# On the Dimension of a Composition Algebra

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Abstract. The possible dimensions of a composition algebra are 1, 2, 4, or 8. We give a tensor categorical argument.

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## I. Introduction

Let $C$ be a composition algebra over a field of characteristic different from 2, let $V$ be its pure subspace (consisting of the vectors orthogonal to 1) and let $d = \dim V$. We show that the following relation holds in the groundfield:

$$d(d-1)(d-3)(d-7) = 0.$$

This is not very surprising since the only possibilities for $C$ are either the ground field, a separable quadratic extension, a quaternion algebra, or an octonion algebra. The proof of the relation given in this note seems to be different from former approaches (cf. [1], [2]). It works on a tensor categorical level. In characteristic 0 one recovers the determination of the possible dimensions of a composition algebra.

Our starting problem was to understand composition algebras from a tensor categorical point of view. Instead of composition algebras we looked at the equivalent notion of vector product algebras. These algebras can be obtained be rewriting the axioms of a composition algebra in terms of the pure vectors. Vector product algebras allow to use diagrammatic tensor calculus in a handy way. Using a graphical technique we found—just by playing around—a proof of the relation on $\dim V$. These notes contain alone the algebraic calculations which were extracted from the graph considerations. After these notes had been written, we noticed an identity in vector product algebras which perhaps makes the result less mysterious. So there is more to

say about the topic than explained in this text. We hope to come back to this at another place. Anyway, the text is completely self-contained and contains an argument on the possible dimensions.
Throughout the paper we assume char $\neq 2$.

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## II. Composition Algebras and Vector Products

We first recall a definition.

(1) Composition algebras.
A composition algebra consists of a vector space $C$ together with

(1.1) a nondegenerate symmetric bilinear form $\langle\,,\rangle$ on $C$,

(1.2) a linear map $C \otimes C \to C, \quad x \otimes y \mapsto x \cdot y$,

(1.3) an element $0 \neq e \in C$,

such that (with $\mathrm{N}(x) = \langle x, x\rangle$)

(1.4) $e \cdot x = x \cdot e = x$,

(1.5) $\mathrm{N}(x \cdot y) = \mathrm{N}(x)\,\mathrm{N}(y)$.

For our purpose we have to consider the following algebraic structure.

(2) Vector product algebras.
A vector product algebra consists of a vector space $V$ together with

(2.1) a nondegenerate symmetric bilinear form $\langle\,,\rangle$ on $V$,

(2.2) a linear map $V \otimes V \to V, \quad x \otimes y \mapsto x \times y$,

such that

(2.3) $\langle x \times y, z\rangle$ is alternating in $x$, $y$, $z$,

(2.4) $(x \times y) \times x = \langle x, x\rangle y - \langle x, y\rangle x$.

The vector product $\times$ is anti-commutative, since (2.3) implies $x \times x = 0$. Therefore $x \times (y \times x) = (x \times y) \times x$. Hence the choice of the arrangement of the brackets in the lefthand side of (2.4) is not essential.

B. Eckmann has considered (continous) vector products in [B. Eckmann, Stetige Lösungen linearer Gleichungssysteme, Comment. Math. Helv. **15** (1942/43), 318–339], see also [B. Eckmann, Continous solutions of linear equations — An old problem, its history and its solution, Expo. Math. **9** (1991), 351–365]. He used the axioms

$$\langle x \times y, x\rangle = \langle x \times y, y\rangle = 0, \qquad N(x \times y) = \det \begin{vmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle y,x\rangle & \langle y,y\rangle \end{vmatrix}.$$

They are perhaps more close to the intuitive idea of a vector product. Under presence of (2.1)–(2.2) they are easily seen to be equivalent to (2.3)–(2.4).

Vector product algebras and composition algebras are equivalent notions.

Namely, given a composition algebra $C$, let $V = \langle e\rangle^{\perp}$ and put

$$\text{(i)} \qquad x \times y = \frac{1}{2}(x \cdot y - y \cdot x).$$

Conversely, given a vector product algebra $V$, put $C = \langle e\rangle \perp V$ and define the product on $C$ by

$$\text{(ii)} \qquad (ae + x) \cdot (be + y) = \big(ab - \langle x, y\rangle\big)e + ay + bx + x \times y.$$

The rewriting formulas (i) and (ii) identify composition algebras and vector product algebras on a "tensor categorical" level. This means that the composition rule (1.5) gives after polarization and decomposition with respect to $C = \langle e\rangle \perp V$ the same tensor equations as (2.3) and the polarization of (2.4).

