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Supplement to "On the Inverse of Monoidal Transformation"

By

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In this note we shall fill a gap in the original proof of the Main Theorem in the previous paper [4], and sharpen the Theorem by showing that the condition (α) alone is sufficient to derive the conclusion of the theorem. This sharpened theorem enables us to solve a problem posed by K. Kodaira in [2].

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§1. Vanishing of Cohomology

In the proof of theorem 1 in [4], we made too easy use of Serre's duality theorem. Let V be a connected paracompact complex analytic manifold of dimension n and E a holomorphic vector bundle over V. Let us denote by $C^{p,q}=C^{p,q}(V, E)$ the space of E-valued differential forms of type (p, q) and of class C^∞, with the topology as given in [5]. ($C^{p,q}$ stands for $A^{p,q}$ in [5].) The space $K_*^{n-p,n-q}$ of E^*-valued currents of type $(n-p, n-q)$ with compact supports is isomorphic to the topological dual of $C^{p,q}$, and the transpose of the sequence

$$C^{p,q-1}\xrightarrow{\bar\partial_1}C^{p,q}\xrightarrow{\bar\partial_2}C^{p,q+1}$$

is given by

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$$K_*^{n-p,\ n-q+1} \xleftarrow{u} K_*^{n-p,\ n-q} \xleftarrow{v} K_*^{n-p,\ n-q-1}$$

where u, v are equal to $\bar{\partial}$ on currents up to signs. To conclude the duality between the cohomology groups $\mathrm{Ker}\,\bar{\partial}_2/\mathrm{Im}\,\bar{\partial}_1$ and $\mathrm{Ker}\,u/\mathrm{Im}\,v$, it is enough to know either $\mathrm{Im}\,\bar{\partial}_1$ and $\mathrm{Im}\,\bar{\partial}_2$ are closed or $\mathrm{Im}\,u$ and $\mathrm{Im}\,v$ are closed, as subspaces of C's or K_*'s respectively. (One can refer, for example, to Komatsu [3] for this point.)

Now under the condition of theorem 1 in [4], we proved $H_K^{n-q}(V, \Omega^n(\mathscr{B}^\varepsilon))=0$ for $q=1, 2, \cdots, n-1$, and $H_K^0(V, \Omega^n(\mathscr{B}^\varepsilon))=0$ is trivial provided V is not compact. Hence for $p=0$, $q=1, 2, \cdots, n-1$, $\mathrm{Im}\,v$ is closed because $\mathrm{Im}\,v=\mathrm{Ker}\,u$. For $q\geqq 2$, $\mathrm{Im}\,u$ is nothing but $\mathrm{Im}\,v$ taken for $q-1$, and is closed. Hence we can conclude $H^q(V, \mathcal{O}(\mathscr{B}^{-\varepsilon}))=0$ for $q=2, \cdots, n-1$. But this method breaks down for $q=1$. What Andreotti and Vesentini proved in [1] shows that the image in $\bar{\partial}$: $\mathscr{D}^{n,n-1}(\mathscr{B}^\varepsilon)\to\mathscr{D}^{n,n}(\mathscr{B}^\varepsilon)$ is closed. ($\mathscr{D}^{p,q}(\mathscr{B}^\varepsilon)$ denotes the space of $\mathscr{B}^\varepsilon$-valued C^∞-forms of type (p, q) with compact supports.) But it seems that known results on duality between pairs of locally convex topological vector spaces cannot be applied directly to secure the Serre duality for pairs of $\mathscr{D}$ and its dual.

Here we shall replace our theorem 1 by the following

Theorem 1′. *Let V be a connected paracompact complex analytic manifold and suppose there exist a plurisubharmonic function Ψ (of class C^∞) and a complex line bundle $\mathscr{B}$ on V with the properties*

(a) *$\Psi\geqq 0$ and for any c with $0<c<\sup_{x\in V}\Psi(x)$. $V_c=\{x\in V\,|\,\Psi(x)<c\}$ is not empty and is relatively compact in V.*

(b) *$\mathscr{B}$ is positive.*

Then the restriction map $H^q(V, \Omega^n(\mathscr{B}))\to H^q(V_c, \Omega^n(\mathscr{B}))$ is the 0-map for $q=1, 2, \cdots, n$.

