (-1)^k \mu_f^k(z) \psi_{f,\pm}^{\mathrm{app}}(z)(1 + \mathcal{O}(\nu^{1/4})) \tag{2.29}$$

for any $z \geq \sigma_1$ and any k.

2.4. Matching at $z = z_c$

It remains to match, at $z = z_c$, the solutions constructed with the WKB method for $z \geq \sigma_1$ with the solutions constructed thanks to Langer's transformation for $z \leq \sigma_1$. We look for constants a and b such that

$$a \frac{\widetilde{\mathrm{Ai}}(z)}{\widetilde{\mathrm{Ai}}(\sigma_1)} + b \frac{\widetilde{\mathrm{Ci}}(z)}{\widetilde{\mathrm{Ci}}(\sigma_1)}$$

and $\psi^{\rm app}_{f,-}/\psi^{\rm app}_{f,-}(\sigma_1)$ and their first derivatives match at $z = \sigma_1$, which leads to

$$a + b = 1,$$
$$a\frac{\partial_z \widetilde{\rm Ai}(\sigma_1)}{\widetilde{\rm Ai}(\sigma_1)} + b\frac{\partial_z \widetilde{\rm Ci}(\sigma_1)}{\widetilde{\rm Ci}(\sigma_1)} = \frac{\partial_z \psi^{\rm app}_{f,-}(\sigma_1)}{\psi^{\rm app}_{f,-}(\sigma_1)}.$$

We now use (2.22), (2.24), and (2.28) to get $a \sim 1$ and $b = \mathcal{O}(\mu_f(\sigma_1)^{-1})$. We then multiply a and b by $\psi^{\rm app}_{f,-}(\sigma_1)$ to get an extension of $\psi^{\rm app}_{f,-}$ from $z > \sigma_1$ to the whole line. The construction is similar to extend $\psi^{\rm app}_{f,+}$.

2.5. From $\mathcal{A}$ to Airy

We have now constructed global approximate solutions $\psi^{\rm app}_{f,\pm}$ to the equation $\mathcal{A}\psi = 0$ that satisfy

$$|\mathcal{A}\psi^{\rm app}_{f,\pm}| \le C\nu^N |\psi^{\rm app}_{f,\pm}|.$$

Recall from (2.1) that Airy $= \mathcal{A}\partial_z^2$. It thus remains to solve

$$\partial_z^2 \phi^{\rm app}_{f,\pm}(z) = \psi^{\rm app}_{f,\pm}(z). \tag{2.30}$$

Let us focus on the $-$ case, the other being similar. For $z \ge \sigma_1$, we look for solutions $\phi^{\rm app}_{f,\pm}$ of the form

$$\phi^{\rm app}_{f,\pm} = h(z)\psi^{\rm app}_{f,\pm} = h(z)e^{\theta(z)/\varepsilon^{1/2}}, \tag{2.31}$$

which leads to

$$h'' + 2h'\theta'(z)\varepsilon^{-1/2} + h\theta''(z)\varepsilon^{-1/2} + h\theta'^2(z)\varepsilon^{-1} = 1.$$

Hence h may be expanded as a series in $\varepsilon^{1/2}$, namely

$$h(z) = \sum_{i=0}^{M} \varepsilon^{i/2} h_i(z)$$

for some arbitrarily large M. The first two terms $h_0(z)$, $h_1(z)$ are defined by

$$h_0(z) = \frac{\varepsilon}{\theta'^2(z)}, \quad h_1(z) = -\frac{1}{\theta'^2(z)}(2h_0'(z)\theta'(z) + h_0\theta''(z)),$$

while the remaining terms $h_i(z)$, $i \ge 2$ are inductively defined by

$$h_i(z) = -\frac{1}{\theta'^2(z)}\big(h''_{i-2}(z) + 2h'_{i-1}(z)\theta'(z) + h_{i-1}\theta''(z)\big).$$

We note that for $z \ge \sigma_1$ (i.e. away from the critical layer z_c), by definition (2.27), the function $\theta(z)$ is bounded away from zero, and so $h_i(z)$ are well defined and uniformly bounded.

As a consequence, we may write a complete WKB expansion for $\phi^{\text{app}}_{f,\pm}$ given by (2.31). In particular, we note that

$$\frac{h(y)}{h(x)} \sim \frac{\theta'^2(x)}{\theta'^2(y)} \sim \frac{U(x)-c}{U(y)-c} \sim 1$$

for $y > x \geq \sigma_1$ (i.e. away from the critical layer z_c). Hence,

$$\frac{\phi^{\text{app}}_{f,\pm}(y)}{\phi^{\text{app}}_{f,\pm}(x)} \lesssim e^{\pm \int_x^y \Re \mu_f(z)\,dz}, \tag{2.32}$$

provided $y > x \geq \sigma_1$.

For $z < \sigma_1$, we integrate (2.30) once, which gives

$$\partial_z \phi^{\text{app}}_{f,-}(z) = \partial_z \phi^{\text{app}}_{f,-}(\sigma_1) - \int_z^{\sigma_1} \psi^{\text{app}}_{f,-}(t)\,dt.$$

Now $\psi^{\text{app}}_{f,-}$ is a combination of $\widetilde{\text{Ai}}$ and $\widetilde{\text{Ci}}$ for $z < \sigma_1$. Let us focus on the $\widetilde{\text{Ai}}$ term. We have to study

$$\int_z^{\sigma_1} \widetilde{\text{Ai}}(t)\,dt = \int_z^{\sigma_1} \frac{1}{\sqrt{g'(t)}} \text{Ai}\big(\gamma(g(t)-z_c)\big)\,dt.$$

Let $s = \gamma(g(t)-z_c)$. Then $ds = \gamma g'(t)\,dt$, hence

$$\int_z^{\sigma_1} \frac{1}{\sqrt{g'(t)}} \text{Ai}\big(\gamma(g(t)-z_c)\big)\,dt = \gamma^{-1} \int_{\gamma(g(z)-z_c)}^{\gamma(g(\sigma_1)-z_c)} \frac{1}{g'(t)^{3/2}} \text{Ai}(s)\,ds.$$

As γ is large, the integral term is equivalent to

$$\frac{\gamma^{-1}}{g'(z)^{3/2}} \int_{\gamma(g(z)-z_c)}^{\gamma(g(\sigma_1)-z_c)} \text{Ai}(s)\,ds \sim \frac{\gamma^{-1}}{g'(z)^{3/2}} \big[\text{Ai}\big(1,\gamma(g(\sigma_1)-z_c)\big) - \text{Ai}\big(1,\gamma(g(z)-z_c)\big)\big],$$

where we introduced the primitive $\text{Ai}(1,x)$ of Ai. This leads to

$$\partial_z \phi^{\text{app}}_{f,-}(z) \sim \frac{\gamma^{-1}}{g'(z)^{3/2}} \text{Ai}\big(1,\gamma(g(z)-z_c)\big). \tag{2.33}$$

