Calabi-Yau Threefolds of Quasi-Product Type

Keiji Oguiso

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Abstract. According to the numerical Iitaka dimension $\nu(X, D)$ and $c_2(X) \cdot D$, fibered Calabi-Yau threefolds $\Phi_{|D|} : X \to W$ $(\dim W > 0)$ are coarsely classified into six different classes. Among these six classes, there are two peculiar classes called of type II_0 and of type III_0 which are characterized respectively by $\nu(X, D) = 2$ and $c_2(X) \cdot D = 0$ and by $\nu(X, D) = 3$ and $c_2(X) \cdot D = 0$. Fibered Calabi-Yau threefolds of type III_0 are intensively studied by Shepherd-Barron, Wilson and the author and now there are a satisfactory structure theorem and the complete classification. The purpose of this paper is to guarantee a complete structure theorem of fibered Calabi-Yau threefolds of type II_0 to finish the classification of these two peculiar classes. In the course of proof, the log minimal model program for threefolds established by Shokurov and Kawamata will play an important role. We shall also introduce a notion of quasi-product threefolds and show their structure theorem. This is a generalization of the notion of hyperelliptic surfaces to threefolds and will have other applicability, too.

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Introduction

Let us start this introduction by recalling a global picture of fibered Calabi-Yau threefolds known at the present and then state the Main Theorem precisely.

Throughout this paper, by a Calabi-Yau threefold, we mean a normal projective complex threefold X with only $\mathbb{Q}$−factorial terminal singularities (so that isolated) and with $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$ and $\pi_1^{alg}(X) = \{1\}$. The last condition is equivalent to $\pi_1^{alg}(X - \mathrm{Sing}\, X) = \{1\}$, because the local fundamental group of three dimensional terminal Gorenstein singularities is trivial ([Kw3]). This also implies $h^1(\mathcal{O}_X) = 0$ ([O1]). We define

$$c_2(X) \cdot D := c_2(X') \cdot \nu^*(D)$$

for any resolution $\nu : X' \to X$ of $\mathrm{Sing}\,(X)$.

It is known by Miyaoka that $c_2(X) \cdot D$ is non-negative if D is nef ([Mi]).

A surjective morphism $\Phi : X \to W$ is called a fibered Calabi-Yau threefold if X is a Calabi-Yau threefold, W is a normal projective variety (of positive dimension) and Φ has connected fibers. Note that Φ is nothing but $\Phi_{|D|}$ if D is the pull back of (any) very ample divisor H on W.

Fibered Calabi-Yau threefolds $\Phi_{|D|} : X \to W$ are divided into six classes by the numerical invariants $\nu(X, D)$ and $c_2 \cdot D$:

Type I_0 : $\nu(X,D) = 1$ and $c_2 \cdot D = 0$; *Type* I_+ : $\nu(X,D) = 1$ and $c_2 \cdot D > 0$;
Type II_0 : $\nu(X,D) = 2$ and $c_2 \cdot D = 0$; *Type* II_+ : $\nu(X,D) = 2$ and $c_2 \cdot D > 0$;
Type III_0 : $\nu(X,D) = 3$ and $c_2 \cdot D = 0$; *Type* III_+ : $\nu(X,D) = 3$ and $c_2 \cdot D > 0$.

The following (more or less tautological) coarse classification is proved in [O1].

THEOREM 1 ([O1]). *Each class of fibered Calabi-Yau threefolds $\Phi(= \Phi_{|D|}) : X \to W$ defined above is characterized as follows.*

Type I_0*: General fibers are smooth Abelian surfaces and $W = \mathbb{P}^1$,*
Type I_+*: General fibers are smooth K3 surfaces and $W = \mathbb{P}^1$,*
Type II_0*: General fibers are smooth elliptic curves and W is a normal projective rational surface with only quotient singularities and with $K_W \equiv 0$,*
Type II_+*: General fibers are smooth elliptic curves and W is a normal projective rational surface with only quotient singularities and with $K_W + \Delta \equiv 0$ for some non-zero effective $\mathbb{Q}$-divisor Δ such that (W, Δ) is klt,*
Type III_0*: Φ is a birational morphism and W is a normal projective threefold with only canonical singularities and with $\mathcal{O}_W(K_W) \simeq \mathcal{O}_W$ and $c_2(W)(:= \Phi_* c_2(X)) = 0$ as a linear form on $Pic(W)$,*
Type III_+*: Φ is a birational morphism and W is a normal projective threefold with only canonical singularities and with $\mathcal{O}_W(K_W) \simeq \mathcal{O}_W$ and $c_2(W) \neq 0$.*

Moreover, if $\Phi : X \to W$ is a fibered Calabi-Yau threefold of type II_0 *and H is a general very ample divisor on W, then the induced elliptic surface $\Phi^{-1}(H) \to H$ has no singular fibers while $\Phi^{-1}(H) \to H$ has at least one singular fiber composed of rational curves if $\Phi : X \to W$ is of type* II_+.

Theorem 1 shows that fibered Calabi-Yau threefolds of type III_0 or of type II_0 have rather special nature.

The following two theorems give a complete picture of fibered Calabi-Yau threefolds of type III_0.

THEOREM 2 ([SW]). *Let $\Phi : X \to \overline{X}$ be a fibered Calabi-Yau threefold of type* III_0. *Then, there exist an Abelian threefold A and a finite Gorenstein automorphism group G of A such that*

(1) *$A^{[G]}$ is a non-empty finite set, and*
(2) *$\overline{X} = A/G$.*

THEOREM 3 ([O3]). *Two fiber spaces* $\Phi_3 : X_3 \to \overline{X_3}$ *and* $\Phi_7 : X_7 \to \overline{X_7}$ *defined in the following (1) and (2) are fibered Calabi-Yau threefolds of type* III_0.

(1) *Let* E_{ζ_3} *be the elliptic curve with period* $\zeta_3 := \exp(2\pi i/3)$. *Setting* $\overline{X_3} := E^3_{\zeta_3}/\langle \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3)\rangle$, *we define* $\Phi_3 : X_3 \to \overline{X_3}$ *to be a unique crepant (toric) resolution of* $\overline{X_3}$.

(2) *Let* A_7 *be the Jacobian threefold of the Klein quintic curve* $C := (x_0x_1^3 + x_1x_2^3 + x_2x_0^3 = 0) \subset \mathbb{P}^2_{[x_0:x_1:x_2]}$ *and* g_7 *the automorphism of* A_7 *induced by the automorphism of* C *given by* $[x_0 : x_1 : x_2] \mapsto [x_0^1 : x_1^2 : x_2^4]$. *Setting* $\overline{X_7} := A_7/\langle g_7\rangle$, *we define* $\Phi_7 : X_7 \to \overline{X_7}$ *to be a unique crepant (toric) resolution of* $\overline{X_7}$.

Conversely, any fibered Calabi-Yau threefold of type III_0 *is isomorphic to either* $\Phi_3 : X_3 \to \overline{X_3}$ *or* $\Phi_7 : X_7 \to \overline{X_7}$ *as fiber spaces.*

In particular, there are exactly two fibered Calabi-Yau threefolds of type III_0 *and both of them are smooth and rigid.*

Now it is interesting to study another peculiar class of fibered Calabi-Yau threefolds called of type II_0.

Base surfaces W of fibered Calabi-Yau threefolds $\Phi : X \to W$ of type II_0 are classified into two classes by the global canonical covering $\pi : T \to W$, for which we have either

(1) T is a smooth Abelian surface, or
(2) T is a (projective) K3 surface with only Du Val singularities.

In case (1) (resp. (2)), a fibered Calabi-Yau threefold $\Phi : X \to W$ of type II_0 is called of type II_0A (resp. of type II_0K).

The following theorem gives a complete classification of fibered Calabi-Yau threefolds of type II_0A.

THEOREM 4 ([O2]).

(1) *Let* $\Phi_3 : X_3 \to E^3_{\zeta_3}/\mathrm{diag}(\zeta_3, \zeta_3, \zeta_3)$ *be as in Theorem 3 and* $p : X_3 \to E^2_{\zeta_3}/\mathrm{diag}(\zeta_3, \zeta_3)$ *the natural map given by the composite of* Φ_3 *and the natural projection* $p_{12} : E^3_{\zeta_3}/\mathrm{diag}(\zeta_3, \zeta_3, \zeta_3) \to E^2_{\zeta_3}/\mathrm{diag}(\zeta_3, \zeta_3)$. *Then, any composite of flops* $f : X_3 \cdots \to X_3'$ *along curves in* $p^{-1}(\mathrm{Sing}(E^2_{\zeta_3}/\mathrm{diag}(\zeta_3, \zeta_3)))$ *gives a fibered Calabi-Yau threefolds* $p \circ f^{-1} : X_3' \to E^2_{\zeta_3}/\mathrm{diag}(\zeta_3, \zeta_3)$ *of type* II_0A. *In this case,* $E^2_{\zeta_3}$ *is nothing but the global canonical cover of the base surface* $E^2_{\zeta_3}/\mathrm{diag}(\zeta_3, \zeta_3)$.

(2) *Conversely, every fibered Calabi-Yau threefolds of type* II_0A *is obtained by the above process up to isomorphisms as fiber spaces. In particular, every fibered Calabi-Yau threefolds of type* II_0A *is smooth and rigid. Moreover, there are exactly 14 different fibered Calabi-Yau threefolds of type* II_0A *up to isomorphism as fiber spaces.*

The purpose of this paper is to show the following structure theorem of fibered Calabi-Yau threefolds of type II_0K. This theorem tells us how to construct all the fibered Calabi-Yau threefolds of type II_0K.

MAIN THEOREM. *Let us prepare*

(i) *a smooth elliptic curve E with a fixed origin 0,*
(ii) *a projective K3 surface S with only Du Val singularities and its minimal resolution $\mu : S' \to S$, and*
(iii) *two groups*
$G \in \{\{1\}, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_8, (\mathbb{Z}_2)^2, (\mathbb{Z}_3)^2, (\mathbb{Z}_4)^2, \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_6\}$,
and
$\langle g \rangle \simeq \mathbb{Z}_I \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6\}$,

such that $\tilde{G} := G \rtimes \langle g \rangle$ (semi-direct product) acts faithfully on both E and S (and then on S' and $E \times S'$) in such a way that

(iv) $G \ni a : E \times S' \to E \times S', (x, y) \mapsto (x + a_E, a_{S'}(y))$ *with* $a_E \in (E)_{\mathrm{ord}\,(a)}$ *and* $a_{S'}^* \omega_{S'} = \omega_{S'}$, *where $\omega_{S'}$ is a nowhere vanishing regular 2 form on S',*
(v) $g : E \times S' \to E \times S', (x, y) \mapsto (\zeta_I^{-1} x, g_{S'}(y))$ *with* $g_{S'}^* \omega_{S'} = \zeta_I \omega_{S'}$, *and*
(vi) $(S')^{[\tilde{G}]} \subset \mathrm{Exc}(\mu)$ *except for finitely many points in $(S')^{[\tilde{G}]}$, that is, $(S)^{[\tilde{G}]}$ is a finite set.*

Note that $\tilde{G}$ is a finite Gorenstein automorphism group of $E \times S'$. Let

$$\nu : Y(E, S, \tilde{G}) \to (E \times S')/\tilde{G}$$

be a crepant resolution (whose existence is now guaranteed by Roan [Ro]) and

$$p : Y(E, S, \tilde{G}) \to S/\tilde{G}$$

the natural projection given by the composite of $\nu : Y(E, S, \tilde{G}) \to (E \times S')/\tilde{G}$, $p_2 : (E \times S')/\tilde{G} \to S'/\tilde{G}$, and $\mu/\tilde{G} : S'/\tilde{G} \to S/\tilde{G}$.

Then,

(1) *any composite of flop $f : Y(E, S, \tilde{G}) \cdots \to Y'$ along curves in $p^{-1}(\mathrm{Sing}\,(S/\tilde{G}))$ gives a fibered Calabi-Yau threefold $p \circ f^{-1} : Y' \to S/\tilde{G}$ of type $\mathrm{II}_0 K$ provided that $\pi_1^{alg}(Y) = \{1\}$. In this case S/G gives the global canonical cover of the base space $S/\tilde{G}$.*
(2) *Conversely, every fibered Calabi-Yau threefold of type $\mathrm{II}_0 K$ is obtained by the above process for some triplet $(E, S, \tilde{G})$ satisfying the conditions (i)-(vi) up to isomorphisms as fiber spaces. In particular, every fibered Calabi-Yau threefold of type $\mathrm{II}_0 K$ is smooth.*

This together with Theorems 2, 3 and 4 will complete the structure theorem of the two peculiar classes of fibered Calabi-Yau threefolds called of types II_0 and III_0.

REMARK. *Investigating the actions of G and $\langle g \rangle$ on E, we easily see that*

(1) *$\tilde{G}$ is uniquely determined by G and $\langle g \rangle$ as an abstract group, and*
(2) *among 52 possibilities of $(G, \langle g \rangle)$ in the Main Theorem, the following 18 combinations do not occur:*
$(\mathbb{Z}_4, \mathbb{Z}_3)$, $(\mathbb{Z}_5, \mathbb{Z}_3)$, $(\mathbb{Z}_6, \mathbb{Z}_3)$, $(\mathbb{Z}_8, \mathbb{Z}_3)$, $(\mathbb{Z}_2 \times \mathbb{Z}_6, \mathbb{Z}_3)$, $(\mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_3)$,
$(\mathbb{Z}_3, \mathbb{Z}_4)$, $(\mathbb{Z}_4, \mathbb{Z}_4)$, $(\mathbb{Z}_6, \mathbb{Z}_4)$, $(\mathbb{Z}_7, \mathbb{Z}_4)$, $(\mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_4)$,
$(\mathbb{Z}_2, \mathbb{Z}_6)$, $(\mathbb{Z}_4, \mathbb{Z}_6)$, $(\mathbb{Z}_5, \mathbb{Z}_6)$, $(\mathbb{Z}_6, \mathbb{Z}_6)$, $(\mathbb{Z}_8, \mathbb{Z}_6)$, $(\mathbb{Z}_2 \times \mathbb{Z}_6, \mathbb{Z}_6)$, $(\mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_6)$.

