Maximal Indexes of Tits Algebras

A. S. Merkurjev[1]

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Abstract. Let G be a split simply connected semisimple algebraic group over a field F and let C be the center of G. It is proved that the maximal index of the Tits algebras of all inner forms of G_L over all field extensions L/F corresponding to a given character χ of C equals the greatest common divisor of the dimensions of all representations of G which are given by the multiplication by χ being restricted to C. An application to the discriminant algebra of an algebra with an involution of the second kind is given.

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Let G be an adjoint semisimple algebraic group defined over a field F, let $\pi : \widetilde{G} \to G$ be the universal covering and let $C = \ker(\pi)$ denote the center of $\widetilde{G}$. In [13] Tits has constructed a homomorphism

$$\beta : C^*(F) \to \mathrm{Br}(F)$$

where $C^*(F)$ is the group of characters of C defined over F and $\mathrm{Br}(F)$ is the Brauer group of F. For any character $\chi \in C^*(F)$ one can choose a central simple algebra A (called the *Tits algebra*), representing the class $\beta(\chi) \in \mathrm{Br}(F)$, in such a way that there is a group homomorphism

$$\widetilde{G} \to \mathbf{GL}_1(A)$$

restricting to the character χ on the center C and inducing an irreducible representation over a separable closure F_{sep} of the field F. It follows from the representation theory of semisimple algebraic groups that the index $\mathrm{ind}(A)$ of the algebra A divides the dimension of any irreducible representation $\rho : \widetilde{G}^q \to \mathbf{GL}(V)$ of a quasisplit inner form $\widetilde{G}^q$ of $\widetilde{G}$ such that the restriction of ρ to the center C^q of $\widetilde{G}^q$ is given by the multiplication by χ (we identify the Galois modules of the character groups C^* and C^{q*}). Therefore, if we denote by $n_\chi(G)$ the greatest common divisor of the dimensions of all such representations, then $\mathrm{ind}(A)$ divides $n_\chi(G)$. The numbers $n_\chi(G)$ depend only on the class of the inner forms of G, i.e. on the Dynkin diagram $D = \mathrm{Dyn}(G_{\mathrm{sep}})$, and the action of the absolute Galois group of F on $\mathrm{Aut}(D)$. In particular, if G is of

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inner type, then the numbers $n_\chi(G)$ depend only on the isomorphism class of G over F_{sep} and were computed in [5].

It was proved in [5], case by case, that, for a group G of inner type, the maximal possible index of the Tits algebra A corresponding to χ reaches its upper bound $n_\chi(G)$. More precisely, there is a field extension E/F and an inner form G' of the group $G \times_F E$ over E such that for any character of the center of the universal covering of G', defined over E, the index of the Tits algebra A corresponding to χ equals $n_\chi(G) = n_\chi(G')$.

We give here a uniform proof of this statement for all adjoint semisimple algebraic groups G (not necessarily of inner type). The field E appears as a function field of a "classifying variety" Y for the corresponding adjoint quasisplit group G^q.

The universal property of the variety Y asserts that any inner form of G over an arbitrary field extension L/F arises from some L-point of Y. Hence, the Tits algebras over the function field $E = F(Y)$ are generic ones, and, therefore, are of maximal index. It follows that, if the index of the Tits algebra A corresponding to χ reaches the upper bound $n_\chi(G)$ over some field extension, then it does so over $F(Y)$.

In the first part of the paper we define, for a group scheme $\mathcal{G}$, the dual group scheme $\mathcal{G}'$ with respect to a $\mathcal{G}$-torsor. This construction is a slight generalization of the corollary of Prop. 34 in [10]. For an adjoint semisimple algebraic group G over a field F we construct a classifying variety Y over F such that the scheme $\mathcal{G}'$, dual to $G \times_F Y$ with respect to a certain torsor, represents the algebraic family of all inner forms of G.

In section 4 we define Tits algebras and give a list of all Tits algebras for all absolutely simple groups of classical types.

The main result is formulated in section 5. The rest of the paper is devoted to the proof of the theorem. In the last section we give an application of the theorem in the case of groups of outer type A_{2n-1} which was not covered in [5].

All the group schemes considered in the paper are assumed to be flat affine of finite type over a Noetherian separated base scheme Y.

For a field F we denote by F_{sep} a separable closure and by Γ the absolute Galois group $\text{Gal}(F_{\text{sep}}/F)$. The split 1-dimensional torus $\text{Spec}F[t, t^{-1}]$ is denoted by $\mathbb{G}_m$.

1. Dual group scheme with respect to a torsor

Let $\mathcal{G}$ be a group scheme over a scheme Y, and let $\pi : X \to Y$ be a (left) $\mathcal{G}$-torsor [7]. Denote by $\text{Aut}_{\mathcal{G}}(X)$ the group of all $\mathcal{G}$-automorphisms of X over Y. If $X = \mathcal{G}$ is a trivial torsor, then the map $\mathcal{G}(Y) \to \text{Aut}_{\mathcal{G}}(X)$ given by the rule $g \mapsto (g' \mapsto g' \cdot g^{-1})$ is clearly a group isomorphism.

Consider the sheaf of groups in the flat topology Y_{fl} on Y:

$$S(Z) = \text{Aut}_{\mathcal{G} \times_Y Z}(X \times_Y Z).$$

Proposition 1.1. *The sheaf S is represented by a group scheme over Y.*

Proof. Since $\pi : X \to Y$ is faithfully flat, it is sufficient to prove that the restriction of S on X is represented by a group scheme (by faithfully flat descent, [7, Th.2.23]). But over X the torsor $\pi \times_Y \text{id} : X \times_Y X \to X$ is trivial, hence for any scheme Z over X we have a canonical isomorphism $S(Z) \xrightarrow{\sim} \mathcal{G}(Z)$ and therefore the restriction of S on X is represented by $\mathcal{G} \times_Y X$. □

We denote by $\mathcal{G}'$ the group scheme over Y representing S and call $\mathcal{G}'$ *the group scheme dual to* $\mathcal{G}$ *with respect to the* $\mathcal{G}$*-torsor* $\pi : X \to Y$. It follows from the proof of proposition 1.1 that the group schemes $\mathcal{G}' \times_Y X$ and $\mathcal{G} \times_Y X$ are isomorphic over X.

By definition the group scheme $\mathcal{G}'$ acts on X over Y.

PROPOSITION 1.2. *The morphism* $\pi : X \to Y$ *is a* $\mathcal{G}'$*-torsor.*

Proof. By faithfully flat descent we may assume that $X = \mathcal{G}$ is a trivial $\mathcal{G}$-torsor. Then the action of $\mathcal{G}' \xrightarrow{\sim} \mathcal{G}$ on X clearly leads to the structure of a trivial $\mathcal{G}'$-torsor on X. □

There is a natural bijection of the set of isomorphism classes of $\mathcal{G}$-torsors $\pi : X \to Y$ over Y and the set $H^1_{\mathrm{fl}}(Y, \mathcal{G})$ (see [7]).