We integrate $\partial_z \phi^{\text{app}}_{f,-}$ once again and introduce $\text{Ai}(2,x)$, the second primitive of Ai, and obtain

$$\phi^{\text{app}}_{f,-}(z) \sim \frac{\gamma^{-2}}{g'(z)^{5/2}} \text{Ai}\big(2,\gamma(g(z)-z_c)\big). \tag{2.34}$$

The study of $\phi_{f,+}$ is similar. As the asymptotic expansion of $\text{Ai}(z)$ is known, we can compute the asymptotic expansions of $\text{Ai}(1,z)$ and $\text{Ai}(2,z)$; see, for instance, [1, Appendix] or [6, Section 4]. For instance, there hold

$$\begin{aligned} |\text{Ai}(k,z)| &\leq C\langle z\rangle^{-k/2-1/4} e^{-2z^{3/2}/3}, \\ |\text{Ci}(k,z)| &\leq C\langle z\rangle^{-k/2-1/4} e^{2z^{3/2}/3}, \end{aligned} \tag{2.35}$$

for $k \in \mathbb{Z}$ and $z \gg 1$.

2.6. End of proof of Proposition 2.1

By construction, we have approximate solutions $\phi_\pm^{\mathrm{app}}$ to the equation $\mathcal{A}\partial_z\phi = 0$. Namely,

$$|\mathcal{A}\partial_z^2\phi_\pm^{\mathrm{app}}| \le C\nu^N|\phi_\pm^{\mathrm{app}}|,$$

which is (2.4). As $\partial_z\phi_{f,+}^{\mathrm{app}}(z)$ is bounded by $C\nu^{-1/4}\phi_{f,+}^{\mathrm{app}}(z)$, (2.4) combined with (2.3) gives (2.5). We now check the estimates stated in Proposition 2.1.

In fact, we first normalize $\phi_{f,\pm}^{\mathrm{app}}$ by multiplying it with γ^2, again denoted by $\phi_{f,\pm}^{\mathrm{app}}$, giving the expansions (1.17) and (1.18) at $z = 0$. Note in particular that

$$\phi_{f,\pm}^{\mathrm{app}}(0) = \mathcal{O}(1). \tag{2.36}$$

Next, the bounds in (2.6) follow directly from the construction and the estimates (2.23)–(2.24) and (2.29) for z near and away from the critical layers, respectively.

It remains to prove (2.7). For $\nu^{1/4} \ll z \le z'$, estimate (2.32) is exactly (2.7). We thus focus on the case when $z \lesssim \nu^{1/4}$. For $z' \gg \nu^{1/4}$, in view of (2.28) we have

$$\phi_{f,\pm}^{\mathrm{app}}(z') \lesssim C\exp\Big(\pm\int_0^{z'}\Re\mu_f(s)\,ds\Big).$$

As $\mu_f(z)$ is of order $\mathcal{O}(\nu^{-1/4})$ for z of order $\nu^{1/4}$, we obtain for any $0 \le z \le z'$,

$$\left|\frac{\phi_{f,+}^{\mathrm{app}}(z')}{\phi_{f,+}^{\mathrm{app}}(z)}\right| \le C\exp\left|\int_z^{z'}\Re\mu_f(s)\,ds\right| \tag{2.37}$$

for some constant C, and similarly for $\phi_{f,-}$, which gives (2.7).

2.7. Green function for Airy

We will now construct an approximate Green function for the Airy operator. We first construct an approximate Green function for the operator $\mathcal{A} = -\varepsilon\partial_x^2 + (U(x) - c)$. Let

$$G^{\mathrm{Ai}}(x,y) = \frac{1}{\varepsilon W^{\mathrm{Ai}}(x)}\begin{cases}\dfrac{\psi_+^{\mathrm{app}}(y)}{\psi_+^{\mathrm{app}}(x)} & \text{if } y < x,\\[2ex] \dfrac{\psi_-^{\mathrm{app}}(y)}{\psi_-^{\mathrm{app}}(x)} & \text{if } y > x,\end{cases}$$

where W^{Ai} is the Wronskian determinant of $\psi_\pm^{\mathrm{app}}(x)$. Note that the Wronskian determinant is independent of x, since there is no first derivative term in $\mathcal{A}$. In addition, we have

$$W^{\mathrm{Ai}}(x) \sim \gamma = \mathcal{O}(\nu^{-1/4}).$$

In particular, we have

$$G^{\mathrm{Ai}}(x,y) = \mathcal{O}(\nu^{-1/2})\exp\Big(-C\Big|\int_x^y\Re\mu_f(z)\,dz\Big|\Big);$$

therefore G^{Ai} is rapidly decreasing in y on both sides of x, within scales of order $\nu^{1/4}$. By construction,

$$\mathcal{A}G^{\mathrm{Ai}}(x,y) = \delta_x + \mathcal{O}(\nu^{3/4})G^{\mathrm{Ai}}(x,y).$$

We then integrate G^{Ai} twice in y to get an approximate Green function for the Airy operator. More precisely, let

$$G^{\mathrm{Ai},1}(x,y) = \int_y^{+\infty} G^{\mathrm{Ai}}(x,z)\,dz$$

and similarly for $G^{\mathrm{Airy}} = G^{\mathrm{Ai},2}$, the primitive of $G^{\mathrm{Ai},1}$, so that $\partial_y^2 G^{\mathrm{Ai},2}(x,y) = G^{\mathrm{Ai}}(x,y)$. We have

$$G^{\mathrm{Ai},1}(x,y) = \mathcal{O}(\nu^{-1/4})\exp\left(-C\left|\int_x^y \Re\mu_f(z)\,dz\right|\right) + \mathcal{O}(\nu^{-1/4})1_{y<x},$$

and similarly for $G^{\mathrm{Ai},2}$,

$$G^{\mathrm{Ai},2}(x,y) = \mathcal{O}(1)\exp\left(-C\left|\int_x^y \Re\mu_f(z)\,dz\right|\right) + \mathcal{O}(\nu^{-1/4})1_{y<x}x.$$

Note that, taking into account the fast decay of G^{Ai} near x,

$$\begin{aligned}\mathrm{Airy}(G^{\mathrm{Ai},2}) &= \delta_x + \mathcal{O}(\nu^{3/4})G^{\mathrm{Ai}}(x,y)\\ &= \delta_x + \mathcal{O}(\nu^{1/4})\exp\left(-C\left|\int_x^y \Re\mu_f(z)\,dz\right|\right)\\ &= \delta_x + \mathcal{O}(\nu^{1/4}).\end{aligned} \tag{2.38}$$

We define the AirySolve operator by

$$\mathrm{AirySolve}(f)(y) = \int_0^{+\infty} G^{\mathrm{Ai},2}(x,y)f(x)\,dx \tag{2.39}$$

and the associated error term,

$$\mathrm{ErrorAiry}(f)(y) = \int_0^{+\infty} \mathcal{O}(\nu^{3/4})G^{\mathrm{Ai}}(x,y)f(x)\,dx, \tag{2.40}$$

the Airy operator acting on the y variable. These operators will be used in Section 3.5.