Proof. We may assume that $\sup\Psi(x)=\infty$. (Otherwise we replace Ψ by $(1-\Psi/\sup\Psi)^{-1}-1$.) Take an open covering $\{V'_\lambda\}$ of V such that $\mathscr{B}$ is trivial on each V'_λ and $\bar{V}'_\lambda$ is contained in a domain of local coordinates (x_λ^j). Then $\mathscr{B}$ can be determined by a system of transition functions $\{e_{\lambda\mu}\}$. Because $\mathscr{B}$ is positive, there exists a system $\{a_\lambda\}$ of positive C^∞-functions a_λ on V'_λ, such that

$$a_\lambda/a_\mu=|e_{\lambda\mu}|^2,$$

(1.1)
$$\left(\frac{\partial^2 \log a_\lambda}{\partial x_\lambda^j \partial \bar{x}_\lambda^k}\right)>0.$$

We introduce in V a Kähler metric $ds_0^2=\sum g_{j\bar{k}}^{(0)}(dx^j, d\bar{x}^k)$ by $g_{j\bar{k}}^{(0)}=\dfrac{\partial^2 \log a_\lambda}{\partial x^j \partial \bar{x}^k}$.

For $\mathscr{B}$-valued differential forms $\varphi=\{\varphi_\lambda\}$, $\psi=\{\psi_\lambda\}$ of type (p, q), we define a function $a_0(\varphi, \psi)$ on V by

(1.2)
$$\frac{1}{a_\lambda}\varphi_\lambda \wedge \overline{*\psi_\lambda}=a_0(\varphi, \psi)\, dv_0,$$

where dv_0 is the volume element with respect to the metric ds_0^2 and $*$ is defined by ds_0^2 too.

Given $c>0$ we set $\Phi=(1-\Psi/c)^{-1}$. $\Phi(x)$ tends to ∞ when x tends to ∂V_c from inside. We also set

(1.3)
$$b_\lambda=e^{\Phi^2}\cdot a_\lambda \qquad \text{on } V_\lambda' \cap V_c.$$

Then, as was proved in [4], **1.2**, $g_{j\bar{k}}=\dfrac{\partial^2 \log b_\lambda}{\partial x_\lambda^j\, \partial \bar{x}_\lambda^k}$ gives a complete Kähler metric $ds^2=\sum g_{j\bar{k}}(dx_\lambda^j, d\bar{x}_\lambda^k)$ on V_c. On the other hand, it is easily seen that $(g_{j\bar{k}})=(g_{j\bar{k}}^{(0)})+(h_{j\bar{k}})$, where $(h_{j\bar{k}})$ is a positive semi-definite matrix and $h_{j\bar{k}}(x)$ remains of order at most Φ^l with an integer l, when x approaches ∂V_c from inside. we take ds^2 as the metric on the base manifold V_c and take $\{b_\lambda\}$ to define the metric on the fibres of $\mathscr{B}$, then if $\varphi \in C^{p,q}(V, \mathscr{B})$ we have

$$\frac{1}{b_\lambda}\varphi_\lambda \wedge \overline{*\varphi_\lambda}=e^{-\Phi^2}f\cdot dv_0,$$

where f is of order Φ^l with some integer l. (Note that $*$ is taken with respect to ds^2, while dv_0 is the volume element of ds_0^2.) This shows that $\varphi \in \mathscr{L}^{p,q}(V_c, \mathscr{B})$.