Let $f : \mathcal{G} \to \mathcal{G}_1$ be a morphism of group schemes over Y, and let $\pi : X \to Y$ be a $\mathcal{G}$-torsor. A $\mathcal{G}_1$-torsor $\pi_1 : X_1 \to Y$ representing the image of the class of π under the map

$$H^1_{\mathrm{fl}}(Y, \mathcal{G}) \to H^1_{\mathrm{fl}}(Y, \mathcal{G}')$$

is called *the image of the* $\mathcal{G}$*-torsor* $\pi : X \to Y$ *under* f. Let $\mathcal{G}'$ (resp. $\mathcal{G}'_1$) be the group scheme dual to $\mathcal{G}$ (resp. $\mathcal{G}_1$) with respect to the torsor $\pi : X \to Y$ (resp. $\pi_1 : X_1 \to Y$). The natural group homomorphism

$$\mathrm{Aut}_{\mathcal{G}}(X) \to \mathrm{Aut}_{\mathcal{G}_1}(X_1)$$

induces a group scheme homomorphism $f' : \mathcal{G}' \to \mathcal{G}'_1$ over Y of the dual group schemes.

2. PGL-TORSORS

Let $p : \mathcal{V} \to Y$ be a vector bundle over Y and $\mathcal{E} = \mathrm{End}_Y(\mathcal{V})$ (viewed as a vector bundle over Y). Consider the group scheme $\mathcal{G} = \mathbf{PGL}(\mathcal{V})$ over Y. Let $\pi : X \to Y$ be a $\mathcal{G}$-torsor. The group scheme $\mathcal{G}$ acts on $\mathcal{E}$ and on X over Y, hence on $\mathcal{E} \times_Y X$. Denote by $\mathrm{Sec}_{\mathcal{G}}(\mathcal{E})$ the $\Gamma(Y, \mathcal{O}_Y)$-algebra of $\mathcal{G}$-invariant sections $X \to \mathcal{E} \times_Y X$ of the vector bundle $\mathcal{E} \times_Y X \to X$. Consider the sheaf T of algebras on Y_{fl}:

$$T(Z) = \mathrm{Sec}_{\mathcal{G} \times_Y Z}(\mathcal{E} \times_Y Z).$$

PROPOSITION 2.1. *The sheaf* T *is represented by the total space of an Azumaya algebra over* Y.

Proof. By faithfully flat descent we may assume that $X = \mathcal{G}$ is a trivial torsor. Then for any scheme Z over Y we have $T(Z) = \mathrm{Mor}_Y(Z, \mathcal{E})$, hence T is represented by $\mathcal{E}$ which is the total space of the associated locally free sheaf $\mathcal{E}nd_Y(\mathcal{V})$ of Azumaya algebras. □

We call an Azumaya algebra $\mathcal{A}$ over Y whose total space represents T *the algebra associated to the* $\mathcal{G}$*-torsor* $\pi : X \to Y$. It follows from the proof of proposition 2.1 that the $\mathcal{O}_X$-algebra $\pi^*\mathcal{A}$ is isomorphic to $\pi^*(\mathcal{E}nd_Y(\mathcal{V}))$.

Consider the sheaf of sets on Y_{fl}:

$$U(Z) = \mathrm{Iso}_{\mathcal{O}_Z-alg}(\lambda^*\mathcal{A}, \lambda^*\mathcal{E}nd_Y(\mathcal{V}))$$

for any $\lambda : Z \to Y$. The group $\mathcal{G}(Z)$ acts naturally on $U(Z)$ making U a $\mathcal{G}$-torsor.

PROPOSITION 2.2. *The sheaf* U *is represented by the* $\mathcal{G}$*-torsor* $\pi : X \to Y$.

Proof. A morphism $\mu : Z \to X$ over Y defines a trivialization of the torsor $X \times_Y Z \to Z$ and, hence, an isomorphism of $\mathcal{O}_Z$-algebras $(\pi\mu)^*\mathcal{A}$ and $(\pi\mu)^*(\mathcal{E}nd_Y(\mathcal{V}))$. Therefore, we get a map $X(Z) \to U(Z)$ which gives rise to a map of sheaves

$$\mathrm{Mor}_Y(*, X) \to U.$$

To prove that this map is a bijection, by faithfully flat descent one may assume that X is a trivial torsor. By the Skolem-Noether theorem in this case the statement is clear. □

Remark 2.3. Proposition 2.2 shows how to reconstruct the $\mathcal{G}$-torsor $\pi : X \to Y$ out of the algebra $\mathcal{A}$. Thus, we have a bijection between the set of isomorphism classes of Azumaya algebras $\mathcal{A}$ over Y such that $\pi^*\mathcal{A} \xrightarrow{\sim} \pi^*(\mathcal{E}nd_Y(\mathcal{V}))$ and the set of isomorphism classes of $\mathbf{PGL}(\mathcal{V})$-torsors over Y.

The group scheme $\mathbf{PGL}_1(\mathcal{A})$ over Y acts naturally on the sheaf U, and the action commutes with that of $\mathcal{G}$. Hence, we have a group scheme homomorphism $\mathbf{PGL}_1(\mathcal{A}) \to \mathcal{G}'$ where $\mathcal{G}'$ is the group scheme dual to $\mathcal{G}$ with respect to the $\mathcal{G}$-torsor $\pi : X \to Y$. To prove that this homomorphism is an isomorphism, by faithfully flat descent, one may consider the split situation in which our statement is clear. Hence,

$$\mathcal{G}' = \mathbf{PGL}(\mathcal{V})' = \mathbf{PGL}_1(\mathcal{A}).$$

Now let $\mathcal{G}$ be an arbitrary group scheme over Y, let $\pi : X \to Y$ a $\mathcal{G}$-torsor and let

$$f : \mathcal{G} \to \mathbf{PGL}(\mathcal{V})$$

be a projective representation over Y, where $\mathcal{V}$ is a vector bundle over Y. Denote by $\mathcal{A}$ the Azumaya algebra on Y associated to the $\mathbf{PGL}(\mathcal{V})$-torsor, which is equal to the image of π under f. We call $\mathcal{A}$ *the algebra associated to the $\mathcal{G}$-torsor π and the projective representation f*. There is a natural group scheme homomorphism

$$f' : \mathcal{G}' \to \mathbf{PGL}_1(\mathcal{A})$$

where $\mathcal{G}'$ is the group scheme dual to $\mathcal{G}$ with respect to π.

3. Inner forms

Let G be a semisimple algebraic algebraic group defined over a field F with center $Z(G)$. Denote by $\overline{G}$ the corresponding adjoint group $G/Z(G)$. An algebraic group G' over F is called *a twisted form of* G if $G'_{\mathrm{sep}} \simeq G_{\mathrm{sep}}$. The set of isomorphism classes of twisted forms of G is in 1–1 correspondence with the set $H^1(F, \mathrm{Aut}(G_{\mathrm{sep}}))$ ([10]). The natural homomorphism

$$\overline{G}(F_{\mathrm{sep}}) \to \mathrm{Aut}(G_{\mathrm{sep}}), \qquad \bar{g} \mapsto (g' \mapsto gg'g^{-1})$$

induces the map

$$\alpha : H^1(F, \overline{G}(F_{\mathrm{sep}})) \to H^1(F, \mathrm{Aut}(G_{\mathrm{sep}})).$$

A twisted form G' of the group G is called an *inner form of* G if the cocycle corresponding to G' belongs to the image of α. The group G is called of *inner type* if G is an inner form of a split group.