Remark. *There are examples of non-rigid fibered Calabi-Yau threefolds of type* II_0K *and the number of fibered Calabi-Yau threefolds of type* II_0K *is not finite any more ([O1]).*

Remark. *It is interesting to compare Theorems 2, 3, 4 and main theorem with the so called Bogomolov decomposition theorem (see for example [Bo]). These look very similar, while our proof is free from the Bogomolov decomposition theorem.*

The Main Theorem and Theorem 4 immediately imply

Corollary. *Let* $\Phi : X \to W$ *is a fibered Calabi-Yau threefold of type* II_0. *Then the global canonical index of* W *is either* 2, 3, 4 *or* 6.

Corollary. *Let* $\Phi : X \to W$ *be a fibered Calabi-Yau threefold of type* II_0K *(resp. of type* II_0A*). Then, there is a composite of flops* $Y \to W$ *of* $\Phi : X \to W$ *over* W *such that* Y *has at least two different fiber space structures,* $Y \to W$ *of type* II_0K *(resp. of type* II_0A*) and* $Y \to \mathbb{P}^1$ *of type* I_+ *(resp. of type* I_0*).*

Very little is known for a fibered Calabi-Yau threefold of type I_0, that is, a Calabi-Yau threefold with an Abelian fibration. However, our main theorem and Theorem 4 show

Corollary. *Let* X *be a Calabi-Yau threefold with at least two different Abelian fibrations. Then,* X *is a Calabi-Yau threefold described as in either the Main Theorem (2) or Theorem 4(2). In particular,* X *is smooth and birational to either a quotient of an Abelian threefolds or that of the product of a K3 surface and an elliptic curve.*

In fact, if $\Phi_{|D_i|} : X \to \mathbb{P}^1$ $(i = 1, 2)$ are two different Abelian fibrations on X, then $\Phi_{|m(D_1+D_2)|} : X \to W$ is of type II_0 for some m.

The outline of this paper is as follows.

In section 1, we introduce the notion of quasi-product threefolds ((1.1)) and show their structure theorem ((1.3)). This plays an important role for our proof of the Main Theorem.

Sections 2 - 4 are devoted to prove the Main Theorem. Since Main Theorem (1) is quite clear, we prove only Main Theorem (2).

Let $\Phi_T : X_T := X \times_W T \to T$ be the base change of a fibered Calabi-Yau threefold $\Phi : X \to W$ of type II_0K to the global canonical cover $\pi : T \to W$. Since Φ always has a two dimensional fibers ([O1]), X_T has very bad singularities and Φ_T itself is a very complicated map in general.

In section 2, we apply the log minimal model program established by Shokurov and Kawamata or Kollár et al. [Sh] and [Kw4] (also [Ko3]) to find a good birational (canonical) model $f : Z \to T$ of $\Phi_T : X_T \to T$ over T such that

(1) $Gal(T/W) := \langle g \rangle$ acts regularly on $f : Z \to T$ and
(2) $\Phi : X \to W$ is birational to the quotient $(f : Z \to T)/\langle g \rangle$.

Moreover applying the result in section 1, we show that there are a smooth elliptic curve E, a normal projective surface S which is either an Abelian surface or a K3 surface with only Du Val singularities, and a finite automorphism group G of the fiber space $p_2 : E \times S \to S$ such that $(f : Z \to T) = (p_2 : E \times S \to S)/G$.

In section 3, we show that the action of $\langle g\rangle$ on $f : Z \to T$ lifts to that on its covering $p_2 : E \times S \to S$ in an equivariant way. This is a rather special phenomenon, because a composite of Galois extensions is not Galois in general.

Till section 3, the main part of our proof of the Main Theorem is completed. It remains only to show the impossibility for S to be a smooth Abelian surface. This problem is treated in section 4. This requires our assumption $\pi_1^{alg}(X) = \{1\}$ and forces rather minute analysis of automorphism groups of an Abelian surface.

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NOTATION AND CONVENTION

Throughout this paper, we work over the complex number field $\mathbb{C}$.

We will employ standard notion and notation in minimal model program ([KMM] or [Ko3]) freely.

By a minimal threefold, we mean a normal projective threefold V with only $\mathbb{Q}-$factorial terminal singularities and with nef canonical (Weil) divisor K_V.

A surjective morphism $\Phi : V \to W$ is said to be relatively minimal if V has only $\mathbb{Q}-$factorial terminal singularities and the canonical divisor K_V is relatively nef with respect to Φ.

We often use the notion of klt (Kawamata log terminal) given in [Ko3]. This is same as the notion of log terminal in [KMM].

By a fiber space on a normal projective variety V, we mean a surjective morphism $\Phi : V \to W$ to a normal projective variety W with connected fibers. Note that Φ is not equi-dimensional in general. By $\Phi^{-1}(w)$ $(w \in W)$, we denote the scheme theoretic fiber over w. We denote its reduction by $\Phi^{-1}(w)_{\mathrm{red}}$. This is in some sense a set theoretical fiber.

Two fiber spaces $\Phi : V \to W$ and $\Phi' : V' \to W'$ are said to be isomorphic if there are isomorphisms $F : V \to V'$ and $f : W \to W'$ such that $\Phi' \circ F = f \circ \Phi$.

For two morphisms $\Phi : V \to W$ and $\pi : T \to W$, we sometimes denote natural morphisms $V \times_W T \to T$ and $V \times_W T \to V$ by $\Phi_T : V_T \to T$ and $\pi_V : V_T(= T_V) \to V$ respectively.

The primitive $n-$th root of unity $exp(2\pi i/n)$ is denoted by ζ_n.

We denote the cyclic group of order n by $\mathbb{Z}_n$.

The elliptic curve with period $\tau \in \mathbb{H}$ is written as E_τ.

The $n-$torsion group of an Abelian variety A with origin 0 is denoted by $(A)_n$. By global coordinates around a point P of an $n-$dimensional Abelian variety A, we mean those of its universal cover $\mathbb{C}^n$ or, equivalently, those of the tangent space $T_{A,P}$.

For a faithful group action of G on a variety V, we set

$$V^{[G]} := \{x \in V \mid \exists g \in G - \{1\}, g(x) = x\},$$

while,

$$H^G := \{v \in H \mid \forall g \in G, g^*(v) = v\}$$

for any cohomology group H of V.

Similarly, for an automorphism g of a variety V, we set

$$V^g := \{x \in V \mid g(x) = x\}.$$

An equivariant action of a finite group G on a fibration $\Phi : V \to W$ induces a new fibration $\Phi(\mathrm{mod}\, G) : X/G \to W/G$. We sometimes abbreviate this fibration by $(\Phi : V \to W)/G$.

We say that G acts on $\Phi : V \to W$ over W if the action of G is equivariant and is trivial on W.

An automorphism group G of a variety V with $\mathcal{O}_V(K_V) \simeq \mathcal{O}_V$ is called Gorenstein if the action of G on $H^0(V, \mathcal{O}_V(K_V))$ is trivial, that is, all elements g of G satisfy $g^*\omega_V = \omega_V$ for a generator ω_V of $H^0(V, \mathcal{O}_V(K_V))$.

For the automorphism group $\mathrm{Aut}\,(V)$ of a variety V and a subset B in V, we often consider the subgroup $\{g \in \mathrm{Aut}\,(V) \mid g(B) = B\}$. We denote this group by $\mathrm{Aut}\,(X, B)$. For example, if A is an Abelian variety with origin 0, then $\mathrm{Aut}\,(A, \{0\})$ is nothing but the so called Lie automorphism group of A.

§1. Quasi-product threefolds

In this preliminary section, we shall introduce the notion of quasi-product threefolds and prove their structure theorem (Theorem (1.3)). This is a rather wide generalisation of the notion of hyperelliptic surfaces to threefolds.

Definition (1.1). *A normal projective threefold V with only rational singularities is called a quasi-product threefold with distinguished morphisms a and f if*

(1) *V has a fiber space structure $a : V \to A$ over a smooth elliptic curve A,*
(2) *V has a fiber space structure $f : V \to T$ over a normal projective surface T with only rational singularities and with $H^1(\mathcal{O}_T) = 0$ such that $f^{-1}(t)_{\mathrm{red}}$ is a smooth elliptic curve for any $t \in T$, and that $f^{-1}(t)$ itself is smooth except at most finitely many points $t \in T$.*

Example (1.2). *Let S be a normal projective surface with only rational singularities and E a smooth elliptic curve. Assume that a finite group of translations G of E acts faithfully on S in such a way that $S^{[G]}$ is finite and $H^1(\mathcal{O}_S)^G = 0$. Then the quotient threefold $(E \times S)/G$ is a quasi-product threefold with distinguished morphisms $p_1 : (E \times S)/G \to E/G$ and $p_2 : (E \times S)/G \to S/G$.*

Conversely, we shall show

Theorem (1.3). *Let V be a quasi-product threefold with two distinguished morphisms $a : V \to A$ and $f : V \to T$. Let S be a general fiber of a.*

Then, there exist an elliptic curve E and a finite subgroup $G \subset E$, that is, a finite group of translations of E (and then is isomorphic to either $\mathbb{Z}_m$ or $\mathbb{Z}_n \times \mathbb{Z}_m$ with $(n|m)$) such that

(1) *there is an injective homomorphism $\iota : G \to \mathrm{Aut}\,(S)$,*
(2) *$V = (E \times S)/G$ under the (free) action of G on $E \times S$ defined by*

$$G \ni g : E \times S \ni (u, v) \mapsto (u + g, \iota(g)v) \in E \times S,$$

(3) *two distinguished morphisms* $a : V \to A$ *and* $f : V \to T$ *are given by the natural projections*

$$p_1 : (E \times S)/G \to E/G$$

and

$$p_2 : (E \times S)/G \to S/\iota(G)$$

respectively.

As a result, S *can be replaced by any fiber of* a*. We set* $G_S := \iota(G)(\simeq G)$*.*
Moreover, if $\mathcal{O}_V(K_V) \simeq \mathcal{O}_V$*, then,*

(4) *any fiber* S *of* a *is either a K3 surface with only Du Val singularities or a smooth Abelian surface,*
(5) G_S *is a finite Gorenstein automorphism of* S*,*
(6) *if* S *is a K3 surface with only Du Val singularities, then* $S^{[G_S]}$ *is a non-empty finite set and* $G_S(\simeq G)$ *is isomorphic to either one of the following groups;*
$\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, $\mathbb{Z}_7$, $\mathbb{Z}_8$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_6$, $\mathbb{Z}_3\times\mathbb{Z}_3$, or $\mathbb{Z}_4\times\mathbb{Z}_4$,
(7) *if* S *is a smooth Abelian surface, then* $S^{[G_S]}$ *is a non-empty finite set and* $G_S(\simeq G)$ *is isomorphic to either one of the following groups;*
$\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$, or $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3$.
In addition, if $G_S \simeq \mathbb{Z}_m$*, then* $G_S \subset \mathrm{Aut}\,(S, \{0\})$ *for an appropriate origin* 0 *of* S*, while, if* $G_S \simeq \mathbb{Z}_n \times \mathbb{Z}_m$ $(n|m)$*, then* $\mathbb{Z}_n \subset (S)_n$ *and* $\mathbb{Z}_m \subset \mathrm{Aut}\,(S, \{0\})$ *for an appropriate origin* 0 *of* S*. Moreover,* $\mathrm{Sing}\,(S/G_S)$ *is described as follows for each* G_S *([Kt]).*
$(G_S, \mathrm{Sing}\,(S/G_S)) = (\mathbb{Z}_2, 16A_1), (\mathbb{Z}_2 \times \mathbb{Z}_2, 16A_1), (\mathbb{Z}_3, 9A_2), (\mathbb{Z}_3 \times \mathbb{Z}_3, 9A_2)$
$(\mathbb{Z}_4, 4A_3 + 6A_1), (\mathbb{Z}_2 \times \mathbb{Z}_4, 4A_3 + 6A_1), (\mathbb{Z}_6, A_5 + 4A_2 + 5A_1)$.

Remark. *Let* $\nu : S' \to S$ *be the minimal resolution of* S*. Then* G *induces an equivariant free action on* $id \times \nu : E \times S' \to E \times S$*. The induced morphism* $(E \times S')/G \to (E \times S)/G$ *gives a resolution of* $(E \times S)/G$*.*

Remark. *Our proof given here basically follows the argument of Bombieri and Mumford for hyperelliptic surfaces([BM]). However, since we work at threefolds, we should keep the following two essential differences in mind:*

(1) f *may not be flat over* T*,*
(2) *three dimensional relatively minimal models are not unique among their birational models (even if they exist) so that rational actions on a relatively minimal model are not necessarily regular in general.*

Proof. Set $B := \{t \in T |$ either $f^{-1}(t)$ is not reduced or T is singular at $t\}$, and denote $C_t := f^{-1}(t)(t \in T)$ and $S_x := a^{-1}(x)$ $(x \in A)$. By our assumption, B is a finite set. Let us fix a general point $0 \in A$ and regard this point as an origin of A. Set $S := S_0$. Then S is a normal surface with only rational singularities. Put $n := (C_t \cdot S)$. This is independent of $t \in T - B$ (because $T - B$ is smooth and $f|_{f^{-1}(T-B)}$ is a smooth morphism over $T - B$.)