The argument in [4], **1.4** applied to $\mathscr{B}$, gives

(1.4) $$L\Lambda-\Lambda L=\square-*^{-1}\square *.$$

(Formula (1.16). Note that χ is the Kähler form of the metric ds^2 on V_c.) Hence $\mathscr{B}$ is $W^{n,q}$-elliptic with the constant $1/q$, that is to say, we have

$$\frac{1}{q}\{(\bar\partial\psi,\ \bar\partial\psi)+(\vartheta\psi,\ \vartheta\psi)\}\geqq(\psi,\ \psi)$$

for $\psi\in\mathscr{D}^{n,q}(V_c,\ \mathscr{B})$.

Suppose now φ is of type $(n,\ q)$ with $q\geqq 1$ and $\bar\partial\varphi=0$, then by virtue of [4], proposition 5 ([1], Theorem 2), we see that φ can be expressed as $\bar\partial\eta$ on V_c, where $\eta\in C^{n,q-1}(V_c,\ \mathscr{B})$. This proves the theorem.

§2. On Condition (β)

We shall supplement [4], theorem 2 by the following

Theorem 2'. *If $\tilde X$, S, M satisfy the condition (α) of the Main Theorem, then for any $a\in M$, we can find a neighbourhood V of L_a in $\tilde X$ and a plurisubharmonic function Ψ on V such that*

(a) *$0\leqq\Psi<1$ on V, $\Psi=0$ on L_a and for any c with $0<c<1$, $V_c=\{x\in V\,|\,\Psi(x)<c\}$ is relatively compact,*

(b) *$[S]_V$ and the canonical bundle $\mathscr{K}$ of V are negative,*

(c) *each fibre L_b of $S\to M$ is either contained in V or does not meet V.*

Proof. We go back to the argument of [4], §2 and use the same notations as there. By the adjunction formula for canonical bundles, we have

$$\mathscr{K}\,|_{V'\cap S}=\mathscr{K}'\otimes[S]_S^{-1}=[e]^{-(r-1)},$$

where $\mathscr{K}'$ denotes the canonical bundle of $V'\cap S\cong D\times\mathbf{P}^{r-1}$, and $V'=\bigcup_\lambda V'_\lambda$ as considered in [4], §2. Therefore we may assume that $\mathscr{K}$ is determined by a system $\{k_{\lambda\mu}\}$ of transition functions with the property

$$k_{\lambda\mu}\,|\,S=\varepsilon_{\lambda\mu}^{-(r-1)}.$$

Take a monomial M of degree $r-1$ in the homogeneous coordinates $\{\eta^{\alpha}\}$ on $\mathbf{P}^{r-1}$, then $\omega=\{\omega_\lambda\}$ defined by $\omega_\lambda=M/_{(\eta^{\tau(\lambda)})^{r-1}}$ is an element of $\Gamma(D\times\mathbf{P}^{r-1}, \mathcal{O}([e]^{r-1}))$. We try to extend ω approximately to a cross section of $\mathscr{K}^{-1}$. We take holomorphic functions w'_λ in V'_λ such that $w'_\lambda \mid S=\omega_\lambda$, then

$$w'_{\lambda\mu}=(w'_\lambda-k_{\lambda\mu}^{-1}w'_\mu)(y_\lambda)^{-1}$$

define an element $w'=\{w'_{\lambda\mu}\}\in Z^1(\mathfrak{V}, \mathcal{O}(\mathscr{K}^{-1}\otimes[S]^{-1}))$. $\{w'_{\lambda\mu}\mid S\}$ can be represented as the coboundary of an element $\theta=\{\theta_\lambda\}\in C^0(\mathfrak{U}, \mathcal{O}([e]^r))$:

$$w'_{\lambda\mu}\mid S=\theta_\lambda-\varepsilon^r_{\lambda\mu}\theta_\mu.$$

We extend θ_λ to a holomorphic function s_λ on V'_λ and set $w''_\lambda=w'_\lambda-y_\lambda s_\lambda$. Then $w''_\lambda\mid S=w_\lambda$ and

