Twisted Pfister Forms

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Received: November 29, 1995
Revised: February 27, 1996

Communicated by Alexander S. Merkurjev

Abstract. Let F be a field of characteristic $\neq 2$. In this paper we investigate quadratic forms φ over F which are anisotropic and of dimension 2^n, $n \geq 2$, such that in the Witt ring WF they can be written in the form $\varphi = \sigma - \pi$ where σ and π are anisotropic n- resp. m-fold Pfister forms, $1 \leq m < n$. We call these forms twisted Pfister forms. Forms of this type with $m = n-1$ are of great importance in the study of so-called good forms of height 2, and such forms with $m = 1$ also appear in Izhboldin's recent proof of the existence of n-fold Pfister forms τ over suitable fields F, $n \geq 3$, for which the function field $F(\tau)$ is not excellent over F. We first derive some elementary properties and try to give alternative characterizations of twisted Pfister forms. We also compute the Witt kernel $W(F(\varphi)/F)$ of a twisted Pfister form φ. Our main focus, however, will be the study of the following problems: For which forms ψ does a twisted Pfister form φ become isotropic over $F(\psi)$? Which forms ψ are equivalent to φ (i.e., the function fields $F(\varphi)$ and $F(\psi)$ are place-equivalent over F)? We also investigate how such twisted Pfister forms behave over the function field of a Pfister form of the same dimension which then leads to a generalization of the result of Izhboldin mentioned above.

1991 Mathematics Subject Classification: Primary 11E04; Secondary 11E81, 12F20.

1 Introduction

Let F be a field of characteristic $\neq 2$. WF denotes the Witt ring of non-degenerate quadratic forms over F (which we will simply call forms over F). P_nF (resp. GP_nF) denotes the set of all forms isometric (resp. similar) to n-fold Pfister forms, i.e., forms

[1] This research has been carried out during stays at the Mathematics Department of the University of Kentucky, Lexington and the Sonderforschungsbereich 343: "Diskrete Strukturen in der Mathematik", Universität Bielefeld. I would like to thank both institutions for their hospitality.

of the type $\langle\langle a_1,\cdots,a_n\rangle\rangle = \langle 1,a_1\rangle\otimes\cdots\langle 1,a_n\rangle$. We say that φ is a Pfister neighbor if there exists $\pi\in P_nF$ for some n such that φ is similar to a subform of π and $\dim\varphi>\frac{1}{2}\dim\pi=2^{n-1}$. In this case, we say that φ is a Pfister neighbor of π. Any $\pi\in P_nF$ can be written as $\pi\simeq\langle 1\rangle\perp\pi'$. The form π' is called the pure part of π and it is uniquely determined up to isometry.

An important part of the algebraic theory of quadratic forms deals with the behavior of forms over F under a field extension K/F. Of particular interest is the case where $K=F(\psi)$ is the function field of a form ψ over F. If ψ is isotropic then $F(\psi)/F$ is purely transcendental. This situation is of not much interest with regard to the questions we will consider since one of our main goals lies in determining whether an anisotropic form φ over F becomes isotropic over K, something which cannot happen if K/F is purely transcendental. The extension K/F is said to be excellent if for any form φ over F the anisotropic part $(\varphi_K)_{\mathrm{an}}$ of φ over K is defined over F, i.e., there exists a form $\tilde\varphi$ over F such that $(\varphi_K)_{\mathrm{an}}\simeq\tilde\varphi_K$. Knebusch has shown in [K 2, Theorem 7.13] that if $F(\psi)/F$ is excellent where ψ is an anisotropic form, then ψ is a Pfister neighbor. As for the converse of this statement, it suffices to consider Pfister forms. This is because if ψ is a Pfister neighbor of τ then $F(\psi)$ and $F(\tau)$ are (place-)equivalent over F which implies that $F(\psi)$ is excellent iff $F(\tau)$ is excellent. So let $K=F(\tau)$ for some anisotropic $\tau\in P_nF$. It is easy to show that K/F is excellent for $n=1$, and for $n=2$ this was shown by Arason in [ELW 1, Appendix II]. It was an open problem whether K/F is always excellent for $n\geq 3$ until recently, when Izhboldin [I] gave a negative answer. In fact, he proved the even stronger result that to any anisotropic $\tau\in P_nF$, $n\geq 3$, there always exists a field extension E/F, some $\sigma\simeq\langle 1\rangle\perp\sigma'\in P_nE$ and some $d\in\dot E=E\setminus\{0\}$ not a square such that $\sigma'\perp\langle d\rangle$ is anisotropic, it becomes isotropic over $E(\tau)$, but its anisotropic part over $E(\tau)$ is not defined over E. In particular, $E(\tau)/E$ is not excellent.

Let us now turn to a seemingly unrelated problem. It is well-known that if φ is an anisotropic form over F then $\varphi_{F(\varphi)}$ is hyperbolic iff $\varphi\in GP_nF$ for some n. Going one step further, what can one say about an anisotropic form φ over F for which $\varphi_1\simeq(\varphi_{F(\varphi)})_{\mathrm{an}}$ does not vanish but where $(\varphi_1)_{F_1(\varphi_1)}$ becomes hyperbolic where $F_1=F(\varphi)$. Such a form is said to be of height 2. By the above, we know that $\varphi_1\in GP_mF_1$ for some $m\geq 1$ and we say that φ has degree m. We call φ good if there exists some $\rho\in P_mF$ such that $\varphi_1\simeq a\rho_{F_1}$ for some $a\in\dot F_1$. If one can choose $a\in\dot F$ already then φ is an excellent form in the sense of Knebusch [K 2, Section 7], and in this case one knows how φ has to look like (cf. [K 2, Lemma 10.1(i)]). An open problem is to classify anisotropic good non-excellent forms of height 2. It is believed that if φ is of that type and of degree $n-1$ then there exists some $\alpha\in P_{n-2}F$ and some 4-dimensional form β over F such that $\varphi\simeq\alpha\otimes\beta$ and $\alpha\otimes\langle\langle -d\rangle\rangle$ is anisotropic where $d=d_\pm\beta$ is the signed discriminant of β. This conjecture has been proved for $n=2$ (cf. [K 2, Theorem 10.3]), $n=3$ (cf. [F 2, Theorem 1.6]), and $n=4$ (cf. [Ka, Théorème 2.12]). (It is easy to show that if φ is of this type $\alpha\otimes\beta$ then φ is good non-excellent of height 2.)

What do these forms $\alpha\otimes\beta$ of height 2 and Izhboldin's examples $\sigma'\perp\langle d\rangle$ have in common? In both cases we are dealing with anisotropic forms φ of dimension 2^n. If $\varphi\simeq\alpha\otimes\beta$ and if we write $\beta\simeq\langle d,u,v,uv\rangle$ (possibly after scaling), then in WF we have

$$\varphi=\alpha\otimes\langle\langle u,v\rangle\rangle-\alpha\otimes\langle\langle -d\rangle\rangle\ .$$

If $\varphi \simeq \sigma' \perp \langle d \rangle$ then in WF we have

$$\varphi = \sigma - \langle\langle -d \rangle\rangle \ .$$

We observe that in these situations φ can be written as the difference of an n-fold Pfister form and an $(n-1)$-fold resp. 1-fold Pfister form. We aim at a unifying concept which includes both types of forms $\alpha \otimes \beta$ and $\sigma' \perp \langle d \rangle$. This leads us quite naturally to what we call *twisted Pfister forms*. φ is said to be a twisted Pfister form if φ is anisotropic of dimension 2^n for some n, such that in WF it can be written as $\varphi = \sigma - \pi$ for some anisotropic forms $\sigma \in P_nF$ and $\pi \in P_mF$, $1 \leq m < n$. The above examples represent twisted Pfister forms at the extreme ends of the spectrum: $m = n-1$ and $m = 1$. These examples also serve as a motivation for our in-depth study of these forms. It should be emphasized that Izhboldin's striking results in [I] and his clever constructions there gave the initial impulse to our present investigations.

As simple as the structure of twisted Pfister forms appears, this class of forms leads in our opinion to a wealth of interesting results and new problems as the above examples indicate. This is somewhat surprising considering their "proximity" to ordinary Pfister forms.

In the next section, we will recall some of the important facts about function fields and generic splitting of quadratic forms which we will need rather extensively in what will follow. Starting with the basic notion of linkage of Pfister forms in Section 3, we will then give the precise definition of twisted Pfister forms and derive some of their fundamental properties as well as some alternative characterizations. For completeness' sake, we included a short Section 4 in which we compute the Witt kernel $W(F(\varphi)/F)$ of a twisted Pfister form φ. These results have been previously obtained by Fitzgerald [F 1]. In Section 5 we attack the problem of determining those forms ψ for which a twisted Pfister form φ becomes isotropic over $F(\psi)$. In Section 6 we will determine in some cases the equivalence class of a twisted Pfister form φ (here, we mean that φ is equivalent to ψ, $\varphi \sim \psi$, if $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic). Some of our results in Sections 5 and 6 apply to an even bigger class of forms than twisted Pfister forms. The results in these two sections can be regarded as an extension and generalization of our earlier work in [H 4]. In Section 7, we consider the case of a twisted Pfister form φ of dimension 2^n and an anisotropic $\tau \in P_nF$. We generalize Izhboldin's results in [I] on when $(\varphi_{F(\tau)})_{\mathrm{an}}$ is defined over F and add some remarks about so-called $F(\tau)$-minimal forms. Finally, in Section 8, we explicitly construct $\tau \in P_nF$, $n \geq 3$, such that $F(\tau)/F$ is not excellent where F is purely transcendental of degree $n-1$ over $\mathbb{Q}$. We also generalize Izhboldin's construction of a field extension E/F such that $E(\tau)/E$ is not excellent where one starts with an arbitrary field F permitting an anisotropic Pfister form $\tau \in P_nF$, $n \geq 3$. Our construction is still based on Izhboldin's original ideas used in [I].

2 Some basic facts

In our notations and terminology we follow Lam's book [L 1] and Scharlau's book [S]. $\varphi \simeq \psi$ denotes isometry of the forms φ and ψ over F, whereas $\varphi = \psi$ stands for equality in the Witt ring WF. We write φ_{an} for the anisotropic part of φ and $i_W(\varphi)$ for its Witt index. Thus, if we denote the hyperbolic plane $\langle 1, -1 \rangle$ by $\mathbb{H}$ and put

$i = i_W(\varphi)$, we have $\varphi \simeq \varphi_{\mathrm{an}} \perp (i \times \mathbb{H})$. If φ is a subform of ψ, i.e., if there exists a form η such that $\psi \simeq \varphi \perp \eta$, then we write $\varphi \subset \psi$ for short.

If K/F is a field extension and if φ is a form over F, then we denote the form which one obtains from φ by scalar extension by φ_K. The Witt kernel $W(K/F)$ is the kernel of the natural map $WF \to WK$ induced by scalar extension. We put $D_K(\varphi) = \{a \in \dot{K} \mid \langle a \rangle \subset \varphi_K\}$ and $G_K(\varphi) = \{a \in \dot{K} \mid a\varphi_K \simeq \varphi_K\}$ (we omit the subscript if $K = F$). K/F is said to be excellent if for any form φ over F there exists a form $\tilde{\varphi}$ over F such that $(\varphi_K)_{\mathrm{an}} \simeq \tilde{\varphi}$, i.e., the anisotropic part of φ over K is defined over F. A form φ over F is called K-minimal if φ is anisotropic, φ_K is isotropic, and η_K is anisotropic for any $\eta \subset \varphi$ with $\dim \eta < \dim \varphi$. Two field extensions K and L of F are called equivalent if there exist F-places $\lambda : K \to L \cup \infty$ and $\mu : L \to K \cup \infty$, we write $K \sim L$ for short. In this situation, K/F is excellent iff L/F is excellent (this follows from [K 1, Proposition 3.1], see also [ELW 1, Corollary 2.8]), and K-minimal forms are exactly the L-minimal forms. φ is said to be round (or multiplicative) if $D(\varphi) = G(\varphi)$. If φ is a Pfister form then φ is multiplicative and either anisotropic or hyperbolic (cf. [L 1, Ch. 10, Corollaries 1.6, 1.7] or [S, Ch. 4, Corollary 1.5]).

Let now φ be a form over F such that $\dim \varphi \geq 2$ and $\varphi \not\simeq \mathbb{H}$. The function field $F(\varphi)$ of φ is the function field of the projective quadric defined by $\varphi = 0$. To avoid case distinctions, we put $F(\varphi) = F$ if $\dim \varphi \leq 1$ or $\varphi \simeq \mathbb{H}$. If $\dim \varphi = n \geq 2$ then $F(\varphi)/F$ is a purely transcendental extension of degree $n-2$ over F followed by a quadratic extension, and $F(\varphi)/F$ is purely transcendental iff φ is isotropic ([S, Ch. 4, Remark 5.2(vi)]). $F(\varphi)$ is a generic zero (or isotropy) field of φ over F, i.e., if K is any field extension of F with φ_K isotropic then there exists a place $\lambda : F(\varphi) \to K \cup \infty$ over F. We say that two forms φ and ψ are equivalent if $F(\varphi) \sim F(\psi)$, and we write $\varphi \sim \psi$. In the following proposition, we collect some more results about function fields of quadratic forms which we will need later on.

Proposition 2.1 *Let φ and ψ be anisotropic forms over F.*

(i) ([K 1, Theorem 3.3].) *$\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are both isotropic iff $F(\varphi) \sim F(\psi)$, i.e., iff $\varphi \sim \psi$.*

(ii) ([S, Ch. 4, Theorem 5.4(i)].) *$\varphi_{F(\varphi)}$ is hyperbolic iff $\varphi \in GP_nF$ for some $n \geq 1$.*

(iii) ([L 1, Ch. 7, Lemma 3.1], [S, Ch. 2, Lemma 5.1].) *If $\dim \psi = 2$ then $\varphi_{F(\psi)}$ is isotropic iff $a\psi \subset \varphi$ for some $a \in \dot{F}$.*

(iv) (Cassels-Pfister subform theorem, [S, Ch. 4, Theorem 5.4(ii)].) *If $\varphi_{F(\psi)}$ is hyperbolic then $a\psi \subset \varphi$ for any $a \in D(\varphi)\cdot D(\psi)$.*

(v) ([S, Ch. 4, Theorem 5.4(iv)].) *If ψ is a Pfister neighbor of the Pfister form π, then $\varphi_{F(\psi)}$ is hyperbolic iff there exists a form γ over F such that $\varphi \simeq \pi \otimes \gamma$. In particular, $W(F(\psi)/F) = \pi WF$.*

(vi) ([H 3, Theorem 1].) *If $\dim \varphi \leq 2^n < \dim \psi$ for some n then $\varphi_{F(\psi)}$ stays anisotropic.*

(vii) ([H 3, Proposition 2].) *If ψ is a Pfister neighbor of the Pfister form π then $\varphi \sim \psi$ iff φ is a Pfister neighbor of π.*

(viii) ([L 2, Theorem 10.1].) *Let ρ be another form over F. If $\varphi_{F(\psi)}$ is isotropic and if $\psi_{F(\rho)}$ is isotropic then $\varphi_{F(\rho)}$ is isotropic.*

(ix) ([K 2, Theorem 7.13] or [ELW 1, Examples 2.2(i)].) *If $F(\psi)/F$ is excellent then ψ is a Pfister neighbor.*

(x) (Cf. part (iii) of this proposition and [ELW 1, Appendix II by Arason].) *If ψ is a Pfister neighbor of an n-fold Pfister form, $n = 1$ or 2, then $F(\psi)/F$ is excellent.*

Let now φ be a form over F which is not hyperbolic. We define inductively fields F_i, $i \geq 0$, and forms φ_i over F_i as follows. Let $\varphi_0 \simeq \varphi_{\mathrm{an}}$ and $F_0 = F$. For $i \geq 1$ we put $F_i = F_{i-1}(\varphi_{i-1})$ and $\varphi_i \simeq ((\varphi_{i-1})_{F_{i-1}})_{\mathrm{an}}$. The smallest h for which $\dim \varphi_h \leq 1$ is called the height of φ. The tower $F_0 \subset F_1 \subset \cdots \subset F_h$ is called a generic splitting tower of φ over F, F_h is a generic splitting field of φ over F, and F_{h-1} is called the leading field of φ over F. φ_j is called the j-th kernel form of φ and $i_W(\varphi_{F_j}) = i_j(\varphi)$ the j-th Witt index of φ. By the splitting pattern of φ we mean the sequence $\{\dim \varphi_0, \dim \varphi_1, \cdots, \dim \varphi_h\}$ (this definition is different from the one given in [HuR]). The degree of φ is defined as follows. If $\dim \varphi$ is odd we put $\deg \varphi = 0$. Otherwise, we know by Proposition 2.1(ii) that $\varphi_{h-1} \in GP_nF_{h-1}$ for some $n \geq 1$. In this case we put $\deg \varphi = n$. Let $\tau \in P_nF_{h-1}$ such that φ_{h-1} is similar to τ. Then τ is called the leading form of φ. If the leading form is defined over F we say that φ is a good form (in this case there actually exists $\sigma \in P_nF$ such that $\tau \simeq \sigma_{F_{h-1}}$, cf. [K 2, Proposition 9.2]).

There are two natural filtrations of the Witt ring. One is given by the n-th powers I^nF of the ideal IF of even-dimensional forms in WF. I^nF is additively generated by the n-fold Pfister forms. One has $I^2F = \{\varphi \in IF \mid d_\pm \varphi = 1 \in \dot{F}/\dot{F}^2\}$, where $d_\pm \varphi$ denotes the signed discriminant of a form φ, and by Merkurjev's theorem [M] one has $I^3F = \{\varphi \in I^2F \mid c(\varphi) = 1\}$ where $c(\varphi)$ denotes the Clifford invariant of φ which is an element in the Brauer group BrF of F. The other filtration is given by the ideals $J_nF = \{\varphi \in WF \mid \deg \varphi \geq n\}$ (cf. [K 1, Theorem 6.4] for the fact that these sets are ideals, see also [S, Ch. 4, Theorem 7.3]). One has $I^nF \subset J_nF$ for all $n \geq 0$ (cf. [K 1, Corollary 6.6], [S, p. 164]). This is essentially the Arason-Pfister Hauptsatz which in its original form states that if $0 \neq \varphi \in I^nF$ is anisotropic then $\dim \varphi \geq 2^n$, and furthermore, if $\varphi \in I^nF$ is anisotropic and $\dim \varphi = 2^n$ then $\varphi \in GP_nF$ (see [AP, Hauptsatz and Korollar 3]). If we define $\deg' \varphi = n$ if $\varphi \in I^nF \setminus I^{n+1}F$, we thus have $\deg' \varphi \leq \deg \varphi$. It is still an open problem whether $I^nF = J_nF$ for all n and all F. This is known to be true for $n \leq 4$ (cf. [Ka, Théorème 2.8] and the references there). We will mainly work with the ideals J_nF.

PROPOSITION 2.2 *Let φ and ψ be forms over F with φ not hyperbolic, and let $F = F_0 \subset F_1 \subset \cdots \subset F_h$ be a generic splitting tower of φ as defined above.*

(i) ([K 1, Proposition 6.9 and Corollary 6.10], see also [S, Ch. 4, Theorem 7.5].) *$I^mFJ_nF \subset J_{m+n}F$ for all $m, n \geq 0$. Furthermore, $\deg(\varphi \otimes \psi) = \deg \varphi$ iff $\dim \psi$ is odd.*

(ii) ([AK, Satz 18].) *If $\deg \varphi_{F(\psi)} > \deg \varphi$ then $\dim \psi \leq 2^n$, and if furthermore $\dim \psi = 2^n$ then $\psi \in GP_nF$ and $\varphi \equiv \psi \pmod{J_{n+1}F}$. In particular, $\psi_{F_{h-1}}$ is similar to the leading form of φ.*

(iii) ([K 1, Corollary 3.9 and Proposition 5.13].) *Let K/F be a field extension. Let $K \cdot F_j$ be the free composite of K and F_j over F. If $i_W(\varphi_K) \geq i_j(\varphi)$ then $K \cdot F_j$ is purely transcendental over K.*

3 Twisted Pfister forms

The following result is well-known (cf. [EL, Theorem 1.4]). Since we will use it quite often without always referring to it explicitly, we will include a proof at this point for the reader's convenience.