CLAIM (1.4). *$a_t := a|_{C_t} : C_t \to A$ is surjective for each $t \in T - B$. In particular, a_t is an isogeny of elliptic curves of degree $n := (C_t \cdot S)$ for each $t \in T - B$ (and then $n > 0$).*

Proof of Claim (1.4). Assume the contrary that $a(C_t)$ is a point on A for some $t \in T - B$. Then, $a(C_{t'})$ must be a point for every $t' \in T - B$ because f is flat over $T - B$. Thus, a induces a morphism $\overline{a} : T - B \to A$. This gives a rational map $\overline{a} : T \cdot\cdot\to A$ with $a = \overline{a} \circ f$. Let $T' \to T$ be a resolution of both singularities of T and indeterminacy of $\overline{a}$. Since T has only rational singularities, we have $h^1(\mathcal{O}_{T'}) = h^1(\mathcal{O}_T) = 0$. Thus, $\overline{a} \circ \nu(T')$ is a point. Hence $\overline{a}$ is a morphism and $\overline{a}(T)$ is a point. Then, $a(V)$ would be a point because $a = \overline{a} \circ f$. But this contradicts the surjectivity of a. q.e.d. for (1.4).

Let t be an arbitrary point on $T - B$. Then, by (1.4), A acts on C_t via the composite of the group homomorphism $A \simeq Pic^0(A) \to Pic^0(C_t)$ given by a_t^* and the natural action of $Pic^0(C_t)$ on C_t. More concretely, this action is written as

$$A \ni x : C_t \ni P \mapsto P + x_1 + ... + x_n - 0_1 - ... - 0_n \in C_t,$$

where $\{x_1, ..., x_n\} := a_t^{-1}(x) = C_t \cap S_x$ and $\{0_1, ..., 0_n\} := a_t^{-1}(0) = C_t \cap S$. Note that f has a local section over $T - B$. Thus, gluing these together, we get a regular action of A on $\cup_{t \in T-B} C_t = f^{-1}(T - B)$ over $T - B$. This gives a rational action on V over T. But, since the possible indeterminacy $f^{-1}(B)$ of this action on V consists of elliptic curves (then no rational curves) and since V has only rational singularities, this action of A on V must be regular. Let us denote this action by $\sigma : A \times V \to V$. By construction, σ stabilizes each fiber of f. Set $\tau := \sigma|_{A \times S} : A \times S \to V$. Since a_t is an isogeny, we have

$$a_t(P + x_1 + ... + x_n - 0_1 - ... - 0_n) = a_t(P) + nx$$

for $t \in T - B$ and $x \in A$. So, once we define a new action of A on A by

$$A \ni x : A \to A; y \mapsto y + nx,$$

that is, by $n \times (translation)$, then A induces an equivariant action on the fibration $V - f^{-1}(B) \to A$. By the same reason as before, this action of A is extended to an equivariant regular action on the whole space $a : V \to A$.

By definition, we have $x(S)(= x(S_0)) = S_{nx}$ ($x \in A$). In particular, $\tau : A \times S \to V$ is surjective. Moreover, the action of the $n-$torsion group $(A)_n$ of A on V stabilizes $S = S_0$. This induces a group homomorphism $\iota : (A)_n \to \mathrm{Aut}\,(S)$.

The following claim ([BM]) is now proved formally.

CLAIM (1.5). *Let (x, v) and (x', v') be points on $A \times S$. Then, the following (1) and (2) are equivalent to one another.*

(1) $\tau(x, v) = \tau(x', v')$,
(2) *(x, v) and (x', v') are in the same orbit of the action*

$$(A)_n \ni k : A \times S \to A \times S; (x, v) \mapsto (x - k, \iota(k)v).$$

Proof of Claim (1.5). Since $\tau(x-k,\iota(k)v) = \sigma(x-k,\sigma(k,v)) = \sigma(x-k+k,v) = \tau(x,v)$, (2) implies (1). We prove the converse. Since $\tau(x,v) \in S_{nx}$ and $\tau(x',v') \in S_{nx'}$, it follows that $nx = nx'$, or equivalently, $k := x - x' \in (A)_n$. We may show that $\iota(k)(v) = v'$. Using $\tau(x,v) = \tau(x',v')$, that is, $\sigma(x,v) = \sigma(x',v')$, we calculate

$$v' = \sigma(-x',\sigma(x',v')) = \sigma(-x',\sigma(x,v)) = \sigma(x-x',v).$$

This is nothing but the desired equality, $\iota(k)(v) = v'$. q.e.d. for (1.5).

By (1.5), we get $V = (A \times S)/(A)_n$. Moreover, just by construction, we see that $f : (A \times S)/(A)_n \to T$ factors through the natural projection $p_2 : (A \times S)/(A)_n \to S/(A)_n$. In fact, f factors through p_2 at least over $T - B$. But, since B is finite and $S/(A)_n$ is normal, this is so over the whole T. Let $\mu : S/(A)_n \to T$ be the induced morphism. Since both f and p_2 have only one dimensional connected fibers, μ must be a finite birational morphism. Thus, by the Zariski main theorem, μ is isomorphism and then $f = p_2$ under the identification $T = S/(A)_n$. Similarly, $a : (A\times S)/(A)_n \to A$ factors through $p_1 : (A \times S)/(A)_n \to A/(A)_n = A$. Now the equality $a = p_2$ is shown by the same argument as before.

It only remains to make ι injective to complete the first half part of (1.3). But this is done as follows. Let $G = (A)_n/Ker\,\iota$. Then, $(A \times S)/(A)_n = (A/(Ker\,\iota) \times S)/G$ and $A/(A)_n = (A/Ker\,\iota)/G$, in which G acts on translation group of an elliptic curve $A/Ker\,\iota$. Now replacing A, $(A)_n$ and ι by $E = A/(Ker\,\iota)$, G, and the injection $\iota \circ (-1) : G \to \mathrm{Aut}\,(S)$, we are done. Here we will compose (-1) only to change the sign $-$ in (1.5) into $+$ as in (1.3).

From now on, we shall prove the latter half part of (1.3). It is obvious that S is either a K3 surface with only Du Val singularities or a smooth Abelian surface. Moreover, since G acts on E as a translation group and $\mathcal{O}_V(K_V) \simeq \mathcal{O}_V$, it follows that G_S must be a Gorenstein automorphism group of S. In the rest we denote G_S simply by G if no confusion seems to arise.

Assume first that S is a K3 surface with only Du Val singularities. Let $S' \to S$ be the minimal resolution of S. Then G gives a commutative Gorenstein action on S'. Now the result follows from the Nikulin's classification ([Ni]). Note that two groups $(\mathbb{Z}_2)^3$ and $(\mathbb{Z}_2)^4$ in his list are excluded because G is isomorphic to either $\mathbb{Z}_n$ or $\mathbb{Z}_n \times \mathbb{Z}_m$ $(n|m)$.

Finally, assuming that S is a smooth Abelian surface, we show that G satisfies the condition in (1.3)(7). Since G is a finite Gorenstein automorphism group of S with $T = S/G$ and since $h^1(T,\mathcal{O}_T) = 0$, it follows that $S^{[G]}$ is a non-empty finite set. Choose an appropriate origin 0 of S and identify S with its translation automorphism group. Set $Aut^0(S) := \{\sigma \in \mathrm{Aut}\,(S) | \sigma^*\omega_S = \omega_S\}$, $Aut^0(S,\{0\}) := \{\sigma \in Aut^0(S) | \sigma(0) = 0\}$, where ω_S is a non-zero global regular two form on S. Then, $Aut^0(S) = S \rtimes Aut^0(S,\{0\})$ and $G \subset Aut^0(S)$. Identifying $Aut^0(S,\{0\}) = Aut^0(S)/S$, we denote the natural projection by $p : Aut^0(S) \to Aut^0(S,\{0\})$. If we choose global coordinates around 0, we can explicitly write down the action of $g \in Aut^0(S)$ in its affine form

$$g(x) = M_g x + t_g, M_g \in SL(2,\mathbb{C}), t_g \in S.$$

Then p is nothing but the map taking the matrix part, that is, $g \mapsto M_g$. It follows from this expression that

(1) as an abstract group, $p(G)$ is independent of the choice of an origin of S,
(2) a finite Gorenstein automorphism $g \in Aut^0(S)$ has a fixed point if and only if g is not a translation.

On the other hand, Katsura's classification ([Kt]) of possible finite subgroups of $Aut^0(S, \{0\})$ shows that the commutative group $p(G)$ is isomorphic to either $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$.

Thus we can choose $g \in G$ and $0 \in S$ such that $p(g)$ generates $p(G)$ and $g(0) = 0$. From now on, we regard this point 0 as the origin of S.

Claim (1.6).

(1) *$H := Ker(p)$ consists of translations in G, that is, $H \subset S$,*
(2) *$\langle g \rangle \simeq p(G)$.*
(3) *G is isomorphic to $H \times \langle g \rangle$.*
(4) *H is a subgroup of S^g (under the inclusion $H \subset S$).*

Proof of (1.6). The assertion (1) follows from $M_h = id$ for $h \in H$. By definition, $p|_{\langle g \rangle} : \langle g \rangle \to p(G)$ is surjective group homomorphism. Let h be an element of $Ker(p|_{\langle g \rangle})$. Then, $h(0) = 0$ and $h \in H$. Combining this with (1), we get $h = id$. Thus, $p|_{\langle g \rangle}$ is isomorphism. This shows that G is a semi-direct product of H and $\langle g \rangle$. Since G is commutative, this must be the direct product. The last statement now directly follows from the relation $gh = hg$ $(h \in H)$. q.e.d. of (1.6).

Claim (1.7). *According to* $\operatorname{ord}(g) = 2, 3, 4, 6$, *$S^g$ is isomorphic to* $(\mathbb{Z}_2)^4$, $(\mathbb{Z}_3)^2$, $(\mathbb{Z}_2)^2$ *and* $\{0\}$.

Proof of (1.7). If $\operatorname{ord}(g) = 2$, then $S^g = (S)_2$. Since $(S)_2 \simeq (\mathbb{Z}_2)^4$, we are done.

Assume that $\operatorname{ord}(g) = 3$. Then, using appropriate global coordinates (x, y) around 0, we can write $g = \operatorname{diag}(\zeta_3, \zeta_3^{-1})$. In particular, $1 + g + g^2 = 0$. Thus, $3p = p + p + p = p + g(p) + g^2(p) = (1 + g + g^2)(p) = 0$ for $p \in (S)^g$. Hence $S^g \subset (S)_3$ and $S^g \simeq (\mathbb{Z}_3)^k$ for some non negative integer k. On the other hand, by the Lefschetz fixed point formula, we have $\sharp S^g = \sum_{i=0}^{4} (-1)^i tr(g^* | H^i(S, \mathbb{C}))$. Recall that

$$H^1(S, \mathbb{C}) = \mathbb{C}dx \oplus \mathbb{C}dy \oplus \mathbb{C}d\overline{x} \oplus \mathbb{C}d\overline{y},$$

and

$$H^i(S, \mathbb{C}) = \wedge^i H^1(S, \mathbb{C}).$$

Now an explicit calculation based on $g = \operatorname{diag}(\zeta_3, \zeta_3^{-1})$ shows $tr(g^* | H^0(S, \mathbb{C})) = 1, -2, 3, -2, 1$ according to $i = 0, 1, 2, 3, 4$. Thus, $\sharp S^g = 9$. This implies $S^g \simeq (\mathbb{Z}_3)^2$.

Assume that $ord(g) = 4$. Since $S^g \subset S^{g^2} \simeq (\mathbb{Z}_2)^4$, it follows that $S^g \simeq (\mathbb{Z}_2)^k$ for some non negative integer k. As in the case of $\operatorname{ord}(g) = 3$, we can choose appropriate global coordinates (x, y) around 0 such that $g = \operatorname{diag}(\zeta_4, \zeta_4^{-1})$. Then, again using the Lefschetz fixed point formula, we calculate $\sharp S^g = 4$. This implies $S^g \simeq (\mathbb{Z}_2)^2$.

Finally assume that $\operatorname{ord}(g) = 6$. Then, it follows from the previous observation that $S^g \subset S^{g^2} \cap S^{g^3} \subset (S)_2 \cap (S)_3 = \{0\}$. q.e.d. of (1.7).

Now Claims (1.6), (1.7) and the fact that G is a finite Abelian group of the form $\mathbb{Z}_n$ or $\mathbb{Z}_n \times \mathbb{Z}_m$ $(n|m)$ together with the fundamental theorem on finite Abelian groups imply the assertion (1.3)(7).

The only remaining problem is to study $\mathrm{Sing}\,(S/G)$ for each G. If G is isomorphic to $\mathbb{Z}_m$, the result follows from Katsura's table ([Kt]). Next, consider the case when $\mathbb{Z}_n \times \mathbb{Z}_m$ for some n and m (with $n|m$). Since $S/G \simeq (S/\mathbb{Z}_n)/\mathbb{Z}_m$ and since $(S/\mathbb{Z}_n)$ is again an Abelian surface, the assertion follows from the first case.

Now we are done. Q.E.D. of (1.3).