3. Rayleigh solutions near critical layers

In this section we construct two approximate solutions $\phi_{s,\pm}^{\mathrm{app}}(z)$ to the Orr–Sommerfeld equation, with slow behaviors as $z \to +\infty$. This together with the approximate solutions $\phi_{f,\pm} = \phi_{f,\pm}^{\mathrm{app}}$ with fast behaviors constructed in the previous section forms a basis of approximate solutions, which are sufficient for the next section to construct the Green function to the Orr–Sommerfeld problem. More precisely, in this section we prove the following lemma.

Lemma 3.1. *For ν small enough, there exist two independent functions $\phi^{\text{app}}_{s,\pm}$ such that*

$$W[\phi^{\text{app}}_{s,+}, \phi^{\text{app}}_{s,-}](z) = 1 + o(1),$$
$$\mathrm{Orr}_{\alpha,c}(\phi^{\text{app}}_{s,+}) = \mathcal{O}(\nu^{1/2}), \quad \mathrm{Orr}_{\alpha,c}(\phi^{\text{app}}_{s,-}) = \mathcal{O}(\nu^{1/2}\log\nu).$$

Furthermore, we have the following expansions in L^∞:

$$\phi^{\text{app}}_{s,-}(z) = e^{-\alpha z}(U - c + \mathcal{O}(\nu^{1/4})),$$
$$\phi^{\text{app}}_{s,+}(z) = e^{-\alpha z}\mathcal{O}(1),$$

as $z \to \infty$. At $z = 0$, there hold

$$\phi^{\text{app}}_{s,-}(0) = -c + \alpha\frac{U_+^2}{U'(0)} + \mathcal{O}(\nu^{1/2}),$$
$$\phi^{\text{app}}_{s,+}(0) = -\frac{1}{U'(0)} + \mathcal{O}(\nu^{1/2}),$$

where $U_+ = \lim_{z\to\infty} U(z)$.

The construction of approximate solutions for the Orr–Sommerfeld equation starts with the construction of approximate solutions for the Rayleigh operator. For small α, the construction of solutions to the Rayleigh equation is a perturbation of the construction for $\alpha = 0$, which is explicit. We will now detail the construction of an inverse of Ray_0 and then of an approximate inverse of Ray_α for small α. For convenience, we recall that

$$\mathrm{Orr}_{\alpha,c}(\phi_\alpha) = -\varepsilon\Delta_\alpha^2\phi_\alpha + (U - c)\Delta_\alpha\phi_\alpha - U''\phi_\alpha,$$
$$\mathrm{Ray}_\alpha(\phi_\alpha) = (U - c)\Delta_\alpha\phi_\alpha - U''\phi_\alpha.$$

In particular, $\mathrm{Ray}_0(\cdot)$ denotes the Rayleigh operator $\mathrm{Ray}_\alpha(\cdot)$ for $\alpha = 0$. Note that

$$\mathrm{Orr}_{\alpha,c}(\phi_\alpha) = \mathrm{Ray}_\alpha(\phi_\alpha) - \varepsilon\Delta_\alpha^2\phi_\alpha. \tag{3.1}$$

3.1. Function spaces

In the next sections we will denote

$$X^\eta = L^\infty_\eta = \left\{ f \mid \sup_{z\ge 0} |f(z)|e^{\eta z} < +\infty \right\}.$$

The highest derivative of the Rayleigh equation vanishes at $z = z_c$, since $U(z_c) = c$. To handle functions which have large derivatives when z is close to $\Re z_c$, we introduce the space Y^η defined as follows. Note that in our analysis, z_c is never real, so $z - z_c$ never vanishes. We are close to a singularity but never reach it.

Precisely, we say that a function f lies in Y^η if for any $z \ge 1$,

$$|f(z)| + |\partial_z f(z)| + |\,\partial_z^2 f(z)| \le Ce^{-\eta z}$$

and if for $z \le 1$,

$$|f(z)| \le C(1 + |z - z_c|\,|\log(z - z_c)|),$$
$$|\partial_z f(z)| \le C(1 + |\log(z - z_c)|),$$
$$|\partial_z^2 f(z)| \le C(1 + |z - z_c|^{-1}).$$

The best constant C in the previous bounds defines the norm $\|f\|_{Y^\eta}$. Note that $Y^\eta \subset X^\eta$.

3.2. Rayleigh equation when $\alpha = 0$

In this section we study the Rayleigh operator Ray_0. More precisely, we solve

$$\mathrm{Ray}_0(\phi) = (U - c)\partial_z^2\phi - U''\phi = f. \tag{3.2}$$

The main observation is that

$$\mathrm{Ray}_0(U - c) = 0.$$

Therefore,

$$\phi_{1,0} = U - c$$

is a first explicit solution. The second one is obtained through the Wronskian equation

$$W[\phi_{1,0}, \phi_{2,0}] = 1.$$

This leads to the following lemma, whose proof is given in [6, Lemma 3.2].

Lemma 3.2 ([6,7]). *Assume that $\Im c \neq 0$. There exist two independent solutions $\phi_{1,0} = U - c$ and $\phi_{2,0}$ of $\mathrm{Ray}_0(\phi) = 0$ with unit Wronskian determinant*

$$W(\phi_{1,0}, \phi_{2,0}) := \partial_z\phi_{2,0}\phi_{1,0} - \phi_{2,0}\partial_z\phi_{1,0} = 1.$$

Furthermore, there exist smooth functions $P(z)$ and $Q(z)$ with $P(z_c) \neq 0$ and $Q(z_c) \neq 0$, so that, near $z = z_c$,

$$\phi_{2,0}(z) = P(z) + Q(z)(z - z_c)\log(z - z_c). \tag{3.3}$$

Moreover,

$$\phi_{2,0}(0) = -\frac{1}{U'(0)}$$

and

$$\partial_z\phi_{2,0}(z) + \frac{1}{U_+} \in Y^{\eta_1} \tag{3.4}$$

for some $\eta_1 > 0$.