Lemma 3.1 *Let $\alpha \in WF$ be a round form, i.e., $G(\alpha) = D(\alpha)$. Let $\varphi \in WF$. If $\alpha \otimes \varphi$ represents $a \in \dot{F}$, then there exists $\psi \in WF$ with $a \in D(\psi)$ such that $\alpha \otimes \varphi \simeq \alpha \otimes \psi$. Furthermore, if $\dim \varphi \geq 2$ and $\alpha \otimes \varphi$ is isotropic, then then there exists an isotropic $\psi \in WF$ such that $\alpha \otimes \varphi \simeq \alpha \otimes \psi$.*

Proof. Let $\varphi \simeq \langle a_1, \cdots, a_n \rangle$ so that $\alpha \otimes \varphi \simeq a_1\alpha \perp \cdots \perp a_n\alpha$. Since $a \in D(\alpha \otimes \varphi)$ there are $x_i \in D(\alpha)$, not all 0, such that $a = a_1x_1 + \cdots + a_nx_n$. Say, $x_1, \cdots, x_m \neq 0$, $m \leq n$. As α is round, $x_i\alpha \simeq \alpha$ for $1 \leq i \leq m$. Thus, $\alpha \otimes \langle a_1, \cdots, a_m \rangle \simeq \alpha \otimes \langle a_1x_1, \cdots, a_mx_m \rangle$. By the above, a is represented by $\langle a_1x_1, \cdots, a_mx_m \rangle$. Hence, $\langle a_1x_1, \cdots, a_mx_m \rangle \simeq \langle a, a_2', \cdots, a_m' \rangle$ and thus $\alpha \otimes \varphi \simeq \alpha \otimes \langle a, a_2', \cdots, a_m', a_{m+1}, \cdots, a_n \rangle$.

Now suppose that $\dim \varphi \geq 2$ and that $\alpha \otimes \varphi$ is isotropic. Write $\varphi \simeq \varphi' \perp \langle -x \rangle$. By assumption, there exists $y \in \dot{F}$ such that y is represented both by $\alpha \otimes \varphi'$ and by $x\alpha$. This is clear if both forms are anisotropic because their difference is isotropic. If either one of them is isotropic, it is universal and therefore represents any non-zero element represented by the other form. By the above, $\alpha \otimes \varphi' \simeq \alpha \otimes \psi'$ with $y \in D(\psi')$ and $x\alpha \simeq y\alpha$. In particular, the form $\psi \simeq \psi' \perp \langle -y \rangle$ is isotropic and $\alpha \otimes \varphi \simeq \alpha \otimes \psi$. □

To gain a better understanding of the definition of twisted Pfister forms which we will give later it seems useful to recall another well-known result due to Elman and Lam [EL, Theorem 4.5].

Lemma 3.2 *Let $\sigma \in P_nF$ and $\pi \in P_mF$ be anisotropic with $m \leq n$. Let $a, b \in \dot{F}$. Then $i := i_W(a\sigma \perp b\pi) = 0$ or 2^r for some integer r with $0 \leq r \leq m$. Furthermore, $i \geq 1$ iff there exists $x \in \dot{F}$ such that $(a\sigma \perp b\pi)_{\mathrm{an}} \simeq x(\sigma \perp -\pi)_{\mathrm{an}}$. If $i = 2^r \geq 1$ then there exist $\alpha \in P_rF$, $\sigma_1 \in P_{n-r}F$, and $\pi_1 \in P_{m-r}F$ such that $\sigma \simeq \alpha \otimes \sigma_1$ and $\pi \simeq \alpha \otimes \pi_1$.*

Proof. We may assume that $i \geq 1$. Then there exist $u \in D(\sigma)$ and $v \in D(\pi)$ such that $au + bv = 0$. The roundness of σ and π implies that $u\sigma \simeq \sigma$ and $v\pi \simeq \pi$. Thus, with $x = au = -bv$, we have

$$a\sigma \perp b\pi \simeq au\sigma \perp bv\pi \simeq x\sigma \perp -x\pi.$$

Thus, $(a\sigma \perp b\pi)_{\mathrm{an}} \simeq x(\sigma \perp -\pi)_{\mathrm{an}}$.

Now if $i = 1$ there is nothing else to show. So let us assume that $i \geq 2$ and let α' be a common Pfister neighbor of σ and π of maximal dimension. Since $i \geq 2$ we have $\dim \alpha' \geq 2$ as both forms have at least a common 2-dimensional form, and every such 2-dimensional form is trivially a Pfister neighbor. Say, α' is a Pfister neighbor of $\alpha \in P_nF$. Since α' becomes isotropic over $F(\alpha)$, it follows that $\sigma_{F(\alpha)}$ and $\pi_{F(\alpha)}$ are also isotropic and hence hyperbolic. By the Cassels-Pfister subform theorem and because 1 is represented by σ, π, and α, there exist forms σ_0 and π_0 such that $\sigma \simeq \alpha \perp \sigma_0$ and $\pi \simeq \alpha \perp \pi_0$, cf. Proposition 2.1(iv). The maximality of $\dim \alpha'$ implies that $\dim \alpha = \dim \alpha'$. Suppose $i > \dim \alpha$. Then $\sigma_0 \perp -\pi_0$ is isotropic and there exists a $w \in \dot{F}$ which is represented both by σ_0 and π_0. In particular, the Pfister neighbor $\alpha \perp \langle w\rangle$ of $\alpha\otimes\langle\langle w\rangle\rangle$ is a common subform of σ and π, a contradiction to the maximality of $\dim\alpha' = \dim\alpha$. Thus, $i = \dim\alpha = 2^r$ for some $r \geq 1$. As for the remaining statement, there is nothing else to show if $\dim\alpha = \dim\sigma$. So suppose $\dim\sigma_0 > 0$ and let $v \in D(\sigma_0)$. Thus, the Pfister neighbor $\alpha \perp \langle v\rangle$ of $\alpha\otimes\langle\langle v\rangle\rangle$ is a subform of σ, and by an argument similar to above, we get that $\sigma \simeq \alpha\otimes\langle\langle v\rangle\rangle \perp \tilde{\sigma}$. The existence of $\sigma_1 \in P_{n-r}F$ now follows by an easy induction on $\dim\sigma_0$. The existence of π_1 can be shown in the same way. $\square$

DEFINITION 3.3 Let σ, π be anisotropic Pfister forms. If $i_W(\sigma \perp -\pi) = 2^r$, $r \geq 0$, then r is called the *linkage number* of σ and π. We write $\ln(\sigma,\pi) = r$. A form $\alpha \in P_rF$ such that $\sigma \simeq \alpha\otimes\sigma_1$ and $\pi \simeq \alpha\otimes\pi_1$ for suitable Pfister forms σ_1, π_1 is called a *link* of σ and π.

It should be remarked that a link α is generally not uniquely determined up to isometry.

We now consider the case of an anisotropic form φ of dimension 2^n such that in WF we have $\varphi = a\sigma + b\pi$, where $a, b \in \dot{F}$ and σ ,π are Pfister forms with $\dim\sigma \geq \dim\pi$. In view of Lemma 3.2, we then have that $\varphi \simeq x(\sigma \perp -\pi)_{\mathrm{an}}$ for some $x \in \dot{F}$. We want to exclude the case where $\varphi \in GP_nF$. An easy check then shows that we may assume $\sigma \in P_nF$ and $\pi \in P_mF$ are both anisotropic with $1 \leq m < n$ and we have $\ln(\sigma,\pi) = m-1$. We now come to the definition of twisted Pfister forms and of what we will call weakly twisted Pfister forms, a type of form which will also appear frequently throughout the paper.

DEFINITION 3.4 (i) Let $1 \leq m < n$. A form φ over F is called a *twisted* (n,m)-*Pfister form* (or simply (n,m)-*Pfister form*) if there exist anisotropic forms $\sigma \in P_nF$ and $\pi \in P_mF$ such that $\ln(\sigma,\pi) = m-1$ and such that $\varphi \simeq (\sigma \perp -\pi)_{\mathrm{an}}$. In this case we say that the (n,m)-Pfister form φ is defined by (σ,π). The set of all forms isometric (resp. similar) to (n,m)-Pfister forms is denoted by $P_{n,m}F$ (resp. $GP_{n,m}F$). φ is called a *twisted Pfister form* if $\varphi \in GP_{n,m}F$ for some (n,m) with $1 \leq m < n$.

(ii) Let $1 \leq m < n$. A form φ over F is called a *weakly twisted* (n,m)-*Pfister form* if φ is anisotropic, $\dim\varphi = 2^n$, and $\varphi \equiv \pi\otimes\eta \pmod{J_nF}$ for some anisotropic $\pi \in P_mF$ and some odd-dimensional $\eta \in WF$. We call π the *twist* of φ. The set of all weakly twisted (n,m)-Pfister forms will be denoted by $P^w_{n,m}F$.

REMARK 3.5 (i) Let $1 \leq m < n$. In view of Lemma 3.2 and by the remarks preceeding the definition, $\varphi \in GP_{n,m}F$ iff φ is anisotropic, $\dim\varphi = 2^n$, and there exist anisotropic $\sigma \in GP_nF$ and $\pi \in GP_mF$ such that $\varphi = \sigma + \pi$ in WF. If this is the case, then $\varphi \notin GP_nF$. In fact, $\varphi \equiv \sigma + \pi \equiv \pi \not\equiv 0 \pmod{J_nF}$ because $\sigma \in GP_nF \subset J_nF$ and $\pi \in GP_mF$ is anisotropic and thus, since $\dim\pi = 2^m < 2^n$ and by the Arason-Pfister Hauptsatz, $\pi \notin J_nF$. Similarly, $\varphi \equiv \pi \not\equiv 0 \pmod{I^nF}$.

(ii) Let now $\varphi \in P_{n,m}F$ be defined by (σ,π). Let α be a link of σ and π, i.e., $\alpha \in P_{m-1}F$ and there exist $\sigma_1 \in P_{n-m+1}F$ and $d \in \dot{F}$ such that $\sigma \simeq \alpha\otimes\sigma_1$ and $\pi \simeq \alpha\otimes\langle\langle -d\rangle\rangle$. Let σ_1' denote the pure part of σ_1, i.e., $\sigma_1 \simeq \langle 1\rangle \perp \sigma_1'$. Then $\varphi \simeq \alpha\otimes(\langle d\rangle \perp \sigma_1')$.

(iii) If $\varphi \in P_{n,m}F$ is defined by (σ,π) then $\varphi \equiv \sigma - \pi \equiv -\pi \pmod{J_nF}$. Hence, $GP_{n,m}F \subset P^w_{n,m}F$. This is generally a proper inclusion if $m \leq n-3$ (see, e.g., Example 5.13), but it is an equality if $1 \leq n-2 \leq m \leq n-1 \leq 3$ (cf. Proposition 3.17 below).

Before we continue, let us mention some of the properties of twisted Pfister forms which will be useful later. In fact, we state these results for a possibly wider class of forms (see also Conjecture 3.9 below).

PROPOSITION 3.6 *Let $1 \leq m < n$. Let $\varphi \in WF$ be anisotropic and $\dim\varphi = 2^n$. Suppose that $\varphi \equiv x\pi \pmod{J_nF}$ for some anisotropic $\pi \in P_mF$ and some $x \in \dot{F}$. Then the following holds.*

(i) *$\varphi_{F(\pi)}$ is anisotropic and in $GP_nF(\pi)$. In particular, if $\varphi \in P_{n,m}F$ is defined by (σ,π), then $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)}$ is anisotropic.*

(ii) *φ is good with leading form defined by π. We have*

$$\mathrm{ht}(\varphi) = \begin{cases} 2 & \text{if} \quad m = n-1, \\ 3 & \text{if} \quad m < n-1. \end{cases}$$

In particular, $i_1(\varphi) = 2^{m-1}$ and the splitting pattern of φ is $\{2^n, 2^{n-1}, 0\}$ if $m = n-1$ and $\{2^n, 2^n - 2^m, 2^m, 0\}$ if $m < n-1$.

These statements hold in particular if $\varphi \in GP_{n,m}F$.

Proof. (i) First note that $\pi_{F(\pi)} = 0$ and thus $\varphi_{F(\pi)} \in J_nF(\pi)$. If $\varphi_{F(\pi)}$ is anisotropic then, since $\dim\varphi = 2^n$, this implies $\varphi_{F(\pi)} \in GP_nF(\pi)$. So suppose $\varphi_{F(\pi)}$ is isotropic. Then $\dim(\varphi_{F(\pi)})_{\mathrm{an}} < 2^n$ and by the Arason-Pfister Hauptsatz, $\varphi_{F(\pi)}$ is hyperbolic. Thus, there exists $\gamma \in WF$, $\dim\gamma = 2^{n-m}$, such that $\varphi \simeq \pi \otimes \gamma$. Since $n > m$ we have that $\dim\gamma$ is even, i.e., $\gamma \in IF$. But $\pi \in P_mF \subset I^mF$. Hence, $\varphi \simeq \pi \otimes \gamma \in I^{m+1}F \subset J_{m+1}F$. But clearly, $\varphi \equiv x\pi \not\equiv 0 \pmod{J_{m+1}F}$, a contradiction.

If $\varphi \in P_{n,m}F$ is defined by (σ,π), then in WF we have $\varphi = \sigma - \pi$ and thus, in $WF(\pi)$, $\varphi_{F(\pi)} = \sigma_{F(\pi)}$. Now $\varphi \equiv -\pi \pmod{J_nF}$ and by Remark 3.5(i) and by the above, it is clear that $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)}$ is anisotropic.

(ii) Since $\dim\varphi = 2^n$ but $\varphi \notin GP_nF$, we have $\mathrm{ht}(\varphi) \geq 2$. Let $F_0 = F$, F_1, and F_2 be the first three fields in a splitting tower of φ, and let φ_1 and φ_2 be the first two kernel forms of φ. Clearly, $0 < \dim\varphi_1 < 2^n$ and $(\varphi_1)_{F_1(\pi)} \equiv 0 \pmod{J_nF_1(\pi)}$. Hence, by the Arason-Pfister Hauptsatz, $(\varphi_1)_{F_1(\pi)}$ is hyperbolic. Thus, there exists $\gamma \in WF_1$ such that $\varphi_1 \simeq \pi_{F_1} \otimes \gamma$. Comparing dimensions shows that $1 \leq \dim\gamma \leq 2^{n-m} - 1$. This shows in particular that $i_1(\varphi) \geq 2^{m-1}$. Define $\psi \in WF_1$ by $\psi \simeq \varphi_1 \perp -x\pi_{F_1} \simeq \pi_{F_1} \otimes (\gamma \perp \langle -x\rangle)$. Note that $\dim\psi \leq 2^n$ and

$$\psi \equiv \varphi_1 - x\pi_{F_1} \equiv \varphi_{F_1} - x\pi_{F_1} \equiv x\pi_{F_1} - x\pi_{F_1} \equiv 0 \pmod{J_nF}.$$

Thus, by the Arason-Pfister Hauptsatz, either ψ is hyperbolic or ψ is anisotropic and in GP_nF_1.

Suppose that ψ is hyperbolic. Let $\mu \simeq (\varphi \perp -x\pi)_{\mathrm{an}}$ over F. By definition, $\mu \equiv 0 \pmod{J_nF}$. Note that φ and π are anisotropic and $\dim\varphi = 2^n > \dim\pi = 2^m$. Hence, $0 < 2^n - 2^m \leq \dim\mu \leq 2^n + 2^m < 2^{n+1}$. Therefore, by the Arason-Pfister Hauptsatz, $2^n \leq \dim\mu < 2^{n+1}$ and we must have $\deg\mu = n$. Over $F_1 = F(\varphi)$ we have $\mu_{F_1} = \varphi_{F_1} - x\pi_{F_1} = \psi_{F_1} = 0$ and thus $\deg\mu_{F_1} = \infty > \deg\mu = n$. Now $\dim\varphi = 2^n$ and Proposition 2.2(ii) yields $\varphi \in GP_nF$, obviously a contradiction.

It follows that $\psi \simeq \varphi_1 \perp -x\pi_{F_1} \in GP_nF_1$ is anisotropic. Furthermore, $\dim\varphi_1 = 2^n - 2^m$. In particular, $\psi_{F_1(\pi)}$ is isotropic and hence hyperbolic and thus $(\varphi_1)_{F_1(\pi)}$ is hyperbolic as well. If $m = n-1$ then $\dim\varphi_1 = \dim\pi = 2^{n-1}$ which immediately yields that φ_1 is similar to π_{F_1}, which in turn implies that φ is good of height 2 with leading form defined by π. Now if $m < n-1$ then $\dim\varphi_1 = 2^n - 2^m > 2^{n-1}$. Thus, φ_1 is a Pfister neighbor with complementary form $-x\pi_{F_1}$. It follows readily that $\varphi_2 \simeq x\pi_{F_2}$ and that φ is a good form of height 3 with leading form defined by π. In fact, the second kernel form is defined by $x\pi$ already over F. □

It should be remarked that the fact that the leading form of φ is defined by π also follows directly from $\varphi \equiv x\pi \equiv \pi \pmod{J_{m+1}F}$ by [K 2, Theorem 9.6]. In our proof, we also wanted to determine the height of φ explicitly.

Corollary 3.7 *Let $1 \leq m < n$. Let $\varphi \in P^w_{n,m}F$ with twist $\pi \in P_mF$. Then $\varphi_{F(\pi)}$ is anisotropic and in $GP_nF(\pi)$, $i_1(\varphi) = 2^{m-1}$, and φ is good with leading form defined by π. Furthermore, if $F(\pi)/F$ is excellent, then there exists $\sigma \in GP_nF$ such that $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)}$.*

Proof. Write $\varphi \equiv \pi \otimes \eta \pmod{J_nF}$ with $\dim\eta$ odd. That $\varphi_{F(\pi)}$ is anisotropic and in $GP_nF(\pi)$ can be shown as in Proposition 3.6, using the fact that $\pi \otimes \eta \not\equiv 0 \pmod{J_nF}$ as π is anisotropic and $\dim\eta$ is odd and hence $\deg(\pi\otimes\eta) = \deg\pi = m < n$ (see Proposition 2.2(i)). Similarly as before, we get that $i_1(\varphi) \geq 2^{m-1}$. We want to show that we have equality and also that φ is good with leading form defined by π. We may assume that after scaling $d_\pm\eta = 1$. It is clear that $\varphi \equiv \pi\otimes\eta \equiv \pi \pmod{J_{m+1}F}$ and it follows from [K 2, Theorem 9.6] that φ is good with leading form defined by π. Let L be the leading field of $\pi \otimes \eta$ and $K = F(\pi)$. Since $(\pi \otimes \eta)_K = 0$ we have that the free composite KL is purely transcendental over K (see Proposition 2.2(iii)). Since φ_K is anisotropic, we therefore have that φ_{KL} is anisotropic and hence φ_L is anisotropic as well. Now $(\pi \otimes \eta)_L = \pi_L$ in WL by [K 1, Proposition 6.12]. Hence, φ_L is anisotropic, $\dim\varphi_L = 2^n$, and $\varphi_L \equiv \pi_L \pmod{J_nL}$. By Proposition 3.8, we have $i_1(\varphi_L) = 2^{m-1}$. But $i_1(\varphi_L) \geq i_1(\varphi) \geq 2^{m-1}$. Hence, $i_1(\varphi) = 2^{m-1}$.

Finally, if $F(\pi)/F$ is excellent, then the existence of some $\sigma \in GP_nF$ such that $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)}$ follows from [ELW 1, Proposition 2.11]. □

Corollary 3.8 *Let $1 \leq m < n$. Let $\varphi \in P^w_{n,m}F$. Let $\psi \subset a\varphi$ for some $a \in \dot{F}$ and $\dim\psi > 2^n - 2^{m-1}$. Then $\psi_{F(\varphi)}$ is isotropic and $\varphi \sim \psi$.*

Proof. We have $i_1(\varphi) = i_W(\varphi_{F(\varphi)}) = 2^{m-1}$ by the previous proposition. Since ψ is similar to a subform of φ and $\dim\psi > \dim\varphi - i_1(\varphi)$, it follows readily that $\psi_{F(\varphi)}$ is isotropic. Clearly, $\psi_{F(\psi)}$ and hence $\varphi_{F(\psi)}$ are isotropic as well. Thus, $\varphi \sim \psi$. □

We finish this section with some conjectures and a characterization of forms in $GP_{n,m}F$. As already remarked, if $\varphi \in P_{n,m}F$ is defined by (σ, π), then $\varphi \equiv -\pi \pmod{J_nF}$. It would be interesting to know whether a converse of this also holds, i.e., is the following conjecture always true?

Conjecture 3.9 *Let $1 \leq m < n$. Let φ be an anisotropic form over F with $\dim\varphi = 2^n$. If there exists an anisotropic $\pi \in GP_mF$ such that $\varphi \equiv \pi \pmod{J_nF}$ then $\varphi \in GP_{n,m}F$.*

This conjecture is related to the following well-known conjecture (see, for example, [Ka, Conj. 9]).

CONJECTURE 3.10 *Let* $\psi \in J_nF$ *be anisotropic and* $\dim\psi < 2^n + 2^{n-1}$. *Then* $\dim\psi = 2^n$ *and* $\psi \in GP_nF$.

By the definition of degree it is clear that $\dim\psi \geq 2^n$ and that $\psi \in GP_nF$ if $\dim\psi = 2^n$. Note also that if $\psi \simeq (\pi_1 \perp -\pi_2)_{\mathrm{an}}$ where $\pi_i \in P_nF$ and $\ln(\pi_1,\pi_2) = n-2$, then $\dim\psi = 2^n + 2^{n-1}$. So the conjecture essentially states that there is a gap in the dimensions of anisotropic forms in J_nF between 2^n and 2^n+2^{n-1}. We have the following results concerning these two conjectures.

PROPOSITION 3.11 (i) *Conjecture 3.10 implies Conjecture 3.9.*
(ii) *Conjecture 3.10 holds for* $n \leq 4$.
(iii) *Conjecture 3.9 holds for* $n \leq 4$.