This equivalence between composition algebras and vector product algebras seems to provide a convenient way to comprise some wellknown rules in composition algebras.

For the associator in $C$ one finds

$$(x \cdot y) \cdot z - x \cdot (y \cdot z) = 2\big((x \times y) \times z - \langle x, z\rangle y + \langle y, z\rangle x\big)$$

for $x,\ y,\ z \in V$.

## III. A Relation for the Contraction of $\langle\ ,\ \rangle$

Let $V$ be a finite-dimensional vector product algebra and let $(e_i)_i$ be an orthonormal basis of $V$ over some algebraic closure. Put

$$d = \sum_i \langle e_i, e_i\rangle.$$

(3) Proposition. *One has the relation*

$$d(d-1)(d-3)(d-7) = 0.$$

In the following we will tacitly apply (2.3) in the formulation

$$\text{(2.3a)} \qquad \langle x \times y, z\rangle = \langle x, y \times z\rangle,$$

$$\text{(2.3b)} \qquad y \times x = -\, x \times y.$$

The relation (2.4) will be used also in the following forms which are obtained by polarizing and from (2.3):

$$(x \times y) \times z + x \times (y \times z) = 2\langle x, z\rangle y - \langle x, y\rangle z - \langle z, y\rangle x, \tag{2.4a}$$

$$\begin{gathered}\langle x \times y, z \times t\rangle + \langle y \times z, t \times x\rangle = \\ 2\langle x, z\rangle\langle y, t\rangle - \langle x, y\rangle\langle z, t\rangle - \langle y, z\rangle\langle t, x\rangle.\end{gathered} \tag{2.4b}$$

Other relations to be used are

$$\sum_i e_i \times (v \times e_i) = \sum_i \langle e_i, e_i\rangle v - \sum_i \langle e_i, v\rangle e_i = dv - v = (d-1)v \tag{3.1}$$

and

$$\sum_{i,j} \langle e_i \times e_j, e_i \times e_j\rangle = \sum_{i,j} \big\langle e_i, e_j \times (e_i \times e_j)\big\rangle = (d-1)\sum_i \langle e_i, e_i\rangle = d(d-1). \tag{3.2}$$

To warm up, we first consider vector product algebras which correspond to associative composition algebras.

(4) PROPOSITION. *Suppose that the following sharpening of* (2.4) *holds:*

$$(x \times y) \times z = \langle x, z\rangle y - \langle y, z\rangle x. \tag{4.1}$$

*Then*

$$d(d-1)(d-3) = 0.$$

*Proof.* Consider

$$A = \sum_{i,j,k} \big\langle e_i \times (e_k \times e_i), e_j \times (e_k \times e_j)\big\rangle.$$

By (3.1) we have

$$A = \sum_k (d-1)^2 \langle e_k, e_k\rangle = d(d-1)^2.$$

On the other hand, using (4.1) and (3.2) one finds

$$\begin{aligned} A &= \sum_{i,j,k} \big\langle \big(e_i \times (e_k \times e_i)\big) \times e_j, e_k \times e_j\big\rangle \\ &= \sum_{i,j,k} \big\langle \langle e_i, e_j\rangle e_k \times e_i - \langle e_k \times e_i, e_j\rangle e_i, e_k \times e_j\big\rangle \\ &= \sum_{i,k} \langle e_k \times e_i, e_k \times e_i\rangle - \sum_{i,j,k} \langle e_k \times e_i, e_j\rangle\langle e_i \times e_k, e_j\rangle \\ &= 2\sum_{i,k} \langle e_k \times e_i, e_k \times e_i\rangle = 2d(d-1). \end{aligned}$$

So

$$0 = A - A = d(d-1)(d-3). \qquad \square$$

Let us start with the proof of Proposition 3.