Assume now that G is an adjoint group, i.e. $\overline{G} = G$. Let X be a G-torsor over F. It corresponds to some element $\xi \in H^1(F, G(F_{\mathrm{sep}}))$ ([10]). It is straightforward to check that the group G', dual to G with respect to the torsor X, corresponds to $\alpha(\xi) \in H^1(F, \mathrm{Aut}(G_{\mathrm{sep}}))$.

We have proved

Proposition 3.1. *Let G and G' be adjoint semisimple algebraic groups over a field F. Then G' is an inner form of G iff there is a G-torsor X over F such that G' is the dual group with respect to the G-torsor X.* □

Remark 3.2. The second condition of proposition 3.1 can be taken as the definition of an inner form of an adjoint group (in order to avoid referring to cocycles).

4. Tits algebras

Let G be an adjoint semisimple algebraic group defined over a field F, let $\widetilde{G} \to G$ be the universal covering, and C the kernel of the covering. It is known that C, being the center of $\widetilde{G}$, is a closed subscheme of $\widetilde{G}$ of multiplicative type (not necessarily reduced) ([2],[12]). Denote by C^* the finite Γ-module $\mathrm{Hom}(C_{\mathrm{sep}}, \mathbb{G}_m)$ of characters.

The group G is an inner form of some quasisplit group defined over F ([1],[12]). By proposition 3.1, there exists a G-torsor X over F such that the group G', dual to G with respect to X, is quasisplit. The choice of a point of X over F_{sep} defines an isomorphism $G_{\mathrm{sep}} \xrightarrow{\sim} G'_{\mathrm{sep}}$ which is uniquely determined up to conjugation. This isomorphism extends uniquely to an isomorphism $\widetilde{G}_{\mathrm{sep}} \xrightarrow{\sim} \widetilde{G}'_{\mathrm{sep}}$ where $\widetilde{G}'$ is the universal covering of G' over F ([12]). Hence, we obtain an isomorphism of the centers $\varphi : C_{\mathrm{sep}} \xrightarrow{\sim} C'_{\mathrm{sep}}$. One can easily see that this isomorphism is defined over F (hence, induces an isomorphism of Γ-modules $\varphi^* : C^* \xrightarrow{\sim} C'^*$) and depends only on the choice of the G-torsor X (which is not unique in general) but not on the point of X over F_{sep}.

Denote by B a Borel subgroup in $\widetilde{G}'$ defined over F, by T a maximal torus in B defined over F and by Λ the subgroup in T^* generated by roots of $\widetilde{G}$ relative to T. The restriction map induces the natural isomorphism of Γ-modules

$$T^*/\Lambda \xrightarrow{\sim} C^*.$$

There is a partial ordering on T^*: we write $\alpha > \beta$ for α, $\beta \in T^*$ if $\alpha - \beta$ is a sum of roots of B. In each coset of T^*/Λ there is a unique minimal element with respect to this ordering called the *minimal weight*.

Choose a character $\chi \in C^*$ defined over F and put $\chi' = \varphi^*(\chi) \in C'^*$. By the representation theory of quasisplit semisimple groups (see [13]) there is an irreducible representation $\tilde{\rho} : \widetilde{G}' \to \mathbf{GL}(V)$ such that the restriction of $\tilde{\rho}$ to C' is given by multiplication by χ'. Consider a central simple F-algebra A associated to the G-torsor X and the projective representation $\rho : G' \to \mathbf{PGL}(V)$ induced by $\tilde{\rho}$ (section 2). The algebra A is called the *Tits algebra of the group G corresponding to the representation ρ.* Its class in $\mathrm{Br}(F)$ depends only on the choice of character $\chi \in C^*$ ([13]) and is called *the Tits class of the group G corresponding to* χ. By construction, the index of A divides $\dim V$. Denote by $n_\chi(G)$ the greatest common divisor of the numbers $\dim V$ for all representations $\tilde{\rho} : \widetilde{G}' \to \mathbf{GL}(V)$ such that the restriction of $\tilde{\rho}$ on C' is given by multiplication by χ'. We have observed that $\operatorname{ind} A$ divides $n_\chi(G)$ (see [5]). If $\chi = 0$, then $n_\chi(G) = 1$.

Let $\chi \in C^*(F) \simeq (T^*/\Lambda)^\Gamma$ and $\mu \in T^*$ be the minimal weight in the coset χ. Let $\tilde{\rho} : \widetilde{G}' \to \mathbf{GL}(V)$ be a representation (unique up to an isomorphism) with the highest weight μ called the *minimal representation.* The Tits algebra corresponding to ρ is called the *minimal Tits algebra of* G and is denoted by A_χ. The algebra A_χ is the canonical representative of the Tits class corresponding to χ. For example, if $\chi = 0$, then $A_\chi = F$. Any Tits algebra is Brauer equivalent to a minimal one.

Remark 4.1. The isomorphism $\varphi : C_{\text{sep}} \xrightarrow{\sim} C'_{\text{sep}}$ depends on the choice of the G-torsor X. Another choice of X changes φ by an automorphism of C' induced by an (outer) automorphism of G', but clearly does not change the numbers $n_\chi(G)$.

Remark 4.2. By definition, the numbers $n_\chi(G)$ depend only on the quasisplit inner form of G and hence do not change if we replace G by any inner form of it. In turn, the class of inner forms of G is uniquely determined by the isomorphism class of G over F_{sep} and the action of Γ on the group of outer automorphisms

$$\operatorname{Out}(G_{\text{sep}}) = \operatorname{Aut}(G_{\text{sep}})/\operatorname{Int}(G_{\text{sep}}) = \operatorname{Aut}(\operatorname{Dyn}(G_{\text{sep}}))$$

of the group G_{sep}. If we change F by a field extension E/F such that F is separably closed in E, the numbers $n_\chi(G)$ do not change. If G is a group of inner type (i.e. G' is a split group, or equivalently, Γ acts trivially on $\operatorname{Out}(G_{\text{sep}})$) then the numbers $n_\chi(G)$ depend only on the isomorphism class of G over F_{sep} and are computed in [5].

We would like to classify the Tits classes of all adjoint semisimple algebraic groups. A Tits algebra of the product of adjoint semisimple groups is the tensor product of the Tits algebras of factors. Since any adjoint semisimple group is the product of the groups $G_1 = R_{L/F}(G)$ where G is an absolutely simple adjoint group over a finite separable field extension L/F ([12]), it suffices to describe the Tits algebras of G_1. If $\widetilde{G} \to G$ is the universal covering of G with kernel C, then

$$\widetilde{G}_1 = R_{L/F}(\widetilde{G}) \to R_{L/F}(G) = G_1$$

is the universal covering of G_1 with kernel $C_1 = R_{L/F}(C)$.

Let $F \subset L \subset F_{\text{sep}}$, $\Gamma_0 = \operatorname{Gal}(F_{\text{sep}}/L) \subset \Gamma$. We have a canonical isomorphism

$$\theta : C^*(L) = (C^*)^{\Gamma_0} \xrightarrow{\sim} (C_1^*)^\Gamma = C_1^*(F),$$

and for any $\chi_0 \in C^*(L)$ the Tits algebra A_χ with $\chi = \theta(\chi_0)$ for the group G_1 equals the corestriction in the extension L/F of the Tits algebra A_{χ_0} of G ([13]). Hence, it is sufficient to classify the Tits classes of absolutely simple adjoint groups.