Proof. (i) Let φ be anisotropic, $\dim\varphi = 2^n$, and $\varphi \equiv \pi \pmod{J_nF}$ for some anisotropic $\pi \in GP_mF$ where $1 \leq m < n$. If $m = n-1$ then, for all $x \in \dot{F}$, $\pi \equiv x\pi \pmod{J_nF}$ so that in this case we may assume (after possibly scaling) that there exists $u \in D(\varphi) \cap D(\pi)$. Consider $\sigma \simeq (\varphi \perp -\pi)_{\mathrm{an}}$. We clearly have $\sigma \in J_nF$. Furthermore, $0 < 2^n - 2^m \leq \dim\sigma \leq 2^n + 2^{n-1} - 2$. The last inequality is obvious if $m < n-1$, and it follows for $m = n-1$ since we assumed that φ and π represent a common element $u \in \dot{F}$. If Conjecture 3.10 holds, we have that $\sigma \in GP_nF$. Hence, in WF, $\varphi = \sigma - \pi$ with $\sigma \in GP_nF$ and $\pi \in GP_mF$. By Remark 3.5(i) we have $\varphi \in GP_{n,m}F$.

(ii) The case $n = 2$ is trivial and the case $n = 3$ is essentially due to Pfister (cf. [P, Satz 14] or [S, Ch. 2, Theorem 14.4], the result is usually given in terms of I^3F). The case $n = 4$ can be found in [H 7], again in terms of I^4F. Here, we use that $I^nF = J_nF$ for $n \leq 4$.

(iii) follows from (ii) and (i). □

The next little result shows that Conjecture 3.9 is at least "stably" true.

PROPOSITION 3.12 *Let* $1 \leq m < n$. *Let* φ *be an anisotropic form over* F *with* $\dim\varphi = 2^n$. *If there exists an anisotropic* $\pi \in GP_mF$ *such that* $\varphi \equiv \pi \pmod{J_nF}$ *then* $\varphi \in GP_{n,m}K$ *for some field extension* K/F.

Proof. Let K be the leading field of $\varphi \perp -\pi$. Since $0 \neq \varphi \perp -\pi \in J_nF$ and $\dim(\varphi \perp -\pi) < 2^{n+1}$, we have that $\deg(\varphi \perp -\pi) = n$, i.e., $((\varphi \perp -\pi)_K)_{\mathrm{an}} \simeq \sigma \in GP_nK$. Since K is obtained by taking function fields of dimension $> 2^n$ (in case $K \neq F$), φ_K and π_K are anisotropic by [H 3, Theorem 1]. Also, $\varphi_K = \sigma + \pi_K$ in WK. It is now obvious by Remark 3.5(i) that $\varphi \in GP_{n,m}K$. □

COROLLARY 3.13 *Let* $1 \leq m < n$. *Let* $\varphi \in P^w_{n,m}F$ *with twist* $\pi \in P_mF$. *Then* $\varphi \in GP_{n,m}K$ *for some field extension* K/F.

Proof. Write $\varphi \equiv \pi \otimes \eta \pmod{J_nF}$ with $\dim\eta$ odd. Without loss of generality, we may assume that $d_{\pm}\eta = 1$. Let L be the leading field of $\pi \otimes \eta$. As in the proof of Corollary 3.7, we have that φ_L is anisotropic and $\varphi_L \equiv \pi_L \pmod{J_nL}$. The claim now follows immediately from Proposition 3.12. □

We know by Proposition 3.11(i) that Conjecture 3.10 implies Conjecture 3.9. It would be interesting to know whether the converse holds as well, i.e., whether these two conjectures are equivalent. At least a partial answer is given by the following.

Proposition 3.14 (i) *If Conjecture 3.9 holds for* $(n, m) = (n, 1)$, $n \geq 3$, *then there are no anisotropic forms of dimension* $2^n + 2$ *in* J_nF.

(ii) *If Conjecture 3.9 holds for* $(n, m) = (n, 1)$ *and for* $(n, m) = (n, 2)$, $n \geq 4$, *then there are no anisotropic forms of dimension* $2^n + 2$ *and* $2^n + 4$ *in* J_nF.

Proof. (i) Let $\psi \in J_nF$ and suppose that $\dim \psi = 2^n + 2$ and that ψ is anisotropic. Write $\psi \simeq \varphi \perp -\pi$ with $\dim \pi = 2$. Obviously, both $\pi \in GP_1F$ and φ are anisotropic, $\dim \varphi = 2^n$ and $\varphi \equiv \pi \pmod{J_nF}$. By assumption, this implies that $\varphi \in GP_{n,1}F$. Thus, there exist $\sigma \in GP_nF$ and $\tau \in GP_1F$ such that in WF we have $\varphi = \sigma + \tau$. Hence, $\psi = \sigma + \tau - \pi \in WF$. Now ψ and $\sigma \in J_nF$. Therefore, $\tau - \pi \in J_nF$. But $\dim \tau + \dim \pi = 4 < 2^n$ which, by the Arason-Pfister Hauptsatz, yields that $\tau - \pi = 0$ in WF. Hence, in WF we have $\psi = \sigma$. But $\dim \sigma = 2^n < \dim \psi$. Thus, ψ is isotropic, a contradiction.

(ii) By part (i), forms of dimension $2^n + 2$ in J_nF are isotropic. So let $\psi \in J_nF$ and suppose that $\dim \psi = 2^n + 4$ and that ψ is anisotropic. Write $\psi \simeq \psi' \perp -\delta$ with $\dim \delta = 2$. Let $d = d_{\pm}\delta$ so that δ is similar to $\langle 1, -d\rangle$. Let $L = F(\sqrt{d})$. We have that $\psi_L = \psi'_L \in J_nL$. Now $\dim \psi' = 2^n + 2$ and by assumption and part (i), we have that ψ'_L is isotropic. Hence, ψ' contains a subform similar to δ, say, $\psi' \simeq \varphi \perp x\delta$. Let $-\pi \simeq \delta \perp x\delta \in GP_2F$. Then we have $\psi \simeq \varphi \perp -\pi$ and thus, $\varphi \equiv \pi \pmod{J_nF}$ with anisotropic $\pi \in GP_2F$, anisotropic φ, $\dim \varphi = 2^n$. By assumption, this implies that $\varphi \in GP_{n,2}F$. With a reasoning analogous to the one in the proof of part (i), we conclude again that ψ is isotropic, a contradiction. $\square$

The following result shows that the existence of a large enough Pfister neighbor as a subform of the form φ in Conjecture 3.9 is equivalent to φ being in $GP_{n,m}F$.

Proposition 3.15 *Let* $1 \leq m < n$. *Let* φ *be an anisotropic form over* F *with* $\dim \varphi = 2^n$. *Suppose there exists an anisotropic* $\pi \in GP_mF$ *such that* $\varphi \equiv \pi \pmod{J_nF}$. *Then* $\varphi \in GP_{n,m}F$ *iff* φ *contains a Pfister neighbor of dimension* $2^{n-1} + 1$.

Proof. Say, $\varphi \in P_{n,m}F$. Then it follows readily from Remark 3.5(ii) (and with the notations there) that φ contains the Pfister neighbor $\alpha \otimes \sigma'_1$ of dimension $2^n - 2^{m-1} \geq 2^{n-1} + 1$.

Conversely, let $\mu \subset \varphi$ be a Pfister neighbor of dimension $2^{n-1} + 1$ of, say, $\sigma \in P_nF$, and let $x \in \dot{F}$ such that $\mu \subset x\sigma$. Define $\psi \simeq (\varphi \perp -x\sigma)_{\mathrm{an}}$. Then $\dim \psi \leq \dim \varphi + \dim \sigma - 2 \dim \mu = 2^n - 2$. Note that $\psi \equiv \varphi - x\sigma \equiv \varphi \equiv \pi \pmod{J_nF}$ and hence $\psi_{F(\pi)} \equiv 0 \pmod{J_nF(\pi)}$. By the Arason-Pfister Hauptsatz, this implies that $\psi_{F(\pi)} = 0$ in $WF(\pi)$ and there exists $\eta \in WF$ such that $\psi \simeq \pi \otimes \eta$. Since $\dim \psi \leq 2^n - 2$ we must therefore have $\dim \psi \leq 2^n - 2^m$. As $\psi \perp -\pi \in J_nF$ and $\dim(\psi \perp -\pi) \leq 2^n$, the Arason-Pfister Hauptsatz yields two cases. Either $\psi \perp -\pi = 0$ in WF. Then $\varphi = \psi + x\sigma = \pi + x\sigma$ in WF and thus $\varphi \in GP_{n,m}F$ by Remark 3.5(i). Or $\psi \perp -\pi \simeq \tau \in GP_nF$ is anisotropic. In this case, $\varphi = \psi - x\sigma = \tau - x\sigma + \pi = \gamma + \pi$, where $\gamma \simeq (\tau \perp -x\sigma)_{\mathrm{an}}$. Now $\dim \varphi = 2^n$ and $\dim \pi = 2^m$, $\tau, x\sigma \in GP_nF$, and φ, π, and γ are all anisotropic. By Lemma 3.2, $\dim \gamma = 0$, 2^n, or $\geq 2^n + 2^{n-1}$. We consider two cases. If $m \leq n-2$ then in order to have $\varphi = \gamma + \pi$, i.e., $\gamma = \varphi - \pi$, we must have $\dim \gamma = 2^n$ as

$$0 < 2^n - 2^m \leq \dim(\varphi \perp -\pi)_{\mathrm{an}} \leq 2^n + 2^m < 2^n + 2^{n-1} \ .$$

But $\gamma \in I^n F$ and thus necessarily $\gamma \in GP_n F$. Since $\varphi = \gamma + \pi$ we therefore have $\varphi \in GP_{n,m}F$. If $m = n-1$ then $\psi \perp -\pi \in GP_n F$ implies that $\psi \simeq y\pi$ for some $y \in \dot{F}$. Hence, $\varphi = \psi + x\sigma = y\pi + x\sigma$ in WF which readily implies $\varphi \in GP_{n,m}F = GP_{n,n-1}F$. $\square$

Let us finish this section by showing that in certain cases "weakly twisted" implies "twisted" as mentioned already in Remark 3.5(iii). First, we show a very easy lemma.

LEMMA 3.16 *Let $n \geq 2$, $\pi \in P_{n-2}F$ and $\tilde{\eta} \in WF$. Then there exists a form $\eta \in WF$ with $\dim \eta \leq 2$ and $\dim \eta \equiv \dim \tilde{\eta} \pmod 2$ such that $\pi \otimes \tilde{\eta} \equiv \pi \otimes \eta \pmod{J_n F}$.*

Proof. We may assume that $\dim \tilde{\eta} \geq 3$. Let us first consider the case where $\dim \tilde{\eta}$ is even. Let $a = d_{\pm}\tilde{\eta}$. Then $\tilde{\eta} \perp -\langle\langle -a \rangle\rangle \in I^2 F$ and thus, since $\pi \in J_{n-2}F$, $\pi \otimes (\tilde{\eta} \perp -\langle\langle -a \rangle\rangle) \in J_n F$ or $\pi \otimes \tilde{\eta} \equiv \pi \otimes \langle\langle -a \rangle\rangle \pmod{J_n F}$ and we put $\eta \simeq \langle\langle -a \rangle\rangle$.

Let us now consider the case where $\dim \tilde{\eta}$ is odd. After scaling, we may assume that $\tilde{\eta} \simeq \langle 1 \rangle \perp \eta'$. Let now $a = d_{\pm}\eta'$. By a similar argument as above, we get

$$\pi \otimes \tilde{\eta} \equiv \pi + \pi \otimes \eta' \equiv \pi + \pi \otimes \langle\langle -a \rangle\rangle \equiv \pi \otimes \langle\langle 1, -a \rangle\rangle + a\pi \equiv a\pi \pmod{J_n F}$$

because $\pi \otimes \langle\langle 1, -a \rangle\rangle \in GP_n F \subset J_n F$. Here, we put $\eta \simeq \langle a \rangle$. $\square$

The final result in this section now follows readily from this lemma together with Proposition 3.11.

PROPOSITION 3.17 *Let $1 \leq n-2 \leq m \leq n-1$. Let $\varphi \in P^w_{n,m}F$ with twist $\pi \in P_m F$. Then there exists $x \in \dot{F}$ such that $\varphi \equiv x\pi \pmod{J_n F}$. In particular, if Conjecture 3.9 holds for (n,m) (m as above) then $P^w_{n,m}F = GP_{n,m}F$. Thus, this equality holds whenever $1 \leq n-2 \leq m \leq n-1 \leq 3$.*

4 THE WITT KERNEL OF THE FUNCTION FIELD OF A TWISTED PFISTER FORM

We already used several times that if $\pi \in P_n F$ is anisotropic and if $\varphi \in W(F(\pi)/F)$ is anisotropic, then there exists a form γ over F such that $\varphi \simeq \pi \otimes \gamma$. In particular, $W(F(\pi)/F)$ is a *strong n-Pfister ideal*, i.e., every anisotropic form in $W(F(\pi)/F)$ is isometric to an orthogonal sum of forms similar to n-fold Pfister forms in $W(F(\pi)/F)$, in this case forms similar to π itself. We will show that if $\varphi \in P_{n,m}F$ then $W(F(\varphi)/F)$ is a strong $(n+1)$-Pfister ideal, and we will determine the $(n+1)$-fold Pfister forms in $W(F(\varphi)/F)$. These results are implicitly contained in the work of Fitzgerald [F 1]. We will nevertheless provide a proof for the reader's convenience.

THEOREM 4.1 *Let $\varphi \in P_{n,m}F$ be defined by (σ, π). Let $\alpha \in P_{m-1}F$ be a link of σ and π and let $d \in \dot{F}$ such that $\pi \simeq \alpha \otimes \langle\langle -d \rangle\rangle$. Let η be an anisotropic form over F. Then $\eta \in W(F(\varphi)/F)$ if and only if there exist an integer $k \geq 1$, $r_i, s_i \in \dot{F}$, $1 \leq i \leq k$, such that $s_i \in D(\langle d \rangle \perp -\alpha)$ and*

$$\eta \simeq \perp_{i=1}^{k} r_i \sigma \otimes \langle\langle s_i \rangle\rangle \ .$$

This theorem follows from the following more general result.

THEOREM 4.2 *Let* $\sigma \simeq \langle 1\rangle \perp \sigma' \in P_nF$, $n \geq 2$, *be anisotropic and let* $\gamma_1, \gamma_2 \in WF$ *such that* $\gamma_1 \subset \sigma'$, $\dim \gamma_1 > 2^{n-1}$, *and* $\gamma_1 \perp \gamma_2 \simeq \sigma$. *Let* $d \in \dot{F}$ *such that* $\psi \simeq \gamma_1 \perp \langle d\rangle$ *is anisotropic and not a Pfister neighbor. Let* η *be an anisotropic form over* F. *Then* $\eta \in W(F(\psi)/F)$ *if and only if there exist an integer* $k \geq 1$, $r_i, s_i \in \dot{F}$, $1 \leq i \leq k$, *such that* $s_i \in D(\langle d\rangle \perp -\gamma_2)$ *and*

$$\eta \simeq \mathop{\perp}_{i=1}^{k} r_i\sigma \otimes \langle\langle s_i\rangle\rangle\ .$$

Proof. To show the "if"-part, it suffices to show that if $s \in D(\langle d\rangle \perp -\gamma_2)$ then $\sigma \otimes \langle\langle s\rangle\rangle \in W(F(\psi)/F)$. Now s being represented by $\langle d\rangle \perp -\gamma_2$ is equivalent to d being represented by $\langle s\rangle \perp \gamma_2$ (Witt cancellation!). Now clearly $s\sigma$ represents s. Hence,

$$\psi \simeq \gamma_1 \perp \langle d\rangle \subset \gamma_1 \perp \gamma_2 \perp \langle s\rangle \subset \sigma \perp s\sigma \simeq \sigma \otimes \langle\langle s\rangle\rangle\ .$$

It is now obvious that $\sigma \otimes \langle\langle s\rangle\rangle$ is isotropic and hence hyperbolic over $F(\psi)$.

As for the converse, let $\eta \in W(F(\psi)/F)$ be anisotropic. Since $\gamma_1 \subset \psi$ we have that η also becomes hyperbolic over $F(\gamma_1)$. But γ_1 is a Pfister neighbor of σ, i.e., $\gamma_1 \sim \sigma$ and thus $\eta \in W(F(\sigma)/F)$. Hence, there exists a form τ over F with $\eta \simeq \sigma \otimes \tau$. After scaling, we may assume that τ represents 1, i.e., $\tau \simeq \langle 1\rangle \perp \tau'$ and $\eta \simeq \sigma \perp \sigma \otimes \tau'$. Now $\dim \psi > 2^{n-1}$ and ψ is not a Pfister neighbor. In particular, ψ is not similar to a subform of $\sigma \in P_nF$ and therefore $\sigma_{F(\psi)}$ stays anisotropic. Hence, we must have $\dim \tau' \geq 1$. As $\eta_{F(\psi)} = 0$, the Cassels-Pfister subform theorem yields that for every $a \in D(\eta)\cdot D(\psi)$ we have $a\psi \subset \eta$. Since ψ and σ and therefore also η have the subform γ_1 in common, they represent common elements. Hence, we may choose $a = 1$ and we get that $\psi \subset \eta$, i.e.,

$$\psi \simeq \gamma_1 \perp \langle d\rangle \subset \eta \simeq \sigma \perp \sigma \otimes \tau' \simeq \gamma_1 \perp \gamma_2 \perp \sigma \otimes \tau'\ .$$

Hence, there exists $u \in D(\gamma_2) \cup \{0\}$ and $s \in D(\sigma \otimes \tau') \cup \{0\}$ such that $d = u + s$. Note that $d \notin D(\gamma_2)$ because otherwise $\psi \simeq \gamma_1 \perp \langle d\rangle \subset \gamma_1 \perp \gamma_2 \simeq \sigma$, i.e., ψ is a Pfister neighbor of σ, in contradiction to the definition of ψ. Hence, we must have $s \neq 0$, i.e., $s \in D(\sigma \otimes \tau')$. By Lemma 3.1, we may in fact assume that $s \in D(\tau')$ so that $\tau' \simeq \langle s\rangle \perp \tau''$. Hence, we get

$$\eta \simeq \sigma \perp s\sigma \perp \sigma \otimes \tau'' \simeq \sigma \otimes \langle\langle s\rangle\rangle \perp \sigma \otimes \tau''\ .$$

Now $s = d - u \in D(\langle d\rangle \perp -\gamma_2)$. We have already shown that in this case, $\sigma \otimes \langle\langle s\rangle\rangle$ becomes hyperbolic over $F(\psi)$. Therefore, $\sigma \otimes \tau''$ also has to become hyperbolic over $F(\psi)$ because η does. The proof can now easily be finished by induction on $\dim \tau$. $\square$

Proof of Theorem 4.1. As in Remark 3.5(ii), we write $\sigma \simeq \alpha \otimes \sigma_1$ for some $\sigma_1 \simeq \langle 1\rangle \perp \sigma_1' \in P_{n-m+1}F$, so that we get $\varphi \simeq \alpha \otimes (\langle d\rangle \perp \sigma_1')$. Let $\psi \simeq \alpha \otimes \sigma_1' \perp \langle d\rangle \subset \varphi$. We have $\dim \psi = 2^n - 2^{m-1} + 1$. Hence, by Corollary 3.8, $\varphi \sim \psi$ and therefore $W(F(\varphi)/F) = W(F(\psi)/F)$. Note that ψ is not a Pfister neighbor because φ is not a Pfister neighbor and the only forms equivalent to Pfister neighbors are Pfister neighbors themselves (cf. Proposition 2.1(vii)). Note also that $\sigma \simeq \alpha \otimes \sigma_1' \perp \alpha$. Now $\alpha \simeq \langle 1\rangle \perp \alpha'$ and hence $\sigma' \simeq \alpha \otimes \sigma_1' \perp \alpha'$ contains $\alpha \otimes \sigma_1'$ as a subform. The assumptions in Theorem 4.2 on ψ are then fulfilled by putting $\gamma_1 \simeq \alpha \otimes \sigma_1'$ and $\gamma_2 \simeq \alpha$. The claim of the theorem now follows immediately from Theorem 4.2 with σ, d, ψ, γ_1 and γ_2 given as above. $\square$

5 Isotropy of twisted Pfister forms over function fields of quadratic forms

Let φ be an anisotropic Pfister form over F and $\psi \in WF$ be anisotropic with $\dim\psi \geq 2$. The fact that Pfister forms are either anisotropic or hyperbolic plus the Cassels-Pfister subform theorem imply that φ becomes isotropic over $F(\psi)$ iff ψ is similar to a subform of φ. Now suppose φ is a twisted Pfister form and ψ is as above. When is φ isotropic over $F(\psi)$? The problem turns out to be considerably more complicated and we are only able to obtain partial results. Let us start with a useful observation.