Let $\phi_{1,0}, \phi_{2,0}$ be constructed as in Lemma 3.2. Then the Green function $G_{R,0}(x,z)$ of the Ray_0 operator can be explicitly defined by

$$G_{R,0}(x,z) = \begin{cases} (U(x) - c)^{-1}\phi_{1,0}(z)\phi_{2,0}(x) & \text{if } z > x, \\ (U(x) - c)^{-1}\phi_{1,0}(x)\phi_{2,0}(z) & \text{if } z < x. \end{cases}$$

The inverse of Ray_0 is explicitly given by

$$\mathrm{RaySolver}_0(f)(z) := \int_0^{+\infty} G_{R,0}(x,z) f(x)\, dx. \tag{3.5}$$

Note that the Green kernel $G_{R,0}$ is singular at z_c. The following lemma asserts that the operator $\mathrm{RaySolver}_0(\cdot)$ is in fact well defined from X^η to Y^0, which in particular shows that $\mathrm{RaySolver}_0(\cdot)$ gains two derivatives, but loses the fast decay at infinity. It transforms a bounded function into a function which behaves like $(z-z_c)\log(z-z_c)$ near z_c.

Lemma 3.3. *Assume that* $\Im c \neq 0$. *For any* $f \in X^\eta$, $\mathrm{RaySolver}_0(f)$ *is a solution to the Rayleigh problem* (3.2). *In addition,* $\mathrm{RaySolver}_0(f) \in Y^0$, *and there holds*

$$\|\mathrm{RaySolver}_0(f)\|_{Y^0} \le C(1 + |\log \Im c|)\|f\|_{X^\eta}$$

for some constant C.

Proof. Using (3.4), it is clear that $\phi_{1,0}(z)$ and $\phi_{2,0}(z)/(1+z)$ are uniformly bounded. Thus, considering the cases $x < 1$ and $x > 1$, we obtain

$$|G_{R,0}(x,z)| \le C \max\big\{(1+x), |x-z_c|^{-1}\big\}. \tag{3.6}$$

That is, $G_{R,0}(x,z)$ grows linearly in x for large x and has a singularity of order $|x-z_c|^{-1}$ when x is near z_c. As $|f(z)| \le e^{-\eta z}\|f\|_{X^\eta}$, the integral (3.5) is well defined and we have

$$\begin{aligned} |\mathrm{RaySolver}_0(f)(z)| &\le C\|f\|_{X^\eta} \int_0^\infty e^{-\eta x} \max\big\{(1+x), |x-z_c|^{-1}\big\}\, dx \\ &\le C(1+|\log \Im c|)\|f\|_{X^\eta}, \end{aligned}$$

in which we used the fact that $\Im z_c \approx \Im c$.

To bound the derivatives, we need to check the order of the singularity for z near z_c. We note that

$$|\partial_z \phi_{2,0}| \le C(1 + |\log(z-z_c)|),$$

and hence

$$|\partial_z G_{R,0}(x,z)| \le C \max\big\{(1+x), |x-z_c|^{-1}\big\}(1 + |\log(z-z_c)|).$$

Thus, $\partial_z \mathrm{RaySolver}_0(f)(z)$ behaves as $1 + |\log(z-z_c)|$ near the critical layer. In addition, from the Ray_0 equation we have

$$\partial_z^2(\mathrm{RaySolver}_0(f)) = \frac{U''}{U-c}\mathrm{RaySolver}_0(f) + \frac{f}{U-c}. \tag{3.7}$$

This proves that $\mathrm{RaySolver}_0(f) \in Y^0$ and gives the desired bound. ∎

3.3. Approximate Green function when $\alpha \ll 1$

Let $\phi_{1,0}$ and $\phi_{2,0}$ be the two solutions of $\mathrm{Ray}_0(\phi) = 0$ that are constructed above, in Lemma 3.2. We now construct an approximate Green function to the Rayleigh equation for $\alpha > 0$. To proceed, let us introduce

$$\phi_{1,\alpha} = \phi_{1,0} e^{-\alpha z}, \quad \phi_{2,\alpha} = \phi_{2,0} e^{-\alpha z}. \tag{3.8}$$

A direct computation shows that their Wronskian determinant equals

$$W[\phi_{1,\alpha}, \phi_{2,\alpha}] = \partial_z \phi_{2,\alpha} \phi_{1,\alpha} - \phi_{2,\alpha} \partial_z \phi_{1,\alpha} = e^{-2\alpha z}.$$

Note that the Wronskian vanishes at infinity since both functions have the same behavior at infinity. In addition,

$$\mathrm{Ray}_\alpha(\phi_{j,\alpha}) = -2\alpha(U - c)\partial_z \phi_{j,0} e^{-\alpha z}. \tag{3.9}$$

We are then led to introduce an approximate Green function $G_{R,\alpha}(x, z)$, defined by

$$G_{R,\alpha}(x, z) = \begin{cases} (U(x) - c)^{-1} e^{-\alpha(z-x)} \phi_{1,0}(z) \phi_{2,0}(x) & \text{if } z > x, \\ (U(x) - c)^{-1} e^{-\alpha(z-x)} \phi_{1,0}(x) \phi_{2,0}(z) & \text{if } z < x. \end{cases}$$

Again, like $G_{R,0}(x, z)$, the Green function $G_{R,\alpha}(x, z)$ is "singular" near z_c. By a view of (3.9),

$$\mathrm{Ray}_\alpha(G_{R,\alpha}(x, z)) = \delta_x + E_{R,\alpha}(x, z) \tag{3.10}$$

for each fixed x, where the error kernel $E_{R,\alpha}(x, z)$ is defined by

$$E_{R,\alpha}(x, z) = \begin{cases} -2\alpha(U(z) - c)(U(x) - c)^{-1} e^{-\alpha(z-x)} \partial_z \phi_{1,0}(z) \phi_{2,0}(x) & \text{if } z > x, \\ -2\alpha(U(z) - c)(U(x) - c)^{-1} e^{-\alpha(z-x)}\ \phi_{1,0}(x) \partial_z \phi_{2,0}(z) & \text{if } z < x. \end{cases}$$

We then introduce an approximate inverse of the operator Ray_α defined by

$$\mathrm{RaySolver}_\alpha(f)(z) := \int_0^{+\infty} G_{R,\alpha}(x, z) f(x)\, dx \tag{3.11}$$

and the related error operator

$$\mathrm{Err}_{R,\alpha}(f)(z) := 2\alpha(U(z) - c) \int_0^{+\infty} E_{R,\alpha}(x, z) f(x)\, dx. \tag{3.12}$$

Lemma 3.4. *Assume that $\Im c > 0$. For any $f \in X^\eta$ with $\alpha < \eta$, the function* $\mathrm{RaySolver}_\alpha(f)$ *is well defined in Y^α, and satisfies*

$$\mathrm{Ray}_\alpha(\mathrm{RaySolver}_\alpha(f)) = f + \mathrm{Err}_{R,\alpha}(f).$$

Furthermore, there hold

$$\|\mathrm{RaySolver}_\alpha(f)\|_{Y^\alpha} \le C(1 + |\log \Im c|)\|f\|_{X^\eta} \tag{3.13}$$

and

$$\|\mathrm{Err}_{R,\alpha}(f)\|_{Y^\eta} \le C|\alpha|(1+|\log(\Im c)|)\|f\|_{X^\eta} \tag{3.14}$$

for some universal constant C.