Below is the list of minimal Tits algebras and numbers $n_\chi(G)$ for absolutely simple adjoint groups. We use the notation and the computations from [4] and [5].

4.1. Type A_n. An adjoint simple algebraic group of the type A_n, defined over F, is isomorphic to the projective unitary group $G = \mathbf{PGU}(B, \tau)$, where B is an Azumaya algebra of degree $n+1$ over an étale quadratic extension L/F with an involution τ of the second kind trivial on F. Its universal covering is the special unitary group $\widetilde{G} = \mathbf{SU}(B, \tau)$

Assume first that L splits, i.e. $L \simeq F \times F$. In this case $B \simeq A \times A^{op}$ with the switch involution τ where A is a central simple algebra of degree $n+1$ over F, where

$\widetilde{G} = \mathbf{SL}_1(A)$ and $G = \mathbf{PGL}_1(A)$. Then $C = \boldsymbol{\mu}_{n+1}$, and $C^* = \mathbb{Z}/(n+1)\mathbb{Z}$ with the trivial Γ-action. For any $i = 0, 1, \dots, n$, consider the natural representation

$$\rho_i : \widetilde{G} \to \mathbf{GL}_1(\lambda^i A)$$

where $\lambda^i A$ are *external powers* of A (see [4]). In the split case, ρ_i is the i-th external power representation known as a minimal representation. Hence, $\lambda^i A$ for $i = 0, 1, \dots, n$ are minimal Tits algebras of G. If $\chi = i + (n+1)\mathbb{Z} \in C^*$, then $n_\chi = (n+1)/gcd(i, n+1)$.

Now let $\widetilde{G} = \mathbf{SU}(B, \tau)$, where B is a central simple algebra of degree $n+1$ with a unitary involution over a quadratic separable field extension L/F. The group Γ acts on $C^* = \mathbb{Z}/(n+1)\mathbb{Z}$ by $x \mapsto -x$ through $\mathrm{Gal}(L/F)$. The only non-trivial element in $C^*(F)$ is $\chi = \frac{n+1}{2} + (n+1)\mathbb{Z}$ (when n is odd). There is a natural homomorphism

$$\rho : \widetilde{G} \to \mathbf{GL}_1(D(B, \tau)),$$

where $D(B, \tau)$ is the discriminant algebra (see section 10). In the split case ρ is the external $\frac{n+1}{2}$-power representation. Hence, the algebra $D(B, \tau)$ is the minimal Tits algebra for the group G corresponding to χ. The number n_χ equals 2 if $(n+1)$ is a 2-power and equals 4 otherwise (see section 10).

4.2. Type B_n. An adjoint simple algebraic group of type B_n, defined over F, is isomorphic to the special orthogonal group $G = \mathbf{O}^+(V, q)$ where (V, q) is a non-degenerate quadratic form of dimension $2n+1$. Its universal covering is the spinor group $\widetilde{G} = \mathbf{Spin}(V, q)$. Then $C = \boldsymbol{\mu}_2$, $C^* = \mathbb{Z}/2\mathbb{Z} = \{0, \chi\}$. The embedding

$$\widetilde{G} \hookrightarrow \mathbf{GL}_1(C_0(V, q)),$$

where $C_0(V, q)$ is the even Clifford algebra of (V, q), is, in the split case, the spinor representation known as a minimal representation. Hence, the even Clifford algebra $C_0(V, q)$ is the minimal Tits algebra A_χ. The number n_χ equals 2^n.

4.3. Type C_n. An adjoint simple algebraic group of type C_n, defined over F, is isomorphic to the group of projective similitudes $G = \mathbf{PGSp}(A, \sigma)$, where A is a central simple algebra of degree $2n$ with a symplectic involution σ. Its universal covering is the symplectic group $\widetilde{G} = \mathbf{Sp}(A, \sigma)$. Then $C = \boldsymbol{\mu}_2$ and $C^* = \mathbb{Z}/2\mathbb{Z} = \{0, \chi\}$. The embedding

$$\widetilde{G} \hookrightarrow \mathbf{GL}_1(A)$$

is, in the split case, a minimal representation. Hence, A is the minimal Tits algebra A_λ. The number n_χ is the largest 2-power which divides $2n$.

4.4. Type D_n. An adjoint simple algebraic group of type D_n, defined over F (of non-trialitarian type if $n = 4$), is isomorphic to the group of proper projective similitudes $G = \mathbf{PGO}^+(A, \sigma, f)$ where A is a central simple, algebra of degree $2n$ with an orthogonal pair (σ, f) (see [4]). Its universal covering is the spinor group $\widetilde{G} = \mathbf{Spin}(A, \sigma, f)$. Then $C^* = \{0, \chi, \chi^+, \chi^-\}$ where χ factors through the special orthogonal group $\mathbf{O}^+(A, \sigma, f)$. The composition

$$\mathbf{Spin}(A, \sigma, f) \to \mathbf{O}^+(A, \sigma, f) \hookrightarrow \mathbf{GL}_1(A)$$

is, in the split case, the standard minimal representation. Hence, A is the minimal Tits algebra A_χ. The number n_χ equals the largest 2-power which divides $2n$.

Assume that the discriminant of σ is trivial (i.e. the center Z of the Clifford algebra $C(A,\sigma,f)$ splits). The group Γ acts trivially on C^*. The natural compositions

$$\mathbf{Spin}(A,\sigma,f) \hookrightarrow \mathbf{GL}_1(C(A,\sigma,f)) \to \mathbf{GL}_1(C^{\pm}(A,\sigma,f))$$

where $C^{\pm}(A,\sigma,f)$ are simple components of $C(A,\sigma,f)$, are, in the split case, semispinor minimal representations. Hence, $C^{\pm}(A,\sigma,f)$ are minimal Tits algebras $A_{\chi^{\pm}}$. The numbers $n_{\chi^{\pm}}$ equal 2^{n-1}.

If the discriminant of σ is not trivial, then Γ interchanges χ^+ and χ^- and χ is the only nontrivial Γ-invariant character.

4.5. Exceptional types.

4.5.1. *Trialitarian type D_4.* The image of the map $\Gamma \to \mathrm{Aut}(C^*)$ contains a subgroup of order 3. It implies that $C^*(F)=0$ and there are no nontrivial characters and Tits algebras.

4.5.2. *Type E_6.* In this case $C^* \simeq \mathbb{Z}/3\mathbb{Z}$ and for a nontrivial character $\chi \in C^*(F)$ one has $n_\chi = 27$.

4.5.3. *Type E_7.* In this case $C^* \simeq \mathbb{Z}/2\mathbb{Z}$ and for a nontrivial character $\chi \in C^*(F)$ one has $n_\chi = 8$.

4.5.4. *Types E_8, F_4 and G_2.* In these cases $C^* = 0$ and there are no nontrivial characters and Tits algebras.