PROPOSITION 5.1 *Let $1 \leq m < n$. Let $\varphi \in P^w_{n,m}F$ with twist $\pi \in P_mF$. Let $\psi \in WF$ be anisotropic with $\dim\psi \geq 2$ and assume that $D(\varphi)\cap D(\psi) \neq \emptyset$. Then $\varphi_{F(\psi)}$ is isotropic iff $\psi_{F(\pi)} \subset \varphi_{F(\pi)}$. In particular, if $\varphi \in P_{n,m}F$ is defined by (σ,π) and if $D(\varphi)\cap D(\psi) \neq \emptyset$ or $D(\sigma)\cap D(\psi)\neq\emptyset$, then $\varphi_{F(\psi)}$ is isotropic iff $\psi_{F(\pi)} \subset \sigma_{F(\pi)}$.*

Proof. The second statement clearly follows from the first one since if $\varphi \in P_{n,m}F$ is defined by (σ,π) then $\varphi \equiv \sigma - \pi \equiv -\pi \pmod{J_nF}$ and $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)}$, see Proposition 3.6(i).

To prove the first statement, we note that by Corollary 3.7 we have that $\varphi_{F(\pi)} \in GP_nF(\pi)$ is anisotropic. If $\varphi_{F(\psi)}$ is isotropic then $\varphi_{F(\pi)(\psi)}$ is also isotropic and hence hyperbolic, and the Cassels-Pfister subform theorem together with $D(\varphi)\cap D(\psi)\neq\emptyset$ implies that $\psi_{F(\pi)} \subset \varphi_{F(\pi)}$.

Conversely, suppose that $\psi_{F(\pi)} \subset \varphi_{F(\pi)}$. Clearly, $\varphi_{F(\pi)(\psi)}$ is isotropic and hence hyperbolic because $\varphi_{F(\pi)} \in GP_nF(\pi)$. Note that $F(\pi)(\psi) \simeq F(\psi)(\pi)$. Suppose $\varphi_{F(\psi)}$ is anisotropic. Then, by Proposition 2.1(v) and since $\varphi_{F(\psi)(\pi)} = 0$, there exists $\gamma \in WF(\psi)$ such that $\varphi_{F(\psi)} \simeq \gamma\otimes\pi_{F(\psi)}$. Now $\dim\gamma = 2^{n-m}$ is even and $\pi \in GP_mF$. In particular, $\gamma\otimes\pi_{F(\psi)} \in J_{m+1}F(\psi)$ by Proposition 2.2(1). Now if we write $\varphi \equiv \pi\otimes\eta \pmod{J_nF}$ with $\dim\eta$ odd, we readily get $\varphi \equiv \pi\otimes\eta \equiv \pi \pmod{J_{m+1}F}$. Hence we have

$$\varphi_{F(\psi)} \equiv \pi_{F(\psi)} \equiv \gamma\otimes\pi_{F(\psi)} \equiv 0 \pmod{J_{m+1}F(\psi)}$$

which yields $\pi_{F(\psi)} = 0$ in $WF(\psi)$. But then $F(\psi)(\pi)/F(\psi)$ is purely transcendental. Thus, the anisotropic form $\varphi_{F(\psi)}$ stays anisotropic over $F(\psi)(\pi)$, a contradiction to $\varphi_{F(\psi)(\pi)} = 0$. $\square$

This result gives us a criterion to decide whether φ becomes isotropic over $F(\psi)$, however, it only works over $F(\pi)$. Although function fields of Pfister forms have a somewhat nicer behavior than function fields of arbitrary forms, it seems desirable to find criteria which, at least in principle, work over F itself. What we would like to have is some sort of descent from $F(\pi)$ to F where π is an anisotropic Pfister form. This can easily be achieved if $F(\pi)/F$ is an excellent field extension which is always the case for $m = 1$ and 2, but generally not for $m \geq 3$, see also Proposition 2.1(x) and Corollary 8.4.

PROPOSITION 5.2 *Let $1 \leq m < n$. Let $\varphi \in P^w_{n,m}F$ with twist $\pi \in P_mF$. Let $\psi \in WF$ be anisotropic with $\dim\psi \geq 2$ and assume that $D(\varphi)\cap D(\psi)\neq\emptyset$. Suppose furthermore that $F(\pi)/F$ is excellent. Then $\varphi_{F(\psi)}$ is isotropic iff there exists a form $\tilde\psi \in WF$, $\dim\tilde\psi = 2^n$, such that $\psi \subset \tilde\psi$ and $\tilde\psi_{F(\pi)} \simeq \varphi_{F(\pi)}$. In particular, if $\varphi \in P_{n,m}F$ is defined by (σ,π) and if $D(\varphi)\cap D(\psi)\neq\emptyset$ or $D(\sigma)\cap D(\psi)\neq\emptyset$, then*

$\varphi_{F(\psi)}$ is isotropic iff there exists a form $\tilde{\psi} \in WF$, $\dim \tilde{\psi} = 2^n$, such that $\psi \subset \tilde{\psi}$ and $\tilde{\psi}_{F(\pi)} \simeq \sigma_{F(\pi)} \simeq \varphi_{F(\pi)}$.

Proof. The "if"-part follows directly from Proposition 5.1 even without the excellence assumption. As for the converse, consider $\varphi \perp -\psi$. Since $\varphi_{F(\psi)}$ is isotropic, we know by Proposition 5.1 that $\psi_{F(\pi)} \subset \varphi_{F(\pi)}$. This plus the excellence of $F(\pi)/F$ imply that there exists $\chi \in WF$, $\dim \chi = 2^n - \dim \psi$, such that $\varphi_{F(\pi)} \simeq \psi_{F(\pi)} \perp \chi_{F(\pi)}$, and with $\tilde{\psi} \simeq \psi \perp \chi$ over F, we get $\tilde{\psi}_{F(\pi)} \simeq \varphi_{F(\pi)}$. $\square$

REMARK 5.3 The previous proposition provides indeed a criterion which, at least in principle, can be checked over F. This is because $\tilde{\psi}_{F(\pi)} \simeq \varphi_{F(\pi)}$ means that $\tilde{\psi} \perp -\varphi \in W(F(\pi)/F)$. In other words, with φ and ψ as above, $\varphi_{F(\psi)}$ is isotropic iff there exist forms $\tilde{\psi}$ and τ in WF with $\dim \tilde{\psi} = 2^n$ such that $\psi \subset \tilde{\psi}$ and $(\tilde{\psi} \perp -\varphi)_{\mathrm{an}} \simeq \pi \otimes \tau$.

We do not know whether Proposition 5.2 holds in general without the assumption on $F(\pi)/F$ being excellent. However, this result at least indicates that in order to decide if φ becomes isotropic over $F(\psi)$, it seems to be important to characterize those forms ψ over F of dimension 2^n for which $\psi_{F(\pi)} \simeq \varphi_{F(\pi)}$. This will be the focus of most of the remainder of this section.

We will eventually be interested in characterizing those forms $\psi \in WF$ of dimension 2^n which become isometric to some $\varphi \in P_{n,m}F$ over $F(\pi)$, where φ is defined by (σ, π). Our aim is to make this description as precise as possible, something which, in general, doesn't seem to be easy and which we will only do in the cases $m = n-1$ and $m = n-2$. In fact, the case $m = n-1$ has been dealt with in [H 4, Theorem 3.3]. We have the following result.

THEOREM 5.4 *Let $\varphi \in P_{n,n-1}F$ be defined by (σ, π). Let $\psi \in WF$ with $\dim \psi = 2^n$. Then the following are equivalent.*

- (i) *$\varphi_{F(\psi)}$ is isotropic.*
- (ii) *$\psi_{F(\pi)}$ is similar to $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)}$.*
- (iii) *Either ψ is similar to φ or ψ is similar to some $\tau \in P_nF$ and φ contains a Pfister neighbor of τ.*

Proof. The equivalence of (i) and (ii) is clear from Proposition 5.1. Clearly, (iii) implies (i). That (iii) follows from any of the other statements was shown in [H 4, Theorem 3.3] under the additional assumption that ψ contains a Pfister neighbor of dimension $2^{n-1} + 2^{n-2}$. By Proposition 5.8, (ii) implies that there exist $\alpha \in P_{n-2}F$ and $\psi_1 \in WF$, $\dim \psi_1 = 4$, such that $\psi \simeq \alpha \otimes \psi_1$. Let $\psi' \subset \psi_1$ with $\dim \psi' = 3$. Then ψ' is a Pfister neighbor of some $\beta \in P_2F$ and $\alpha \otimes \psi' \subset \psi$ is a Pfister neighbor of dimension $2^{n-1} + 2^{n-2}$ of $\alpha \otimes \beta \in P_nF$ and we can apply [H 4, Theorem 3.3] as desired. $\square$

COROLLARY 5.5 *Let $\varphi \in P_{n,n-1}F$ be defined by (σ, π) and suppose that $F(\pi)/F$ is excellent (which always holds if $n-1 = 1$ or 2). Let $\psi \in WF$ with $\dim \psi \geq 2$. Then $\varphi_{F(\psi)}$ is isotropic iff ψ is similar to a subform of φ or ψ is similar to a subform of some $\tau \in P_nF$ and φ contains a Pfister neighbor of τ.*

Proof. This is a direct consequence of Proposition 5.2 and the previous theorem. □

Part (iii) of the previous theorem tells us that in order to decide whether $\varphi_{F(\psi)}$ is isotropic (where $\dim\psi = 2^n$), it suffices to look only at φ and ψ and how they relate to each other over F. The form π isn't really needed explicitly. It turns out that if $\varphi \in P_{n,m}F$ with $m < n-1$ then the situation is not quite so nice anymore as the form π will play a more prominent role. Let us start with a simple observation.

Proposition 5.6 *Let $1 \leq m < n$. Let $\varphi \in P^w_{n,m}F$ with twist $\pi \in P_mF$. Let $\psi \in WF$ be anisotropic with $\dim\psi = 2^n$. Suppose that $\varphi_{F(\psi)}$ is isotropic. Then there exists $\mu \in WF$ such that $\psi \equiv \pi \otimes \mu \pmod{J_nF}$. In particular, $\deg\psi \geq m$, and $\deg\psi = m$ iff $\dim\mu$ is odd (i.e., iff $\psi \in P^w_{n,m}F$ with twist π). Furthermore, If $m = n-1$ (resp. $m = n-2$) then there exists such μ with $\dim\mu \leq 1$ (resp. $\dim\mu \leq 2$).*

Proof. Write $\varphi \equiv \pi \otimes \eta \pmod{J_nF}$ with $\dim\eta$ odd. After scaling, we may assume that $D(\psi) \cap D(\varphi) \neq \emptyset$. Since $\varphi_{F(\psi)}$ is isotropic we have by Proposition 5.1 that $\psi_{F(\pi)} \simeq \varphi_{F(\pi)}$. Let $\chi \simeq (\psi \perp -\varphi)_{\mathrm{an}}$. Then $\chi \in W(F(\pi)/F)$ and there exists $\tilde{\mu} \in WF$ with $\chi \simeq \pi \otimes \tilde{\mu}$. Let us put $\mu \simeq \tilde{\mu} \perp \eta$. Then we have

$$\psi \equiv \chi + \varphi \equiv \pi \otimes \tilde{\mu} + \pi \otimes \eta \equiv \pi \otimes \mu \pmod{J_nF}\ .$$

We have $\deg(\pi \otimes \mu) = \deg\pi = m$ if $\dim\mu$ is odd (cf. Proposition 2.2(i)), in which case $\deg\psi = \deg(\pi \otimes \mu) = m$ as $m < n$. If $\dim\mu$ is even, we have $\deg(\pi \otimes \mu) \geq m+1$. Since $n \geq m+1$ we thus also have $\deg\psi \geq m+1$.

The remaining statements for $n-2 \leq m \leq n-1$ follow readily from Lemma 3.16. □

Remark 5.7 In the above proof, we have $\dim\chi \leq 2^{n+1}-2$ and one obtains $\dim\tilde{\mu} \leq 2^{n+1-m}-1$ or $\dim\mu \leq 2^{n+1-m}-1+\dim\eta$. If $\varphi \in GP_{n,m}F$ or if $F(\pi)/F$ is excellent, then we know that there exists $\sigma \in GP_nF$ such that $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)} \in GP_nF(\pi)$ (cf. Corollary 3.7 in the case where $F(\pi)/F$ is excellent). We can slightly improve the estimate of $\dim\mu$ for general m in this case. After scaling, we may assume that $\sigma \in P_nF$ and $D(\sigma) \cap D(\psi) \neq \emptyset$. Since $\varphi_{F(\psi)}$ is isotropic we then have by Proposition 5.1 that $\psi_{F(\pi)} \simeq \sigma_{F(\pi)}$. Let $\chi \simeq (\psi \perp -\sigma)_{\mathrm{an}}$. Then $\dim\chi \leq 2^{n+1}-2$ as $D(\psi) \cap D(\sigma) \neq \emptyset$, and also $\chi \in W(F(\pi)/F)$. Hence, there exists $\mu \in WF$ with $\chi \simeq \pi \otimes \mu$. Since $2^m\dim\mu = \dim\chi \leq 2^{n+1}-2$ we have $\dim\mu \leq 2^{n+1-m}-1$. Furthermore, $\sigma \in P_nF$ and we get

$$\chi \equiv \psi - \sigma \equiv \psi \equiv \pi \otimes \mu \pmod{J_nF}\ .$$

In the case where φ is a twisted Pfister form, we can be more precise about how ψ has to look like.

Proposition 5.8 *Let $\varphi \in P_{n,m}F$ be defined by (σ,π). Let $\alpha \in P_{m-1}F$, $\sigma_1 \in P_{n-m+1}F$, and $d \in \dot{F}$ such that $\sigma \simeq \alpha \otimes \sigma_1$ and $\pi \simeq \alpha \otimes \langle\langle -d\rangle\rangle$ (see Remark 3.5(ii)). Let $\psi \in WF$ with $\dim\psi = 2^n$. Then the following holds.*

(i) *If $\psi_{F(\pi)} \simeq \varphi_{F(\pi)}$ then there exists $\psi_1 \in WF$, $\dim\psi_1 = 2^{n-m+1}$, such that $\psi \simeq \alpha \otimes \psi_1$. In particular, $\psi \in I^mF$, i.e., $\deg'\psi \geq m$.*

(ii) *If $\psi \in P_nF$ and $\psi_{F(\pi)} \simeq \varphi_{F(\pi)}$ then there exist $s \in \dot{F}$, σ_2, $\psi_2 \in P_{n-m}F$, such that $\sigma \simeq \alpha \otimes \langle\langle s\rangle\rangle \otimes \sigma_2$ and $\psi \simeq \alpha \otimes \langle\langle s\rangle\rangle \otimes \psi_2$. In particular, $\ln(\psi,\sigma) \geq m$ and $\ln(\psi,\pi) = m-1$.*

Proof. First, let us recall that $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)}$, so that in both parts we actually assume that $\psi_{F(\pi)} \simeq \sigma_{F(\pi)}$ and therefore, we may assume that ψ represents 1 already over F after possibly scaling (this is of course always true if $\psi \in P_nF$).

(i) If $m = 1$, i.e., $\dim\alpha = 1$, there is nothing to show. So let us assume that $m \geq 2$ so that we have $\alpha_{F(\alpha)} = 0$. Since $\psi_{F(\pi)} \simeq \sigma_{F(\pi)}$ we clearly have $\psi_{F(\alpha)(\pi)} \simeq \sigma_{F(\alpha)(\pi)} \simeq (\alpha\otimes\sigma_1)_{F(\alpha)(\pi)} = 0$. Similarly, $\pi_{F(\alpha)} = 0$ which implies that $F(\alpha)(\pi)/F(\alpha)$ is purely transcendental. Hence, we must already have $\psi_{F(\alpha)} = 0$. Note that ψ is anisotropic as $\psi_{F(\pi)} \simeq \varphi_{F(\pi)}$ is anisotropic. Hence, by Proposition 2.1(v), there exists $\psi_1 \in WF$, $\dim\psi_1 = 2^{n-m+1}$, such that $\psi \simeq \alpha \otimes \psi_1$. Since $\alpha \in I^{m-1}F$ and $\dim\psi_1$ even, i.e., $\psi_1 \in IF$, we have $\psi \in I^mF$, i.e., $\deg'\psi \geq m$.

(ii) Let $\gamma \simeq (\psi \perp -\sigma)_{\mathrm{an}} \in I^nF$. By assumption, $\gamma \in W(F(\pi)/F)$. Also, $\dim\gamma \leq 2^{n+1} - 2$ as $1 \in D(\psi) \cap D(\sigma)$. Thus, by Proposition 2.1(v), there exists $\beta \in WF$, $\dim\beta < 2^{n+1-m}$, such that $\gamma \simeq \pi \otimes \beta$. Since $\gamma \in I^nF$ and $\pi \in P_mF$ with $1 \leq m < n$, we must have that $\dim\beta$ is even. Therefore, $\dim\beta \leq 2^{n+1-m} - 2$, i.e., $\dim\gamma \leq 2^{n+1} - 2^{m+1}$. Hence, $i_W(\psi \perp -\sigma) \geq 2^m$ and the existence of $s \in \dot{F}$, σ_2, $\psi_2 \in P_{n-m}F$ such that $\sigma \simeq \alpha \otimes \langle\langle s\rangle\rangle \otimes \sigma_2$ and $\psi \simeq \alpha \otimes \langle\langle s\rangle\rangle \otimes \psi_2$ now follows from Lemma 3.2 (cf., in particular, the proof of Lemma 3.2 and use the fact that we already have $\sigma \simeq \alpha \otimes \sigma_1$ and $\psi \simeq \alpha \otimes \psi_1$ by part (i)).

Clearly, $\ln(\psi,\sigma) \geq m$ since $\alpha \otimes \langle\langle s\rangle\rangle \in P_mF$ divides both ψ and σ. It is also obvious that $m \geq \ln(\psi,\pi) \geq m-1$ as $\alpha \in P_{m-1}F$ divides both ψ and π. Now $\ln(\psi,\pi) = m$ would imply that $\pi \subset \psi$ and thus $\psi_{F(\pi)} \in P_nF(\pi)$ would be hyperbolic, a contradiction to $\psi_{F(\pi)} \simeq \varphi_{F(\pi)}$ being anisotropic. $\square$

Before we state our theorem about forms in $P_{n,n-2}F$ which parallels in a certain sense Theorem 5.4, we will provide a lemma which we will need in the proof of this theorem.

Lemma 5.9 *Let $1 \leq m \leq n-2$ and let $\varphi \in WF$ be anisotropic with $\dim\varphi = 2^n$ such that $\varphi \equiv \pi \pmod{J_nF}$ for some anisotropic $\pi \in GP_mF$. Assume furthermore that $\varphi \simeq \alpha \otimes \beta$ for some $\alpha \in P_{m-1}F$ and some $\beta \in WF$, $\dim\beta = 2^{n-m+1}$. If Conjecture 3.9 holds for $(n, m+1)$ then $\varphi \in GP_{n,m}F$.*

Proof. After scaling, we may assume that $\pi \in P_mF$. First, we note that there exists $d \in \dot{F}$ such that $\pi \simeq \alpha\otimes\langle\langle -d\rangle\rangle$. This is obvious if $m = 1$, i.e., $\alpha \simeq \langle 1\rangle \in P_0F$. If $m > 1$ we have that $\alpha_{F(\alpha)} = 0$, thus $\varphi_{F(\alpha)} = 0$ as well, which in turn yields $\pi_{F(\alpha)} \in J_nF(\alpha)$. Since $\pi \in P_mF$ and $m < n$ we must have $\pi_{F(\alpha)} = 0$. The existence of $d \in \dot{F}$ such that $\pi \simeq \alpha \otimes \langle\langle -d\rangle\rangle$ follows immediately from Proposition 2.1(v). Write $\beta \simeq \langle x\rangle \perp \beta'$ and define $\tilde{\beta} \simeq \langle xd\rangle \perp \beta'$ and $\tilde{\varphi} \simeq \alpha \otimes \tilde{\beta}$. Then $\dim\tilde{\varphi} = \dim\varphi = 2^n$ and $\tilde{\varphi} = \varphi - x\pi$ in WF. In particular, one gets $\tilde{\varphi}_{F(\pi)} \simeq \varphi_{F(\pi)}$ which is anisotropic by Proposition 3.6. Hence, $\tilde{\varphi}$ is anisotropic. Furthermore,

$$\tilde{\varphi} \equiv \varphi - x\pi \equiv \pi - x\pi \equiv \pi \otimes \langle\langle -x\rangle\rangle \pmod{J_nF} .$$

We have two cases. If $\pi \otimes \langle\langle -x\rangle\rangle$ is isotropic and hence hyperbolic, then $\tilde{\varphi} \in J_nF$ and thus $\tilde{\varphi} \in GP_nF$ as $\dim\tilde{\varphi} = 2^n$. In WF, we get $\varphi = \tilde{\varphi} + x\pi$ which readily implies that $\varphi \in GP_{n,m}F$ as $\tilde{\varphi} \in GP_nF$ and $x\pi \in GP_mF$ (cf. Remark 3.5(i)). If $\pi \otimes \langle\langle -x\rangle\rangle$ is anisotropic then by our assumption $\tilde{\varphi} \in GP_{n,m+1}F$ and there exists $\sigma \in GP_nF$ and $\rho \in GP_{n,m+1}F$ such that $\tilde{\varphi} = \sigma + \rho$ in WF. In particular, $\tilde{\varphi} \equiv \rho \equiv \pi \otimes \langle\langle -x\rangle\rangle$

(mod J_nF), and it readily follows that there exists $y \in \dot{F}$ such that $\rho \simeq y\pi \otimes \langle\langle -x \rangle\rangle$ (recall that $m+1 < n$). Hence, in WF,

$$\begin{aligned} \varphi - \pi &= \tilde{\varphi} + x\pi - \pi = \sigma + y\pi \otimes \langle\langle -x \rangle\rangle + x\pi - \pi \\ &= \sigma + \pi \otimes \langle y, -xy, x, -1 \rangle = \sigma - \pi \otimes \langle\langle -x, -y \rangle\rangle \ . \end{aligned}$$

Suppose first that $m+2 < n$. Then $\varphi - \pi \equiv 0 \equiv -\pi \otimes \langle\langle -x, -y \rangle\rangle \pmod{J_nF}$, i.e., $\pi \otimes \langle\langle -x, -y \rangle\rangle \in J_nF$, which implies $\pi \otimes \langle\langle -x, -y \rangle\rangle = 0$ as $\pi \otimes \langle\langle -x, -y \rangle\rangle \in P_{m+2}F$ and $m+2 < n$. We then have $\varphi = \sigma + \pi$ with $\sigma \in GP_nF$ and $\pi \in P_mF$ which yields that $\varphi \in GP_{n,m}F$ as desired.