Put

$$h(u,v) = \sum_i (u \times e_i) \times (e_i \times v).$$

The following formula has been introduced by T. A. Springer.

$$h(u,v) = (d-4)\, u \times v. \tag{3.3}$$

To check it one uses (2.4a) with $x = u$, $y = e_i$ and $z = e_i \times v$ and finds

$$\begin{aligned} h(u,v) &= -\sum_i u \times \big(e_i \times (e_i \times v)\big) + 2\sum_i \langle u, e_i \times v\rangle e_i \\ &\qquad - \sum_i \langle u, e_i\rangle e_i \times v - \sum_i \langle e_i \times v, e_i\rangle u \\ &= (d-1) u \times v + 2\sum_i \langle v \times u, e_i\rangle e_i \\ &\qquad - u \times v - \sum_i \langle v, e_i \times e_i\rangle u \\ &= (d-1)u \times v - 2u \times v - u \times v - 0 = (d-4)u \times v. \end{aligned}$$

Formulas (3.3) and (3.2) make it easy to compute the sum

$$\begin{aligned} B &= \sum_{i,k} \big\langle h(e_i, e_k), h(e_k, e_i)\big\rangle \\ &= (d-4)^2 \sum_{i,k} \langle e_i \times e_k, e_k \times e_i\rangle = -d(d-1)(d-4)^2 \end{aligned}$$

We next compute $B$ in a different way. One has

$$B = \sum_{i,j,k,l} \big\langle (e_i \times e_j) \times (e_j \times e_k), (e_k \times e_l) \times (e_l \times e_i)\big\rangle.$$

Applying (2.4b) shows

$$B + B' = 2C - D - D',$$

where

$$\begin{aligned} B' &= \sum_{i,j,k,l} \big\langle (e_j \times e_k) \times (e_k \times e_l), (e_l \times e_i) \times (e_i \times e_j)\big\rangle, \\ C &= \sum_{i,j,k,l} \langle e_i \times e_j, e_k \times e_l\rangle \langle e_j \times e_k, e_l \times e_i\rangle, \\ D &= \sum_{i,j,k,l} \langle e_i \times e_j, e_j \times e_k\rangle \langle e_k \times e_l, e_l \times e_i\rangle, \\ D' &= \sum_{i,j,k,l} \langle e_j \times e_k, e_k \times e_l\rangle \langle e_l \times e_i, e_i \times e_j\rangle. \end{aligned}$$

By reindexing one finds $B = B'$ and $D = D'$. Therefore

$$B = C - D.$$

We compute $C$ and $D$:

$$\begin{aligned}
C &= \sum_{i,j,k,l} \big\langle e_i, e_j \times (e_k \times e_l)\big\rangle \big\langle (e_j \times e_k) \times e_l, e_i\big\rangle \\
&= \sum_{j,k,l} \big\langle e_j \times (e_k \times e_l), (e_j \times e_k) \times e_l\big\rangle \\
&= \sum_{j,k,l} \big\langle \big(e_j \times (e_k \times e_l)\big) \times (e_j \times e_k), e_l\big\rangle \\
&= -\sum_{k,l} \big\langle h(e_k \times e_l, e_k), e_l\big\rangle = -(d-4)\sum_{k,l} \big\langle (e_k \times e_l) \times e_k, e_l\big\rangle \\
&= -(d-1)(d-4)\sum_{l} \langle e_l, e_l\rangle = -d(d-1)(d-4), \\
D &= \sum_{i,j,k,l} \big\langle e_i, e_j \times (e_j \times e_k)\big\rangle \big\langle (e_k \times e_l) \times e_l, e_i\big\rangle \\
&= \sum_{j,k,l} \big\langle e_j \times (e_j \times e_k), (e_k \times e_l) \times e_l\big\rangle \\
&= \sum_{k} (d-1)(d-1)\langle e_k, e_k\rangle = d(d-1)^2.
\end{aligned}$$

Hence

$$B = -d(d-1)(d-4) - d(d-1)^2 = -d(d-1)(2d-5).$$

Finally

$$\begin{aligned}
0 = B - B &= -d(d-1)(2d-5) + d(d-1)(d-4)^2 \\
&= d(d-1)(d^2 - 10d + 21) = d(d-1)(d-3)(d-7). \qquad \square
\end{aligned}$$


## References

[1] Hurwitz, A., Über die Komposition der quadratischen Formen, *Math. Ann.* 88, 1-25.

[2] Jacobson, N., Basic Algebra I, W. H. Freeman and Co., San Francisco, 1974.



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