§2. Good model over the global canonical covering

Let us fix a fibered Calabi-Yau threefold $\Phi : X \to W$ of type II_0K. Define $I := min\{n \in \mathbb{N}|\mathcal{O}_W(nK_W) \simeq \mathcal{O}_W\}$ and denote the global canonical cover of W by $\pi : T \to W$ ([Kw1, Z]). By our assumption, T is a projective K3 surface with only Du Val singularities. Set $W_0 := W - \mathrm{Sing}\,(W)$. It is well known by [Kw1, Z] that $\pi : T \to W$ is a cyclic Galois covering of order $I(W)$ and is étale over W_0. Moreover, there is a generator g of the Galois group $Gal(T/W)$ such that $g^*\omega_T = \zeta_I \omega_T$, where ω_T is a nowhere vanishing regular two form on T, that is, a generator of $H^0(\mathcal{O}_T(K_T))$.

We fix these notation till the end of Section 4.

Set $\Phi_T : X_T := X \times_W T \to T$. Then, the Galois group $Gal(T/W) = \langle g \rangle$ acts on this fibration by $g : (x, y) \mapsto (x, g(y))$ and induces an isomorphism

$$(\Phi : X \to W) \simeq (\Phi_T / : X_T \to T)/\langle g \rangle.$$

However, X_T itself has very bad singularities in general.

The goal of this section is to prove the following

KEY LEMMA (2.1). *There is a normal projective threefold Z such that*

1. *Z has only $\mathbb{Q}-$factorial canonical singularities with $\mathcal{O}_Z(K_Z) \simeq O_Z$,*
2. *Z is a quasi-product threefold ((1.1)) with two distinguished morphisms $f : Z \to T$ and $a : Z \to A$, where the latter map is the Albanese morphism of Z (see [Kw2] for the definition of the Albanese variety and the Albanese morphism for varieties with rational singularities), and*
3. *there is a regular action of the Galois group of $\langle g \rangle$ on the fibration $f : Z \to T$ such that $W = T/\langle g \rangle$ and $(\Phi : X \to W)$ is birational to $(f : Z \to T)/\langle g \rangle$ over $W = T/\langle g \rangle$. Moreover, these are isomorphic over $W - \mathrm{Sing}\,(W)$.*

The plan of proof of Key Lemma is as follows. First, applying the log minimal model program, we find a birational model $f : Z \to T$ of $\Phi_T : X_T \to T$ with property (1) in (2.1). Then, we check that $f : Z \to T$ also satisfies (2) and (3).

In order to carry out this plan, we start by observing some general lemmas.

PROPOSITION (2.2). *Let $\varphi : V \to S$ be a surjective morphism from a normal projective $\mathbb{Q}-$factorial threefold V to a normal projective surface S. Let $\{E_i\}_{i \in I}$ be the set of all two-dimensional irreducible components in fibers of φ. Set $E = \Sigma_{i \in I} E_i$. Assume that*

1. *V is not covered by rational curves,*

(2) $K_V = \Sigma_{i\in I} a_i E_i$ *(as a Weil divisor on V) for some* $a_i \in \mathbb{Z}_{\geq 0}$,
(3) $(V, \epsilon E)$ *is klt for some positive small rational number* ϵ.

Then, there are a normal projective threefold $V^{(n)}$ *and a surjective morphism* $\varphi^{(n)}$: $V^{(n)} \to S$ *such that*

(4) $V^{(n)}$ *has only* $\mathbb{Q}-$*factorial canonical singularities with* $\mathcal{O}_{V^{(n)}}(K_{V^{(n)}}) \simeq \mathcal{O}_{V^{(n)}}$,
(5) $\varphi^{(n)} : V^{(n)} \to S$ *is birational to* $\varphi : V \to S$ *over* S *and is isomorphic except over a finite set* $\varphi(E)$, *and*
(6) $\varphi^{(n)} : V^{(n)} \to S$ *is an equi-dimensional elliptic fibration.*

Proof. First, we remark

Claim (2.3). $K_V + \epsilon E$ *is not nef unless* $E = 0$ *as a divisor.*

Proof of (2.3). Let H be a general very ample divisor on V. Then H is a normal surface and the restriction $\varphi|_H : H \to S$ is surjective. Since $(K_V + \epsilon E)|_H \equiv \Sigma_{i\in I}(a_i + \epsilon)E_i|_H$ and since $E_i|_H$ are contracted by φ_H, we get

$$((K_V + \epsilon E)^2 \cdot H) = ((K_V + \epsilon E)|_H)^2 = (\Sigma_{i\in I}(a_i + \epsilon)E_i|_H)^2 < 0$$

unless $E = 0$. q.e.d. of (2.3).

Let us apply the log minimal model program for a klt divisor $K_V + \epsilon E$. If $E \neq 0$, then $K_V + \epsilon E$ is not nef by (2.3). Thus, there is a log extremal ray R such that $(K_V + \epsilon E) \cdot C < 0$ for any curve C belonging to R. Let $cont_R : V \to W$ be the contraction morphism associated to R. This is a birational morphism by our assumption (1). Since $0 > (K_V + \epsilon E) \cdot C = \Sigma(a_i + \epsilon)(E_i \cdot C)$, there is a prime divisor E_i such that $E_i \cdot C < 0$. This implies $C \subset E_i$. Thus $cont_R$ is defined over S. Let $\phi : W \to S$ be the induced morphism.

If $cont_R$ is a divisorial contraction, setting $V^{(1)} := W$, $\varphi^{(1)} := \phi$ and changing E by its strict transform $E^{(1)}$ on $V^{(1)}$, we see that $\varphi^{(1)} : V^{(1)} \to S$ and $E^{(1)}$ satisfy all the assumptions in (2.2) (without any change of coefficients).

If $cont_R$ is a small contraction, then we apply a log flip for $cont_R$ to get $cont_R^+$: $V^+ \to W$.

The existence of log flips for threefolds is guaranteed by [Sh].

Now, setting $V^{(1)} := V^+$, $\varphi^{(1)} := \phi \circ cont_R^+$ and changing E by its strict transform $E^{(1)}$ on $V^{(1)}$, we see that $\varphi^{(1)} : V^{(1)} \to S$ and $E^{(1)}$ also satisfy all the assumptions in (2.2).

Putting $V^{(0)} := V$, $\varphi^{(0)} := \varphi$ and $E^{(0)} := E$ and repeating this process, say, for $n(\geq 0)$ times, we finally get $\varphi^{(n)} : V^{(n)} \to S$ and the strict transform $E^{(n)}$ of E to $V^{(n)}$ such that

(1) $\varphi^{(n)} : V^{(n)} \to S$ and $E^{(n)}$ satisfy all the assumptions in (2.2), and
(2) $K_{V^{(n)}} + \epsilon E^{(n)}$ is nef.

This is due to the termination of log flips for threefolds shown by [Kw4].

Then $E^{(n)} = 0$ by (2.3). This implies the equi-dimensionality of $\varphi^{(n)}$. Note that all modifications are done over $\varphi(E)$. Thus $\varphi^{(n)} : V^{(n)} \to S$ and $\varphi : V \to S$ coincide over $S - \varphi(E)$. Set $V_0 := V - E - \mathrm{Sing}\,(V)$. Then the assumption (2) implies $\mathcal{O}_{V_0}(K_{V_0}) \simeq \mathcal{O}_{V_0}$. Let $\nu : V \cdot\cdot\to V^{(n)}$ be the birational map obtained by the above process. Since $\nu|_{V_0} : V_0 \to \nu(V_0)$ is an isomorphism, we have $\mathcal{O}_{\nu(V_0)}(K_{\nu(V_0)}) \simeq \mathcal{O}_{\nu(V_0)}$.

Since the codimension of $V^{(n)} - \nu(V_0)$ in $V^{(n)}$ is at least two by $E^{(n)} = 0$ and since $V^{(n)}$ is normal, this isomorphism gives $\mathcal{O}_{V^{(n)}}(K_{V^{(n)}}) \simeq \mathcal{O}_{V^{(n)}}$. Note that $V^{(n)}$ has only rational singularities, because $(V^{(n)}, E^{(n)}) = (V^{(n)}, 0)$ is klt. Thus $V^{(n)}$ has only rational Gorenstein singularities, that is, canonical singularities of index one. Now the remaining assertion is obvious. Q.E.D. of(2.2).

The next two lemmas are concerned with singular fibers of certain elliptic threefolds.

Lemma (2.4). *Let $\varphi : V \to S$ be a fiber space such that*

(1) *V is a normal projective threefold with only $\mathbb{Q}$−factorial terminal singularities and with $K_V \equiv 0$,*
(2) *S is a normal projective surface with only quotient singularities and with $K_V \equiv 0$.*

Then, $\varphi^{-1}(s)$ is a smooth elliptic curve if $s \in S - \mathrm{Sing}\,(S)$. In particular, φ is a smooth morphism over $S - \mathrm{Sing}\,(S)$.

Proof. We make use of the following theorem due to Nakayama.

Theorem (2.5)([Na1 also Na2]). *Let $f : V_{\Delta^2} \to \Delta^2$ be a relatively minimal projective elliptic fibration over a two-dimensional (small) polydisk*

$$\Delta^2 := \{(x, y) \in \mathbb{C}^2 \mid |x| < \epsilon, |y| < \epsilon\}.$$

Assume that f has (singular) fibers of type I_a $(a \geq 0)$ over $(x = 0) - \{(0,0)\}$ and those of type I_b $(b \geq 0)$ over $(y = 0) - \{(0,0)\}$. (Here we employed Kodaira's notation.) Then $f^{-1}((0,0))$ is a (singular) fiber of type I_{a+b}. In particular, if f is smooth over $\Delta^2 - \{(0,0)\}$, then $f^{-1}((0,0))$ is a smooth elliptic curve and f is a smooth morphism over the whole Δ^2.

First, we show

Claim (2.6). *$\varphi : V \to S$ is an elliptic fibration and has singular fibers only over a finite set of points of S.*

Proof of (2.6). Note that a general fiber of φ is a smooth elliptic curve. Let H be a general very ample divisor on S. Set $V_H := \varphi^{-1}(H)$. Since V has only isolated singularities and since H is general, we may assume that $H \cap (\mathrm{Sing}\,(S) \cup \varphi(\mathrm{Sing}\,(V))) = \phi$ and both H and V_H are smooth. Let $\varphi|_{V_H} : V_H \to H$ be the induced elliptic fibration. Using the adjunction formula, we calculate $K_H \equiv H|_H$ and $K_{V_H} = (K_V + V_H)|_{V_H} \equiv \varphi^*(K_H)$. Comparing this with the canonical bundle formula of an elliptic surface (for example see [BPV]), we find that $\varphi|_{V_H}$ is a smooth morphism. This implies the result. q.e.d of (2.6).

Let $s \in S$ be an arbitrary smooth point of S and take a sufficiently small polydisk $\Delta^2 \subset S$ around s. By (2.6), φ is smooth over $\Delta^2 - \{s\}$. Now applying (2.5) for an elliptic fibration $\varphi|_{\varphi^{-1}(\Delta^2)} : \varphi^{-1}(\Delta^2) \to \Delta^2$, we get (2.4). Q.E.D. of (2.4).

Lemma (2.7). *Let $\varphi : V \to S$ be a fiber space such that*

(1) *V is a normal projective threefold with only canonical singularities and with $\mathcal{O}_V(K_V) \simeq \mathcal{O}_V$,*
(2) *S is a normal projective surface with only Du Val singularities and with $\mathcal{O}_S(K_S) \simeq \mathcal{O}_S$,*
(3) *φ is an equi-dimensional fibration, and*
(4) *φ is smooth except over a finite set of points of S.*

Then, the reduction of each fiber $\varphi^{-1}(s)_{\mathrm{red}}$ $(s \in S)$ is a smooth elliptic curve. Moreover, if s is a smooth point of S, then, $\varphi^{-1}(s)$ itself is a smooth elliptic curve. In particular, φ is a smooth morphism over $S - \mathrm{Sing}\,(S)$.

Proof. Let $s \in S$ be an arbitrary point of S. Since S has only Du Val singularities, we can choose a small neighborhood U around s such that

$$U = \Delta^2/G, s = (0,0)(\mathrm{mod}\, G).$$

Here Δ^2 is a two dimensional small polydisk and G is a finite Gorenstein automorphism group of Δ^2 each of whose element fixes only the origin $(0,0)$. We may also assume by (4) that φ is smooth over $U - \{s\}$.

Letting $\varphi_U : V_U \to U$ be the restriction of φ, we consider the fiber product

$$\varphi_{\Delta^2} : V_{\Delta^2} := V_U \times_U \Delta^2 \to \Delta^2.$$

Since $\Delta^2 \to U$ is étale over $U - \{s\}$ and φ_U is smooth over $U - \{s\}$, it follows that $\varphi_{\Delta^2} : V_{\Delta^2} \to \Delta^2$ is smooth over $\Delta^2 - \{(0,0)\}$.

Take a resolution $\nu : V^{(1)} \to V_{\Delta^2}$ of V_{Δ^2} and set $\varphi^{(1)} := \varphi \circ \nu : V^{(1)} \to \Delta^2$. Note that φ and $\varphi^{(1)}$ coincide over $\Delta^2 - \{(0,0)\}$.