Finally, if $m+2 = n$, we have that σ, $\pi \otimes \langle\langle -x, -y \rangle\rangle \in GP_nF$. By Lemma 3.6 we then get $\dim(\sigma \perp -\pi \otimes \langle\langle -x, -y \rangle\rangle)_{\mathrm{an}} = 0$, 2^n, or $\geq 2^n + 2^{n-1}$. On the other hand,

$$\begin{aligned} 0 < 2^n - 2^m = \dim \varphi - \dim \pi \leq \dim(\varphi \perp -\pi)_{\mathrm{an}} \leq \\ \leq \dim \varphi + \dim \pi = 2^n + 2^m < 2^n + 2^{n-1} \ . \end{aligned}$$

Now $(\sigma \perp -\pi \otimes \langle\langle -x, -y \rangle\rangle)_{\mathrm{an}} \simeq (\varphi \perp -\pi)_{\mathrm{an}}$ and we therefore must have $\dim(\sigma \perp -\pi \otimes \langle\langle -x, -y \rangle\rangle)_{\mathrm{an}} = 2^n$. As $\sigma \perp -\pi \otimes \langle\langle -x, -y \rangle\rangle \in J_nF$ it follows that $(\sigma \perp -\pi \otimes \langle\langle -x, -y \rangle\rangle)_{\mathrm{an}} \simeq \tau \in GP_nF$. Hence, $\varphi = \tau + \pi$ with $\tau \in GP_nF$ and $\pi \in P_mF$ and again $\varphi \in GP_{n,m}F$. □

THEOREM 5.10 *Suppose that Conjecture 3.9 holds for $(n, n-1)$. Let $\varphi \in P_{n,n-2}F$ be defined by (σ, π). Let $\psi \in WF$ with $\dim \psi = 2^n$. Then the following are equivalent.*

(i) *$\varphi_{F(\psi)}$ is isotropic.*

(ii) *$\psi_{F(\pi)}$ is similar to $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)}$.*

(iii) *There exists $\tau \in P_nF$ such that $\tau_{F(\pi)} \simeq \sigma_{F(\pi)}$ and either*
  - *ψ is similar to τ, or*
  - *there exist $x \in \dot{F}$ and $\rho \in P_{n,n-1}F$ such that ρ is defined by $(\tau, \pi \otimes \langle\langle x \rangle\rangle)$ and ψ is similar to ρ, or*
  - *there exists $\chi \in P_{n,n-2}F$ such that χ is defined by (τ, π) and ψ is similar to χ.*

Proof. The equivalence of (i) and (ii) is clear from Proposition 5.1. One readily checks that (iii) implies that $\psi_{F(\pi)}$ is similar $\tau_{F(\pi)} \simeq \sigma_{F(\pi)}$ and we are in (ii). Finally, (i) implies by Proposition 5.6 that $\psi \equiv \pi \otimes \mu \pmod{J_nF}$ for some $\mu \in WF$, $0 \leq \dim \mu \leq 2$. If $\dim \mu \in \{0, 2\}$ then $\psi \in GP_nF$ or $\psi \in GP_{n,n-1}F$ (the latter only if $\pi \otimes \mu \neq 0$ and because we assumed that Conjecture 3.9 holds for $(n, n-1)$). If $\dim \mu = 1$ we have $\psi \in GP_{n,n-2}F$ by Lemma 5.9 together with Proposition 5.8(i). All this together with the fact that $\psi_{F(\pi)}$ is similar to $\sigma_{F(\pi)}$ readily imply (iii) and we leave the details to the reader. □

COROLLARY 5.11 *Let $\varphi \in P_{n,n-2}F$ be defined by (σ, π). Let $\psi \in WF$ with $\dim \psi = 2^n$. Suppose that $n \leq 4$ or that ψ contains a Pfister neighbor of dimension $2^{n-1}+1$. Then the following are equivalent.*

(i) *$\varphi_{F(\psi)}$ is isotropic.*

(ii) *$\psi_{F(\pi)}$ is similar to $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)}$.*

(iii) *There exists* $\tau \in P_nF$ *such that* $\tau_{F(\pi)} \simeq \sigma_{F(\pi)}$ *and either*
- ψ *is similar to* τ, *or*
- *there exist* $x \in \dot{F}$ *and* $\rho \in P_{n,n-1}F$ *such that* ρ *is defined by* $(\tau, \pi \otimes \langle\langle x \rangle\rangle)$ *and* ψ *is similar to* ρ, *or*
- *there exists* $\chi \in P_{n,n-2}F$ *such that* χ *is defined by* (τ, π) *and* ψ *is similar to* χ.

Proof. This is an immediate consequence of the previous theorem and Propositions 3.11 and 3.15. □

COROLLARY 5.12 *Suppose that Conjecture 3.9 holds for* $(n, n-1)$. *Let* $\varphi \in P_{n,n-2}F$ *be defined by* (σ, π) *and suppose that* $F(\pi)/F$ *is excellent. Let* $\psi \in WF$ *with* $\dim \psi \geq 2$. *Then the following are equivalent.*

(i) $\varphi_{F(\psi)}$ *is isotropic.*

(ii) $\psi_{F(\pi)}$ *is similar to a subform of* $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)}$.

(iii) *There exists* $\tau \in P_nF$ *such that* $\tau_{F(\pi)} \simeq \sigma_{F(\pi)}$ *and*
- ψ *is similar to a subform of* τ, *or*
- *there exist* $x \in \dot{F}$ *and* $\rho \in P_{n,n-1}F$ *such that* ρ *is defined by* $(\tau, \pi \otimes \langle\langle x \rangle\rangle)$ *and* ψ *is similar to a subform of* ρ, *or*
- *there exists* $\chi \in P_{n,n-2}F$ *such that* χ *is defined by* (τ, π) *and* ψ *is similar to a subform of* χ.

In particular, the equivalence of (i), (ii) *and* (iii) *always holds for* $n \leq 4$.

Proof. This is an immediate consequence of Theorem 5.10 and Proposition 5.2. Furthermore, if $n \leq 4$ then Conjecture 3.9 holds by Proposition 3.11, and $F(\pi)/F$ is excellent since π is of fold ≤ 2. □

Corollaries 5.5 and 5.12 give us a fairly complete picture for which forms $\psi \in WF$ a given form $\varphi \in P_{n,m}F$, which is defined by (σ, π), becomes isotropic over $F(\psi)$ in the cases $(n, m) \in \{(2,1), (3,1), (3,2), (4,2)\}$. In a certain sense, we know this in general in the case $(n, m) = (n, 1)$ or $(n, 2)$ by Propositions 5.1 and 5.2. It comes down to characterizing those forms $\tilde{\psi}$ of dimension 2^n for which $\tilde{\psi}_{F(\pi)} \simeq \sigma_{F(\pi)}$. In the cases $(2,1)$ and $(3,2)$ we have a very precise description by Corollary 5.5. In the cases $(3,1)$ and $(4,2)$ we can essentially reduce this problem to the determination of those $\tau \in P_nF$ with $\tau_{F(\pi)} \simeq \sigma_{F(\pi)}$ or $\tau \perp -\sigma \in W(F(\pi)/F)$. This narrows down the set of forms we have to look at quite considerably.

The following example shows that if $\varphi \in P_{n,m}F$ with $n - m \geq 3$ and if $\psi \in WF$, $\dim \psi = 2^n$, then $\varphi_{F(\psi)}$ being isotropic does generally not imply that ψ is similar to a Pfister form or a twisted Pfister form, something which cannot happen in the cases considered above.

EXAMPLE 5.13 Let $F = \mathbb{R}(t)$ be the rational function field in one variable t over the reals. Let $m \geq 1$ and $n - m \geq 3$. Let $\sigma \simeq \langle\langle 1, \cdots, 1 \rangle\rangle \in P_nF$ and $\pi \simeq \langle\langle 1, \cdots, 1, -t \rangle\rangle \in P_mF$. We then have

$$\varphi \simeq (\sigma \perp -\pi)_{\text{an}} \simeq \langle \underbrace{1, \cdots, 1}_{2^n - 2^{m-1}}, \underbrace{t, \cdots, t}_{2^{m-1}} \rangle \in P_{n,m}F \ .$$

Let

$$\psi \simeq (\sigma \perp -\langle 1,1,1\rangle \otimes \pi)_{\mathrm{an}} \simeq \langle \underbrace{1,\cdots,1}_{2^n-3\cdot 2^{m-1}}, \underbrace{t,\cdots,t}_{3\cdot 2^{m-1}}\rangle \ .$$

One easily sees that φ and ψ are anisotropic (for example by passing to the power series field $\mathbb{R}((t)) \supset F$ and applying Springer's theorem [L 1, Ch. 6, Proposition 1.9], [S, Ch. 6, Corollary 2.6(i)]). Clearly, $\psi_{F(\pi)} \simeq \sigma_{F(\pi)} \simeq \varphi_{F(\pi)}$. Thus, by Proposition 5.1, $\varphi_{F(\psi)}$ is isotropic. We claim that ψ is neither similar to a Pfister form nor to a twisted Pfister form. First, using that σ, $\langle\langle 1,1\rangle\rangle \otimes \pi \in J_{m+2}F$, we note that

$$\psi \equiv \sigma - \langle 1,1,1\rangle \otimes \pi \equiv -\langle 1,1,1\rangle \otimes \pi + \langle\langle 1,1\rangle\rangle \otimes \pi \equiv \pi \not\equiv 0 \pmod{J_{m+2}F} \ .$$

Hence, $\deg \psi = \deg \pi = m$. Clearly, ψ is not similar to a Pfister form. Furthermore, ψ is also not similar to twisted Pfister form. For otherwise, since $\deg \psi = m$, we have $\psi \in GP_{n,m}F$ and by definition, there exist anisotropic forms $\tau \in GP_nF$ and $\rho \in GP_mF$ such that $\psi = \tau + \rho$ in WF. Thus, $\tau + \rho = \sigma - \langle 1,1,1\rangle \otimes \pi$ or $\rho + \langle 1,1,1\rangle \otimes \pi = \sigma - \tau$ in WF and we get $\dim(\rho \perp \langle 1,1,1\rangle \otimes \pi)_{\mathrm{an}} = \dim(\sigma \perp -\tau)_{\mathrm{an}}$. Now $\dim(\sigma \perp -\tau)_{\mathrm{an}} = 0$ or $\geq 2^n$ by Lemma 3.2. We also have $\dim \rho = 2^m$ and $\dim \langle 1,1,1\rangle \otimes \pi = 3\cdot 2^m$. Thus,

$$\begin{aligned} 0 < 2^{m+1} = \dim \langle 1,1,1\rangle \otimes \pi - \dim \rho \leq \dim(\rho \perp \langle 1,1,1\rangle \otimes \pi)_{\mathrm{an}} \leq \\ \leq \dim \langle 1,1,1\rangle \otimes \pi + \dim \rho = 2^{m+2} < 2^n \ . \end{aligned}$$

This obviously yields a contradiction. Note, however, that $\psi \in P^w_{n,m}F$. $\square$

6 The equivalence class of a twisted Pfister form

Recall that two forms φ and ψ over F are called equivalent, we write $\varphi \sim \psi$, if $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are isotropic. Since the function field of an isotropic form is purely transcendental over the ground field and since anisotropic forms stay anisotropic over purely transcendental extensions, the question whether $\varphi \sim \psi$ holds is of interest only in the case of anisotropic forms. Let us denote the equivalence class of a form φ over F with respect to "$\sim$" by $\mathrm{Equiv}(\varphi)$.

We know by Proposition 2.1(vii) that if φ is an anisotropic Pfister form then $\mathrm{Equiv}(\varphi) = \{\psi \in WF \,|\, \psi \text{ is a Pfister neighbor of } \varphi\}$. The equivalence classes of forms of dimension ≤ 5 and certain forms of dimension 6, 7, and 8 have been determined in [W], [H 1], [H 2], [H 4], [Lag]. Furthermore, for forms in $P_{n,n-1}F$ we have the following result (cf. [H 4, Corollary 3.4, Theorem 4.4]).

THEOREM 6.1 *Let $n \geq 2$ and let $\varphi \in P_{n,n-1}F$.*
(i) *Let $\psi \in WF$ with $\dim \psi = 2^n$. Then $\varphi \sim \psi$ iff ψ is similar to φ.*
(ii) *Let $n \leq 3$. Then*

$$\mathrm{Equiv}(\varphi) = \{\psi \in WF \,|\, x\psi \subset \varphi \textit{ for some } x \in \dot{F} \textit{ and } \dim \psi > 2^n - 2^{n-2}\} \ .$$

In view of part (ii) of this theorem, the following conjecture seems natural (see also [H 4, Conjecture 4.3]).

CONJECTURE 6.2 *Let $n \geq 2$ and $\varphi \in P_{n,n-1}F$. Then*

$$\mathrm{Equiv}(\varphi) = \{\psi \in WF \,|\, x\psi \subset \varphi \textit{ for some } x \in \dot{F} \textit{ and } \dim \psi > 2^n - 2^{n-2}\} \ .$$

Let us right away state what we propose as the corresponding conjecture for forms in $P_{n,n-2}F$ and which we will prove to be correct in the cases $n \le 4$ (see Corollary 6.11).

CONJECTURE 6.3 *Let $n \ge 3$ and let $\varphi \in P_{n,n-2}F$ be defined by (σ, π). Let $\psi \in WF$. Then the following statements are equivalent.*

(i) $\psi \in \mathrm{Equiv}(\varphi)$.

(ii) *There exists $\chi \in P_{n,n-2}F$ such that*

- *χ is defined by (τ, π) for some $\tau \in P_nF$ with $\tau_{F(\pi)} \simeq \sigma_{F(\pi)}$,*
- *$x\psi \subset \chi$ for some $x \in \dot{F}$, and*
- $\dim \psi > 2^n - 2^{n-3}$.

It will be crucial to determine first those $\psi \in WF$ of dimension 2^n such that $\psi \sim \varphi$, and we will start with some more general results.

THEOREM 6.4 *Let $1 \le m < n$. Let $\varphi \in P^w_{n,m}F$ with twist $\pi \in P_mF$. Let $\psi \in WF$ be anisotropic with $\dim \psi = 2^n$. Then the following are equivalent.*

(i) $\psi \in \mathrm{Equiv}(\varphi)$ *(i.e., $\psi \sim \varphi$).*

(ii) *$\psi_{F(\pi)}$ is similar to $\varphi_{F(\pi)}$ and $\psi \in P^w_{n,m}F$ with twist π.*

(iii) *$\psi_{F(\pi)}$ is similar to $\varphi_{F(\pi)}$ and $\deg \psi = m$.*

Proof. We clearly may assume that ψ is anisotropic.

(ii)⇒(i). Since $\psi_{F(\pi)}$ is similar to $\varphi_{F(\pi)}$, and since $\varphi \equiv \pi \otimes \eta \pmod{J_nF}$ and $\psi \equiv \pi \otimes \mu \pmod{J_nF}$ with $\dim \eta \equiv \dim \mu \equiv 1 \pmod 2$, it follows directly from Proposition 5.1 and the symmetry of the situation that $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are both isotropic. Hence, $\varphi \sim \psi$.

(i)⇒(ii). Let now $\varphi \sim \psi$. Then, because $\varphi_{F(\psi)}$ is isotropic, $\psi_{F(\pi)}$ is similar to $\varphi_{F(\pi)}$ by Proposition 5.1 and $\psi \equiv \pi \otimes \mu \pmod{J_nF}$ for some $\mu \in WF$ by Proposition 5.6. Suppose $\dim \mu$ is even so that we have $\deg(\pi \otimes \mu) \ge m+1$. Let $K = F(\pi)$ and let L be the generic splitting field of $\pi \otimes \mu$ as defined in Section 2. Then $(\pi \otimes \mu)_K = 0$ in WK and it follows that the free composite $M = KL$ is purely transcendental over K (cf. Proposition 2.2(iii)). Since φ_K is anisotropic we have that φ_{KL} is also anisotropic and thus, φ_L is anisotropic as well. Since $\varphi \sim \psi$, it follows that ψ_L stays also anisotropic. But $\psi_L \equiv (\pi \otimes \mu)_L \equiv 0 \pmod{J_nL}$ and $\dim \psi = 2^n$. This yields that $\psi \in GP_nL$. Now $\varphi \sim \psi$ also implies that $\varphi_L \sim \psi_L$ and we conclude that φ_L is similar to ψ_L, in particular, $\varphi_L \in GP_nL$ and $\deg \varphi_L = n > m = \deg \varphi$. But [AK, Satz 20] implies that $\deg \varphi_L = \deg \varphi = m$ because L is a generic splitting field of $\pi \otimes \mu$ and $\deg(\pi \otimes \mu) \ge m+1 > m$ since $\dim \mu$ is even. This is clearly a contradiction and we therefore have that $\dim \mu$ is odd.

(ii)⇔(iii). The condition that $\psi_{F(\pi)}$ is similar to $\varphi_{F(\pi)}$, which appears in both statements, implies that $\varphi_{F(\psi)}$ is isotropic by Proposition 5.1, and thus we get $\psi \equiv \pi \otimes \mu \pmod{J_nF}$ for some $\mu \in WF$ in both (ii) and (iii). The equivalence of (ii) and (iii) now follows from the easy observation that $\deg \psi = m$ iff $\deg(\pi \otimes \mu) = m$ iff $\dim \mu$ is odd. □

Corollary 6.5 *Let $n \geq 3$. Let $\varphi \in P_{n,n-2}F$ be defined by (σ, π). Let $\psi \in WF$ be anisotropic with $\dim \psi = 2^n$. Then $\varphi \sim \psi$ iff $\psi_{F(\pi)}$ is similar to $\varphi_{F(\pi)}$ and $\psi \equiv x\pi \pmod{J_n F}$ for some $x \in \dot{F}$.*

In particular, if Conjecture 3.9 holds for $(n, n-2)$ or $(n, n-1)$ (which is fulfilled if $n \leq 4$), or if ψ contains a Pfister neighbor of dimension $2^{n-1}+1$, then $\varphi \sim \psi$ iff ψ is similar to some $\chi \in P_{n,n-2}F$ which is defined by (τ, π) such that $\tau_{F(\pi)} \simeq \sigma_{F(\pi)}$.

Proof. This follows from Theorem 6.4 together with Propositions 3.6, 3.11, 3.15, 5.6, 5.8 and Lemma 5.9. We leave the details to the reader. □

Definition 6.6 Let $1 \leq m < n$. Let $\varphi \in P^w_{n,m}F$ with twist $\pi \in P_m F$. We define $\mathfrak{E}(\varphi)$ to be the set of all $\psi \in WF$ with $\dim \psi > 2^n - 2^{m-1}$ for which there exist $\tilde{\psi} \in P^w_{n,m}F$ with twist π such that

- $\psi \subset \tilde{\psi}$, and
- $\tilde{\psi}_{F(\pi)}$ is similar to $\varphi_{F(\pi)}$.

In view of Theorem 6.4 and Corollary 3.8, we conjecture the following.

Conjecture 6.7 *Let $1 \leq m < n$. Let $\varphi \in P^w_{n,m}F$ with twist $\pi \in P_m F$. Then* $\mathrm{Equiv}(\varphi) = \mathfrak{E}(\varphi)$.

Proposition 6.8 *Let $1 \leq m < n$. Let $\varphi \in P^w_{n,m}F$ with twist $\pi \in P_m F$. Then* $\mathfrak{E}(\varphi) \subset \mathrm{Equiv}(\varphi)$.