Proof. The proof follows that of Lemma 3.3. Indeed, since

$$G_{R,\alpha}(x,z) = e^{-\alpha(z-x)} G_{R,0}(x,z),$$

the behavior near the critical layer $z = z_c$ is the same for these two Green functions, and hence the proof of (3.13) and (3.14) near the critical layer identically follows from that of Lemma 3.3.

Let us check the behavior at infinity. We can normalize to assume $\|f\|_{X^\eta} = 1$. Using (3.6), we get

$$|G_{R,\alpha}(x,z)| \le C e^{-\alpha(z-x)} \max\big\{(1+x), |x-z_c|^{-1}\big\}.$$

Hence, by definition,

$$|\mathrm{RaySolver}_\alpha(f)(z)| \le C e^{-\alpha z} \int_0^\infty e^{\alpha x} e^{-\eta x} \max\big\{(1+x), |x-z_c|^{-1}\big\}\, dx,$$

which is bounded by $C(1+|\log \Im c|)e^{-\alpha z}$, upon recalling that $\alpha < \eta$. This proves the right exponential decay of $\mathrm{RaySolver}_\alpha(f)(z)$ at infinity, for all $f \in X^\eta$.

The estimates on $\mathrm{Err}_{R,\alpha}$ are the same, once we notice that $(U(z)-c)\partial_z \phi_{2,0}$ has the same bound as that for $\phi_{2,0}$, and similarly for $\phi_{1,0}$. ∎

Remark 3.5. For $f(z) = (U-c)g(z)$ with $g \in X^\eta$, the same proof as done for Lemma 3.4 yields

$$\begin{aligned} \|\mathrm{RaySolver}_\alpha(f)\|_{Y^\alpha} &\le C\|g\|_{X^\eta}, \\ \|\mathrm{Err}_{R,\alpha}(f)\|_{Y^\eta} &\le C|\alpha|\|g\|_{X^\eta}, \end{aligned} \tag{3.15}$$

which are slightly better estimates than (3.13) and (3.14).

3.4. Construction of $\phi_{s,-}^{\mathrm{app}}$

Let us start with the decaying solution $\phi_{s,-}$. We note that

$$\psi_0 = e^{-\alpha z}(U-c)$$

is only an $\mathcal{O}(\alpha)$ smooth approximate solution to the Rayleigh equation, leaving an error of approximation

$$e_0 := \mathrm{Ray}_\alpha(\psi_0) = -2\alpha(U-c)U' e^{-\alpha z},$$

which is of order α. Similarly, a direct computation (see (3.1)) shows that

$$\mathrm{Orr}_{\alpha,c}(\psi_0) = e_0 - \varepsilon \Delta_\alpha^2 \psi_0 = \mathcal{O}(\alpha + |\varepsilon|) = \mathcal{O}(\nu^{1/4})$$

upon recalling $\varepsilon = \nu / i\alpha$, with $\alpha = \mathcal{O}(\nu^{1/4})$. This is not sufficient for our purposes, and we have to go to the next order. We therefore introduce

$$\psi_1 = -\operatorname{RaySolver}_\alpha(e_0).$$

Note that ψ_1 is of order $\mathcal{O}(\alpha)$ in Y^η, and behaves like $\alpha(z - z_c)\log(z - z_c)$ near z_c. In particular, ψ_1 is not a smooth function near z_c. Its fourth-order derivative behaves like $\alpha/(z - z_c)^3$ in the critical layer. We have

$$\operatorname{Orr}_{\alpha,c}(\psi_1) = -\varepsilon(\partial_z^2 - \alpha^2)^2\psi_1 + \operatorname{Ray}_\alpha(\psi_1),$$

hence

$$\operatorname{Orr}_{\alpha,c}(\psi_0 + \psi_1) = -\varepsilon(\partial_z^2 - \alpha^2)^2\psi_1 - \varepsilon\Delta_\alpha^2\psi_0 + \operatorname{Err}_{R,\alpha}(e_0). \tag{3.16}$$

Note that, using (3.14), we have

$$\operatorname{Err}_{R,\alpha}(e_0) = O(\alpha^2|\log(\alpha)|)_{Y^\eta}, \tag{3.17}$$

where $\log\alpha$ corresponds to the log loss of $\log\Im c$, with $|\Im c| \ge \sigma_0\nu^{1/4}$. Moreover, using the Rayleigh equation,

$$(\partial_z^2 - \alpha^2)\psi_1 = \frac{\operatorname{Ray}_\alpha(\psi_1) - U''\psi_1}{U - c},$$

hence we compute

$$\varepsilon(\partial_z^2 - \alpha^2)^2\psi_1 = \varepsilon(\partial_z^2 - \alpha^2)\Big\{\frac{\operatorname{Ray}(\psi_1) - U''\psi_1}{U - c}\Big\}. \tag{3.18}$$

In view of Remark 3.5, $\operatorname{Ray}_\alpha(\psi_1)$ and $U''\psi_1$ are of order $\mathcal{O}(\alpha)$ in X^η. We thus have

$$\varepsilon\alpha^2\Big|\frac{\operatorname{Ray}(\psi_1) - U''\psi_1}{U - c}\Big| \le C\frac{\varepsilon\alpha^2}{|z - z_c|} \le C\frac{\varepsilon\alpha^2}{|\Im c|} \le C\varepsilon\alpha = \mathcal{O}(\nu)_{X^\eta}.$$

Next we expand ∂_z^2 in (3.18) which gives the three terms

$$\varepsilon\frac{\partial_z^2\operatorname{Ray}(\psi_1) - \partial_z^2(U''\psi_1)}{U - c} - 2\varepsilon U'\frac{\partial_z\operatorname{Ray}(\psi_1) - \partial_z(U''\psi_1)}{(U - c)^2}$$
$$+ \varepsilon(\operatorname{Ray}(\psi_1) - U''\psi_1)\partial_z^2\frac{1}{U - c}.$$

We start with the first term. As $\operatorname{Ray}_\alpha(\psi_1)$ and ψ_1 are of order $\mathcal{O}(\alpha)$ in Y^η, this first quantity is bounded by

$$C\varepsilon\Big(1 + \frac{\alpha|\log\Im c|}{|z - z_c|} + \frac{\alpha}{|z - z_c|^2}\Big) \le C\frac{\varepsilon\alpha}{|\Im c|^2} = \mathcal{O}(\alpha^2). \tag{3.19}$$