Applying a relatively minimal model program with respect to $K_{V^{(1)}}$ over Δ^2 ([Mo]), we get a relatively minimal model

$$\varphi^{(2)} : V^{(2)} \to \Delta^2$$

of $\varphi^{(1)} : V^{(1)} \to \Delta^2$. Since each fiber of $\varphi^{(1)}$ over $\Delta^2 - \{(0,0)\}$ is a smooth elliptic curve, $\varphi^{(2)}$ coincides with $\varphi^{(1)}$ (and then φ_{Δ^2}) over $\Delta^2 - \{(0,0)\}$. This together with (2.5) implies that $(\varphi^{(2)})^{-1}((0,0))$ is also a smooth elliptic curve and that $\varphi^{(2)}$ is smooth over whole Δ^2. In particular, $V^{(2)}$ is also smooth. Since φ_{Δ^2} and $\varphi^{(2)}$ are birational over Δ^2, the natural action of G on $\varphi_{\Delta^2} : V_{\Delta^2} \to \Delta^2$ induces a rational action on

$$\varphi^{(2)} : V^{(2)} \to \Delta^2.$$

On the other hand, since each fiber of $\varphi^{(2)}$ is an elliptic curve, it follows that $\varphi^{(2)}$ is a unique relatively minimal model. Thus this action of G on $\varphi^{(2)} : V^{(2)} \to \Delta^2$ is regular and induces

$$\overline{\varphi^{(2)}} : V^{(2)}/G \to \Delta^2/G = U.$$

This is birational to $\varphi_U : V_U \to U$ over U and is isomorphic over $U - \{s\}$. Denote this birational map over U by

$$\mu : V_U \cdots\to V^{(2)}/G.$$

Then, μ gives an isomorphism

$$V_U - \varphi_U^{-1}(s) \simeq V^{(2)}/G - (\overline{\varphi^{(2)}})^{-1}(s).$$

Since $\mathcal{O}_{V_U - \varphi_U^{-1}(s)}(K_{V_U}) \simeq \mathcal{O}_{V_U - \varphi_U^{-1}(s)}$ by our assumption (1) and since $(\overline{\varphi^{(2)}})^{-1}(s)$ is of codimension two in a normal variety it follows that

$$\mathcal{O}_{V^{(2)}/G}(K_{V^{(2)}/G}) \simeq \mathcal{O}_{V^{(2)}/G}.$$

This shows that the action of G on $V^{(2)}$ is Gorenstein. Since each element of G fixes the origin $(0,0)$ of Δ^2, G stabilizes a smooth elliptic curve $E := (\varphi^{(2)})^{-1}((0,0))$. Since G is also Gorenstein on Δ^2, so is on E. That is, G acts on E as a translation group. Thus $(\overline{\varphi^{(2)}})^{-1}(s)_{\mathrm{red}} = E/G$ is a smooth elliptic curve.

Now, in order to complete the first part of (2.7), it is enough to show that $\mu : V_U \cdots \to V^{(2)}/G$ is actually an isomorphism. But, now, this immediately follows from the facts that V_U has only rational singularities and that $V^{(2)}/G$ is $\mathbb{Q}$–factorial.

If s is a smooth point of S, then we can take $G = \{1\}$ and then $V_U = V^{(2)}$ over $U = \Delta^2$. This implies the last half of (2.7). Q.E.D. of (2.7).

The next lemma is a slight generalization of Kollár's result (in the three dimensional case), which should be known by specialists. However, because of the lack of suitable references, we give here a brief proof based on the Kollár's original result.

LEMMA (2.8). *Let $\varphi : V \to S$ be a fiber space such that*

(1) *V is a normal projective threefold with only canonical singularities,*
(2) *S is a normal surface with only Du Val singularities.*

Let ω_V and ω_S be the dualizing sheaves on V and S. Then, $R^1\varphi_*\omega_V \simeq \omega_S$. Assume furthermore that

(3) *$\mathcal{O}_V(K_V) \simeq \mathcal{O}_V$ and*
(4) *S is a K3 surface with only Du Val singularities.*

Then $h^1(\mathcal{O}_V) = 1$.

REMARK. Kollár proved the first part of (2.8) under the assumption that both V and S are smooth ([Ko1]).

Proof. We want to reduce our proof to the smooth case.

Consider the following commutative diagram,

$$\begin{tikzcd} V' \arrow[r, "\nu"] \arrow[d, "\Phi"'] & V \arrow[d, "\varphi"] \\ S' \arrow[r, "\mu"'] & S \end{tikzcd}$$

where $\mu : S' \to S$ is the minimal resolution of S' and $\nu : V' \to V$ is a resolution of both the singularities of V and indeterminacy of $\mu^{-1} \circ \varphi$.

Then $R^i\nu_*\omega_{V'} = 0$ for $i > 0$. Moreover, $\nu_*\omega_{V'} = \omega_V$ because V has only canonical singularities. Thus, from the Leray spectral sequence

$$R^p\varphi_*(R^q\nu_*\omega_{V'}) \Rightarrow R^{p+q}(\varphi\circ\nu)_*\omega_{V'}$$

we get

$$R^p\varphi_*\omega_V \simeq R^p(\varphi\circ\nu)_*\omega_{V'} \simeq R^p(\mu\circ\Phi)_*\omega_{V'}.$$

In particular,

$$R^1\varphi_*\omega_V \simeq R^1(\mu\circ\Phi)_*\omega_{V'}.$$

On the other hand, the edge sequence of another Leray spectral sequence

$$R^p\mu_*(R^q\Phi_*\omega_{V'}) \Rightarrow R^{p+q}(\mu\circ\Phi)_*\omega_{V'}$$

gives an exact sequence

$$0 \to R^1\mu_*(\Phi_*\omega_{V'}) \to R^1(\mu\circ\Phi)_*\omega_{V'} \to \mu_*(R^1\Phi_*\omega_{V'}) \to R^2\mu_*(\Phi_*\omega_{V'}).$$

Note that $R^2\mu_*(\Phi_*\omega_V) = 0$ and that $R^1\mu_*(\Phi_*\omega_{V'})$ is a torsion sheaf, because $\mu : S' \to S$ is a birational morphism between surfaces.

On the other hand, since V' is smooth, $R^1(\mu\circ\Phi)_*\omega_{V'}$ is a torsion free sheaf by [Ko1]. Then, chasing the above exact sequence, we get

$$R^1\mu_*(\Phi_*\omega_{V'}) = 0$$

and

$$R^1(\mu\circ\Phi)_*\omega_{V'} \simeq \mu_*(R^1\Phi_*\omega_{V'}).$$

Since V' and S' are smooth, Kollár's original result implies

$$R^1\Phi_*\omega_{V'} \simeq \omega_{S'}.$$

Thus,

$$R^1(\mu\circ\Phi)_*\omega_{V'} \simeq \mu_*\omega_{S'}.$$

Moreover, since S has only canonical singularities, it follows that

$$\mu_*\omega_{S'} \simeq \omega_S.$$

Thus,

$$R^1(\mu\circ\Phi)_*\omega_{V'} \simeq \omega_S.$$

Combining these, we get

$$R^1(\mu\circ\Phi)_*\omega_{V'} \simeq \omega_S.$$

This completes the proof of the first part.

We show the second part. Since $\omega_V \simeq \mathcal{O}_V$ and $\omega_S \simeq \mathcal{O}_S$, the first part of (2.8) gives

$$R^1\varphi_*\mathcal{O}_Z \simeq \mathcal{O}_S.$$

Substituting this into the edge sequence of the Leray spectral sequence

$$H^p(R^q\varphi_*\mathcal{O}_V) \Rightarrow H^{p+q}(\mathcal{O}_V),$$

we get an exact sequence

$$0 \to H^1(\mathcal{O}_S) \to H^1(\mathcal{O}_V) \to H^0(\mathcal{O}_S).$$

This implies

$$h^1(\mathcal{O}_V) \leq h^1(\mathcal{O}_S) + h^0(\mathcal{O}_S) = 0 + 1 = 1.$$

We show that $h^1(\mathcal{O}_V) \geq 1$. Considering the pullback of the regular two forms by Φ and using Hodge theory, we calculate

$$h^2(\mathcal{O}_{V'}) = h^{2,0}(V') \geq h^{2,0}(S') = 1.$$

On the other hand, using the fact that V has only rational singularities and the Serre duality, we see that

$$h^2(\mathcal{O}_{V'}) = h^2(\mathcal{O}_V) = h^1(\mathcal{O}_V).$$

Combining these, we get the desired inequality $h^1(\mathcal{O}_V) \geq 1$. Q.E.D. of (2.8).

We return back to Key Lemma (2.1). This is now proved by a simple combination of the previous lemmas.

Proof of Key Lemma.

Set $W_0 := W - \mathrm{Sing}\,(W)$ as before and denote the restrictions of $\Phi : X \to W$ and $\pi : T \to W$ to W_0 by

$$\Phi_0 : X_0 := \Phi^{-1}(W_0) \to W_0$$

and

$$\pi_0 : T_0 := \pi^{-1}(W_0) \to W_0.$$

Note that Φ_0 is a smooth morphism by (2.4) and π_0 is an étale morphism by definition.

We consider the Cartesian product defined by Φ and π

$$\begin{array}{ccc} X_T := X \times_W T & \xrightarrow{\pi_X} & X \\ \Phi_T \downarrow & & \downarrow \Phi \\ T & \xrightarrow[\pi]{} & W \end{array}$$

and its restriction over W_0

$$\begin{array}{ccc} (X_T)_0 := X_0 \times_{W_0} T_0 & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ T_0 & \longrightarrow & W_0 \end{array}$$

Since W_0 is smooth and since each morphism in the second diagram is smooth or étale, it follows that

$$\mathrm{Sing}\,(X) \subset \Phi^{-1}(W - W_0),$$

and

$$\mathrm{Sing}\,(X_T) \subset \pi_X^{-1}(\mathrm{Sing}\,(X)) \subset (\pi_X \circ \Phi)^{-1}(W - W_0) = \Phi_T^{-1}(T - T_0).$$

In what follows, we apply several birational modifications on the first diagram keeping everything in the second diagram invariant.

Since all singularities in the first diagram are supported over $W - W_0$, we find a commutative diagram

$$\begin{array}{ccc} X'_T & \xrightarrow{\pi'_X} & X' \\ {\scriptstyle \nu_{X_T}}\downarrow & & \downarrow{\scriptstyle \nu_X} \\ X_T & \xrightarrow[\pi_X]{} & X \end{array}$$

such that

(1) X' and X'_T are smooth,
(2) $\nu_X : X' \to X$ is a birational modification only over $W - W_0$, and that
(3) $\nu_{X_T} : X'_T \to X_T$ is a birational modification only over $T - T_0$.

Let $\{E_i\}_{i \in I}$ be the set of all the two dimensional irreducible components of fibers of $\Phi'_T := \Phi_T \circ \nu_{X_T} : X'_T \to X_T \to T$. Set $E := \Sigma_{i \in I} E_i$. By construction, E is supported only over $T - T_0$.

Claim (2.10).

(1) *X'_T is not covered by rational curves.*
(2) *$K_{X'_T} = \Sigma_{i \in I} a_i E_i$ for some non-negative integers a_i.*
(3) *$(X'_T, \epsilon E)$ is klt if $\epsilon > 0$ is sufficiently small.*

Proof of (2.10). The assertions (1) and (3) are clear. We show the assertion (2). Since X has only terminal singularities, $\mathrm{Sing}\,(X) \subset X - X_0$, and $K_X = 0$ as a divisor, we see that

$$K_{X'} = \Sigma c_j E'_j,$$

where c_j are some positive integers and E'_j are some irreducible divisors supported in $\nu_X^{-1}(X - X_0)$.

On the other hand, since $\pi'_{X_T} : X'_T \to X'$ ramifies only at E, the ramification formula gives

$$K_{X'_T} = (\pi'_{X_T})^*(K_{X'}) + \Sigma_{i \in I} b_i E_i,$$

for some non-negative integers b_i. Since $(\pi'_{X_T})^* E'_i$ are effective divisors supported in E, substituting the first equality into the second, we get the result. q.e.d. of (2.10).

Now we can apply (2.2) for $\Phi'_T : X'_T \to T$ to get a fiber space $f : Z \to T$ such that

(1) Z has only $\mathbb{Q}$−factorial canonical singularities with $\mathcal{O}_Z(K_Z) \simeq \mathcal{O}_Z$,
(2) $f : Z \to T$ is birational to $\Phi_T : X_T \to T$ over T and is isomorphic over T_0,
(3) $f : Z \to T$ is an equi-dimensional elliptic fibration.

Recall that T is a K3 surface with only Du Val singularities, and that Φ_T is smooth over T_0.
Now using (2.7) and (2.8), we see that

(4) $f^{-1}(t)_{\mathrm{red}}$ is a smooth elliptic curve for each $t \in T$,

(5) $f^{-1}(t)$ itself is smooth if t is a smooth point of T (in particular, if $t \in T_0$),
(6) $h^1(\mathcal{O}_Z) = 1$.

Thus, it follows from (1) and (6) and [Kw2] that

(7)$A := \mathrm{Alb}\,(Z)$ is a smooth elliptic curve and the Albanese morphism $a : Z \to A$ is a fiber space.

By (2), the natural action of $\langle g \rangle$ on $\Phi_T : X_T \to T$ induces a rational action of G on $f : Z \to T$ which is regular over T_0. By virtue of (1) and (4), we can apply the same argument as in the last part of the proof of (2.7) to conclude

(8) $\langle g \rangle$ induces a regular action on $f : Z \to T$ and
(9) $(f : Z \to T)/\langle g \rangle$ is birational to $\Phi : X \to W$ and is isomorphic over $W_0 = T_0/\langle g \rangle$.

Now these statements (1) - (9) imply the Key Lemma. Q.E.D. of Key Lemma.

§3. Lifting the group action on a fiber space to its covering

In this section, we continue to employ the same notation given at the beginning of Section 2.

Let $f : Z \to T$ be the quasi-product threefold found in (2.1) for a fibered Calabi-Yau threefold $\Phi : X \to W$ of type $\mathrm{II}_0 K$.

Then $(f : Z \to T) \simeq (p_2 : E \times S \to S)/G$, where

(1) E is a smooth elliptic curve,
(2) S is either a (projective) K3 surface with only Du Val singularities or a smooth Abelian surface, given as (any) fiber of the Albanese morphism $a : Z \to A$,
(3) G is a finite commutative Gorenstein automorphism group of $E \times S$ as is described in Theorem (1.3).