Proof. Let $\psi \in \mathfrak{E}(\varphi)$. Then there exist $\tilde{\psi}, \mu \in WF$ with $\dim \tilde{\psi} = 2^n$, $\dim \mu$ odd, such that $\tilde{\psi}_{F(\pi)}$ is similar to $\varphi_{F(\pi)}$ and $\tilde{\psi} \equiv \pi \otimes \mu \pmod{J_n F}$. By Theorem 6.4 this implies $\tilde{\psi} \sim \varphi$. Furthermore, $\tilde{\psi}$ has the property that $\psi \subset \tilde{\psi}$, and we also have $\dim \psi > 2^n - 2^{m-1}$. Hence, $\psi \sim \tilde{\psi}$ by Corollary 3.8 and therefore $\psi \sim \tilde{\psi} \sim \varphi$, i.e., $\psi \in \mathrm{Equiv}(\varphi)$. □

The main result of this section is the following.

Theorem 6.9 *Conjecture 6.7 holds for $m \leq 2$.*

Before we prove the theorem, we consider a special case.

Lemma 6.10 *Let $\varphi \in P_{n,m}F$ be defined by (σ, π) and let $\psi \subset \varphi$ with $\dim \psi \leq 2^n - 2^{m-1}$. Then there exists a field extension K/F such that $\varphi_K \in P_{n,m}K$ is defined by (σ_K, π_K) and $\psi_K \subset \sigma_K$. In particular, $\psi_{F(\varphi)}$ is anisotropic. If $m = 1$ such a K can be chosen to be of the form $K = F(\beta)$ for some $\beta \in P_2 F$.*

Proof. Let K/F be a field extension. If φ_K, σ_K and π_K all stay anisotropic then one easily concludes that one still has $\varphi_K \in P_{n,m}K$ and that it is defined by (σ_K, π_K). To prove this lemma, we may assume that $\dim \psi = 2^n - 2^{m-1}$. Let $\psi' \in WF$, $\dim \psi' = 2^{m-1}$ such that $\varphi \simeq \psi \perp \psi'$. Then, in WF, we have $\varphi = \sigma - \pi = \psi + \psi'$ or $\sigma \perp -\psi = \psi' \perp \pi$. Note that $\dim \psi' = 2^{m-1} = \frac{1}{2} \dim \pi = 2^m$. By [H 3, Remark 1 and Theorem 4], there exists a field K in the generic splitting tower of $\psi' \perp \pi$ such that $i_W((\psi' \perp \pi)_K) = 2^{m-1}$, i.e., $-\psi'_K \subset \pi_K$, and π_K is anisotropic (see also [HuR, Corollaries 1.9 and 1.12]). In particular, $\dim((\psi' \perp \pi)_K)_{\mathrm{an}} = 2^{m-1}$. By comparing dimensions, we get $\sigma_K \simeq \psi_K \perp ((\psi' \perp \pi)_K)_{\mathrm{an}}$ and hence $\psi_K \subset \sigma_K$.

Note that if $m = 1$ then $\dim(\psi' \perp \pi) = 3$, so $\psi' \perp \pi$ is a Pfister neighbor of some $\beta \in P_2 F$. In our construction, the field K in the splitting tower of $\psi' \perp \pi$ is either F itself if $\psi' \perp \pi$ is already isotropic in which case $F(\beta)/F$ is purely transcendental, or it is $F(\psi' \perp \pi)$ if $\psi' \perp \pi$ is anisotropic. In this case, the field $F(\psi' \perp \pi)$ is equivalent to $F(\beta)$ since $\psi' \perp \pi$ is a Pfister neighbor of β. In any case, the field in the splitting tower which we consider is equivalent to $F(\beta)$ and thus we may as well choose $K = F(\beta)$.

It remains to show that φ_K and σ_K are anisotropic. Since $\dim \psi' = 2^{m-1} < \dim \pi$, it follows from [H 3, Theorem 1] that $\psi'_{F(\pi)}$ is anisotropic. Now $\pi_{F(\pi)} = 0$ and thus $((\psi' \perp \pi)_{F(\pi)})_{\mathrm{an}} \simeq \psi'_{F(\pi)}$ and we have $i_W((\psi' \perp \pi)_{F(\pi)}) = 2^{m-1} = i_W((\psi' \perp \pi)_K)$. By Proposition 2.2(iii), we have that $L = K{\cdot}F(\pi)$ is purely transcendental over $F(\pi)$. Now $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)}$ is anisotropic. Hence, $\varphi_L \simeq \sigma_L$ stays anisotropic which clearly implies that φ_K and σ_K are anisotropic. By our remark at the beginning, we have that $\varphi_K \in P_{n,m}K$ is defined by (σ_K, π_K). Since $\varphi_K \in P_{n,m}K$ we have that φ_K cannot be similar to a subform of σ_K. Therefore, $\sigma_{K(\varphi)}$ stays anisotropic and thus, $\psi_{K(\varphi)}$ stays also anisotropic. This obviously yields that $\psi_{F(\varphi)}$ is anisotropic. $\square$

We added the additional statement in the case $m = 1$ because we will need this particular fact later on in the proof of Proposition 7.8

Proof of Theorem 6.9. Let $1 \le m \le 2$ and $m < n$. Let $\varphi \in WF$ be anisotropic and $\dim \varphi = 2^n$. Suppose that $\varphi \equiv \pi \otimes \eta \pmod{J_n F}$ for some anisotropic $\pi \in P_m F$ and some $\eta \in WF$ with $\dim \eta$ odd. By Proposition 6.8, it remains to show that $\mathrm{Equiv}(\varphi) \subset \mathfrak{E}(\varphi)$.

So let $\psi \in WF$ with $\psi \sim \varphi$. Clearly, $\dim \psi \ge 2$. Now $\psi \sim \varphi$ implies that $\varphi_{F(\psi)}$ is isotropic. Since $m \le 2$ we have that $F(\pi)/F$ is excellent. Proposition 5.2 implies that then there exists $\tilde{\psi} \in WF$, $\dim \tilde{\psi} = 2^n$, such that $\psi \subset \tilde{\psi}$ and, possibly after scaling, $\tilde{\psi}_{F(\pi)} \simeq \varphi_{F(\pi)}$. By Proposition 5.1, we have that $\varphi_{F(\tilde{\psi})}$ is isotropic. Now $\tilde{\psi}_{F(\psi)}$ is isotropic as $\psi \subset \tilde{\psi}$. We also have that $\psi_{F(\varphi)}$ is isotropic because $\psi \sim \varphi$. Hence, $\tilde{\psi}_{F(\varphi)}$ is isotropic as well (cf. Proposition 2.1(viii)), and therefore $\varphi \sim \tilde{\psi}$. By Theorem 6.4, there exists $\mu \in WF$, $\dim \mu$ odd, such that $\tilde{\psi} \equiv \pi \otimes \mu \pmod{J_n F}$. By Corollary 3.13, there exists a field extension K/F such that $\tilde{\psi}_K \in GP_{n,m}K$. We have already seen that $\psi \sim \varphi \sim \tilde{\psi}$. In particular, $\psi_{F(\tilde{\psi})}$ is isotropic which clearly yields that $\psi_{K(\tilde{\psi})}$ is also isotropic. Now $\psi_K \subset \tilde{\psi}_K \in GP_{n,m}K$. Lemma 6.10 implies that $\dim \psi > 2^n - 2^{m-1}$. This completes the proof. $\square$

Corollary 6.11 *Conjecture 6.3 holds for $n \le 4$.*

Proof. This is an immediate consequence of Corollary 6.5 and Theorem 6.9. $\square$

Example 6.12 We return to the forms φ and ψ over $F = \mathbb{R}(t)$ which we defined in Example 5.13. We had $\varphi \in P_{n,m}F$ being defined by (σ, π) where $n - m \ge 3$. We showed that $\psi_{F(\pi)} \simeq \varphi_{F(\pi)}$ and by our construction we had that $\psi \equiv -\langle 1,1,1 \rangle \otimes \pi \pmod{J_n F}$. Hence, by Theorem 6.4, $\varphi \sim \psi$. However, we also showed that $\psi \notin GP_{n,m}F$. This shows that if $\varphi \in P_{n,m}F$ and $\psi \sim \varphi$ with $\dim \psi = \dim \varphi = 2^n$ then generally this does not imply $\psi \in GP_{n,m}F$ if $n - m \ge 3$. $\square$

We know that two anisotropic Pfister forms are equivalent iff they are isometric. To finish this section, we would like to say something about equivalence of twisted Pfister forms. Since their dimensions are always 2-powers, equivalent twisted Pfister forms must be of the same dimension. We have the following.

PROPOSITION 6.13 *Let $1 \leq m, \ell < n$. Let $\varphi \in P_{n,m}F$ be defined by (σ, π), and let $\psi \in P_{n,\ell}F$ be defined by (τ, ρ). Then $\varphi \sim \psi$ iff $\pi \simeq \rho$ and $\sigma_{F(\pi)} \simeq \tau_{F(\pi)}$. Furthermore, if this is the case then we have the following.*

(i) *If $m = n-1$ then φ is similar to ψ.*

(ii) *If $m \leq n-2$ then φ is similar to ψ iff $\sigma \simeq \tau$.*

Proof. We have $\deg \varphi = \deg \pi = m$, $\deg \psi = \deg \rho = \ell$ and $\varphi \equiv -\pi \pmod{J_n F}$, $\psi \equiv -\rho \pmod{J_n F}$. If $\varphi \sim \psi$ then, by Theorem 6.4, $m = \ell$ and $\pi \equiv \varphi \equiv \psi \equiv \rho \pmod{J_{m+1} F}$ which readily yields $\pi \simeq \rho$. Also by Theorem 6.4, we have that $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)}$ is similar to $\psi_{F(\pi)} \simeq \tau_{F(\pi)}$ (here, we already use $\pi \simeq \rho$), which immediately implies $\sigma_{F(\pi)} \simeq \tau_{F(\pi)}$. This shows the "only if" part. The converse follows also easily from Theorem 6.4.

Let us now assume that $\varphi \sim \psi$. If $m = n-1$ we know from Theorem 6.1 that φ is similar to ψ. So let us finally assume that $m \leq n-2$. If $\sigma \simeq \tau$ then obviously $\varphi \simeq \psi$ by definition of a twisted Pfister form. Conversely, suppose that $\varphi \simeq a\psi$ for some $a \in \dot{F}$. Then, in WF,

$$\begin{aligned} 0 &= \varphi - a\psi = \sigma - \pi - a(\tau - \pi) \\ &= \sigma - a\tau - \pi \otimes \langle\langle -a \rangle\rangle\ . \end{aligned}$$

Now σ, $\tau \in P_n F$ and $\pi \otimes \langle\langle -a \rangle\rangle \in P_{m+1} F$ with $m+1 < n$. We therefore get $0 \equiv -\pi \otimes \langle\langle -a \rangle\rangle \pmod{I^n F}$ and the Arason-Pfister Hauptsatz implies that $\pi \otimes \langle\langle -a \rangle\rangle = 0$. Hence, $0 = \sigma - a\tau$ or $\sigma \simeq a\tau$ which implies that $\sigma \simeq \tau$ as σ and τ are both n-fold Pfister forms. □

This little result has a nice application. It is of interest to determine $\mathrm{Equiv}(\varphi)$ for a given anisotropic form $\varphi \in WF$, $\dim \varphi \geq 2$. Clearly, if ψ is similar to φ then $\psi \sim \varphi$. More generally, if $a\psi \subset \varphi$ for some $a \in \dot{F}$ and $\dim \psi > \dim \varphi - i_1(\varphi)$, then $\psi_{F(\varphi)}$ is easily seen to be isotropic. Obviously, so is $\varphi_{F(\psi)}$. Hence, $\psi \sim \varphi$. Even more generally, if there exists an anisotropic $\gamma \in WF$ such that $a\varphi \subset \gamma$ and $b\psi \subset \gamma$ for some $a, b \in \dot{F}$ such that $\dim \varphi, \dim \psi > \dim \gamma - i_1(\gamma)$, then by the same reasoning as above, $\varphi \sim \gamma \sim \psi$.

Another situation where we have $\varphi \sim \psi$ (both forms anisotropic) is when $\dim \varphi = \dim \psi \geq 2$ and there exists $a \in \dot{F}$ such that $\varphi \perp a\psi$ is similar to some $\pi \in P_n F$. Clearly, we have $\dim \varphi = \dim \psi = 2^{n-1}$. Then π is isotropic and hence hyperbolic over $F(\varphi)$ and $F(\psi)$. In particular, $\varphi_{F(\varphi)} \simeq -a\psi_{F(\varphi)}$ and $\varphi_{F(\psi)} \simeq -a\psi_{F(\psi)}$. Comparing dimensions and Witt indices, we conclude that $\varphi_{F(\psi)}$ and $\psi_{F(\varphi)}$ are both isotropic, i.e., $\varphi \sim \psi$.

This leads to the following definitions.

DEFINITION 6.14 Let φ, $\psi \in WF$ be anisotropic. Then φ and ψ are *neighbors* if there exists an anisotropic $\gamma \in WF$, $\dim \gamma \geq 2$, such that φ and ψ are similar to subforms of γ and $\dim \varphi, \dim \psi > \dim \gamma - i_1(\gamma)$.

φ and ψ are called *conjugate* if $\dim \varphi = \dim \psi$ and there exists $a \in \dot{F}$ such that $\varphi \perp a\psi \in GP_n F$ for some n.

If in the definition of neighbor the form γ is a similar to a Pfister form, then we have that φ and ψ are both Pfister neighbors of the same Pfister form. So this definition of neighbor is a natural generalization of a Pfister neighbor. Our definition of conjugate forms is slightly more general than the definition of conjugate forms in [K 2, Definition 8.7].

Remark 6.15 Let φ and ψ be anisotropic forms over F.

(i) If φ and ψ are similar, say, $\varphi \simeq a\psi$, then φ and ψ are neighbors. If in addition $\dim\varphi = \dim\psi = 2^n$ then φ and ψ are also conjugate. This is because $\varphi \perp -a\psi$ is isometric to the hyperbolic $(n+1)$-fold Pfister form.

(ii) Suppose that φ and ψ are neighbors and that $\dim\varphi = \dim\psi = 2^n$. If $\dim\gamma > 2^n$ then $\varphi_{F(\gamma)}$ and $\psi_{F(\gamma)}$ are anisotropic, see Proposition 2.1(vi). A form γ as in the definition above with $\dim\gamma > 2^n$ can therefore not exists. So if γ is such a form as in the definition, we must have $\dim\gamma = 2^n$ which immediately implies that φ is similar to ψ. Hence, two anisotropic forms of dimension 2^n are similar iff they are neighbors.

(iii) Suppose that $\dim\varphi = \dim\psi = 2^n$. Then φ and ψ are similar or conjugate iff there exists an $a \in \dot{F}$ such that $\varphi \perp a\psi \in W(F(\varphi)/F) \cap W(F(\psi)/F)$ (cf. [K 2, Theorem 8.8]).

Generally, conjugate forms are not similar. In a forthcoming paper, we will investigate such examples and the relationship between conjugacy and similarity.

The first examples known to us of forms φ and ψ with $\varphi \sim \psi$ but where φ and ψ are neither neighbors nor conjugate were given by twisted Pfister forms.

Proposition 6.16 *Let $n \geq 3$ and $1 \leq m \leq n-2$. Let φ, $\psi \in P_{n,m}F$ be defined by (σ,π) and (τ,π), respectively, and assume that $\sigma_{F(\pi)} \simeq \tau_{F(\pi)}$ but $\sigma \not\simeq \tau$. Then $\varphi \sim \psi$ but φ and ψ are neither neighbors nor conjugate.*

Proof. By Proposition 6.13, we know that $\varphi \sim \psi$ and that φ is not similar to ψ. By Remark 6.15(ii), φ and ψ are not neighbors.

Suppose that φ and ψ are conjugate, i.e., $\varphi \perp a\psi \in GP_{n+1}F$ for some $a \in \dot{F}$. Then

$$0 \equiv \varphi + a\psi \equiv \sigma - \pi + a(\tau - \pi) \equiv -\pi \otimes \langle\langle a\rangle\rangle \pmod{I^nF}$$

because $\varphi \perp a\psi \in GP_{n+1}F \subset I^nF$ and $\sigma,\tau \in P_nF \subset I^nF$. Since $\dim(\pi \otimes \langle\langle a\rangle\rangle) = 2^{m+1} < 2^n$, the Arason-Pfister Hauptsatz implies $\pi\otimes\langle\langle a\rangle\rangle = 0$ and thus $\varphi \perp a\psi = \sigma \perp a\tau$ in WF. Comparing dimensions and because $\varphi \perp a\psi \simeq \rho \in GP_{n+1}F$ we get that $\sigma \perp a\tau \simeq \rho \in GP_{n+1}F$. Hence, $\rho_{F(\sigma)}$ becomes isotropic and therefore hyperbolic. Thus, in $WF(\sigma)$,

$$0 = \rho_{F(\sigma)} = \sigma_{F(\sigma)} + a\tau_{F(\sigma)} = a\tau_{F(\sigma)}$$

which yields that σ is similar to a subform of τ. This in turn implies that $\sigma \simeq \tau$, a contradiction. □

In the last section, we will construct examples of forms σ, τ, and π which satisfy the conditions in Proposition 6.16. Let us conclude this section with another example of equivalent forms which are neither neighbors nor conjugate.

Example 6.17 Let $F = \mathbb{R}(t)$ and let φ and ψ be the anisotropic forms in Example 5.13. Then $\varphi \sim \psi$ but φ and ψ are neither neighbors nor conjugate. We leave the details to the reader. □

7 Twisted Pfister forms over the function field of a Pfister form

Let $\varphi \in P_{n,m}F$ be defined by (σ, π), i.e., $\sigma \in P_nF$ and $\pi \in P_mF$ are anisotropic, $\ln(\sigma,\pi) = m-1$, and $\varphi \simeq (\sigma \perp -\pi)_{\mathrm{an}}$. Let $\tau \in P_nF$. We know by Proposition 5.1 that $\sigma_{F(\tau)}$ is isotropic iff $\sigma_{F(\pi)} \simeq \tau_{F(\pi)}$. So let us from now on assume that $\sigma_{F(\tau)}$ is isotropic, i.e., $\sigma_{F(\pi)} \simeq \tau_{F(\pi)}$. This also implies by Proposition 5.8 that there exists $\alpha \in P_{m-1}F$ which divides π, σ, and τ.

Izhboldin [I] used twisted Pfister forms in his construction of Pfister forms which yield non-excellent function field extensions. More precisely, he essentially showed that for φ and τ as above and in the particular case where $m=1$ and $\ln(\sigma,\tau)=1$, then $(\varphi_{F(\tau)})_{\mathrm{an}}$ is *not* defined over F. It is our aim to generalize this result. First, let us note the following.

Lemma 7.1 *Let $\varphi \in P_{n,m}F$ be defined by (σ,π) and let $\tau \in P_nF$ with $\sigma_{F(\pi)} \simeq \tau_{F(\pi)}$, i.e., $\varphi_{F(\tau)}$ is isotropic. Then*

$$\dim(\varphi_{F(\tau)})_{\mathrm{an}} = \begin{cases} 2^m & \text{if } \sigma \simeq \tau \\ 2^n - 2^m & \text{if } \sigma \not\simeq \tau \end{cases}$$

Proof. Suppose first that $\sigma \simeq \tau$. Then $\sigma_{F(\tau)} = 0$ and $\pi_{F(\tau)}$ is anisotropic because $m < n$, and in $WF(\tau)$ we have $\varphi_{F(\tau)} = \sigma_{F(\tau)} - \pi_{F(\tau)} = -\pi_{F(\tau)}$. It follows immediately that $(\varphi_{F(\tau)})_{\mathrm{an}} \simeq -\pi_{F(\tau)}$ and $\dim(\varphi_{F(\tau)})_{\mathrm{an}} = 2^m$.

Now suppose that $\sigma \not\simeq \tau$. Then $\sigma_{F(\tau)}$ is anisotropic and thus we have, using $\varphi = \sigma - \pi$ in WF and $\varphi_{F(\tau)}$ isotropic, that $2^n > \dim(\varphi_{F(\tau)})_{\mathrm{an}} \geq \dim\sigma - \dim\pi = 2^n - 2^m$. By Proposition 3.6 we therefore have $\dim(\varphi_{F(\tau)})_{\mathrm{an}} = 2^n - 2^m$. □

We now come to the main result of this section

Theorem 7.2 *Let $\varphi \in P_{n,m}F$ be defined by (σ,π) and let $\tau \in P_nF$ with $\sigma_{F(\pi)} \simeq \tau_{F(\pi)}$, i.e., $\varphi_{F(\tau)}$ is isotropic. Then the following are equivalent.*

(i) *There exists a Pfister neighbor χ of τ such that $\chi \subset \varphi$.*

(ii) *There exists a Pfister neighbor χ of τ with $\dim\chi = 2^{n-1} + 2^{m-1}$ such that $\chi \subset \varphi$.*

(iii) *$(\varphi_{F(\tau)})_{\mathrm{an}}$ is defined over F.*

(iv) $\ln(\sigma,\tau) \geq n-1$.

