Compact Complex Manifolds with Numerically Effective Cotangent Bundles

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Abstract. We prove that a projective manifold of dimension $n = 2$ or 3 and Kodaira dimension 1 has a numerically effective cotangent bundle if and only if the Iitaka fibration is almost smooth, i.e. the only singular fibres are multiples of smooth elliptic curves ($n = 2$) resp. multiples of smooth Abelian or hyperelliptic surfaces ($n = 3$). In the case of a threefold which is fibred over a rational curve the proof needs an extra assumption concerning the multiplicities of the singular fibres. Furthermore, we prove the following theorem: let X be a complex manifold which is hyberbolic with respect to the Carathéodory-Reiffen-pseudometric, then any compact quotient of X has a numerically effective cotangent bundle.

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Introduction

It is a natural question in algebraic geometry to classify manifolds by positivity properties of their tangent resp. cotangent bundles. The first result of this kind was obtained by Mori who solved the Hartshorne-Frankel conjecture [Mo]: every projective n-dimensional manifold with ample tangent bundle is isomorphic to the complex projective space $\mathbb{P}_n$. A degenerate condition of ampleness is numerical effectivity. A line bundle L on a projective manifold X is called numerically effective (abbreviated "nef") if $L.C \geq 0$ for all curves $C \subset X$. A vector bundle E is said to be nef if the tautological quotient line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$, the projective bundle of hyperplanes in the fibres of E, is nef.

Taking the Hartshorne-Frankel conjecture as a guideline, Campana and Peternell considered projective manifolds whose tangent bundles are nef and classified them in dimension 2 and 3 [CP]. For dimension 3 this has been done by Zheng [Zh] too. In general, for arbitrary compact complex manifolds the "nefness" of the tangent bundle leads to strong structural constraints [DPS].

The purpose of this paper is to investigate some aspects of manifolds X whose cotangent bundles Ω^1_X are nef. In the first part we will give a characterization of 2 and 3 dimensional manifolds with Kodaira dimension $\kappa(X) = 1$ and nef cotangent bundle. We will prove:

THEOREM 1 *Let X be a minimal projective manifold of dimension $n = 2$ or 3 with $\kappa(X) = 1$ and let $\pi : X \to C$ be the Iitaka fibration of X. Then the following conditions are equivalent:*
(i) Ω^1_X *is nef.*
(ii) π *is almost smooth, in the sense that the only singular fibres of* π *are multiples of smooth elliptic curves* ($n = 2$) *resp. Abelian or hyperelliptic surfaces* ($n = 3$).
• *Exception: To prove* (ii)$\Rightarrow$(i) *in the case* $n = 3$ *and* $g(C) = 0$ *we need the assumption that* $\sum \frac{m_i - 1}{m_i} \geq 2$, *where the* m_i *are the multiplicities of the singular fibres.*
• *The equivalence of* (i) *and* (ii) *holds also for compact Kähler surfaces.*

This theorem generalizes a result of Fujiwara [Fu] who worked in arbitrary dimension but under the stronger assumption that Ω^1_X is semi-ample, i.e. that some power of $\mathcal{O}_{\mathbb{P}(\Omega^1_X)}(1)$ is globally generated. The implication (i) $\Rightarrow$ (ii) relies on the topological constraints, namely the Chern class inequalities, which hold, when the cotangent bundle is nef. To prove (ii) $\Rightarrow$ (i) we will proceed in two steps. First, we will show that the assertion is true for a smooth fibration. This follows basically from Griffiths's theory on the variation of the Hodge structure. Then, we will study the base-change which reduces an almost smooth fibration to a smooth one and show that this process allows to carry over the "nefness" of the cotangent bundle.
In fact, we will prove in any dimension that a projective manifold has a nef cotangent bundle if (a) it admits a smooth Abelian fibration over a manifold with nef cotangent bundle or (b) it admits an almost smooth Abelian fibration over a curve C such that either (i) $g(C) \geq 1$ or (ii) $g(C) = 0$ and $\sum \frac{m_i - 1}{m_i} \geq 2$.
We remark that the fibres F of the Iitaka fibrations in Theorem 1 are paraAbelian varieties, i.e. there exists an unramified cover $T \to F$ where T is an Abelian variety. In view of this, we expect in any dimension that a manifold with Kodaira dimension 1 has a nef cotangent bundle if and only if the Iitaka fibration is almost smooth with para-Abelian fibres.
In the second part of this paper we consider complex manifolds X which are hyperbolic with respect to the Carathéodory-Reiffen pseudometric. We will show :