5. The classifying variety of a group

Let G be an adjoint semisimple algebraic group over a field F and Y be a scheme over F. Consider the group scheme $\mathcal{G} = G \times_F Y$ over Y, and an arbitrary $\mathcal{G}$-torsor $\pi : X \to Y$. Denote by $\mathcal{G}'$ the dual scheme with respect to this torsor. For any rational point $y \in Y(F)$ the fiber $\mathcal{G}'_y$ of $\mathcal{G}'$ over y is dual to $\mathcal{G}_y = G$ with respect to the G-torsor $\pi_y : X_y \to \operatorname{Spec} F$. Hence, by proposition 3.1, an algebraic group $\mathcal{G}'_y$ is an inner form of G. So, we can view the scheme $\mathcal{G}'$ as the algebraic family of inner forms of G.

Now we take a specific scheme Y. Let $G \hookrightarrow \mathbf{GL}_n$ be any faithful representation over F. Consider the homogeneous variety $Y = \mathbf{GL}_n / G$ and the canonical G-torsor $\pi : \mathbf{GL}_n \to Y$. The variety Y is called the *classifying variety of* G. The universal property of Y asserts that any inner form of G is a member of the algebraic family $\mathcal{G}'$ over Y:

Proposition 5.1. *For any inner form G' of G over F there exists a rational point $y \in Y(F)$ such that $G' \simeq \mathcal{G}'_y$ over F.*

Proof. This follows from Hilbert's Theorem 90 and the exact sequence of pointed sets ([7],[10])

$$Y(F) \to H^1(F, G(F_{\mathrm{sep}})) \to H^1(F, \mathbf{GL}_n(F_{\mathrm{sep}}))$$

induced by the exact sequence

$$1 \to G(F_{\mathrm{sep}}) \to \mathbf{GL}_n(F_{\mathrm{sep}}) \to Y(F_{\mathrm{sep}}) \to 1. \quad \square$$

Now let G_1 be any adjoint semisimple algebraic group over F, and G be its quasisplit inner form. Consider the classifying variety $Y = \mathbf{GL}_n / G$ and the group scheme $\mathcal{G}'$ dual to $\mathcal{G} = G \times_F Y$ with respect to the $\mathcal{G}$-torsor $\pi : \mathbf{GL}_n \to Y$. By proposition 5.1, we have $G_1 \simeq \mathcal{G}'_y$ for some $y \in Y(F)$. Let $\xi \in Y$ be the generic point. The generic fiber $\mathcal{G}'_\xi$ is an adjoint semisimple algebraic group over the function field $F(Y)$. The $\mathcal{G}$-torsor π enables us to identify the character module C^* of the center of the universal coverings of the groups G_1, G and $\mathcal{G}'_\xi$.

Now we formulate the main result.

Theorem 5.2. *For any character $\chi \in C^*(F)$, the index of the Tits class of the group $\mathcal{G}'_\xi$ corresponding to χ equals $n_\chi(G_1) = n_\chi(G) = n_\chi(\mathcal{G}'_\xi)$.*

Corollary 5.3. *For any adjoint semisimple algebraic group G_1 over a field F there exists a field extension E/F and an inner form G_2 of the group $G_1 \otimes_F E$ over E such that F is separably closed in E and for any character χ of the center of the universal covering of G_2, with χ defined over E, the index of the Tits class of the group G_2 corresponding to χ equals $n_\chi(G_1) = n_\chi(G_2)$.* □

6. G-modules

Let $\mathcal{G}$ be a group scheme over a scheme Y. Assume that $\mathcal{G}$ acts on a scheme X over Y. The morphism of the $\mathcal{G}$-action on X we denote by

$$\theta : \mathcal{G} \times_Y X \to X.$$

A *$\mathcal{G}$-module $\mathcal{F}$ on X* is a quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ together with an isomorphism of $\mathcal{O}_{\mathcal{G}\times_Y X}$-modules

$$\varphi : \theta^*\mathcal{F} \xrightarrow{\sim} p_2^*\mathcal{F}$$

(where $p_2 : \mathcal{G} \times_Y X \to X$ is the projection), satisfying the cocycle condition

$$p_{23}^*(\varphi) \circ (\mathrm{id} \times \theta)^*(\varphi) = (m \times \mathrm{id})^*(\varphi)$$

where $m : \mathcal{G} \times_Y \mathcal{G} \to \mathcal{G}$ is the multiplication.

Giving a $\mathcal{G}$-module structure on a quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ is equivalent to giving, naturally in Y-schemes Z, a homomorphism of the group $\mathcal{G}(Z)$ into the automorphism group of the pair $(X \times_Y Z, \mathcal{F} \otimes_Y Z)$ ([8],[11]).

Assume that $\mathcal{G}$ acts on an Azumaya algebra $\mathcal{B}$ over X, i.e. the structure of $\mathcal{G}$-module $\mathcal{B}$ is given by an $\mathcal{O}_{\mathcal{G}\times_Y X}$-algebra isomorphism

$$\psi : \theta^*\mathcal{B} \xrightarrow{\sim} p_2^*\mathcal{B}.$$

Denote by $\underline{\underline{M}}(\mathcal{G}, X, \mathcal{B})$ the abelian category of $\mathcal{G}$-modules $\mathcal{F}$ on X, which are also left $\mathcal{B}$-modules and coherent $\mathcal{O}_X$-modules, such that the following diagram commutes:

$$\begin{array}{ccc} \theta^*\mathcal{B} \otimes \theta^*\mathcal{F} & \longrightarrow & \theta^*\mathcal{F} \\ {\scriptstyle \psi\otimes\varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ p_2^*\mathcal{B} \otimes p_2^*\mathcal{F} & \longrightarrow & p_2^*\mathcal{F}, \end{array}$$

where the horizontal maps are given by the action of $\mathcal{B}$ on $\mathcal{F}$. Morphisms in the category are morphisms of $\mathcal{B}$- and $\mathcal{G}$-modules.

If the algebra $\mathcal{B}$ is trivial, i.e. $\mathcal{B} = \mathcal{O}_X$, then the category is simply denoted by $\underline{\underline{M}}(\mathcal{G}, X)$.

Let $\mathcal{A}$ be an Azumaya algebra on Y. Consider the Azumaya algebra $\mathcal{B} = \pi^*\mathcal{A}$ on X, where $\pi : X \to Y$ is the structure morphism, and the category $\underline{\underline{M}}(Y, \mathcal{A})$ of left $\mathcal{A}$-modules which are coherent $\mathcal{O}_Y$-modules. For $\mathcal{M} \in \underline{\underline{M}}(Y, \mathcal{A})$ the $\mathcal{O}_X$-module

$\mathcal{F} = \pi^*\mathcal{M}$ has a natural structure of a $\mathcal{B}$-module. Since $\pi\theta = \pi p_2$, it follows that we also have a natural $\mathcal{G}$-module structure on $\mathcal{F}$ given by the isomorphisms

$$\varphi : \theta^*\mathcal{F} \simeq (\pi\theta)^*\mathcal{M} = (\pi p_2)^*\mathcal{M} \simeq p_2^*\mathcal{F}.$$

Thus, we have obtained a functor

$$\pi^* : \underline{\underline{M}}(Y, \mathcal{A}) \to \mathcal{M}(\mathcal{G}, X, \mathcal{B}), \qquad \mathcal{M} \mapsto (\pi^*\mathcal{M}, \varphi).$$