Proof. (i)⇒(ii). Let $a \in \dot{F}$ and $\chi \subset \varphi$ with $\dim\chi \geq 2^{n-1}+1$ such that $\chi \subset a\tau$. Let $\eta \simeq (\varphi \perp -a\tau)_{\mathrm{an}}$. Then $\dim\eta \leq \dim\varphi + \dim\tau - 2\dim\chi \leq 2^n - 2$. Note that we have $\eta \equiv \varphi - a\tau \equiv \sigma - \pi - a\tau \equiv -\pi \pmod{I^nF}$ as σ, $\tau \in P_nF \subset I^nF$. Thus, we get $\eta_{F(\pi)} \equiv 0 \pmod{I^nF(\pi)}$ and the Arason-Pfister Hauptsatz implies $\eta_{F(\pi)} = 0$ or $\eta_{F(\pi)} \in W(F(\pi)/F)$. Hence, by Proposition 2.1(v), there exists $\mu \in WF$ such that $\eta \simeq \mu \otimes \pi$. Thus, $\dim\pi = 2^m$ divides $\dim\eta$ and therefore $\dim\eta = \dim(\varphi \perp -a\tau)_{\mathrm{an}} \leq 2^n - 2^m$ or $i_W(\varphi \perp -a\tau) \geq 2^{n-1} + 2^{m-1}$. In particular, φ and $a\tau$ have a common subform of dimension $2^{n-1}+2^{m-1}$.

(ii)⇒(iii). If $\sigma \simeq \tau$ we have already seen in the proof of Lemma 7.1 that $(\varphi_{F(\tau)})_{\mathrm{an}} \simeq -\pi_{F(\tau)}$ and we are done. Hence, we may assume that $\sigma \not\simeq \tau$ and thus $\dim(\varphi_{F(\tau)})_{\mathrm{an}} = 2^n - 2^m$ by Lemma 7.1. Let $\chi \subset \varphi$ such that $\dim\chi = 2^{n-1} + 2^{m-1}$

and $\chi \subset a\tau$ for some $a \in \dot{F}$. Write $\varphi \simeq \chi \perp \tilde{\varphi}$ and $a\tau \simeq \chi \perp \tilde{\tau}$ for suitable $\tilde{\varphi}$, $\tilde{\tau} \in WF$. In $WF(\tau)$, we have

$$\varphi_{F(\tau)} = (\varphi \perp -a\tau)_{F(\tau)} = (\tilde{\varphi} \perp -\tilde{\tau})_{F(\tau)} \ .$$

Now an easy check shows that $\dim(\tilde{\varphi} \perp -\tilde{\tau}) = 2^n - 2^m = \dim(\varphi_{F(\tau)})_{\rm an}$. Therefore, we must have $(\varphi_{F(\tau)})_{\rm an} \simeq (\tilde{\varphi} \perp -\tilde{\tau})_{F(\tau)}$ and we see that $(\varphi_{F(\tau)})_{\rm an}$ is defined over F by $\tilde{\varphi} \perp -\tilde{\tau}$.

(iii)$\Rightarrow$(iv). Let $\eta \in WF$ such that $(\varphi_{F(\tau)})_{\rm an} \simeq \eta_{F(\tau)}$. By Lemma 7.1 we have $2^m \leq \dim \eta \leq 2^n - 2^m$ and thus $0 < \dim(\varphi \perp -\eta)_{\rm an} < 2^{n+1}$. Also, by our choice of η, $(\varphi \perp -\eta)_{\rm an} \in W(F(\tau)/F)$ and by Proposition 2.1(v) there exists $a \in \dot{F}$ such that $(\varphi \perp -\eta)_{\rm an} \simeq a\tau$. Hence, in WF we get $a\tau = \varphi - \eta = \sigma - \pi - \eta$ or $\sigma \perp -a\tau = \pi \perp \eta$. Now $\dim(\pi \perp \eta) \leq 2^m + 2^n - 2^m = 2^n$ and we get that $i_W(\sigma \perp -a\tau) \geq 2^{n-1}$. By Lemma 3.2, $\ln(\sigma, \tau) \geq n-1$.

(iv)$\Rightarrow$(i) This is rather obvious in the case where $\sigma \simeq \tau$, i.e., $\ln(\sigma,\tau) = n$.

So let us assume that $\ln(\sigma,\tau) = n-1$ and let $\rho \simeq (\sigma \perp -\tau)_{\rm an}$. Then $\dim \rho = 2^n$ and in fact $\rho \in GP_nF$ as $\rho \in I^nF$. Let $\psi \simeq (\varphi \perp -\tau)_{\rm an}$. Then, in WF,

$$\psi = \varphi - \tau = \sigma - \pi - \tau = \rho - \pi \ .$$

This yields

$$2^n - 2^m = \dim \rho - \dim \pi \leq \dim \psi \leq \dim \rho + \dim \pi = 2^n + 2^m \ .$$

On the other hand, in $WF(\pi)$,

$$\psi_{F(\pi)} = \varphi_{F(\pi)} - \tau_{F(\pi)} = \sigma_{F(\pi)} - \tau_{F(\pi)} - \pi_{F(\pi)} = 0$$

as $\sigma_{F(\pi)} \simeq \tau_{F(\pi)}$ and $\pi_{F(\pi)} = 0$. Thus, there exists $\gamma \in WF$ with $\psi \simeq \gamma \otimes \pi$ by Proposition 2.1(v). Now $\dim \gamma$ must be odd for otherwise $\psi \in I^{m+1}F$, but $\psi \equiv \sigma - \pi - \tau \equiv -\pi \not\equiv 0 \pmod{I^{m+1}F}$. Hence, there are two cases. Either $\dim \psi = 2^n - 2^m$ or $\dim \psi = 2^n + 2^m$. If $\dim \psi = 2^n - 2^m$ then by the definition of ψ we get $i_W(\varphi \perp -\tau) = \frac{1}{2}(\dim \varphi + \dim \tau - \dim \psi) = 2^{n-1} + 2^{m-1}$. Thus, φ and τ have a common subform of dimension $2^{n-1} + 2^{m-1}$ and we are done.

So let us finally assume that $\dim \psi = 2^n + 2^m$ so that in fact $\psi \simeq (\varphi \perp -\tau)_{\rm an} \simeq \rho \perp -\pi$. Now π divides ψ and we have that π also divides ρ. But $\rho \in GP_nF$. Hence, there exist $\delta \in P_{n-m}F$ and $x \in \dot{F}$ such that $\rho \simeq x\pi \otimes \delta$. Thus,

$$x\psi \simeq \pi \otimes (\delta \perp \langle -x \rangle) \subset \pi \otimes \delta \otimes \langle\langle -x \rangle\rangle \in P_{n+1}F \ .$$

This shows that ψ is a Pfister neighbor of $\beta \simeq \pi \otimes \delta \otimes \langle\langle -x \rangle\rangle \in P_{n+1}F$. Since ψ is anisotropic, β is anisotropic as well. Also, $\psi_{F(\tau)} = \varphi_{F(\tau)} - \tau_{F(\tau)} = \varphi_{F(\tau)}$ in $WF(\tau)$. Comparing dimensions, we conclude that $\psi_{F(\tau)}$ is isotropic and that therefore also $\beta_{F(\tau)}$ is isotropic and hence hyperbolic, which in turn implies that $\beta \simeq \tau \otimes \langle\langle t \rangle\rangle$ for some $t \in \dot{F}$. Since $x\psi \subset \beta$ we get $\psi \subset x\beta \simeq \tau \otimes \langle x, xt \rangle$. Now $\psi \perp \tau = \varphi$ in WF and by comparing dimensions we see that $\psi \perp \tau$ is isotropic. In particular, there exists $y \in D(\psi) \perp D(-\tau)$. Since $y \in D(\psi) \subset D(\tau \otimes \langle x, xt \rangle)$ and since $\tau \in P_nF$, we may assume by Lemma 3.1 that for suitable $z \in \dot{F}$ we have $\tau \otimes \langle x, xt \rangle \simeq \tau \otimes \langle y, z \rangle$.

But $-y \in D(\tau) = G(\tau)$. If we write $\tau \otimes \langle x, xt\rangle \simeq \psi \perp \mu$ for suitable $\mu \in WF$ with $\dim \mu = 2^{n+1} - \dim \psi = 2^n - 2^m$, we obtain

$$\psi \perp \mu \simeq \tau \otimes \langle x, xt\rangle \simeq y\tau \perp z\tau \simeq -\tau \perp z\tau$$

and thus, in WF,

$$\mu = z\tau - \tau - \psi = z\tau - \varphi$$

(here we use $\psi = \varphi - \tau$.) In particular, $(z\tau \perp -\varphi)_{\mathrm{an}} \simeq \mu$ (recall that $\psi \perp \mu$ is similar to the anisotropic Pfister form β and that therefore μ is also anisotropic), which implies that $i_W(z\tau \perp -\varphi) = \frac{1}{2}(\dim\tau + \dim\varphi - \dim\mu) = 2^{n-1} + 2^{m-1}$, i.e., φ and $z\tau$ have a common subform of dimension $2^{n-1} + 2^{m-1}$ which is obviously a Pfister neighbor of τ. □

REMARK 7.3 If φ contains a Pfister neighbor of τ of dimension $> 2^{n-1} + 2^{m-1}$ then $\sigma \simeq \tau$. For in this case, there exists $a \in \dot{F}$ such that $i_W(\varphi \perp -a\tau) > 2^{n-1} + 2^{m-1}$ or $\dim(\varphi \perp -a\tau)_{\mathrm{an}} < 2^n - 2^m$. But in WF we have $\varphi \perp -a\tau = \sigma \perp -a\tau \perp -\pi$ and we must necessarily have $\dim(\sigma \perp -a\tau)_{\mathrm{an}} < 2^n$ which, by Lemma 3.1, implies $\ln(\sigma, \tau) = n$, in other words $\sigma \simeq \tau$.

Conversely, if $\sigma \simeq \tau$ then the largest Pfister neighbor of τ contained in φ has dimension $2^n - 2^{m-1}$. That φ contains such a Pfister neighbor of this dimension follows readily from Remark 3.5(ii) (using the notation there, one may take $\alpha \otimes \sigma_1'$). On the other hand φ does not contain *any* Pfister neighbor of dimension $> 2^n - 2^{m-1}$. For suppose otherwise. Let $\chi \subset \varphi$ be such a Pfister neighbor of some $\tilde{\chi} \in P_nF$ with $\dim \chi > 2^n - 2^{m-1}$. Then $\tilde{\chi} \sim \chi \sim \varphi$, the first equivalence because χ is a Pfister neighbor of $\tilde{\chi}$, the second one by Corollary 3.8. But this is absurd since φ is clearly not a Pfister neighbor of $\tilde{\chi}$.

COROLLARY 7.4 *Let $\varphi \in P_{n,n-1}F$ and $\tau \in P_nF$. Then $(\varphi_{F(\tau)})_{\mathrm{an}}$ is defined over F.*

Proof. If $\varphi_{F(\tau)}$ stays anisotropic then there is nothing to show. So let us assume that $\varphi_{F(\tau)}$ is isotropic. By Proposition 5.1, we have that $\varphi_{F(\pi)} \simeq \sigma_{F(\pi)} \simeq \tau_{F(\pi)}$. It follows immediately from Proposition 5.8(ii) that $\ln(\sigma, \tau) \geq n - 1$. The desired result follows now from Theorem 7.2. □

Statements (i) resp. (ii) of Theorem 7.2 essentially say that the obstruction to $(\varphi_{F(\tau)})_{\mathrm{an}}$ being defined over F is the non-existence of a Pfister neighbor of τ as a subform of φ, and by Corollary 7.4 this can only happen if $n \geq 3$ and $(n, m) \neq (n, n-1)$. This is not at all obvious as the case of the function field of a 2-fold Pfister form η shows. There are many examples of fields F with anisotropic forms $\psi \in WF$ and $\eta \in P_2F$ such that $\psi_{F(\eta)}$ is isotropic but ψ does not contain a Pfister neighbor of η (for such examples we refer to [LVG], [HLVG], [HVG]). However, since $F(\tau)/F$ is excellent we have, by definition of excellence, that $(\psi_{F(\eta)})_{\mathrm{an}}$ is defined over F. Conversely, if $\tau \in P_nF \geq 3$ such that $F(\tau)/F$ is not excellent then there might still be many forms ψ which contain Pfister neighbors of τ but where $(\psi_{F(\tau)})_{\mathrm{an}}$ is not defined over F. For example, let φ be such that $(\varphi_{F(\tau)})_{\mathrm{an}}$ is not defined over F and put $\psi \simeq \tau \perp \varphi$. Then $(\psi_{F(\tau)})_{\mathrm{an}} \simeq (\varphi_{F(\tau)})_{\mathrm{an}}$ is not defined over F, but ψ contains τ itself as a subform.

Twisted Pfister forms also yield new non-trivial examples of $F(\tau)$-minimal forms where $\tau \in P_nF$, $n \geq 3$. Recall that for a field extension K/F we say that φ is

K-minimal if φ is anisotropic, φ_K is isotropic, and if $\eta \subset \varphi$ with $\dim \eta < \dim \varphi$ then η_K is anisotropic. We are interested in the case where $K = F(\tau)$ for some anisotropic $\tau \in P_nF$. If $n = 1$ the K-minimal forms are exactly the scalar multiples of τ, cf. Proposition 2.1(iii). For $n = 2$, one can show that K-minimal forms are always of odd dimension ≥ 3. One can even construct a field F with some $\tau \in P_2F$ such that to each odd integer $m \geq 3$ there exists a K-minimal form of dimension m, cf. [HVG].

Not much is known about K-minimal forms in the case $K = F(\tau)$ with $\tau \in P_nF$, $n \geq 3$. The following is known.

Theorem 7.5 ([H 3, Theorem 3], [H 2, Corollary 4.2].) *Let $\tau \in P_nF$ be anisotropic and $K = F(\tau)$.*

(i) *The K-minimal forms of dimension $\leq 2^{n-1}+1$ are exactly the Pfister neighbors of τ of dimension $2^{n-1}+1$.*

(ii) *If $n \leq 3$ then the K-minimal forms of dimension $\leq 2^{n-1}+2$ are exactly the Pfister neighbors of τ of dimension $2^{n-1}+1$. In particular, if $\varphi \in WF$ is anisotropic with $\dim \varphi \leq 2^{n-1}+2$, then φ_K is isotropic iff φ contains a Pfister neighbor of τ.*

In view of this result, Lemma 7.1 and Theorem 7.2, we get the following.

Proposition 7.6 *Let $n \geq 3$ and $1 \leq m \leq n-2$. Let $\varphi \in P_{n,m}F$ be defined by (σ, π) and let $\tau \in P_nF$ such that $\tau_{F(\pi)} \simeq \sigma_{F(\pi)}$ and $\ln(\sigma, \tau) \leq n-2$. Then each $F(\tau)$-minimal form χ contained in φ has dimension $2^{n-1}+2 \leq \dim \chi \leq 2^n - 2^{m-1}+1$. Moreover, if $(n,m) = (3,1)$ then $7 \leq \dim \chi \leq 8$.*

Proof. By Lemma 7.1, $i_W(\varphi_{F(\tau)}) = 2^{m-1}$. Thus, any subform of φ of dimension $2^n - 2^{m-1}+1$ becomes isotropic over $F(\tau)$. Hence, if $\chi \subset \varphi$ is $F(\tau)$-minimal we must necessarily have $\dim \chi \leq 2^n - 2^{m-1}+1$. We know by Theorem 7.2 that φ does not contain any Pfister neighbor of τ. Therefore, by Theorem 7.5(i), we must have $\dim \chi \geq 2^{n-1}+2$, and if $n = 3$ then Theorem 7.5(ii) even implies that $\dim \chi \geq 7$. □

In fact, Izhboldin [I] showed that with φ, σ, τ as in Proposition 7.6, if $m = 1$ and if $\ln(\sigma, \tau) = 1$ then φ itself is $F(\tau)$-minimal. This leads us to conjecture the following.

Conjecture 7.7 *Let $n \geq 3$ and $1 \leq m \leq n-2$. Let $\varphi \in P_{n,m}F$ be defined by (σ, π) and let $\tau \in P_nF$ such that $\tau_{F(\pi)} \simeq \sigma_{F(\pi)}$ and $\ln(\sigma, \tau) = m$. Let $\chi \subset \varphi$. Then χ is $F(\tau)$-minimal iff $\dim \chi = 2^n - 2^{m-1}+1$.*

Note that $\tau_{F(\pi)} \simeq \sigma_{F(\pi)}$ implies that $\ln(\sigma, \tau) \geq m$, cf. Proposition 5.8. In our conjecture, we require that the linkage of σ and τ is at the lower end, i.e., $\ln(\sigma, \tau) = m$. This will be needed in the proof of the conjecture in the case $m = 1$ and it is for this reason that we imposed this condition in the conjecture.

Proposition 7.8 (Izhboldin [I].) *Let $n \geq 3$. Let $\varphi \in P_{n,1}F$ be defined by (σ, π) and let $\tau \in P_nF$ such that $\tau_{F(\pi)} \simeq \sigma_{F(\pi)}$ and $\ln(\sigma, \tau) = 1$. Then φ is $F(\tau)$-minimal.*

Proof. $\varphi_{F(\tau)}$ is isotropic by Proposition 5.1. To prove that φ is $F(\tau)$-minimal it suffices to show that if $\eta \subset \varphi$ and $\dim \eta = 2^n - 1$, then $\eta_{F(\tau)}$ stays anisotropic.

By Lemma 6.10, there exists $\beta \in P_2F$ such that for $K = F(\beta)$ we have that σ_K is anisotropic and $\eta_K \subset \sigma_K$. Suppose $\eta_{F(\tau)}$ is isotropic. Then $\eta_{K(\tau)}$ is isotropic and hence, $\sigma_{K(\tau)}$ is isotropic and therefore hyperbolic. This implies that τ_K is similar and thus isometric to σ_K. Let $\psi \simeq (\sigma \perp -\tau)_{\mathrm{an}}$. Clearly, $\psi \in W(K/F)$ and hence there exists $\gamma \in WF$ such that $\psi \simeq \beta \otimes \gamma$. Now $\ln(\sigma,\tau) = 1$ and thus $\dim\psi = 2^{n+1} - 4$. Hence, $\dim\gamma = 2^{n-1} - 1$ is odd and one readily concludes that $\psi \equiv \beta \otimes \gamma \equiv \beta \not\equiv 0 \pmod{I^3F}$ as $\beta \in P_2F$ is anisotropic (if β were isotropic, the anisotropic form ψ would stay anisotropic over $K = F(\beta)$). But $\psi = \sigma - \tau \in I^nF$ with $n \geq 3$, a contradiction. □

REMARK 7.9 The reason why this proof works so smoothly in the case $m = 1$ is that the field K from Lemma 6.10 is of a very nice form which just fits the situation. One would hope that with the field K from Lemma 6.10 one could give a similar proof also for $m > 1$. Consider the situation in Conjecture 7.7. To show that the conjecture is true it suffices to show that if $\eta \subset \varphi$ with $\dim\eta = 2^n - 2^{m-1}$ then $\eta_{F(\tau)}$ is anisotropic. One might want to proceed as in the proof above. Let K be as in Lemma 6.10 such that $\eta_K \subset \sigma_K$ and φ_K, σ_K, π_K stay anisotropic. The problem is to show that $\tau_K \not\simeq \sigma_K$. This worked for $m = 1$ because one can choose $K = F(\beta)$ for some $\beta \in P_2F$. If $m > 1$ then our construction in the proof of Lemma 6.10 generally leads to a field K for which it is not so clear why $\tau_K \not\simeq \sigma_K$ should hold.

8 CONSTRUCTIONS OF TWISTED PFISTER FORMS

In this section we explicitly construct examples of $\varphi \in P_{n,m}F$ defined by (σ, π), and $\tau \in P_nF$ such that $\ln(\sigma,\tau) = k$ for some $m \leq k \leq n-2$ such that $\sigma_{F(\pi)} \simeq \tau_{F(\pi)}$. By Proposition 5.8 we then have $\ln(\tau,\pi) = m-1$ and thus, we get a form $\psi \in P_{n,m}F$ defined by (τ,π) simply by putting $\psi \simeq (\tau \perp -\pi)_{\mathrm{an}}$. By Proposition 6.16 this shows the existence of $\varphi, \psi \in P_{n,m}F$ such that $\varphi \sim \psi$ but φ and ψ are neither neighbors nor conjugate, and by Theorem 7.2 it also shows the existence of $\varphi \in P_{n,m}F$ and $\tau \in P_nF$ such that $(\varphi_{F(\tau)})_{\mathrm{an}}$ is not defined over F. Note that by the symmetry of the situation we also have that $(\psi_{F(\sigma)})_{\mathrm{an}}$ is not defined over F. Hence, $F(\tau)/F$ and $F(\sigma)/F$ are both non-excellent field extensions.

In the first example, we will achieve this over a purely transcendental extension of the rationals $\mathbb{Q}$, and in the second example we will actually generalize Izhboldin's approach in [I].