The second term is treated similarly, while the third term in the expansion of (3.18) is

$$\varepsilon\big[\operatorname{Ray}(\psi_1) - U''\psi_1\big](z - z_c)^{-3},$$

which is bounded by $\mathcal{O}(\alpha)$. Thus, putting these into (3.16), we get

$$\mathrm{Orr}_{\alpha,c}(\psi_0 + \psi_1) = E, \tag{3.20}$$

in which we can write the error term as

$$E = E_1 + E_2, \quad E_1 = \mathcal{O}(\alpha^2), \quad E_2 \le C\varepsilon\alpha|z - z_c|^{-3}.$$

This error term E_2 is therefore too large for our purposes. However, it is located near $z = z_c$, namely in the critical layer. We therefore correct $\psi_0 + \psi_1$ by ψ_2 by approximately inverting the Airy operator in this layer. More precisely, let

$$\psi_2 = -\,\mathrm{AirySolve}(E_2),$$

which will create an error term

$$\begin{aligned} E_3 &= \mathrm{Orr}_{\alpha,c}(\psi_2) + E_2 \\ &= \mathrm{Airy}(\psi_2) + \mathrm{OrrAiry}(\psi_2) + E_2 \\ &= \mathrm{OrrAiry}(\psi_2) + \mathrm{ErrorAiry}(E_2). \end{aligned}$$

Let us now bound ψ_2. Using (2.39), we have

$$|\psi_2(y)| \le C\varepsilon\alpha \int_0^{+\infty} |x - z_c|^{-3}\big(e^{-|\int_x^y \Re\mu_f(z)\,dz|} + \mathcal{O}(\nu^{-1/4})1_{y<x}x\big)\,dx.$$

Writing $1_{y<x}x = 1_{y<x}(x - z_c) + 1_{y<x}z_c$, we thus have

$$\begin{aligned} |\psi_2(y)| &\le C\varepsilon\alpha \int_0^{+\infty} \big(|x - z_c|^{-3} + \nu^{-1/4}|x - z_c|^{-2}\big)\,dx \\ &\le C\varepsilon\alpha\big(|\Im c|^{-2} + \nu^{-1/4}|\Im c|^{-1}\big) = \mathcal{O}(\alpha^2). \end{aligned}$$

This together with (2.3) yields $\mathrm{OrrAiry}(\psi_2) = \mathcal{O}(\alpha^2)$. Similarly, using (2.38), we get

$$\mathrm{ErrorAiry}(E_2)(z) \le C\varepsilon\alpha \int_0^{+\infty} |x - z_c|^{-3}\mathcal{O}(\nu^{1/4})\,dx = \mathcal{O}(\alpha^3).$$

Therefore, we have

$$\mathrm{Orr}_{\alpha,c}(\psi_0 + \psi_1 + \psi_2) = \mathcal{O}(\alpha^2).$$

We define

$$\phi_{s,-}^{\mathrm{app}} = \psi_0 + \psi_1 + \psi_2.$$

To end this section we compute $\psi(0)$. By definition,

$$\begin{aligned} \psi_1(0) &= -\,\mathrm{RaySolver}_\alpha(e_0)(0) = -\phi_{2,\alpha}(0) \int_0^{+\infty} e^{2\alpha x}\phi_{1,\alpha}(x)\frac{e_0(x)}{U(x) - c}\,dx \\ &= -2\alpha\phi_{2,0}(0) \int_0^{+\infty} U'(U - c)\,dz = \alpha\phi_{2,0}(0)\big[(U - c)^2\big]_0^{+\infty} \\ &= -\alpha\phi_{2,0}(0)\big[(U_+ - c)^2 - c^2\big] = \alpha\frac{U_+}{U'(0)}(U_+ - 2c). \end{aligned}$$

From the definition we have

$$\phi_{s,-}(0) = U_0 - c + \psi_1(0) + \mathcal{O}(\alpha^2).$$

This proves the lemma, using that $U_0 - c = \mathcal{O}(z_c)$.

3.5. Construction of $\phi_{s,+}^{\mathrm{app}}$

We first start with $\phi_{2,\alpha} = \phi_{2,0}e^{-\alpha z}$, which is an approximate solution of the Rayleigh equation, up to an $\mathcal{O}(\alpha)$ error term. Precisely, noting $\mathrm{Ray}_0(\phi_{2,0}) = 0$, we compute the error of approximation

$$e_1 = \mathrm{Ray}_\alpha(\phi_{2,\alpha}) = -2\alpha(U-c)\partial_z\phi_{2,0}e^{-\alpha z} = \mathcal{O}(\alpha),$$

in which there is no logarithmic loss $\partial_z\phi_{2,0}$, since $U - c$ vanishes at $z = z_c$. Next we introduce

$$\phi_3 = -\mathrm{RaySolver}_\alpha(e_1).$$

Then, using (3.14),

$$\mathrm{Ray}_\alpha(\phi_{2,\alpha} + \phi_3) = -\mathrm{Err}_{R,\alpha}(e_1) = \mathcal{O}(\alpha^2). \tag{3.21}$$

Let us set

$$\phi_{s,+}^{\mathrm{app}} = \phi_{2,\alpha} + \phi_3.$$

By construction, $\phi_{2,0}$ is bounded in Y^η, and so is $\phi_{2,\alpha} = \phi_{2,0}e^{-\alpha z}$. On the other hand, using Lemma 3.4 and the bound (3.13), the function $\phi_3 = -\mathrm{RaySolver}_\alpha(e_1)$ is of order α in Y^η. That is, $\phi_{s,+}^{\mathrm{app}}$ is bounded in Y^η, and thus behaves like $(z - z_c)\log(z - z_c)$ near z_c, due to $\phi_{2,0}(z)$. In addition, using (3.21) we have

$$\mathrm{Orr}_{\alpha,c}(\phi_{s,+}^{\mathrm{app}}) = -\varepsilon(\partial_z^2 - \alpha^2)^2\phi_{s,+}^{\mathrm{app}} + \mathcal{O}(\alpha^2).$$

Note that away from $z = z_c$, the right-hand side is of order $\mathcal{O}(|\varepsilon| + \alpha^2) = \mathcal{O}(\nu^{1/2})$. Near $z = z_c$, we again use the Rayleigh equation (3.21) to get

$$(\partial_z^2 - \alpha^2)\phi_{s,+}^{\mathrm{app}} = \frac{U''}{U-c}\phi_{s,+}^{\mathrm{app}} + \mathcal{O}(\alpha^2),$$

which gives

$$(\partial_z^2 - \alpha^2)^2\phi_{s,+}^{\mathrm{app}} = (\partial_z^2 - \alpha^2)\Big(\frac{U''}{U-c}\phi_{s,+}^{\mathrm{app}}\Big).$$