THEOREM 2 *Let X be a complex manifold which is hyperbolic with respect to the Carathéodory-Reiffen pseudometric and let Q be a compact quotient of X with respect to a subgroup of the automorphism group of X which operates fixpointfree and properly discontinuously. Then* Ω^1_Q *is nef.*

In particular, any compact quotient of a bounded domain $G \subset \mathbb{C}^n$ possesses a nef cotangent bundle. Since the canonical bundle of such a quotient is ample, this yields a class of manifolds with maximal Kodaira dimension and nef cotangent bundle.
To prove theorem 2 we apply the technique of singular hermitian metrics which was developed by Demailly. The Carathéodory-Reiffen pseudometric of X defines a Finsler structure on the tangent bundle of Q and this gives us a singular hermitian metric on $\mathcal{O}_{\mathbb{P}(\Omega^1_Q)}(1)$. The hyperbolicity of X guarantees that this metric is continuous and that the associated curvature current is positive. These conditions imply that $\mathcal{O}_{\mathbb{P}(\Omega^1_Q)}(1)$ is nef.

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1 BASIC DEFINITIONS AND PROPERTIES

Let X and Y be compact complex manifolds and let L be a holomorphic line bundle on X.

DEFINITION 1 (i) *When X is projective, L is said to be nef, if $L \cdot C = \int_C c_1(L) \geq 0$ for every curve C in X.*
(ii) *Let X be an arbitrary compact complex manifold equipped with a hermitian metric ω. Then L is said to be nef, if for all $\epsilon > 0$ there exists a smooth hermitian metric h_ϵ on L such that the associated curvature form satisfies*

$$\Omega_{h_\epsilon}(L) \geq -\epsilon \cdot \omega.$$

(iii) *Let E be a holomorphic vector bundle on X and $\mathbb{P}(E)$ the projective bundle of hyperplanes in the fibres of E. Then we call E nef over X, if the tautological quotient line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef over $\mathbb{P}(E)$.*

We will frequently use the following propositions which are proved in [DPS].

PROPOSITION 1 *Let $f : Y \to X$ be a holomorphic map and let E be a holomorphic vector bundle over X. Then E nef implies f^*E nef, and the converse is true if f is surjective and has equidimensional fibres.*

PROPOSITION 2 *Let E and F be holomorphic vector bundles. Then*
(i) *E, F nef $\Rightarrow E \otimes F$ nef.*
(ii) *E nef $\Rightarrow \det(E)$ nef.*

PROPOSITION 3 *Let $0 \to F \to E \to Q \to 0$ be an exact sequence of holomorphic vector bundles. Then*
(i) *E nef $\Rightarrow Q$ nef.*
(ii) *F, Q nef $\Rightarrow E$ nef.*

Proposition 1 immediately implies

PROPOSITION 4 *Let Y be a finite unramified covering of X. Then Ω^1_X is nef if and only if Ω^1_Y is nef.*

A fibration of X over Y is a surjective holomorphic map $\pi : X \to Y$ whose fibres are connected. A point $x \in X$ is said to be critical if the tangent map $D\pi(x)$ has not maximal rank. The images $\pi(x) \in Y$ of the critical points are the critical values of π. They form a proper analytic subset of Y, i.e. in the case, where Y is a curve, a finite subset $\{a_1, \ldots, a_l\}$.
Let $y \in Y$ and let $\mathcal{J}$ be the ideal sheaf of y in $\mathcal{O}_Y$. Then the fibre X_y is the complex subspace $(\pi^{-1}(y), \mathcal{O}_X/\pi^*(\mathcal{J}) \cdot \mathcal{O}_X)$ of X, and a fibre X_y is singular if and only if y is a critical value. A fibration, for which $D\pi$ has maximal rank everywhere, is called smooth.
When we consider a fibration $\pi : X \to C$ over a curve C, we will always assume that C is smooth. Such a fibration is said to be almost smooth, if the only singular fibres of π are multiples of smooth irreducible subvarieties. Their multiplicities will be denoted by m_i with $1 \leq i \leq l$, so that the singular fibres are $X_{a_i} = m_i F_i$, where the F_i are smooth irreducible subvarieties.

We will denote the Kodaira dimension of X by $\kappa(X)$. Let X be a projective manifold with $\kappa(X) \geq 1$ for which a power of the canonical bundle is globally generated. Then for m big enough the $m-$canonical map gives us a holomorphic map $\pi : X \to Z$ where Z is a projective variety with $\dim Z = \kappa(X)$. Such a map π is called Iitaka fibration (cf. [Ue]).