$$w''_\lambda-k_{\lambda\mu}^{-1}w''_\mu=(y_\lambda)^2 w''_{\lambda\mu},$$

where $\{w''_{\lambda\mu}\}\in Z^1(\mathfrak{V}, \mathcal{O}(\mathscr{K}^{-1}\otimes[S]^{-2}))$. Restrict $\{w''_{\lambda\mu}\}$ to S and write it as a coboundary, ... In this way, we see that for all monomials $\{M^\rho\}$ in η of degree $r-1$ and for any integer $l>0$, we find holomorphic functions w^ρ_λ in V'_λ such that

$$w^\rho_\lambda\mid S=\omega^\rho_\lambda=M^\rho/(\eta^{\tau(\lambda)})^{r-1},$$

$$w^\rho_\lambda-k_{\lambda\mu}^{-1}w^\rho_\lambda=(y_\lambda)^l h^\rho_{\lambda\mu}$$

$h^\rho_{\lambda\mu}$ are holomorphic in $V'_\lambda\cap V'_\mu$.

Then we take C^∞-functions π^ρ_λ in V'_λ such that

$$h^\rho_{\lambda\mu}=\pi^\rho_\lambda-k_{\lambda\mu}^{-1}(y_\mu/y_\lambda)^l\pi^\rho_\mu,$$

and set $W^\rho_\lambda=w^\rho_\lambda-(y_\lambda)^l\pi^\rho_\lambda$, $B_\lambda=\sum_\rho|W^\rho_\lambda|^2$. We have $W^\rho_\lambda=k_{\lambda\mu}^{-1}W^\rho_\mu$ and $B_\lambda=|k_{\lambda\mu}|^{-2}B_\mu$. Since B_λ reduces to $\sum_\rho|\omega^\rho_\lambda|^2$ on S, we see

$$\left(\frac{\partial^2 \log B_\lambda}{\partial x^\alpha_\lambda\partial\bar{x}^\beta_\lambda}\right)>0 \quad \text{on } V,$$

if $V=\{\mathbf{x}\in V'\mid\psi(\mathbf{x})<\delta\}$ and δ is small enough. Then $\{e^{-m\psi}B_\lambda^{-1}\}$ gives a Hermitian metric on the fibres of $\mathscr{K}$ whose curvature form is negative.

This, combined with the proof of [4], theorem 2, gives theorem 2'.

To achieve our Main Theorem, we replace [4], proposition 9 by the following

Proposition 9'. *Let V be an n-dimensional complex analytic manifold and S a submanifold of V of codimension 1. Suppose that S is analytically homeomorphic to $D\times\mathbf{P}^{r-1}$, where D is a domain in $\mathbf{C}^m$ and $m+r=n$, and that $[S]_S=[e]^{-1}$, $[e]$ being the complex line bundle on S defined by $D\times$ (hyperplane). If there exist an open subset V_c of V such that $V_c\cap S\cong D'\times\mathbf{P}^{r-1}$ and the restriction map $H^1(V, \mathcal{O}([S]^{-\varepsilon}))\to H^1(V_c, \mathcal{O}([S]^{-\varepsilon}))$ is the 0-map for $\varepsilon=1, 2$, then for any point $a\in D'$, there exist a neighbourhood W of L_a (=the submanifold of S corresponding to $a\times\mathbf{P}$) in V, and a holomorphic map π from W to $\Delta=\{(z, y)\in\mathbf{C}^m\times\mathbf{C}^r \mid |z^j|<\delta, |y^\alpha|<\delta\}$ such that (W, π) is the monoidal transform of Δ with centre Γ (=the linear variety defined by $y^1\cdots=y^r=0$). We can identify Γ with a neighbourhood of a in D', and the restriction of π to S corresponds to the canonical projection $D'\times\mathbf{P}\to D'$ by these identifications.*

We have only to make a slight change in the proof of Proposition 9. We note that the obstructions to extending ζ's and η's to V appear in $H^1(V, \mathcal{O}([S]^{-\varepsilon}))$ ($\varepsilon=1, 2$), and hence they disappear in V_c.