Proposition 6.1. *If $\pi : X \to Y$ is a $\mathcal{G}$-torsor then π^* is an equivalence of categories.*

Proof. Under the isomorphisms

$$\mathcal{G} \times_Y X \xrightarrow{\sim} X \times_Y X, \qquad (g, x) \mapsto (gx, x)$$

$$\mathcal{G} \times_Y \mathcal{G} \times_Y X \xrightarrow{\sim} X \times_Y X \times_Y X, \qquad (g_1, g_2, x) \mapsto (g_1g_2x, g_2x, x)$$

the action morphism θ is identified with the first projection $p_1 : X \times_Y X \to X$ and morphisms $m \times \mathrm{id}$, $\mathrm{id} \times \theta$ are identified with the projections $p_{13}, p_{12} : X \times_Y X \times_Y X \to X \times_Y X$. Hence, the isomorphism φ giving a $\mathcal{G}$-module structure on an $\mathcal{O}_X$-module $\mathcal{F}$ can be identified with descent data, i.e. with an isomorphism

$$\psi : p_1^*\mathcal{F} \xrightarrow{\sim} p_2^*\mathcal{F}$$

of $\mathcal{O}_{X\times_Y X}$-modules satisfying the usual cocycle condition

$$(p_{23}^*\psi) \circ (p_{12}^*\psi) = p_{13}^*\psi.$$

The statement follows now by faithfully flat descent ([7, Prop.2.22]). □

7. Modules under groups of multiplicative type

Let C be a diagonalizable group scheme over a field F, and let $C^* = \mathrm{Hom}(C, \mathbb{G}_m)$ be the character group. It is known that $C = \mathrm{Spec}F[C^*]$, where $F[C^*]$ is the group algebra of C^* over F, and the comorphism

$$\overline{m} : F[C^*] \to F[C^*] \otimes_F F[C^*]$$

of the multiplication is given by the formula $\overline{m}(\chi) = \chi \otimes \chi$ ([2]).

To introduce an action of C on an affine scheme $X = \mathrm{Spec}A$ over F is the same as to give a C^*-graded structure on the F-algebra A ([3]):

$$A = \coprod_{\chi \in C^*} A_\chi.$$

The comorphism of the action of C on X,

$$\overline{\theta} : A \to F[C^*] \otimes_F A$$

is given by the formula

$$\overline{\theta}(\sum_{\chi \in C^*} a_\chi) = \sum_{\chi \in C^*} (\chi \otimes a_\chi).$$

The trivial action corresponds to the trivial graded structure: $A_\chi = 0$ for $\chi \neq 0$.

Let M be an A-module. A C-module structure of the associated $\mathcal{O}_X$-module $\mathcal{F} = \widetilde{M}$ is given by an isomorphism of $F[C^*] \otimes_F A$-modules

$$\overline{\varphi} : (F[C^*] \otimes_F A) \otimes_{A,\overline{\theta}} M \xrightarrow{\sim} (F[C^*] \otimes_F A) \otimes_{A,\overline{p_2}} M,$$

satisfying the cocycle condition. Let $\overline{\varphi}(1 \otimes 1 \otimes m) = \sum_{\chi \in C^*}(\chi \otimes 1 \otimes m_\chi)$, where $m_\chi, m \in M$. Since $(e \otimes \mathrm{id})^*\varphi = \mathrm{id}$ ([11]), where $e : \mathrm{Spec} F \to G$ is the group unit, it follows that

$$m = \sum_{\chi \in C^*} m_\chi. \tag{$*$}$$

It is easy to check that the cocycle condition implies that $(m_\chi)_\rho$ equals m_χ if $\chi = \rho$ and equals 0 if $\chi \neq \rho$. Hence, the equality $(*)$ gives rise to the direct sum decomposition

$$M = \coprod_{\chi \in C^*} M_\chi$$

making M a C^*-graded A-module. Therefore, the category $\underline{\underline{M}}(C, X)$ is equivalent to the category of finitely generated C^*-graded modules.

Let an algebraic group G over a field F act on an affine scheme X over F and on an Azumaya algebra $\mathcal{B}$ on X. Assume that a closed central group subscheme $C \subset G$ of multiplicative type acts trivially on X and $\mathcal{B}$. Denote by C^* the Γ-module of characters $\mathrm{Hom}(C_{\mathrm{sep}}, \mathbb{G}_m)$. Since the group C_{sep} is diagonalizable, it follows that for any $\mathcal{F} \in \underline{\underline{M}}(G, X, \mathcal{B})$ we have a decomposition

$$\mathcal{F}_{\mathrm{sep}} = \coprod_{\chi \in C^*} (\mathcal{F}_{\mathrm{sep}})^\chi \tag{$**$}$$

into a direct sum of G_{sep}-submodules $(\mathcal{F}_{\mathrm{sep}})^\chi$ on X_{sep} (since C is central and acts trivially on X and $\mathcal{B}$).

Choose any Γ-invariant character $\chi \in C^*$ (defined over F). Clearly, $(\mathcal{F}_{\mathrm{sep}})^\chi$ and its direct complement in $(**)$ are defined over F, hence we have a canonical decomposition

$$\mathcal{F} = \mathcal{F}^\chi \oplus \mathcal{F}_\chi$$

into a direct sum of G-submodules on X. In other words, these submodules are uniquely determined by the property that $c - \chi(c)$ is trivial on $\mathcal{F}^\chi$ and invertible on $\mathcal{F}_\chi$ for all c in C.

Consider the full subcategories $\underline{\underline{M}}^\chi(G, X, \mathcal{B})$ and $\underline{\underline{M}}_\chi(G, X, \mathcal{B})$ in $\underline{\underline{M}}(G, X, \mathcal{B})$ consisting of all G-modules $\mathcal{F}$ such that $\mathcal{F} = \mathcal{F}^\chi$ and $\mathcal{F} = \mathcal{F}_\chi$ respectively. It is clear that

$$\underline{\underline{M}}(G, X, \mathcal{B}) \simeq \underline{\underline{M}}^\chi(G, X, \mathcal{B}) \times \underline{\underline{M}}_\chi(G, X, \mathcal{B}).$$

If $\chi = 0$ is the trivial character then the category $\underline{\underline{M}}^\chi(G, X, \mathcal{B})$ is equivalent to the category $\underline{\underline{M}}(G/C, X, \mathcal{B})$.

8. Equivariant algebraic K-theory

The K-groups of the category $\underline{\underline{M}}(\mathcal{G}, X, \mathcal{B})$ (see section 6) we denote by $K_*(\mathcal{G}, X, \mathcal{B})$. These groups are clearly contravariant with respect to flat $\mathcal{G}$-morphisms in X. If $\mathcal{B} = \mathcal{O}_X$ is the trivial algebra we simply write $K_*(\mathcal{G}, X)$.

Let G be an algebraic group over F acting on a scheme X over F. We will need the following particular cases of the localization theorem [11, th. 2.7] and the homotopy invariance theorem [11, cor. 4.2] in equivariant algebraic K-theory.