We want to lift the action of $\langle g \rangle$ on $f : Z \to T$ to one on $p_2 : E \times S \to S$ in an equivariant way.

Lemma (3.1). *There is a point* 0 *on* A *such that* $\langle g \rangle$ *stabilizes* $a^{-1}(0)$.

Proof. Since the Albanese morphism is an intrinsically and uniquely defined object, $\langle g \rangle$ acts on the Albanese morphism $a : Z \to A$. This induces a fibration

$$\overline{a} : Z/\langle g \rangle \to A/\langle g \rangle.$$

On the other hand, since X and $Z/\langle g \rangle$ are birational and since both of them have only rational singularities, it follows that $h^1(\mathcal{O}_{Z/\langle g \rangle}) = h^1(\mathcal{O}_X) = 0$. This implies $A/\langle g \rangle = \mathbb{P}^1$. Thus, $A^{\langle g \rangle} \neq \phi$. Since A is an elliptic curve, this is equivalent to $A^g \neq \phi$. Hence we can choose such a point 0 in A^g. Q.E.D. of (3.1).

Let us take $a^{-1}(0)$ as S. Then g induces an action $g_S := g|_S : S \to S$. Since g acts on the fiber space $f : Z \to T$, $\langle g_S \rangle$ and $\langle g \rangle$ give an equivariant action on $q_T := f|_S : S \to T$. Note that q_T is nothing but the quotient map $S \to T = S/G$.

Lemma (3.2). *$g_S^*\omega_S = \zeta_I\omega_S$, where ω_S is a nowhere vanishing regular two form on S, that is, a generator of $H^0(S, \mathcal{O}_S(K_S))$.*

Proof. Let ω_T be a nowhere vanishing regular two form on T. Then, $\omega_S := q_T^*\omega_T$ is a nowhere vanishing regular two form on S. Thus,

$$g_S^*\omega_S = g_S^* \circ q_T^*\omega_T = q_T^* \circ g^*\omega_T = q_T^*\zeta_I\omega_T = \zeta_I\omega_S.$$

This implies the result. Q.E.D. of (3.2).

Lemma (3.3). *There is an automorphism $g_{E\times S}$ of $E \times S$ such that $g_{E\times S}$, g_S and g give an equivariant action on the commutative diagram*

$$\begin{array}{ccc} E \times S & \xrightarrow{p_2} & S \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q_T} \\ Z & \xrightarrow[f]{} & T \end{array}$$

where q and q' are natural quotient maps.

Proof. Let us consider the fiber product

$$\begin{array}{ccc} Z \times_T S & \xrightarrow{p_2} & S \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q_T} \\ Z & \xrightarrow[f]{} & T \end{array}$$

Define the action of $\langle g' \rangle$ on $Z \times_T S$ by

$$g' : Z \times_T S \ni (u, v) \mapsto (g(u), g_S(v)) \in Z \times_T S.$$

Then, g', $\langle g_S \rangle$ and $\langle g \rangle$ give an equivariant action on this fiber product.

By the definition of fiber product, there is a surjective morphism $\nu : E \times S \to Z \times_T S$ which factors through the quotient map $q : E \times S \to Z = (E \times S)/G$ and the second projection $p_2 : E \times S \to S$.

Claim (3.4). *$\nu : E \times S \to Z \times_T S$ is the normalization of $Z \times_T S$.*

Proof of (3.4). Obvious. q.e.d. of (3.4).

Since normalization is an intrinsically and uniquely defined notion, the action $\langle g' \rangle$ on $Z \times_T S$ lifts to the action $\langle g_{E\times S} \rangle$ on $E \times S$ equivariantly with respect to $\nu : E \times S \to Z \times_T S$. This gives a desired action on $E \times S$. Q.E.D. of (3.3).

Corollary (3.5). $\mathrm{ord}\,(g_S) = \mathrm{ord}\,(g_{E\times S}) = I (:= \mathrm{ord}\,(g))$.

Proof. Since g_S is a restriction of g, it follows that $\mathrm{ord}\,(g_S) \leq \mathrm{ord}\,(g)$. On the other hand, since $\tau : S \to T$ is surjective and since g_S and g induce an equivariant action on τ, we see that $\mathrm{ord}\,(g_S) \geq \mathrm{ord}\,(g)$. This implies $\mathrm{ord}\,(g_S) = \mathrm{ord}\,(g)$. Now it follows from the construction of $g_{E\times S}$ that $\mathrm{ord}\,(g_{E\times S}) = \mathrm{ord}\,(g') = \mathrm{ord}\,(g)$. Q.E.D. of (3.5).

Define $\tilde{G}$ to be the subgroup of $\mathrm{Aut}\,(E \times S)$ generated by G and $g_{E\times S}$ found in (3.3). Then $\tilde{G}$ acts on the fiber space $p_2 : E \times S \to S$. Thus, there is a (unique) group homomorphism $\rho : \tilde{G} \to \mathrm{Aut}\,(S)$ such that $p_2 \circ h = \rho(h) \circ p_2$. By construction, we have $\rho(G) = G_S$ and $\rho(g_{E\times S}) = g_S$. Corollary (3.5) shows that $\rho|_{\langle g_{E\times S}\rangle} : \langle g_{E\times S}\rangle \to \langle g_S\rangle$ is a group isomorphism as is $\rho|_G : G \to G_S$. Set $\tilde{G_S} = \rho(\tilde{G})$.

Lemma (3.6).

(1) *G_S is a normal subgroup of $\tilde{G_S}$.*
(2) $\tilde{G_S} = G_S \rtimes \langle g_S\rangle$.
(3) *G is a normal subgroup of $\tilde{G}$.*
(4) $\tilde{G} = G \rtimes \langle g_{E\times S}\rangle$.
(5) *$\rho : \tilde{G} \to \tilde{G_S}$ is an isomorphism.*

Proof. For the assertion (1), it is enough to show that there is an $h' \in G_S$ such that $g_S \circ h = h' \circ g_S$ for each $h \in G_S$. Let $s \in S$ be a point on S such that $g_S(s) \notin S^{G_S}$. Using $g \circ q_T = q_T \circ g_S$ and $T = S/G_S$, we calculate

$$q_T \circ g_S \circ h(s) = g \circ q_T \circ h(s) = g \circ q_T(s) = q_T \circ g_S(s).$$

Thus, for each $s \in S$, there is $h_s \in G_S$ such that $g_S \circ h(s) = h_s \circ g_S(s)$. Such an h_s is uniquely determined by s because $g_S(s) \notin S^{G_S}$. Thus, we find a continuous map $S - R \to G_S$ defined by $s \mapsto h_s$. Since G_S is discrete, the image must be one point, say h'. Then, $g_S \circ h = h' \circ g_S$ over $S - g_S^{-1}(S^{G_S})$. Taking the closure, we find that $g_S \circ h = h' \circ g_S$ whole over S. This finishes the proof of (1).

Applying the same argument for $E \times S \to (E \times S)/G = Z$ (instead of $T = S/G_S$), we can also show assertion (3).

We show assertion (2). By (1), we have $\tilde{G_S}/G_S = \langle g_S (\mathrm{mod}\, G_S)\rangle$. Consider the natural representation $\tilde{G_S}$ on $H^0(S, \mathcal{O}_S(K_S))$

$$\zeta : \tilde{G} \to \mathbb{C}^\times, h \mapsto \zeta(h)$$

defined by $h^*\omega_S = \zeta(h)\omega_S$. Since G_S is a Gorenstein automorphism group of S, this factors

$$\overline{\zeta} : \tilde{G_S}/G_S = \langle g_S (\mathrm{mod}\, G_S)\rangle \to \mathbb{C}^\times.$$

Since $\overline{\zeta}(g_S(\mathrm{mod}\, G_S)) = \zeta(g_S) = \zeta_I$ by (3.3), it follows that $\mathrm{ord}\,(g_S(\mathrm{mod}\, G_S)) \geq I = \mathrm{ord}\,(g_S)$. Thus, the natural surjective group homomorphism $\langle g_S\rangle \to \langle g_S(\mathrm{mod}\, G_S)\rangle$ must be isomorphism. This implies the assertion (2).

Finally, we show assertions (4) and (5).

By (3), we see that $\tilde{G}/G \simeq \langle g_{E\times S}(\mathrm{mod}\, G)\rangle$. Combining this with (3.5), we get

$$\sharp\tilde{G} = (\sharp G) \cdot (\sharp\langle g_{E\times S}(\mathrm{mod}\, G)\rangle) \leq (\sharp G) \cdot (\sharp\langle g\rangle).$$

On the other hand, by (2) and (3.5), we have

$$\sharp\tilde{G_S} = (\sharp G_S) \cdot (\sharp\langle g_S\rangle) = (\sharp G) \cdot (\sharp\langle g\rangle).$$

However, since $\tilde{G_S}$ is an image of $\tilde{G}$, it follows that

$$\sharp\tilde{G} \geq \sharp\tilde{G_S}.$$

Combining these three we get $\sharp\tilde{G} = \sharp\tilde{G_S}$. This implies that the surjective group homomorphism $\rho : \tilde{G} \to \tilde{G_S}$ is an isomorphism. Combining this together with (2), we get $\tilde{G} = G \rtimes \langle g_{E\times S}\rangle$. This completes the proof. Q.E.D. of (3.6).

From now on, we denote the equivariant actions $\tilde{G}$ and $\tilde{G_S}$ on the fiber space $p_2 : E \times S \to S$ simply by $\tilde{G}$. We also set $\tilde{g} := g_{E\times S}$ for consistency of notation. If no confusion seems to arise, we also identify g_S and G_S with $\tilde{g}$ and G (under the isomorphism ρ).

The following corollary is an immediate consequence of Lemma (3.6).

Corollary (3.7).

$$(f : Z \to T)/\langle g\rangle = (p_2 : E \times S \to S)/\tilde{G}.$$

Thus, the fiber space $\Phi : X \to W$ is birational to $(p_2 : E \times S \to S)/\tilde{G}$ over $W = S/\tilde{G}$ and is isomorphic over W_0.

Now this together with the next lemma and the corollary completes the proof of Main Theorem (2) modulo impossibility for S to be a smooth Abelian surface.

Lemma (3.8). *Assume that S is a K3 surface with only Du Val singularities. Then, the action of $\tilde{g}$ on $E \times S$ is written as follows:*

$$\tilde{g} : E \times S \ni (x, y) \mapsto (\zeta_I^{-1}x, g_S(y)) \in E \times S$$

for an appropriate origin 0 of E.

Proof. Since $\langle\tilde{g}\rangle$ acts on $p_2 : E \times S \to S$, there is a homomorphic map

$$c : S \to \mathrm{Aut}\,(E) = E \rtimes \mathrm{Aut}\,(E, \{0\})$$

defined by $s \mapsto (p_1((x, s)) \mapsto p_1(\tilde{g}(x, s)))$.

On the other hand, since $h^1(\mathcal{O}_S) = 0$ and S has only Du Val singularities, the Albanese variety of S is trivial. Thus c must be constant map. That is, $\tilde{g} = (g_E, g_S)$ for some $g_E \in \mathrm{Aut}\,(E)$. Since X is isomorphic to $(E \times S)/\tilde{G}$ over W_0 and since $(E \times S)/\tilde{G} \to W$ is equidimensional, $\mathcal{O}_X(K_X) \simeq \mathcal{O}_X$ implies $\mathcal{O}_{(E\times S)/\tilde{G}}(K_{(E\times S)/\tilde{G}}) \simeq \mathcal{O}_{(E\times S)/\tilde{G}}$. This means $\tilde{G}$ is a Gorenstein automorphism of $E \times S$. In particular, so is $\tilde{g}$. Combining this with $g_S^*\omega_S = \zeta_I\omega_S$, we get $g_E^*\omega_E = \zeta_I^{-1}\omega_E$. In particular, $E^{g_E} \neq \phi$. Now, choosing the origin 0 of E in E^{g_E}, we get the desired expressions of $\tilde{g}$. This completes the proof of (3.8). Q.E.D.

Combining (3.8) and (3.7), we get

Corollary(3.9). *Assume that S is a K3 surface with only Du Val singularities. Then,*

1. *the global canonical index $I = I(W)$ of W is either 2, 3, 4, or 6,*
2. *if $\nu : S' \to S$ is a minimal resolution of S, then the action $\langle\tilde{g}\rangle$ on $E \times S$ lifts to $E \times S'$ in an equivariant way and $\Phi : X \to W$ is birational to*

$$(p_2 \circ (id. \times \nu) : E \times S' \to S)/\tilde{G}$$

over $W = S/\tilde{G}$ and is isomorphic over W_0.

§4. Impossibility for S to be a smooth abelian surface

We continue to employ the same notation given in the previous sections 2 and 3. In this section, we show that each surface S (found at the beginning of section 3) is not a smooth abelian surface if $\Phi : X \to W$ is a Calabi-Yau threefold of type $\mathrm{II}_0 K$. This completes the proof of Main Theorem (2).

Thoughout this section, assuming the contrary that S is a smooth abelian surface, we shall derive a contradiction.

For simplicity, we denote $\tilde{G_S}$, G_S and g_S by $\tilde{G}$, G and $\tilde{g}$ respectively. Under this notation, we have $T = S/G$, $W = T/\langle g \rangle = S/\tilde{G}$ and $I = \mathrm{ord}\,(g) = \mathrm{ord}\,(\tilde{g})$. As before, we denote by $q_T : S \to T$ the natural quotient morphism. This has an equivariant action of $\langle \tilde{g} \rangle$ and $\langle g \rangle$. Recall also that all the possibilities of G are listed up in (1.3)(4).