The worst term in the right-hand side is

$$\Big[\partial_z^2\Big(\frac{1}{U-c}\Big)\Big]U''\phi_{s,+}^{\mathrm{app}},$$

which is of order $(\Im z_c)^{-3}$ times $\phi_{s,+}^{\mathrm{app}}$, near $z = z_c$. Hence, recalling $|\Im c| \geq \sigma_0\nu^{1/4}$, $\mathrm{Orr}_{\alpha,c}(\phi_{s,+}^{\mathrm{app}})$ is of order

$$\frac{\varepsilon}{(\Im z_c)^3}\phi_{s,+}^{\mathrm{app}} \sim \frac{\nu}{\alpha}\frac{1}{\nu^{3/4}}\phi_{s,+}^{\mathrm{app}} \sim \phi_{s,+}^{\mathrm{app}} = \phi_{2,\alpha} + \phi_3,$$

which is of order $(z - z_c)\log(z - z_c)$, coming from $\phi_{2,0}(z)$. That is, similarly to (3.20), we obtain

$$\mathrm{Orr}_{\alpha,c}(\phi_{s,+}^{\mathrm{app}}) = E_1 + E_2$$

with $E_1 = \mathcal{O}(\alpha^2)$, while $E_2 = \mathcal{O}(\varepsilon)(z - z_c)^{-3}\phi_{2,0}$, which is a $\log\alpha$ loss as compared to (3.20) for the construction of $\phi_{s,-}^{\mathrm{app}}$. The remaining construction to correct the approximation near the critical layer by approximately inverting the Airy operator follows identically to the previous section.

4. Green function for the Orr–Sommerfeld equations

Having constructed slow and fast approximate modes $\phi_{s,\pm}^{\mathrm{app}}$ and $\phi_{f,\pm}^{\mathrm{app}}$ in the previous two sections, we are now ready to construct an approximate Green function G^{app}. We will decompose this Green function into two components

$$G^{\mathrm{app}} = G_i^{\mathrm{app}} + G_b^{\mathrm{app}},$$

where G_i^{app} takes care of the source term δ_x and where G_b^{app} takes care of the boundary conditions.

4.1. Interior approximate Green function

We look for $G_i^{\mathrm{app}}(x, y)$ of the form

$$G_i^{\mathrm{app}}(x, y) = a_+(x)\phi_{s,+}^{\mathrm{app}}(y) + b_+(x)\frac{\phi_{f,+}^{\mathrm{app}}(y)}{\phi_{f,+}^{\mathrm{app}}(x)} \quad \text{for } y < x,$$
$$G_i^{\mathrm{app}}(x, y) = a_-(x)\phi_{s,-}^{\mathrm{app}}(y) + b_-(x)\frac{\phi_{f,-}^{\mathrm{app}}(y)}{\phi_{f,-}^{\mathrm{app}}(x)} \quad \text{for } y > x,$$

where $\phi_{f,\pm}^{\mathrm{app}}(x)$ play the role of normalization constants. Let

$$F_\pm = \phi_{f,\pm}^{\mathrm{app}}(x)$$

and let

$$v(x) = (-a_-(x), a_+(x), -b_-(x), b_+(x)).$$

By definition of a Green function, G^{app}, $\partial_y G^{\mathrm{app}}$, and $\partial_y^2 G^{\mathrm{app}}$ are continuous at $x = y$, whereas $-\varepsilon\partial_y^3 G^{\mathrm{app}}$ has a unit jump at $x = y$. Let

$$M = \begin{pmatrix} \phi_{s,-} & \phi_{s,+} & \phi_{f,-}/F_- & \phi_{f,+}/F_+ \\ \partial_y\phi_{s,-}/\mu_f & \partial_y\phi_{s,+}/\mu_f & \partial_y\phi_{f,-}/(F_-\mu_f) & \partial_y\phi_{f,+}/(F_+\mu_f) \\ \partial_y^2\phi_{s,-}/\mu_f^2 & \partial_y^2\phi_{s,+}/\mu_f^2 & \partial_y^2\phi_{f,-}/(F_-\mu_f^2) & \partial_y^2\phi_{f,+}/(F_+\mu_f^2) \\ \partial_y^3\phi_{s,-}/\mu_f^3 & \partial_y^3\phi_{s,+}/\mu_f^3 & \partial_y^3\phi_{f,-}/(F_-\mu_f^3) & \partial_y^3\phi_{f,+}/(F_+\mu_f^3) \end{pmatrix}, \tag{4.1}$$

where the functions $\phi_{s,\pm} = \phi_{s,\pm}^{\text{app}}$ and $\phi_{f,\pm} = \phi_{f,\pm}^{\text{app}}$ and their derivatives are evaluated at $y = x$, and where the various factors μ_f are introduced to renormalize the lines of M. Then

$$M v = (0, 0, 0, -1/(\varepsilon \mu_f^3)). \tag{4.2}$$

We will evaluate M^{-1} using the following block structure. Let A, B, C, and D be the two-by-two matrices defined by

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We will prove that C is small, that D is invertible, and that A is related to Rayleigh equations. This will allow the construction of an explicit approximate inverse, and by iteration, of the inverse of M. Let us detail these points.

Let us first study D. Following (2.6), for $z \gg \nu^{1/4}$,

$$D = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + o(1),$$

hence D is invertible and

$$D^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} + o(1).$$

For z of order $\nu^{1/4}$, we note that F_+ and F_- are of order $\mathcal{O}(1)$,

$$\partial_y^2 \phi_{f,-} = \gamma^2 \frac{1}{(g')^2(x)} \frac{\text{Ai}(\gamma g(x))}{\text{Ai}(2, \gamma g(x))} + \mathcal{O}(\gamma),$$

and similarly for $\partial_y \phi_{f,-}$ and $\partial_y \phi_{f,+}$. Note that γ^2/μ_f^2, γ^3/μ_f^3, $\text{Ai}(2, \gamma g(x))$, and $\text{Ci}(2, \gamma g(x))$ are of order $\mathcal{O}(1)$. As $g'(z_c) = 1$, up to normalization of lines and columns, D is close to

$$\begin{pmatrix} \text{Ai} & \text{Ci} \\ \text{Ai}' & \text{Ci}' \end{pmatrix},$$

which is invertible by definition of the special Airy functions Ai and Ci.