EXAMPLE 8.1 Let $1 \leq m \leq k \leq n-2$. To simplify notations, let $\ell = n-2-k$ so that $k+\ell+2 = n$. Let $F = \mathbb{Q}(x_1,\cdots,x_k,y_0,\cdots,y_\ell)$ be the rational function field in the $k+\ell+1 = n-1$ variables x_i and y_j over the rationals $\mathbb{Q}$. Let $p_0,\cdots,p_\ell$ be distinct prime numbers with $p_i \equiv 7 \pmod 8$. We now define Pfister forms $\sigma,\tau \in P_nF$ and $\pi \in P_mF$ as follows:

$$\begin{array}{rcl}\sigma & \simeq & \langle\langle 1,x_1,\cdots,x_k,y_0,\cdots,y_\ell\rangle\rangle\ ;\\ \tau & \simeq & \langle\langle 2,x_1,\cdots,x_k,p_0y_0,\cdots,p_\ell y_\ell\rangle\rangle\ ;\\ \pi & \simeq & \langle\langle x_1,\cdots,x_{m-1},-x_m\rangle\rangle\ .\end{array}$$

One easily sees that σ, τ, and π are anisotropic (for instance by passing to the iterated power series field in the variables x_i, y_j, and then repeatedly applying Springer's theorem [L 1, Ch. 6, Proposition 1.9], [S, Ch. 6, Corollary 2.6(i)]).

Claim 1: $\ln(\sigma,\tau)=k$.
Proof.

$$\sigma\perp-\tau\simeq\langle\langle x_1,\cdots,x_k\rangle\rangle\otimes\underbrace{(\langle\langle 1,y_0,\cdots,y_\ell\rangle\rangle\perp-\langle\langle 2,p_0y_0,\cdots,p_\ell y_\ell\rangle\rangle)}_{\gamma}\ .$$

For $\emptyset\neq I\subset\{0,\cdots,\ell\}$ we define $Y_I=\prod_{i\in I}y_i$ and $P_I=\prod_{i\in I}p_i$. We find that

$$\gamma\simeq\langle 1,1,-1,-2\rangle\perp\mathop{\perp}_{\emptyset\neq I\subset\{0,\cdots,\ell\}}Y_I\langle 1,1,-P_I,-2P_I\rangle\ .$$

By Springer's theorem, we get

$$i_W(\gamma)=i_W(\langle 1,1,-1,-2\rangle)+\sum_{\emptyset\neq I\subset\{0,\cdots,\ell\}}i_W(\langle 1,1,-P_I,-2P_I\rangle)\ ,$$

where the Witt indices of the forms on the right hand side is computed over $\mathbb{Q}$. Now $i_W(\langle 1,1,-1,-2\rangle)=1$ as $\langle 1,1,-1,-2\rangle\simeq\mathbb{H}\perp\langle 1,-2\rangle$ and $\langle 1,-2\rangle$ is anisotropic over $\mathbb{Q}$. By passing to the local field $\mathbb{Q}_{p_i}$ for some $i\in I$, we get for the Legendre symbols $\left(\frac{-2}{p_i}\right)\neq 1$ and $\left(\frac{-1}{p_i}\right)\neq 1$ as $p_i\equiv 7\pmod 8$. Hence, $\langle 1,1\rangle$ and $\langle 1,2\rangle$ are anisotropic over $\mathbb{Q}_{p_i}$ and thus also $\langle 1,1,-P_I,-2P_I\rangle$ because p_i divides P_I exactly to the first power (note that all the p_j's in the product P_I are distinct!). Hence, $i_W(\langle 1,1,-P_I,-2P_I\rangle)=0$ and we have $i_W(\gamma)=1$. Again by Springer's theorem, we readily conclude that

$$i_W(\sigma\perp-\tau)=i_W(\langle\langle x_1,\cdots,x_k\rangle\rangle\otimes\gamma)=2^k i_W(\gamma)=2^k\ .$$

Hence, $\ln(\sigma,\tau)=k$.

Claim 2: $\ln(\sigma,\pi)=\ln(\tau,\pi)=m-1$. *In particular,* $\varphi\simeq(\sigma\perp-\pi)_{\mathrm{an}}$, $\psi\simeq(\tau\perp-\pi)_{\mathrm{an}}\in P_{n,m}F$.
Proof. This can be shown in a similar way as before.

$$\sigma\perp-\pi\simeq\langle\langle x_1,\cdots,x_{m-1}\rangle\rangle\otimes(\langle\langle 1,x_m,\cdots,x_k,y_0,\cdots,y_\ell\rangle\rangle\perp-\langle\langle -x_m\rangle\rangle)\ ,$$

and by Springer's theorem we obtain that

$$\langle\langle 1,x_m,\cdots,x_k,y_0,\cdots,y_\ell\rangle\rangle\perp-\langle\langle -x_m\rangle\rangle\simeq\mathbb{H}\perp\langle\langle 1,x_m,\cdots,x_k,y_0,\cdots,y_\ell\rangle\rangle'\perp\langle x_m\rangle$$

has Witt index 1 as $\langle\langle 1,x_m,\cdots,x_k,y_0,\cdots,y_\ell\rangle\rangle'\perp\langle x_m\rangle$ is anisotropic (here, ρ' denotes the pure part of a Pfister form ρ), and that therefore

$$i_W(\sigma\perp-\pi)=\dim\langle\langle x_1,\cdots,x_{m-1}\rangle\rangle=2^{m-1}$$

which in turn implies that $\ln(\sigma,\pi)=m-1$. A similar argument shows that $\ln(\tau,\pi)=m-1$ and we omit the details. It is now obvious that $\varphi\simeq(\sigma\perp-\pi)_{\mathrm{an}}$ and $\psi\simeq(\tau\perp-\pi)_{\mathrm{an}}$ have dimension 2^n and are in $P_{n,m}F$.

Claim 3: $\sigma_{F(\pi)}\simeq\tau_{F(\pi)}$.
Proof. Let $K=F(\pi)$. We have

$$0=\pi_K=\langle\langle x_1,\cdots,x_{m-1}\rangle\rangle_K\perp-x_m\langle\langle x_1,\cdots,x_{m-1}\rangle\rangle_K$$

and therefore $\langle\langle x_1,\cdots,x_{m-1}\rangle\rangle_K \simeq x_m\langle\langle x_1,\cdots,x_{m-1}\rangle\rangle_K$. Hence,

$$\langle\langle x_1,\cdots,x_{m-1},x_m\rangle\rangle_K \simeq \langle\langle x_1,\cdots,x_{m-1},1\rangle\rangle_K \ .$$

Note also that $\langle\langle 1,2\rangle\rangle \simeq \langle\langle 1,1\rangle\rangle$ and that $\langle\langle 1,1\rangle\rangle \simeq a\langle\langle 1,1\rangle\rangle$ for any positive $a\in\dot{\mathbb{Q}}$. In particular, $\langle\langle 1,1,y_i\rangle\rangle \simeq \langle\langle 1,1,p_iy_i\rangle\rangle$. All this together yields

$$\begin{array}{rcl}
\sigma_K & \simeq & \langle\langle 1,x_1,\cdots,x_{m-1},x_m,x_{m+1},\cdots,x_k,y_0,\cdots,y_\ell\rangle\rangle_K\\
& \simeq & \langle\langle 1,x_1,\cdots,x_{m-1},1,x_{m+1},\cdots,x_k,y_0,\cdots,y_\ell\rangle\rangle_K\\
& \simeq & \langle\langle 1,x_1,\cdots,x_{m-1},1,x_{m+1},\cdots,x_k,p_0y_0,\cdots,p_\ell y_\ell\rangle\rangle_K\\
& \simeq & \langle\langle 2,x_1,\cdots,x_{m-1},1,x_{m+1},\cdots,x_k,p_0y_0,\cdots,p_\ell y_\ell\rangle\rangle_K\\
& \simeq & \langle\langle 2,x_1,\cdots,x_{m-1},x_m,x_{m+1},\cdots,x_k,p_0y_0,\cdots,p_\ell y_\ell\rangle\rangle_K\\
& \simeq & \tau_K\ . \ \square
\end{array}$$

This completes our first example. Similar examples can be constructed also if one replaces $\mathbb{Q}$ by any global field $\mathbb{K}$ of characteristic $\neq 2$ and with the iterated power series field $F=\mathbb{K}((x_1))\cdots((x_k))((y_0))\cdots((y_\ell))$. For the u-invariant of such a field we get $u(F)=2^{k+\ell}u(\mathbb{K})=4\cdot 2^{n-1}=2^{n+1}$. Thus, we can construct examples of (non-formally real) fields F with $u(F)=2^{n+1}$ and $\tau\in P_nF$ such that $F(\tau)/F$ is not excellent. If F is non-formally real and $u(F)<2^n+2^{n-1}$ then examples of the type we constructed above cannot exist. For in order for examples of this type to exist one needs two n-fold Pfister forms σ, τ such that $\ln(\sigma,\tau)\leq n-2$ or $\dim(\sigma\perp-\tau)_{\mathrm{an}}\geq 2^n+2^{n-1}$. So the question is: Are there (non-formally real) fields F with $u(F)<2^n+2^{n-1}$ such that there exists $\tau\in P_nF$ with $F(\tau)/F$ not excellent? In [H 5] it was shown that if F is linked (which implies that $u(F)\in\{0,1,2,4,8\}$) then $F(\tau)/F$ is excellent for all Pfister forms τ over F (here, F may be formally real or non-formally real). Among the many other results in [H 5] let us only mention that if τ is a Pfister form over a field F and if the Hasse number of F, $\tilde{u}(F)$, is ≤ 6 or if $\dim\tau\geq 2\tilde{u}(F)$, then $F(\tau)/F$ is excellent. This is of course mainly of interest in the case where F is formally real. For if F is non-formally real then there are no anisotropic forms of dimension $>\tilde{u}(F)$.

COROLLARY 8.2 *To each $n\geq 3$ there exists a field F such that there are anisotropic n-fold Pfister forms ρ, σ over F with $F(\rho)/F$ excellent and $F(\sigma)/F$ not excellent.*

Proof. We only show this for $n=3$ to keep the notations simple. Let $F=\mathbb{Q}((x))((y))$. The previous example shows that for $\sigma\simeq\langle\langle 1,x,y\rangle\rangle$ we have that $F(\sigma)/F$ is not excellent. Let $\rho\simeq\langle\langle 1,1,1\rangle\rangle$. Since ρ is defined over $\mathbb{Q}$, it is not hard to see that the field $E=F(\rho)=\mathbb{Q}((x))((y))(\rho)$ is contained in $L=\mathbb{Q}(\rho)((x))((y))$. Let ψ be an anisotropic form over F. By Springer's theorem, we can write $\psi\simeq\psi_0\perp x\psi_1\perp y\psi_2\perp xy\psi_3$ where the ψ_i are forms over $\mathbb{Q}$ which are uniquely determined up to isometry over $\mathbb{Q}$. Let $K=\mathbb{Q}(\rho)\subset E$. It is known that function fields of Pfister forms over global fields are always excellent (cf. [ELW 2], [H 5], see also the remarks preceding this corollary). Hence, there are forms μ_i defined over $\mathbb{Q}$ such that $(\mu_i)_K\simeq((\psi_i)_K)_{\mathrm{an}}$. In WE we obviously have

$$(\psi_0)_E\perp x(\psi_1)_E\perp y(\psi_2)_E\perp xy(\psi_3)_E=(\mu_0)_E\perp x(\mu_1)_E\perp y(\mu_2)_E\perp xy(\mu_3)_E.$$

The right hand side is defined over F by $\mu_0\perp x\mu_1\perp y\mu_2\perp xy\mu_3$. To show excellence, it remains to show that $(\mu_0)_E\perp x(\mu_1)_E\perp y(\mu_2)_E\perp xy(\mu_3)_E$ is anisotropic. Indeed, this form is anisotropic over the bigger field L. This is because $L = K((x))((y))$, the $(\mu_i)_K$ are anisotropic and by Springer's theorem we have that $(\mu_0)_L \perp x(\mu_1)_L \perp y(\mu_2)_L \perp xy(\mu_3)_L$ is anisotropic. Thus, we have shown that $(\psi_{F(\rho)})_{\mathrm{an}}$ is defined over F by $\mu_0 \perp x\mu_1 \perp y\mu_2 \perp xy\mu_3$, which in turn proves the excellence of $F(\rho)/F$. □

The next example generalizes Izhboldin's construction in [I] which was carried out only in the case $m = \ln(\sigma,\tau) = 1$. However, we will show that his arguments can be applied, after some minor modifications, to the more general situation where m can be any positive integer $\leq n-2$ and $\ln(\sigma,\tau) = k$ can be any integer with $m \leq k \leq n-2$.

Example 8.3 Let $n \geq 3$ and let $1 \leq m \leq k \leq n-2$. Let $\tau \in P_nF$ be anisotropic. We will construct a unirational field extension E/F such that there exist anisotropic forms $\pi \in P_mE$ and $\sigma \in P_nE$ with $\ln(\sigma,\pi) = \ln(\tau_E,\pi) = m-1$ and $\ln(\sigma,\tau_E) = k$ (note that τ will stay anisotropic over any unirational field extension), and furthermore $\sigma_{E(\pi)} \simeq \tau_{E(\pi)}$. We then get a form $\varphi \simeq (\sigma \perp -\pi)_{\mathrm{an}} \in P_{n,m}E$, and by Theorem 7.2 we have that $(\varphi_{E(\tau)})_{\mathrm{an}}$ is not defined over E. In particular, $E(\tau)/E$ is not excellent. Our construction will involve various field extensions of F. Their relations among each other are shown in a diagram below.

As for the construction of E, let this time $\ell = n-k \geq 2$ and write

$$\tau \simeq \langle\langle a_1,\cdots,a_k,b_1,\cdots,b_\ell\rangle\rangle$$

for suitable $a_i,b_j \in \dot{F}$. Let $F_0 = F(y_1,\cdots,y_\ell)$ and $F_1 = F_0(x) = F(x,y_1,\cdots,y_\ell)$ be rational function fields in the variables $x,y_1,\cdots,y_\ell$ over F. Let

$$\begin{aligned} \sigma &\simeq \langle\langle a_1,\cdots,a_k,y_1,\cdots,y_\ell\rangle\rangle \in P_nF_0 \\ \pi &\simeq \langle\langle x,a_2,\cdots,a_m\rangle\rangle \in P_mF_1 \end{aligned}$$

After passing to the iterated power series field in the variables $x,y_1,\cdots,y_\ell$ and by repeatedly applying Springer's theorem, one readily checks that τ_{F_1}, σ_{F_1}, π_{F_1} are anisotropic and that $\ln(\sigma_{F_1},\tau_{F_1}) = k$ and $\ln(\sigma_{F_1},\pi_{F_1}) = \ln(\tau_{F_1},\pi_{F_1}) = m-1$. We leave the details to the reader. The aim is to construct E/F_1 such that $\sigma_{E(\pi)} \simeq \tau_{E(\pi)}$, such that this form and π_E stay anisotropic and such that $\ln(\sigma_E,\tau_E) = k$. Note that we will have $m \geq \ln(\sigma_E,\pi_E) \geq \ln(\sigma_{F_1},\pi_{F_1}) = m-1$. Now $\ln(\sigma_E,\pi_E) = m$ implies that π_E divides σ_E and thus $\sigma_{E(\pi)} = 0$, a contradiction to its anisotropy. Hence, we will still have $\ln(\sigma_E,\pi_E) = m-1$ and similarly $\ln(\tau_E,\pi_E) = m-1$.

To get this field E, we first define the following forms over F_1 which again are easily seen to be anisotropic:

$$\begin{aligned} \tilde{\tau} &\simeq \langle\langle x,a_2,\cdots,a_k,b_1,\cdots,b_\ell\rangle\rangle \\ \tilde{\sigma} &\simeq \langle\langle x,a_2,\cdots,a_k,y_1,\cdots,y_\ell\rangle\rangle\ . \end{aligned}$$

Let E be the generic splitting field of the anisotropic form defined by $(\sigma-\tau)-(\tilde{\sigma}-\tilde{\tau})$ in WF_1. Then, in WE, $(\sigma-\tau)_E - (\tilde{\sigma}-\tilde{\tau})_E = 0$ or $(\sigma-\tau)_E = (\tilde{\sigma}-\tilde{\tau})_E$. As π divides both $\tilde{\sigma}$ and $\tilde{\tau}$, we get that $\tilde{\sigma}_{E(\pi)} = \tilde{\tau}_{E(\pi)} = 0$, hence, $(\sigma-\tau)_{E(\pi)} = 0$, i.e., $\sigma_{E(\pi)} \simeq \tau_{E(\pi)}$.

We first show that $\sigma_{E(\pi)} \simeq \tau_{E(\pi)}$ is anisotropic. Let $F_2 = F_1(\sqrt{-x})$. Then F_2/F_0 is purely transcendental and thus σ_{F_2} and τ_{F_2} stay anisotropic and we still have $\ln(\sigma_{F_2},\tau_{F_2}) = \ln(\sigma_{F_0},\tau_{F_0}) = k$. Furthermore, $\langle\langle x\rangle\rangle_{F_2} = \langle\langle -1\rangle\rangle_{F_2} = 0$ and hence

$\pi_{F_2} = \tilde{\sigma}_{F_2} = \tilde{\tau}_{F_2} = 0$ in WF_2. Let K be the generic splitting field over F_2 of $(\sigma \perp -\tau)_{F_2}$. Clearly, $(\sigma - \tau)_K = 0$ in WK, i.e., $\sigma_K \simeq \tau_K$. We claim that $\sigma_K \simeq \tau_K$ is anisotropic. Now $\ln(\sigma_{F_2}, \tau_{F_2}) = k < n$ and thus $\sigma_{F_2} \not\simeq \tau_{F_2}$, i.e., $(\sigma \perp -\tau)_{F_2} \neq 0$. Clearly, $\deg(\sigma \perp -\tau)_{F_2} \geq n$. Suppose that $\sigma_K \simeq \tau_K = 0$. Then by [AK, Satz 20] it follows that $\deg(\sigma \perp -\tau)_{F_2} = n$ and $(\sigma \perp -\tau)_{F_2} \equiv \sigma_{F_2} \pmod{J_{n+1}F_2}$. Hence, $-\tau_{F_2} \equiv 0 \pmod{J_{n+1}F_2}$ and the Arason-Pfister Hauptsatz implies that $\tau_{F_2} = 0$, a contradiction.

Obviously, $\tilde{\sigma}_{K(\pi)} = \tilde{\tau}_{K(\pi)} = 0$. Thus, $(\sigma - \tau)_{K(\pi)} - (\tilde{\sigma} - \tilde{\tau})_{K(\pi)} = 0$. Since E/F_1 is the generic splitting field of $(\sigma - \tau) - (\tilde{\sigma} - \tilde{\tau}) \in WF_1$, we have by Proposition 2.2(iii) that $E\cdot K(\pi)/K(\pi)$ is purely transcendental. But $K(\pi)/K$ is purely transcendental as well because $\pi_K = 0$. Hence, $E\cdot K(\pi)/K$ is purely transcendental and therefore $\sigma_{E\cdot K(\pi)} \simeq \tau_{E\cdot K(\pi)}$ is anisotropic because $\sigma_K \simeq \tau_K$ is anisotropic. Since $E(\pi) \subset E\cdot K(\pi)$ we have that $\sigma_{E(\pi)} \simeq \tau_{E(\pi)}$ is anisotropic.

Let now $F_3 = F_1(\sqrt{a_1 X})$. Again, we clearly have that F_3/F_0 is purely transcendental. Furthermore, $\sigma_{F_3} \simeq \tilde{\sigma}_{F_3}$ and $\tau_{F_3} \simeq \tilde{\tau}_{F_3}$ as $a_1 = X$ in $\dot{F}_3/\dot{F}_3^2$. Hence, $(\sigma - \tau)_{F_3} - (\tilde{\sigma} - \tilde{\tau})_{F_3} = 0$ in WF_3 and we have that $E\cdot F_3/F_3$ is purely transcendental by the same reason as before. Hence, $E\cdot F_3/F_0$ is purely transcendental as well and thus, since $F_0 \subset E \subset E\cdot F_3$, we conclude that E/F_0 is unirational. Therefore, $\ln(\sigma_E, \tau_E) = \ln(\sigma_{F_0}, \tau_{F_0}) = k$ as desired. Obviously, E/F is also unirational as F_0/F is purely transcendental. This completes Izhboldin's construction. $\square$

$$\begin{tikzcd}
 & E\cdot K(\pi) \arrow[dd, dash] \arrow[dr, dash] & \\
 & & K(\pi) \arrow[dd, dash] \\
 & E(\pi) \arrow[dd, dash] & \\
E\cdot F_3 \arrow[dr, dash] \arrow[dd, dash] & & K \arrow[dd, dash] \\
 & E \arrow[dd, dash] & \\
F_3 = F_1(\sqrt{a_1 x}) \arrow[dr, dash] & & F_2 = F_1(\sqrt{-x}) \arrow[dl, dash] \\
 & F_1 = F_0(x) \arrow[d, dash] & \\
 & F_0 = F(y_1, \cdots, y_\ell) \arrow[d, dash] & \\
 & F &
\end{tikzcd}$$

COROLLARY 8.4 (Izhboldin [I].) *Let $\tau \in P_nF$ be anisotropic, $n \geq 3$. Then there exists a unirational field extension E/F such that $E(\tau)/E$ is not excellent.*

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