In the third of his five "Acts," on Curvature, there is too much for my taste on the curvature of curves (as in many texts). On the other hand, I miss the interpretation of curvature and mean curvature as the rate of change of length or area under smooth deformations.

In my opinion the final Act V on Forms does not belong in this book, which is already long, despite the value of Cartan's moving frames. Needham himself writes:

> … our aim is to make Forms accessible to the widest possible range of readers, even if their primary interest is not Differential Geometry.

So perhaps give them a separate nice little volume. Actually I think that Act V is not only an unnecessary distraction, but also of lower quality than the first four Acts. In my opinion, it would better start with something familiar, such as the kind of integrand $f\,dx + g\,dy$ that occurs in line integrals and Green's Theorem, and explain it as a covector field. And I found it confusing the way Needham conflates covectors and covector fields. And I found the development too slow.

There are lots of nice exercises throughout, though no solutions. The index is very helpful.

Through it all flows a generous geometric spirit, a yearning after geometry fulfilled. This magnificent book, one of a kind, merits the close attention and tender appreciation of all true scholars and lovers of geometry.

Tristan Needham, *Visual Differential Geometry and Forms. A Mathematical Drama in Five Acts*. Princeton University Press, 2021, 584 pages, Hardback ISBN 978-0-6912-0369-0.

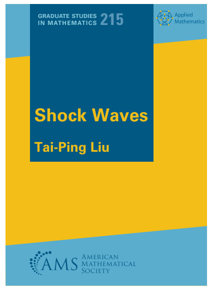
Frank Morgan is Atwell Professor of Mathematics, Emeritus, Williams College. Frank Morgan's proof with colleagues and students of the Double Bubble Conjecture is featured at the National Science Foundation Discoveries site. He has 200 publications, including six books: *Geometric Measure Theory*; *Calculus*; *Riemannian Geometry*; *The Math Chat Book*, based on his live, call-in Math Chat TV show and Math Chat column; *Real Analysis*; and *Real Analysis and Applications*. He has served as vice president of the Mathematical Association of America and of the American Mathematical Society and as editor-in-chief of *Notices of the American Mathematical Society*. He currently lives on the beach in Ocean City, New Jersey, and goes in the ocean every day, summer and winter.

fmorgan@williams.edu

DOI 10.4171/MAG/110

Shock Waves by Tai-Ping Liu

Reviewed by Denis Serre



Tai-Ping Liu, a prominent researcher in Hyperbolic Conservation Laws, offers his own view of the field in this 430-page long monograph. This text comes after several other ones that have been available on the market after the seminal *Shock waves and reaction-diffusion equations* by his advisor J. Smoller (Springer, 1983). Each one has its own merit, and Liu's one stands out with an original approach, driven by his research interests. It is not just another book on a well-known domain, but a rather personal account of the topic. The title itself singles out this text among the existing literature, by focusing on solutions and their qualitative aspects, rather than on the governing structures, conservation laws or PDEs. This is in tune with T.-P. Liu's perception of mathematical activity, which is closer to V. I. Arnold's[1] than to S. P. Novikov's[2].

Liu has been working for about fifty years about various topics, such as general Riemann problem, Glimm scheme, well-posedness of the Cauchy problem, time asymptotics, stability of nonlinear viscous waves, Boltzman shocks, relaxation and so on. The present book of course covers some of this material, but does not pretend to be exhaustive.

At first glance, the table of contents looks classical, with Chapters 3–5 covering scalar conservation laws and 6–9 devoted to systems, all this in one space dimension. Chapters 10–14 address important questions that could have deserved entire books of their own – let us think of multi-dimensional gas flow, which was the object of the famous Courant–Friedrichs *Supersonic flows and shock waves*. These latter chapters are useful introductions which can serve as starting points for further studies, though not being tailored to support some graduate course.

The originality of the approach is evident from the very beginning, in Chapter 3 devoted to scalar convex conservation laws. Instead of following a traditional strategy, where the equation is perturbed one way or another, Liu constructs exact solutions of the Cauchy problem when the initial data are step functions with finitely many jumps. This step involves repeatedly the analysis of wave interactions, one of Liu's trademarks. The corresponding global-in-time solutions remain piecewise smooth. A density argu-

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[1] Arnold was sometimes provocative. He claimed that mathematics is a subset of physics; it is the part in which the experiments are cheap. Of course, Liu doesn't go that far.

[2] Novikov and his collaborators studied the first-order conservation laws, which they called *systems of hydrodynamic type*, from a geometrical point of view, ignoring the notion of weak solutions.

ment allows him to extend the well-posedness to rather general data. The strategy has some practical rewards; in order to check quantitative properties such as L^1-contraction or entropy dissipation – including that of a *generalized entropy functional*, – it is enough to verify them at the level of piecewise smooth solutions, hence to focus on shock waves and carry out some simple calculations. Somehow, everything must follow from the Rankine–Hugoniot condition and the associated entropy inequalities. That this method does not apply in higher space dimensions compels the author to come back to the more traditional vanishing viscosity method in the multi-D case, though only sketching the proof of the existence Theorem 6.3[3] of Chapter 5 (mind that the inequality (6.6) used to prove contraction is incorrect).

Liu's choice of considering piecewise smooth solutions and step functions data is of course motivated by the technique employed later in the study of the Cauchy problem for systems of conservation laws. The climax is achieved in the long and dense ninth chapter entitled *Well-posedness theory*. Approximate solutions are constructed through the Glimm scheme. As far as the existence is concerned, one only needs to control the BV-norm as the mesh size vanishes. This is done as usual with the help of the Glimm functional. The stability/uniqueness part however requires a finer tool, a functional elaborated by Liu and Yang (an alternate approach by homotopy was initiated by Bressan and his collaborators). The chapter culminates with the qualitative analysis of the solutions (asymptotic behaviour, N-waves, local regularity) involving the use of generalized characteristics. These materials are perhaps the most remarkable contributions of the author, among many other ones.

Overall this is a deep and quite technical book. It will not be easy to digest at the first reading. Some proofs are complete, while some others are only sketched, the reader being supposed to convert the ideas into details. This is a price to pay in order to keep such a huge topic within a reasonable length. Each chapter ends with historical notes, which put the results in perspective. The bibliography is a little narrow and the reader will sometimes like to consult that of C. Dafermos' book *Hyperbolic conservation laws in continuum physics*.

This book is recommended primarily to researchers and doctoral students. It is a unique reference about the well-posedness theory of the Cauchy problem for hyperbolic systems of conservation laws. It gives a unified presentation of the program carried out by Liu and his collaborators (often former students of him), which was so far disseminated into dozens papers, if not hundreds.

Tai-Ping Liu, *Shock Waves.* American Mathematical Society,
Graduate Studies in Mathematics 215, 2021, 437 pages, Paperback
ISBN 978-1-4704-6625-1.

[3] The statements and remarks are numbered according to the sections, not the chapters. This makes the cross-references a bit odd.

Denis Serre is emeritus professor of mathematics at the École Normale Supérieure de Lyon. He wrote two books on hyperbolic systems of conservation laws, one of them in collaboration with S. Benzoni-Gavage. He translated into French D. Knuth's book *3:16, Bible Illuminated*.

denis.serre@ens-lyon.fr

DOI 10.4171/MAG/111