The next Lemma is shown by [O2].

LEMMA (4.1). *I is either 2, 3, 4, 6, or 12.*

By virtue of this Lemma, the next two Claims will give a contradiction.

KEY CLAIM (4.2). *I is not divided by 2.*

KEY CLAIM (4.3). *$I \neq 3$.*

The following obvious lemma and its corollaries will be frequently used to prove these claims.

LEMMA (4.4). *Let $q : S_1 \to S_2$ be a surjective finite morphism between normal projective surfaces with $K_{S_1} \equiv 0$ and $K_{S_2} \equiv 0$. Then q ramifies only at finitely many points.*

COROLLARY (4.5). *The quotient map $S \to W (= S/\tilde{G})$ ramifies only at finitely many points. In particular, $S^{\tilde{G_S}}$ is a finite set.*

COROLLARY (4.6). *Let h be a non-Gorenstein involution in $\tilde{G}$. Then, $S^h = \phi$. In particular, if $I = 2k$ is even, then $S^{\tilde{g}^k} = \phi$ and $S^{\tilde{g}} = \phi$.*

Proof. Assuming $S^h \neq \phi$, we take a point P in S^h. Since h is an involution with $h^*\omega_S = -\omega_S$, it follows that $h = \mathrm{diag}\,(-1, 1)$ under appropriate coordinates (x, y) of S around P. But then h would have a fixed curve $(x = 0)$, contradiction. q.e.d. of (4.6).

COROLLARY (4.7). *If I is either 2, 3, or 4, then $T^g \neq \phi$. If $I = pq$ where $p = 2$ or 4 and $q = 3$, then $T^{g^p} \neq \phi$ and $T^{g^q} \neq \phi$. Moreover, if I is either 2 or 4, then $(\phi \neq) T^g \subset \mathrm{Sing}\,(T)$.*

Proof. Since I is the least common multiple of the local canonical indices of W, the first part of the assertion is obvious. Assume that I is either 2 or 4. The first half part shows $T^g \neq \phi$. Assume the contrary that there is a smooth point Q in T^g. Then, arguing similarly as in (4.6), we see that $g^{I/2} = \mathrm{diag}\,(-1, 1)$ under appropriate local coordinates around P. Then, $g^{I/2}$ has a fixed curve. On the other hand, Lemma (4.4) shows $T \to W (= T/\langle g \rangle)$ has no ramification divisor, contradiction. q.e.d. of (4.7).

We return back to the key claims (4.2) and (4.3).

Proof of Key Claim (4.2).

Assume the contrary that $I = 2k$ for some integer k. We set $h := \tilde{g}^k$. Then h is a non-Gorenstein involution on S. Dividing into the following five cases, we shall derive a contradiction:
Case 1. $G \simeq \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$,
Case 2. $G \simeq \mathbb{Z}_6$,
Case 3. $G \simeq \mathbb{Z}_2$,
Case 4. $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$,
Case 5. $G \simeq \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$.

Case 1. Since g acts on the set B consisting of nine singular points of type A_3 on T ((1.3)(4)), $\langle \tilde{g} \rangle$ acts on $q_T^{-1}(B)$. Since $\sharp q_T^{-1}(B)$ is either 9 or 27, h has a fixed points. This contradicts (4.6).

Case 2. Consider the unique singular point Q of type A_5 on T ((1.3)(4)). Then, $q_T^{-1}(Q)$ consists of one point, say, P. Since $g(Q) = Q$, it follows that $\tilde{g}(P) = P$. But this contradicts (4.6).

Case 3. By (4.7), $T^{g^k} \neq \phi$. On the other hand, since g^k is a non-Gorenstein involution on T, the same argument as in (4.7) implies that $T^{g^k} \subset \mathrm{Sing}\,(T)$. Let $Q \in T^{g^k}$. Then Q is a singular point of type A_1 and then $q_T^{-1}(Q)$ consists of one point, say, P ((1.3)(4)). But then $h(P) = P$, contradiction.

Case 4. The same argument as in case 3 shows that $T^{g^k} \neq \phi$ and $T^{g^k} \subset \mathrm{Sing}\,(T)$. Let $Q \in T^{g^k}$. Then, Q is a singular point of type A_1 and $q_T^{-1}(Q)$ is written as $\{P, r(P)\}$ for some point P and a translation r in G ((1.3)(4)). Since h acts on this set, we have either $h(P) = P$ or $h(P) = r(P)$. The first equality contradicts (4.6). Consider the second case. Set $h' := r \circ h$. Then $h^*\omega_S = -\omega_S$. Since the translation subgroup of G is just $\langle r \rangle$ and since $h^{-1} \circ r \circ h$ is a translation in G (because G is a normal subgroup of $\tilde{G}$), it follows that $h^{-1} \circ r \circ h \in \langle r \rangle$ and then $\langle r, h \rangle = \langle r \rangle \times \langle h \rangle \simeq (\mathbb{Z}_2)^2$. Thus h' is a non-Gorenstein involution with $h'(P) = P$. But this contradicts (4.6).

Case 5. We treat the following three cases separately:
Case 5a. $3|I$, *Case 5b.* $I = 4$, and *Case 5c.* $I = 2$.

Case 5a. In this case, $I = 6m$ for some integer m. Set $j := \tilde{g}^m$. This is of order 6. Since g acts on the set consisting of 4 singular points of type A_3 on T ((1.3)(4)), j^2 acts on the inverse image of these points. This consists of either 4 or 8 points. Thus, j^2 has a fixed point among these points. Let P be such a fixed point. Then, $j^2(P) = P$. Since $(j^2)^*\omega_S = \zeta_3\omega_S$ and j^2 has at most finite fixed points by (4.5), an easy coordinate calculation shows that $j^2 = \mathrm{diag}\,(\zeta_3^2, \zeta_3^2)$ under appropriate global coordinates (x, y) around P. Thus, the eigen value of the matrix part of j is in $\{\zeta_3, -\zeta_3\}$. Thus, j has a fixed point on S, say Q. Since $h = j^3$, Q is also a fixed point of h. But this contradicts (4.6).

Case 5b. By (4.7), we can take a point Q in T^g. Again by (4.7) and (1.3)(4), Q is either a singular point of type A_3 or of type A_1.

If Q is a singular point of type A_3, then $q_T^{-1}(Q)$ is written as $\{P\}$ (in the case when $G \simeq \mathbb{Z}_4$) and $\{P, r(P)\}$ for a translation r in G (in the case when $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_4$). In the first case, we have $\tilde{g}(P) = P$. But this contradicts (4.6). In the second case, we have either $\tilde{g}(P) = P$ or $\tilde{g}(P) = r(P)$. Since r is of order two, in each case, we get $h(P) = \tilde{g}^2(P) = P$, contradiction.

If Q is a singular point of type A_1, then $q_T^{-1}(Q)$ is written as $\{P, u^2(P)\}$ (if $G = \langle u \rangle \simeq \mathbb{Z}_4$) and $\{P, u^2(P), r(P), r \circ u^2(P)\}$ (if $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_4$). In the second case, r is the unique translation in G and u is some (suitable) generator of G.

In anyway, we have $\tilde{g}(P) = P$ or $\tilde{g}(P) = t(P)$, where t is an involution in G. Thus, $h(P) = \tilde{g}^2(P) = P$, contradiction.

Case 5c. First consider the case $G = \langle u \rangle \simeq \mathbb{Z}_4$.

Since $\tilde{G} = \langle u \rangle \rtimes \langle \tilde{g} \rangle$ is of order 8, elementary group theory shows that $\tilde{G}$ is isomorphic to either

(1) D_8, the dihedral group of order 8, or
(2) $\mathbb{Z}_4 \times \mathbb{Z}_2$.

Assume first that $\tilde{G} \simeq D_8$. Then, $\tilde{g} \circ u$ is a non-Gorenstein involution. Take a point Q in T^g. Then, Q is a singular point either of type A_3 or of type A_1.

If Q is of type A_3, then $q_T^{-1}(Q) = \{P\}$, a one point set. But then $\tilde{g}(P) = P$, contradiction.

If Q is of type A_1, then $q_T^{-1}(Q)$ is written as $\{P, u(P)\}$ and $\tilde{g}$ stabilizes this set. If $\tilde{g}(P) = P$, then we get the same contradiction as before. If $\tilde{g}(P) = u(P)$, then $\tilde{g} \circ u(P) = P$. Since $\tilde{g} \circ u$ is a non-Gorenstein involution, we again get a contradiction. In any case, we found a contradiction if $\tilde{G} \simeq D_8$.

Next consider the case when $\tilde{G} \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$, that is, $\tilde{G} = \langle u \rangle \times \langle \tilde{g} \rangle$. Then $\langle u \rangle \simeq \tilde{G}/\langle \tilde{g} \rangle$ acts on $\overline{p_2} : (E \times S)/\langle \tilde{g} \rangle \to S/\langle \tilde{g} \rangle$. Note that $(E \times S)/\langle \tilde{g} \rangle$ is also a smooth threefold, because $S^{[\langle \tilde{g} \rangle]} = \phi$ by (4.6) so that $(E \times S)^{[\langle \tilde{g} \rangle]} = \phi$.

CLAIM. $(E \times S/\langle \tilde{g} \rangle)^{[\langle u \rangle]} = \phi$.

Proof of Claim. Since u is of order 4, it is sufficient to show that

$$(E \times S/\langle \tilde{g} \rangle)^{u^2} = \phi.$$

Assume the contrary that $P \in (E \times S/\langle \tilde{g} \rangle)^{u^2}$. Set $\overline{p_2}(P) = Q$. Then $u^2(Q) = Q$. Thus u^2 acts on the fiber $E_Q := (\overline{p_2})^{-1}(Q)$. On the other hand, the fiber of $E \times S \to (E \times S/\langle \tilde{g} \rangle)$ over Q is written as $\{R, \tilde{g}(R)\}$ and u^2 also acts on this set. If $u^2(R) = \tilde{g}(R)$, then $u^2 \circ \tilde{g}(R) = R$ on S. But, since $u^2 \circ \tilde{g}$ is a non-Gorenstein involution on S, this contradicts (4.6). Thus $u^2(R) = R$. Let E_R be the fiber of $p_2 : E \times S \to S$ over R. Then the natural projection $E \times S \to E \times S/\langle \tilde{g} \rangle$ (of degree two) induces an isomorphism $E_R \simeq E_Q$, because $E_{\tilde{g}(R)}$ is also mapped to E_Q. Since u^2 gives an equivariant action on this isomorphism and since u^2 acts on E_R as a translation of order two by (1.3), we see that u^2 also acts on E_Q as a translation of order two. Thus $E_Q^{u^2} = \phi$. But this is absurd, because $P \in E_Q$ is a fixed point of u^2. q.e.d. of Claim.

Thus $Y := ((E \times S)/\langle \tilde{g} \rangle)/\langle u \rangle = (E \times S)/\tilde{G}$ is also a smooth threefold (with $\mathcal{O}_Y(K_Y) \simeq \mathcal{O}_Y$). Since X is birational to Y, X is connected with Y by flops. Then

X is also smooth and $\pi_1(X) \simeq \pi_1(Y)$ ([Ko2]). Thus X has a non-trivial finite étale covering, because so does Y. But this contradicts our assumption $\pi_1^{alg}(X) = \{1\}$. Therefore, we get a contradiction even in the case $G \simeq \mathbb{Z}_4$.

We consider the remaining case $G = \langle t \rangle \times \langle u \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_4$. Reducing to the previous case $G \simeq \mathbb{Z}_4$, we find a contradiction.

Since the translation group of G is just $\langle t \rangle$ and since G is a normal subgroup of $\tilde{G}$, the same argument as before shows $\langle t \rangle$ is a normal subgroup of $\tilde{G}$. Thus $\tilde{G}/\langle t \rangle \simeq \langle u_1 \rangle \rtimes \langle \tilde{g_1} \rangle$, where $u_1 := u(\bmod \langle t \rangle)$ and $\tilde{g_1} := \tilde{g}(\bmod \langle t \rangle)$. Observe that u_1 is of order four and $\tilde{g_1}$ is of order two.

On the other hand, since $\langle t \rangle$ acts on $p_2 : E \times S \to S$, we get a new fiber space

$$\overline{p_2} : (E \times S)/\langle t \rangle \to S/\langle t \rangle,$$

on which $\langle u_1 \rangle \times \langle \tilde{g}_1 \rangle$ gives an equivariant action. Since $\langle t \rangle$ is a translation group on both $E \times S$ and S, it follows that $(E \times S)/\langle t \rangle$ is an Abelian threefold and $S/\langle t \rangle$ is an Abelian surface. Set $S_1 := S/\langle t \rangle$ and $V := (E \times S)/\langle t \rangle$. Then, $T = S_1/\langle u_1 \rangle$ and $W = S_1/\langle u_1, \tilde{g}_1 \rangle$.

Observe that $\tilde{g}_1^* \omega_{S_1} = -\omega_{S_1}$, $u_1^* \omega_{S_1} = \omega_{S_1}$ and that u_1 acts on each fiber over $S_1^{u_1} (\neq \phi)$ as a translation of order 4. The last statement follows from (1.3) and a similar argument as is given in the last claim. Thus we can apply the same argument as in the previous case ($G \simeq \mathbb{Z}_4$) for $\overline{p_2} : (E \times S)/\langle t \rangle \to S/\langle t \rangle$ and $S_1 \to T \to W$ to get a contradiction. This finishes the proof of case 5c.

Now we have completed the proof of (4.2). Q.E.D. of (4.2).

Proof of Key Claim (4.3).