The rest of the proof in [4], § 3 now works to show the Main Theorem.

Remark. We can dispense with the discussion of negativity of $\mathscr{K}$. In fact what we want is to extend holomorphic cross sections of $[e]^\varepsilon$ on $V\cap S$ to those of $[S]^{-\varepsilon}$ on V. Approximate extension (in the sense of ascending power in y_λ) is always possible and the obstruction appears in $H^1(V, \mathcal{O}([S]^{-l}))\cong H^1(V, \Omega^n(\mathscr{K}^{-1}\otimes[S]^{-l}))$. Since $[S]$ is positive on V and V_c is relatively compact in V, we can find l such that $\mathscr{K}^{-1}\otimes[S]^{-l}$ is positive on $V_{c'}$ with $c'>c$. This remark would be of use when we consider other kind of extension problems.

§3. Deformation of a Monoidal Transform

Suppose we have paracompact connected complex analytic manifolds $\mathscr{W}$ and B and a holomorphic mapping p from $\mathscr{W}$ onto B, such that the rank of the jacobian matrix of p is equal to the dimension of B at every point of $\mathscr{W}$. Then we say that $\mathscr{W}$ is a complex fibre manifold over B. In this case $W_u = p^{-1}(u)$ is an analytic submanifold of $\mathscr{W}$ for each point $u \in B$. We shall assume that W_u is connected. If, moreover, p is proper so that each W_u is a compact complex analytic manifold, then $(\mathscr{W}, B, p)$ is an analytic family of compact complex analytic manifolds.

In his paper [2], K. Kodaira has shown the following: suppose $(\mathscr{W}, B, p)$ is a complex fibre manifold and suppose, for a given point O of B, W_0 contains a *compact* submanifold S_0 of codimension 1, with the following properties:

(1) S_0 has a structure of analytic fibre bundle $\pi: S_0 \to M_0$, with a projective space $\mathbf{P}$ as the standard fibre.

(2) The line bundle $[S_0]$ on W_0 gives, when restricted to each fibre L of $S_0 \to M_0$, the line bundle $[e]^{-1}$, where e is a hyperplane on $L(\cong \mathbf{P})$.

Then there exists a neighbourhood N of O on B, an analytic submanifold $\mathscr{S}$ of $p^{-1}(N)$ of codimension 1, and an analytic manifold $\mathscr{M}$ with a holomorphic mapping q onto N, such that

(a) $(\mathscr{S}, N, p)$, $(\mathscr{M}, N, q)$ are analytic families of compact complex manifolds and

$$p^{-1}(0) = S_0,\ q^{-1}(0) = M_0.$$

(b) There exists a holomorphic mapping $\tilde{\pi}$, from $\mathscr{S}$ onto $\mathscr{M}$, such that $p = q \circ \tilde{\pi}$, $\tilde{\pi}|_{S_0} = \pi$ and $\tilde{\pi}: \mathscr{S} \to \mathscr{M}$ is an analytic fibre bundle with the standard fibre $\mathbf{P}$.

We can apply our Main Theorem to $p^{-1}(N) = \mathscr{W}' \supset \mathscr{S} \to \mathscr{M}$ and conclude *that there exists an analytic manifold $\mathscr{X}$ containing $\mathscr{M}$ as a submanifold so that $\mathscr{W}'$ is the monoidal transform of $\mathscr{X}$ with centre $\mathscr{M}$.* As is easily seen, $p: \mathscr{W} \to N$ *determines a holomorphic map* $p_1: \mathscr{X} \to N$ *and makes* $\mathscr{X}$ *a* *complex fibre manifold over* N. *For any* $t \in N$, W_t *is the monoidal transform of* $X_t = p_1^{-1}(t)$ *with centre* M_t.

In short we can say "*A small deformation of a monoidal transform is a monoidal transform.*"

Note added in proof: After this paper was finished, we have proved that $H^q(V_c, \Omega^n(\mathcal{B}))$ itself vanishes under the condition of Theorem 1′.

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