2 MANIFOLDS WITH $\kappa = 1$ AND NEF COTANGENT BUNDLE

We will now prove

THEOREM 3 *Let X be a minimal projective manifold of dimension $n = 2$ or 3 with $\kappa(X) = 1$ and let $\pi : X \to C$ be the Iitaka fibration of X. Then the following conditions are equivalent:*
(i) *Ω^1_X is nef.*
(ii) *π is almost smooth, in the sense that the only singular fibres of π are multiples of smooth elliptic curves $(n = 2)$ resp. Abelian or hyperelliptic surfaces $(n = 3)$.*
• *Exception: To prove* (ii)⇒(i) *in the case $n = 3$ and $g(C) = 0$ we need the assumption that $\sum \frac{m_i - 1}{m_i} \geq 2$, where the m_i are the multiplicities of the singular fibres.*
• *The equivalence of* (i) *and* (ii) *holds also for compact Kähler surfaces.*

Proof: (i) ⇒ (ii) If X is an n-dimensional projective manifold with Ω^1_X nef, it satisfies the Chern class inequality $c_1(X)^2 \geq c_2(X) \geq 0$, i.e.

$$c_1(X)^2 \cdot H_1 \cdot \ldots \cdot H_{n-2} \geq c_2(X) \cdot H_1 \cdot \ldots \cdot H_{n-2} \geq 0$$

for all ample divisors H_i (cf. [DPS], Thm. 2.5). For $n = 2$ and 3 the abundance conjecture holds which means that a power of the canonical bundle of X has to be globally generated so that we get from $\kappa(X) = 1$ that $c_1(X)^2 \equiv 0$ and hence $c_1(X)^2 \equiv c_2(X) \equiv 0$. Here $\equiv$ denotes numerical equivalence.
So for $n = 2$ we have an elliptic surface X whose topological Euler characteristic is $e(X) = c_2(X) = 0$. On the other hand, if $\pi : X \to C$ is the Iitaka fibration of X and X_{a_i} are the singular fibres $(1 \leq i \leq l)$, we calculate $e(X) = \sum e(X_{a_i})$. But now the assertion follows, because $e(X_{a_i}) \geq 0$ and $e(X_{a_i}) = 0$ if and only if the fibre X_{a_i} is a multiple of a smooth elliptic curve (cf. [BPV], Chap. III, Prop. 11.4). This argument remains true for a compact Kähler surface.
For $n = 3$ we have a minimal threefold with the extremal Chern classes $c_1(X)^2 \equiv 3c_2(X) \equiv 0$ and the assertion follows from [PW], Theorem 2.1.
(ii) ⇒ (i) We will prove this direction by reducing it to the case of a smooth fibration.

2.1 SMOOTH FIBRATIONS

We will consider smooth Abelian fibrations first:

PROPOSITION 5 *Let X and Y be projective manifolds and let $\pi : X \to Y$ be a smooth fibration, whose fibres are Abelian varieties. Then the relative cotangent bundle $\Omega^1_{X/Y}$ is nef. If Ω^1_Y is nef, Ω^1_X is nef too.*

Proof: (1) We claim that $\pi^*(\pi_* \Omega^1_{X/Y}) = \Omega^1_{X/Y}$. For all $y \in Y$ the cotangent bundle of the fibre $\Omega^1_{X_y}$ is trivial, so that $\pi_* \Omega^1_{X/Y}$ is locally free of rank equal to the dimension

of the fibres (cf. [Ha], Chap. III, Cor. 12.9). Moreover for all $y \in Y$ we have $(\pi_*\Omega^1_{X/Y})_y \cong H^0(X_y, \Omega^1_{X_y})$ and thus $(\pi^*(\pi_*\Omega^1_{X/Y}))_x \cong H^0(X_y, \Omega^1_{X_y})$ for $\pi(x) = y$. Now, the canonical homomorphism $\alpha : \pi^*(\pi_*\Omega^1_{X/Y}) \to \Omega^1_{X/Y}$ is described stalkwise by $\alpha_x : \sigma \mapsto \sigma(x)$ with $\sigma \in H^0(X_y, \Omega^1_{X_y})$. Since $\Omega^1_{X/Y} \mid_{X_y}$ is globally generated, α_x is surjective and hence bijective.
(2) Any smooth fibration $\pi : X \to Y$ of projective manifolds gives rise to a variation of the Hodge structure in its fibres X_y $(y \in Y)$. From this Griffiths deduces [Gr], Cor. 7.8

THEOREM 4 *For all $n \in \{1, \ldots, \dim_{\mathbb{C}} X_y\}$ the bundles $R^n\pi_*(\mathcal{O}_X)$ are seminegative in the sense of Griffiths.*