Assuming the contrary that $I = 3$ and dividing into the following five cases, we shall derive a contradiction.

Case 1. $G \simeq \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$,
Case 2. $G \simeq \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$,
Case 3. $G \simeq \mathbb{Z}_6$,
Case 4. $G \simeq \mathbb{Z}_3$,
Case 5. $G \simeq \mathbb{Z}_3 \times \mathbb{Z}_3$.

Case 1. Since g acts on the set of singular points of type A_3 and since this set consists of 4 points, g has a fixed point, say Q, in this set. Then, $\tilde{g}$ acts on $q_T^{-1}(Q)$. Since $q_T^{-1}(Q)$ consists of one or two points, $\tilde{g}$ has a fixed point in $q_T^{-1}(Q)$. Denote this point by 0. Since $\tilde{g}^* \omega_S = \zeta_3 \omega_S$, $\tilde{g}(0) = 0$ and since $\tilde{g}$ has only finitely many fixed points, we can apply [CC, also O2] to get $S \simeq E_{\zeta_3}^2$ and $\tilde{g} = \zeta_3^2$, the scalar multiplication by ζ_3^2. On the other hand, the stabilizer of 0 in G is a cyclic group of order 4. We denote this group by $\langle u \rangle$. Then $u = \mathrm{diag}\,(\zeta_4, \zeta_4^{-1})$ under appropriate global coordinates around 0. Set $H := \langle u, \tilde{g} \rangle$. Then, $H \subset \mathrm{Aut}\,(S, \{0\})$. Moreover H is a cyclic group of order 12, because $\tilde{g} = \zeta_3^2$ so that $u \circ \tilde{g} = \tilde{g} \circ u$. In particular $H \ni -1$. But this is impossible by Fujiki's classification ([Fu, Table 6]).

Case 2. Just by the same argument as in case 1, we see that $\tilde{g}$ has a fixed point 0 (over some singular point of type A_1 of T) and then $S = E_{\zeta_3}^2$ and $\tilde{g} = \zeta_3^2$. Set $\mathrm{Stab}_{\{0\}}(G) = \langle u \rangle$. This is a cyclic group of order two and $u = \mathrm{diag}\,(-1, -1)$ under appropriate global coordinates around 0. Thus $u \circ \tilde{g} = \tilde{g} \circ u$. Since $\tilde{G}$ gives an equivariant action on $p_2 : E \times S \to S$, $\tilde{g}$ and u act on the fiber $E := p_2^{-1}(0)$. Since $\tilde{g}$ is a Gorenstein automorphism of $E \times S$, the matrix part of $\tilde{g}$ on E is ζ_3^2 so that $\tilde{g}$ acts on E by

$$\tilde{g} : E \ni x \mapsto \zeta_3^2 x \in E,$$

if we fix an origin 0_E of E in $E^{\tilde{g}} (\neq \phi)$. On the other hand, by (1.3), the action of u on E is written as

$$u : E \ni x \mapsto x + P \in E,$$

where $P \in (E)_2 - \{0\}$. Since $u \circ \tilde{g} = \tilde{g} \circ u$ in $\tilde{G}$, we calculate

$$\tilde{g}(x) + \tilde{g}(P) = \tilde{g} \circ u(x) = u \circ \tilde{g}(x) = \tilde{g}(x) + P.$$

Thus, $P \in E^{\tilde{g}} = E^{\zeta_3} \subset (E)_3$. But this is impossible because $(E)_3 \cap ((E)_2 - \{0\}) = \phi$.

Case 3. Let Q be the unique singular point of type A_5 on T. Then, $q_T^{-1}(Q)$ consists of one point, say, 0. Since $g(Q) = Q$, it follows that $\tilde{g}(0) = 0$. Thus, just by the same argument as before, we get $\tilde{g} = \zeta_3^2$. Set $\mathrm{Stab}_{\{0\}}(G) = \langle u \rangle$. This is a cyclic group of order 6 and $u = \mathrm{diag}\,(\zeta_6, \zeta_6^{-1})$ under an appropriate global coordinates (x, y) around 0. it follows that $\tilde{g} \circ u^2 = \mathrm{diag}\,(1, \zeta_3)$. Then $\tilde{g} \circ u^2$ has a fixed curve $(y = 0)$, contradiction.

Case 4. Set $G = \langle u \rangle$. Since $\tilde{G} = \langle u \rangle \rtimes \langle \tilde{g} \rangle$ is of order 9, it follows that $\tilde{G} = \langle u \rangle \times \langle \tilde{g} \rangle$. Let Q be a point in T^g. Then $\sharp q_T^{-1}(Q)$ is either one or three. If $q_T^{-1}(Q) = \{P\}$, a one point set, then $\tilde{g}(P) = P$. If $q_T^{-1}(Q) = \{P_1, P_2, P_3\}$, then, $\tilde{g}(P_1) = P_j$ for some $j = 1, 2$, or 3. Since $\langle u \rangle$ acts on $\{P_1, P_2, P_3\}$ transitively, we find that $u^i(P_1) = P_j$ for some i. Set $h := u^{-i} \circ \tilde{g}$. Then, $h(P_1) = P_1$. Note that h is of order 3 and satisfies $h^* \omega_S = \zeta_3 \omega_S$ and $\tilde{G} = \langle u \rangle \times \langle h \rangle$. In addition, h and g give an equivariant action on $q_T : S \to T$. Thus, we may replace $\tilde{g}$ by h in the second case. Then $\tilde{g}(P_1) = P_1$ in each case. We regard this point P_1 as an origin of S and denote it by 0_S.

Since $\tilde{g}$ has only isolated fixed points ((4.5)), the same argument as before shows that $S = E_{\zeta_3}^2$ and $\tilde{g} = \zeta_3^2$. This implies $(S)^{\tilde{g}} \cap (S)^u = \phi$. (In fact, otherwise, choosing a point P in $(S)^{\tilde{g}} \cap (S)^u$, we find appropriate coordinates (x, y) around P such that $u = \mathrm{diag}\,(\zeta_3, \zeta_3^{-1})$. Then, $\tilde{g} \circ u = \mathrm{diag}\,(1, \zeta_3)$ has a fixed curve $(y = 0)$, contradiction.)

Since $\tilde{G}$ is a Gorenstein automorphism of $E \times S$ and gives an equivariant action on $p_2 : E \times S \to S$, $\tilde{g}$ induces an automorphism on the fiber $E := p_2^{-1}(0_S)$ whose matrix part is ζ_3^2. Thus $E = E_{\zeta_3}$ and then $E \times S = E_{\zeta_3}^3$. Moreover, choosing an origin 0_E of E in $E^{\tilde{g}}$, we get $\tilde{g} = \zeta_3^2$ on E. Now, taking $0 := (0_S, 0_E)$ as an origin of $E \times S = E_{\zeta_3}^3$, we have $\tilde{g} = \zeta_3^2$ on $E_{\zeta_3}^3$. Let us consider the quotient threefolds $(E_{\zeta_3})^3 / \langle \tilde{g} \rangle$ and its crepant resolution $\nu : Y \to (E_{\zeta_3})^3 / \langle \tilde{g} \rangle$. Note that $\langle u \rangle \simeq \tilde{G} / \langle \tilde{g} \rangle$ acts on $(E_{\zeta_3})^3 / \langle \tilde{g} \rangle$. Note also that ν is unique. (In fact, one of such ν is given by replacing each of 27 singular points of type $1/3(1,1,1)$ of $(E_{\zeta_3})^3 / \langle \tilde{g} \rangle$ by $\mathbb{P}^2$ and then has no flopping curves in the exceptional divisor.) Thus, $\langle u \rangle$ induces a regular action on Y.

Claim. *$\langle u \rangle$ acts freely on Y.*

Proof of Claim. Since $\mathrm{ord}\,(u) = 3$, it is sufficient to show that $Y^u = \phi$. Assume the contrary that $P \in Y^u$. Put $Q := \nu(P)$. Then $u(Q) = Q$. Denote the natural quotient map $E_{\zeta_3}^3 \to (E_{\zeta_3})^3 / \langle \tilde{g} \rangle$ by τ. Then, $Q \notin \tau((E_{\zeta_3}^3)^{\tilde{g}})$. (In fact, otherwise,

$\tau^{-1}(Q) = \{R\}(\subset (E^3_{\zeta_3})^{\tilde{g}})$, a one point set. Thus, $u(R) = R$ and $\tilde{g}(R) = R$ on $(E_{\zeta_3})^3$. Set $R' := p_2(R)$. Then, $u(R') = R'$ and $\tilde{g}(R') = R'$, because $\tilde{G}$ gives an equivariant action on $p_2 : E \times S \to S$. But this contradicts $(S)^{\tilde{g}} \cap (S)^u = \phi$.)

Thus, $\tau^{-1}(Q)$ consists of three points, say, R_1, R_2 and R_3. Since $u(Q) = Q$, u acts on $\{R_1, R_2, R_3\}$. Since $\langle u \rangle$ acts freely on $E^3_{\zeta_3}$ by (1.3), we may assume without loss of generality that $u(R_1) = R_2$. On the other hand, $\{R_1, R_2, R_3\}$ is the orbit space of R_1 by $\langle \tilde{g} \rangle$, it follows that $\tilde{g}^i(R_1) = R_2$ for some $i = 1, 2$. Set again $R' := p_2(R_1)$. Then, $\tilde{g}^i(R') = u(R')(= p_2(R_2))$ so that $u^{-1} \circ \tilde{g}^i(R') = R'$. Since the matrix part of u^{-1} is $\mathrm{diag}\,(\zeta_3, \zeta_3^{-1})$ under some appropriate global coordinates around R', we calculate $u^{-1} \circ \tilde{g}^i = \mathrm{diag}\,(1, \zeta_3)$. Thus $u^{-1} \circ \tilde{g}^i$ has a fixed curve $(y = 0)$, contradiction. q.e.d. of Claim.

By this claim $Y/\langle u \rangle$ is a smooth threefold with $\mathcal{O}_{Y/\langle u \rangle}(K_{Y/\langle u \rangle}) \simeq \mathcal{O}_{Y/\langle u \rangle}$ and with non-trivial étale covering. On the other hand, by construction, our original Calabi-Yau threefold X is birational to Y and then is connected with Y by flops. Thus X is also smooth and $\pi_1(X) \simeq \pi_1(Y)$ by [Ko2]. This implies that X has also non-trivial finite étale covering. But this contradicts our assumption $\pi_1^{alg}(X) = \{1\}$.

Case 5. As in case (5c) in Claim (4.2), reducing to the previous case 4, we find a contradiction. Set $G = \langle t \rangle \times \langle u \rangle$, where t is a translation of order 3. Since the translation group of G is just $\langle t \rangle$, and G is a normal subgroup of $\tilde{G}$, the same argument as in case 4 in Claim (4.2) implies that $\langle t \rangle$ is a normal subgroup of $\tilde{G}$. Thus, $\tilde{G}/\langle t \rangle = \langle u_1 \rangle \times \langle \tilde{g_1} \rangle \simeq (\mathbb{Z}_3)^2$, where $u_1 := u(\mathrm{mod}\,\langle t \rangle)$ and $\tilde{g_1} := \tilde{g}(\mathrm{mod}\,\langle t \rangle)$.

By the way, since $\langle t \rangle$ acts on $p_2 : E \times S \to S$, we get a new fiber space

$$\overline{p_2} : (E \times S)/\langle t \rangle \to S/\langle t \rangle,$$

on which $\langle u_1 \rangle \times \langle \tilde{g}_1 \rangle$ gives an equivariant action. Since $\langle t \rangle$ is a translation group on both $E \times S$ and S, $(E \times S)/\langle t \rangle$ is an Abelian threefold and $S/\langle t \rangle$ is an Abelian surface. Set $S_1 := S/\langle t \rangle$ and $V := (E \times S)/\langle t \rangle$. Then, $T = S_1/\langle u_1 \rangle$ and $W = S_1/\langle u_1, \tilde{g}_1 \rangle$. Moreover $\tilde{g}_1^* \omega_{S_1} = \zeta_3 \omega_{S_1}$ while $u_1^* \omega_{S_1} = \omega_{S_1}$. Now applying the same argument as in case 4 for $S_1 \to T \to W$, we find that $S_1 = E^2_{\zeta_3}$ and $\tilde{g}_1 = \zeta_3^2$ (after replacing $\tilde{g}_1$ by $u_1^i \circ \tilde{g}_1$ so that $S_1^{\tilde{g}_1} \neq \phi$ and then fixing the origin 0 of S_1 in $S_1^{\tilde{g}_1}(\neq \phi)$). Note that $\langle u_1, \tilde{g}_1 \rangle$ gives a Gorenstein action on V. Then letting $E := \overline{p_2}^{-1}(0)$ and applying the same argument as in case 4, we see that $E = E_{\zeta_3}$ and the action of $\tilde{g}_1$ on E is $\tilde{g}_1 = \zeta_3^2$ (after fixing an origin 0_E of E in $E^{\tilde{g}_1}(\neq \phi)$). Thus, regarding 0_E as an origin 0 of V, we get $\tilde{g}_1 = \zeta_3^2$ under appropriate global coordinates around 0. This together with [CC also O2] implies $V = E^3_{\zeta_3}$. Now again applying the same argument as in case 4 for $\overline{p_2} : V \to S_1$, we finally get a contradiction that X is birational to a smooth threefold Y with non-trivial finite étale covering.

Now this completes the proof of Claim (4.3).

Now we are done. Q.E.D. of Main Theorem (2).

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Keiji Oguiso
Department of Mathematical Sciences
University of Tokyo
Komaba, Tokyo
Japan
oguiso@ms.u-tokyo.ac.jp