Let us turn to C. The worst term in C is that involving $\phi_{s,+}$ because of its logarithmic singularity. More precisely, $\partial_y^k \phi_{s,+}$ behaves like $(z - z_c)^{1-k}$ and is bounded by $|\Im c|^{1-k} \sim \nu^{(1-k)/4}$ for $k = 2, 3$. Hence, as $\mu_f^{-1} = \mathcal{O}(\nu^{1/4})$,

$$C = \begin{pmatrix} \mathcal{O}(\nu^{1/2}) & \mathcal{O}(\nu^{1/2}(z - z_c)^{-1}) \\ \mathcal{O}(\nu^{3/4}) & \mathcal{O}(\nu^{3/4}(z - z_c)^{-2}) \end{pmatrix}.$$

Note that $A = A_1 A_2$ with

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \mu_f^{-1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} \phi_{s,-}^{\text{app}} & \phi_{s,+}^{\text{app}} \\ \partial_y \phi_{s,-}^{\text{app}} & \partial_y \phi_{s,+}^{\text{app}} \end{pmatrix}.$$

We have

$$A_2^{-1} = \frac{1}{\det(A_2)} \begin{pmatrix} \partial_y \phi_{s,+}^{\mathrm{app}} & -\phi_{s,+}^{\mathrm{app}} \\ -\partial_y \phi_{s,-}^{\mathrm{app}} & \phi_{s,-}^{\mathrm{app}} \end{pmatrix}.$$

The determinant A_2 is the Wronskian of $\phi_{s,\pm}^{\mathrm{app}}$ and hence a perturbation of the Wronskian of $\phi_{1,\alpha}$ and $\phi_{2,\alpha}$, which equals $e^{-\alpha x}$. We distinguish between $x < \alpha^{1/2}$ and $x > \alpha^{1/2}$. In the second case, $\mathrm{Orr}_{c,\alpha}$ is a small perturbation of a constant-coefficient fourth-order operator. The Green function may therefore be explicitly computed. We will not detail the computations here and focus on the case where $x < \alpha^{1/2}$. In this case the Wronskian is of order $\mathcal{O}(1)$. As a consequence,

$$A_2^{-1} = \begin{pmatrix} \mathcal{O}(\log|z - z_c|) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(z - z_c) \end{pmatrix}$$

and

$$A^{-1} = \begin{pmatrix} \mathcal{O}(\log|z - z_c|) & \mathcal{O}(\mu_f) \\ \mathcal{O}(1) & \mathcal{O}(\mu_f(z - z_c)) \end{pmatrix}.$$

We now observe that the matrix M has an approximate inverse

$$\widetilde{M} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}$$

in the sense that $M\widetilde{M} = \mathrm{Id} + N$, where

$$N = \begin{pmatrix} 0 & 0 \\ CA^{-1} & -CA^{-1}BD^{-1} \end{pmatrix}.$$

Now a direct calculation shows that

$$CA^{-1} = \mathcal{O}(\nu^{1/4})$$

since $\Im z_c = \mathcal{O}(\nu^{1/4})$. As D^{-1} and B are uniformly bounded, $N = \mathcal{O}(\nu^{1/4})$. In particular, $(\mathrm{Id} + N)^{-1}$ is well defined and

$$\begin{aligned} M^{-1} &= \widetilde{M}(\mathrm{Id} + N)^{-1} \\ &= \widetilde{M} \sum_{n \geq 0} N^n = \widetilde{M} \sum_{n \geq 0} \mathcal{O}(\nu^{n/4}). \end{aligned}$$

Note that the two first lines of N^n vanish. The other lines are at most of order $\mathcal{O}(\nu^{1/4})$. Therefore,

$$(\mathrm{Id} + N)^{-1}(0, 0, 0, 1/\nu\mu_f^3) = \big(0, 0, \mathcal{O}(1/\nu\mu_f^4), 1/\nu\mu_f^3\big).$$

As D^{-1} is bounded and $A^{-1}BD^{-1}$ is of order $\mathcal{O}(\mu_f)$, we obtain that $a_\pm$ and $b_\pm$ are respectively bounded by $C/\nu\mu_f^2$ and $C/\nu\mu_f^3$.

4.2. Boundary approximate Green function

We now add to G_i^{app} another Green function G_b^{app} to handle the boundary conditions. We look for G_b^{app} in the form

$$G_b^{\text{app}}(y) = d_s \phi_{s,-}(y) + d_f \frac{\phi_{f,-}(y)}{\phi_{f,-}(0)},$$

where $\phi_{f,-}(0)$ in the denominator is a normalization constant, and look for d_s and d_f such that

$$G_i^{\text{app}}(x,0) + G_b^{\text{app}}(0) = \partial_y G_i^{\text{app}}(x,0) + \partial_y G_b^{\text{app}}(0) = 0. \tag{4.3}$$

Let

$$M = \begin{pmatrix} \phi_{s,-} & \phi_{f,-}/\phi_{f,-}(0) \\ \partial_y \phi_{s,-} & \partial_y \phi_{f,-}/\phi_{f,-}(0) \end{pmatrix},$$

the functions being evaluated at $y = 0$. Then (4.3) can be rewritten as

$$Md = -(G_i^{\text{app}}(x,0), \partial_y G_i^{\text{app}}(x,0)),$$

where $d = (d_s, d_f)$. Note that

$$(G_i^{\text{app}}(x,0), \partial_y G_i^{\text{app}}(x,0)) = Q(a_+, b_+),$$

where

$$Q = \begin{pmatrix} \phi_{s,+}(0) & 1 \\ \partial_y \phi_{s,+}(0) & \partial_y \phi_{f,+}(0)/\phi_{f,+}(0) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(1) & 1 \\ \mathcal{O}(\log(\nu)) & \mathcal{O}(\nu^{-1/4}) \end{pmatrix}.$$

By construction,

$$d = -M^{-1} Q(a_+, b_+). \tag{4.4}$$

We have

$$M^{-1} = \frac{1}{\det(M)} \begin{pmatrix} \partial_y \phi_{f,-}(0)/\phi_{f,-}(0) & -1 \\ -\partial_y \phi_{s,-}(0) & \phi_{s,-}(0) \end{pmatrix}.$$

The determinant of M equals

$$\det M = \frac{W[\phi_{s,-}, \phi_{f,-}](0)}{\phi_{f,-}(0)}$$

and does not vanish by assumption. Therefore

$$M^{-1} = \begin{pmatrix} \mathcal{O}(\nu^{-1/4}) & -1 \\ \mathcal{O}(1) & \mathcal{O}(\nu^{1/4}) \end{pmatrix}.$$

As a consequence,

$$M^{-1} Q = \begin{pmatrix} \mathcal{O}(\nu^{-1/4}) & \mathcal{O}(\nu^{-1/4}) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix}.$$

4.3. Exact Green function

Once we have an approximate Green function, we obtain the exact Green function by a standard iterative scheme, following the strategy developed in [8]. The stated bounds follow from those obtained for the approximate Green function.

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