Proposition 8.1. *Let $U \subset X$ be an open G-equivariant subscheme. Then the restriction homomorphism $K_0(G, X) \to K_0(G, U)$ is surjective.* □

Proposition 8.2. *Assume that G acts linearly on an affine space $\mathbb{A}^n_F$ over F. Then the structure morphism $p : \mathbb{A}^n_F \to \mathrm{Spec}F$ induces an isomorphism*

$$p^* : K_*(G, \mathrm{Spec}F) \xrightarrow{\sim} K_*(G, \mathbb{A}^n_F). \quad \square$$

The category $\underline{\underline{M}}(G, \mathrm{Spec}F)$ is equivalent to the category of finite dimensional representations of G over F. The group $K_0(G, \mathrm{Spec}F)$ we denote by $R(G)$.

Assume that G acts on an Azumaya algebra $\mathcal{B}$ over X and contains a closed central subscheme C over F of multiplicative type, acting trivially on X and $\mathcal{B}$. For $\chi \in C^*(F)$ the K-groups of the category $\underline{\underline{M}}^\chi(G, X, \mathcal{B})$ we denote by $K^\chi_*(G, X, \mathcal{B})$. Since $K^\chi_*(G, X, \mathcal{B})$ is a canonical direct summand of $K_*(G, X, \mathcal{B})$ (section 7), it follows that the statements of propositions 8.1 and 8.2 still hold if we replace K_* by K^χ_*.

The group $K^\chi_0(G, \mathrm{Spec}F)$ we simply denote by $R^\chi(G)$. It is generated by the classes of all representations $\rho : G \to \mathbf{GL}(V)$ such that the restriction of ρ to C is given by χ.

9. Proof of the theorem

Let G_1 be an adjoint semisimple group over a field F, let G be the quasisplit inner form of G_1 with universal covering $\widetilde{G} \to G$, and let C be the kernel of the covering.

Choose a faithful representation $G \hookrightarrow \mathbf{GL}_n$ over F and consider the classifying variety $Y = \mathbf{GL}_n/G$ over F and the group scheme $\mathcal{G}'$ over Y dual to $\mathcal{G} = G \times_F Y$ with respect to the $\mathcal{G}$-torsor $\pi : \mathbf{GL}_n \to Y$. Let ξ be the generic point of Y. The $\mathcal{G}$-torsor π enables us to identify the character modules C^* and C'^*, where C' is the kernel of the universal covering of $\mathcal{G}'_\xi$. Choose a character $\chi' \in C'^*$ defined over $F(Y)$ and denote by $\chi \in C^*$ the corresponding character over F.

Consider a representation $\tilde{\rho} : \widetilde{G} \to \mathbf{GL}(V)$ such that the restriction of $\tilde{\rho}$ to C is given by χ. Consider also the Azumaya algebra $\mathcal{A}$ on Y associated to the $\mathcal{G}$-torsor π and the projective representation $\rho : G \to \mathbf{PGL}(V)$ induced by $\tilde{\rho}$ (section 2). We know that there is an isomorphism of G-algebras

$$\pi^*(\mathcal{A}) \simeq \pi^*(\mathcal{E}nd(V \times_F Y))$$

on $\mathbf{GL}_n$ (section 2) and that $\mathcal{A}_\xi$ is the Tits algebra corresponding to the character χ' (section 4). We have to show that $\mathrm{ind}\,\mathcal{A}_\xi = n_\chi(G)$.

Consider the homomorphism

$$\delta : K_0(\mathcal{A}^{op}_\xi) \to \mathbb{Z},$$

taking an A^{op}_ξ-module M to $\dim_{F(Y)} M$. It is easy to see that

$$\mathrm{im}(\delta) = \mathrm{ind}\,\mathcal{A}_\xi \cdot \deg \mathcal{A} \cdot \mathbb{Z}.$$

Consider also the homomorphism $\gamma : R^\chi(\widetilde{G}) \to \mathbb{Z}$, taking a representation space U to $\dim_F U$. It is clear that $\mathrm{im}(\gamma) = n_\chi(G) \cdot \mathbb{Z}$. For the proof of the theorem it is sufficient to find a surjective homomorphism

$$\alpha : R^\chi(\widetilde{G}) \to K_0(\mathcal{A}^{op}_\xi)$$

such that the composition $\delta \circ \alpha$ equals $\deg \mathcal{A} \cdot \gamma = \dim V \cdot \gamma$. The homomorphism α will be found as a composite of seven epimorphisms $\alpha_1, \alpha_2, \dots, \alpha_7$.

Consider $\mathbf{GL}_n$ as an open subvariety of the affine space $\mathbb{A} = \mathbb{A}^{n^2}_F$ of all $n \times n$-matrices over F on which the group G (and hence $\widetilde{G}$) acts linearly. The open embedding $\mathbf{GL}_n \hookrightarrow \mathbb{A}$ is clearly $\widetilde{G}$-equivariant. By proposition 8.2 (see also a remark at the end of section 8) the structure morphism $\mathbb{A} \to \mathrm{Spec}F$ induces an isomorphism

$$\alpha_1 : R^\chi(\widetilde{G}) = K_0^\chi(\widetilde{G}, \mathrm{Spec}F) \xrightarrow{\sim} K_0^\chi(\widetilde{G}, \mathbb{A}).$$

By proposition 8.1, the restriction homomorphism

$$\alpha_2 : K_0^\chi(\widetilde{G}, \mathbb{A}) \to K_0^\chi(\widetilde{G}, \mathbf{GL}_n)$$

is surjective.

Denote by $\mathcal{B}$ the algebra $\pi^*\mathcal{E}nd(V \times_F Y) = \mathcal{O}_{\mathbf{GL}_n} \otimes_F \mathrm{End}\, V$ on $\mathbf{GL}_n$. The group $\widetilde{G}$ clearly acts on $\mathcal{B}$. Consider two functors

$$\underline{\underline{M}}^\chi(\widetilde{G}, \mathbf{GL}_n) \underset{v}{\overset{u}{\rightleftarrows}} \underline{\underline{M}}^0(\widetilde{G}, \mathbf{GL}_n, \mathcal{B}^{op}),$$

$$u(\mathcal{F}) = V^* \otimes_F \mathcal{F}, \qquad v(\mathcal{M}) = V \otimes_{\mathrm{End}\, V^*} \mathcal{M},$$

where V^* is the F-vector space dual to V. The canonical isomorphisms $V \otimes_{\mathrm{End}\, V^*} V^* \simeq F$ and $V^* \otimes_F V \simeq \mathrm{End}\, V^*$ show that u and v are mutually inverse equivalences of categories. Hence, the functor u induces an isomorphism

$$\alpha_3 : K_0^\chi(\widetilde{G}, \mathbf{GL}_n) \xrightarrow{\sim} K_0^0(\widetilde{G}, \mathbf{GL}_n, \mathcal{B}^{op}).$$

Since the center C of $\widetilde{G}$ acts trivially on $\mathbf{GL}_n$ and $\mathcal{B}$, it follows that the categories $\underline{\underline{M}}^0(\widetilde{G}, \mathbf{GL}_n, \mathcal{B}^{op})$ and $\underline{\underline{M}}(G, \mathbf{GL}_n, \mathcal{B}^{op})$ are equivalent. Hence, we have an isomorphism

$$\alpha_4 : K_0^0(\widetilde{G}, \mathbf{GL}_n, \mathcal{B}^{op}) \xrightarrow{\sim} K_0(G, \mathbf{GL}_n, \mathcal{B}^{op}).$$

The isomorphism $G \times_F X \simeq \mathcal{G} \times_Y X$ shows that the categories $\underline{\underline{M}}(G, \mathbf{GL}_n, \mathcal{B}^{op})$ and $\underline{\underline{M}}(\mathcal{G}, \mathbf{GL}_n, \mathcal{B}^{op})$ are equivalent. Hence, we have an isomorphism

$$\alpha_5 : K_0(G, \mathbf{GL}_n, \mathcal{B}^{op}) \xrightarrow{\sim} K_0(\mathcal{G}, \mathbf{GL}_n, \mathcal{B}^{op}).$$

Since $\pi : \mathbf{GL}_n \to Y$ is a $\mathcal{G}$-torsor and $\mathcal{B} \simeq \pi^*\mathcal{A}$, it follows from proposition 6.1 that the functor

$$\pi^* : \underline{\underline{M}}(Y, \mathcal{A}^{op}) \to \underline{\underline{M}}(\mathcal{G}, \mathbf{GL}_n, \mathcal{B}^{op})$$

is an equivalence of categories. Hence, π^* induces an isomorphism

$$\alpha_6 : K_0(\mathcal{G}, \mathbf{GL}_n, \mathcal{B}^{op}) \xrightarrow{\sim} K_0(Y, \mathcal{A}^{op}).$$

By localization (Proposition 8.1), the functor

$$\underline{\underline{M}}(Y, \mathcal{A}^{op}) \to \underline{\underline{M}}(\mathcal{A}_\xi^{op}), \quad \mathcal{F} \mapsto \text{stalk of } \mathcal{F} \text{ at the generic point } \xi$$

induces an epimorphism

$$\alpha_7 : K_0(Y, \mathcal{A}^{op}) \to K_0(\mathcal{A}_\xi^{op}).$$

It can be easily checked that the composition $\alpha = \alpha_7 \circ \alpha_6 \circ \cdots \circ \alpha_1$ takes the class of a representation space U of the group $\widetilde{G}$ to the generic stalk $\mathcal{F}_\xi$ where

$$\pi^*\mathcal{F} = V^* \otimes_F U \otimes_F \mathcal{O}_{\mathbf{GL}_n}$$

and hence satisfies the desired condition. □

10. Examples

Let L/F be a Galois quadratic field extension, $\Pi = \mathrm{Gal}(L/F)$, and let B be a central simple algebra over L of degree $2n$ with involution τ of the second kind trivial on F.

Consider the special unitary group $\widetilde{G} = \mathbf{SU}(B,\tau)$ over F. The group $\widetilde{G}(F)$ of F-points of $\widetilde{G}$ consists of all elements $b \in B^\times$ such that $\tau(b)\cdot b = 1$ and $\mathrm{Nrd}(b) = 1$ where Nrd is the reduced norm homomorphism. The Galois group Γ acts on $C^* \simeq \mathbb{Z}/2n\mathbb{Z}$ through its factor group $\Pi = \{1,\pi\}$ by $\pi(k+2n\mathbb{Z}) = -k+2n\mathbb{Z}$ (see section 4). The Tits algebra corresponding to the only nontrivial character $\chi = n+2n\mathbb{Z} \in C^*(F)$ can be constructed as follows (see [4],[5]).

Consider the Severi-Brauer variety X over L corresponding to the algebra B and the canonical locally free sheaf J of rank $2n$ on X, so $B = \mathrm{End}_X(J)$ [9]. The canonical nondegenerate bilinear form on the n^{th}-exterior power of J

$$\Lambda^n J \otimes \Lambda^n J \to \Lambda^{2n} J \simeq \mathcal{O}_X$$

induces in the usual way an involution σ of the first kind on the algebra $\lambda^n B = \mathrm{End}_{\mathcal{O}_X}(\lambda^n J)$ over L. One can check that the involutions σ and $\tau' = \lambda^n\tau$ on $\lambda^n B$ commute. Therefore, the set $\{x \in \lambda^n B : \sigma(x) = \tau'(x)\}$ is a central simple algebra over F. We denote this algebra by $D(B,\tau)$ and call it *the discriminant algebra of* (B,τ) ([4]). It is the Tits algebra corresponding to the character χ.

The discriminant algebra enjoys the following properties:

1. The degree of $D(B,\tau)$ equals $\binom{2n}{n}$.

2. The restriction of σ to $D(B,\tau)$ is an involution of the first kind. In particular, the exponent of $D(B,\tau)$ divides 2.

3. $D(B,\tau)\otimes_F L \simeq \lambda^n B \sim B^{\otimes n}$. Since $\exp(B^{\otimes n})$ divides 2, it follows that $\mathrm{ind}(B^{\otimes n})$ also divides 2, and hence $\mathrm{ind}\, D(B,\tau)$ divides 4.

Let $\widetilde{G}'$ be the quasisplit inner form of $\widetilde{G}$. It is the special unitary group of the hyperbolic hermitian form over the quadratic extension L/F ([12]). Since $\widetilde{G}'_{\mathrm{sep}} \simeq \mathbf{SL}_{2n}(F_{\mathrm{sep}})$ it follows that

$$R(\widetilde{G}'_{\mathrm{sep}}) \simeq \mathbb{Z}[t_1, t_2, \ldots, t_{2n-1}]$$

where t_i is the class of the i^{th}-exterior power of the standard representation of $\mathbf{SL}_{2n}$. This ring is $C^* = \mathbb{Z}/2n\mathbb{Z}$-graded, the degree of t_i being equal to $i \pmod{2n}$. The rank map $R(\widetilde{G}'_{\mathrm{sep}}) \to \mathbb{Z}$ takes t_i to $\binom{2n}{i}$. The action of the Galois group Π on $R(\widetilde{G}'_{\mathrm{sep}})$ is given by $\pi(t_i) = t_{2n-i}$. We have also ([13]):

$$R(\widetilde{G}') \simeq \mathbb{Z}[t_1, t_2, \ldots, t_{2n-1}]^{\Pi}.$$

Using this description of the ring $R(\widetilde{G}')$ and the fact that the image of the map $R^\chi(\widetilde{G}') \to \mathbb{Z}$, taking a representation space U of the group $\widetilde{G}'$ to $\dim_F U$, equals $n_\chi\cdot\mathbb{Z}$, one can easily compute the number $n_\chi(G)$ for $G = \widetilde{G}/C$ (see [6]): $n_\chi(G)$ is equal to 2 if n is a 2–power and equals 4 otherwise. Hence, the corollary of the theorem gives in this case the following

Proposition 10.1. *For any Galois quadratic field extension L/F and $n \in \mathbb{N}$ there is a field extension E/F and a central simple algebra B of degree $2n$ over $E\otimes_F L$ with involution τ of the second kind trivial on E such that* $\mathrm{ind}\, D(B,\tau) = 2$ *if n is a 2-power and equals* 4 *otherwise.* □

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A. S. Merkurjev
Sankt-Petersburg State University
and
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D 33501 Bielefeld
Germany
merkurev@mathematik.uni-bielefeld.de