Now the bundle $E = R^n\pi_*(\mathcal{O}_X)$ is conjugate linear to $\bar{E} = \pi_*(\Omega^n_{X/Y})$ so that the curvature matrices with respect to unitary bases behave as

$$\Omega_{\bar{E}} = \bar{\Omega}_E = -\Omega^t_E.$$

Since the transposition of the curvature matrix does not change its positivity properties, the preceding theorem can equivalently be formulated as

THEOREM 5 *For all $n \in \{1, \ldots, \dim_{\mathbb{C}} X_y\}$ the bundles $\pi_*(\Omega^n_{X/Y})$ are semipositive in the sense of Griffiths.*

In particular, since semipositivity implies "nefness", $\pi_*(\Omega^n_{X/Y})$ is nef and hence for a smooth Abelian fibration $\Omega^1_{X/Y} = \pi^*(\pi_*\Omega^1_{X/Y})$ is nef too. The second assertion follows immediately from the relative cotangent sequence and Proposition 3.

Remark: Proposition 5 holds also for compact elliptic surfaces $\pi : X \to C$, because for a smooth π one knows from the study of the period map that $\deg(\pi_*\omega_{X/C}) = 0$ (cf. [BPV], Chap. III, Thm. 18.2).
We have a similar proposition for smooth hyperelliptic fibrations:

PROPOSITION 6 *Let X be a projective 3-dimensional manifold and let $\pi : X \to C$ be a smooth fibration, whose fibres are hyperelliptic surfaces. Furthermore, let $g(C) \geq 1$. Then Ω^1_X is nef.*

Proof: We consider the relative Albanese factorization of π, i.e. the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{A_\pi} & A(X/C) \\ & {\scriptstyle\pi} \searrow & \downarrow {\scriptstyle Alb(\pi)} \\ & & C, \end{array}$$

where $A(X/C)$ is a smooth fibration over C whose fibres over $a \in C$ are the Albanese tori $\mathrm{Alb}(X_a)$ of the fibres X_a of π. The existence of such a relative Albanese diagram is proved in [Ca]. Since the tangent bundle of a hyperelliptic surface is nef, the Albanese map $A_\pi \mid_{X_a}: X_a \to \mathrm{Alb}(X_a)$ is a surjective submersion with smooth elliptic curves as fibres ([DPS], Prop. 3.9.). But also A_π is smooth: let $x \in X, \pi(x) = a$ and $A_\pi(x) = y$, then both tangent directions of $TA(X/C)_y$ lie in the image of $DA_\pi(x)$. First, we can find a tangent vector $v \in (TA(X/Y) \mid_{\mathrm{Alb}(X_a)})_y$ in the image of $DA_\pi(x) \mid_{X_a}$ (because $A_\pi \mid_{X_a}$ is smooth). Now let (x_1, x_2, x_3) be a coordinate system centered in x and let z_1 be a coordinate centered in a, such that $D\pi(x).\frac{\partial}{\partial x_1} = \frac{\partial}{\partial z_1}$. Using the commutativity of the relative Albanese diagram, we get

$$0 \neq D\pi(x).\frac{\partial}{\partial x_1} = DAlb(\pi)(y) \circ DA_\pi(x).\frac{\partial}{\partial x_1}.$$

In particular, $w := DA_\pi(x).\frac{\partial}{\partial x_1} \neq 0$, and since $DAlb(\pi)(y).v = 0$ the vectors v and w have to be linear independent.
We can now apply Proposition 5 twice to conclude that Ω^1_X is nef: $Alb(\pi) : A(X/C) \to C$ is a smooth fibration of projective manifolds whose fibres are elliptic curves and by assumption $g(C) \geq 1$, so that $\Omega^1_{A(X/C)}$ has to be nef. Since $A_\pi : X \to A(X/C)$ is a smooth elliptic fibration too, Ω^1_X is also nef.

2.2 ALMOST SMOOTH FIBRATIONS

Let X be a compact complex manifold of dimension n and let $\pi : X \to C$ be an almost smooth fibration over a smooth curve C. As above we will denote the critical values of π by $a_1, \dots, a_l$ and their multiplicities bym_i where $1 \leq i \leq l$, so that the singular fibres are $X_{a_i} = m_i F_i$, where the F_i are smooth irreducible subvarieties.
To get rid of the multiple fibres we will now perform a base change which was introduced by Kodaira for elliptic surfaces ([Kod], Thm 6.3), but may be used in this general context as well. Let m_0 be the lowest common multiple of the multiplicities and let d be their product. Then we choose a finite covering $\sigma : C' \to C$, which has $\frac{d}{m_i}$ ramification points of order $m_i - 1$ over the points a_i where $0 \leq i \leq l$. Remark that we have to add one extra point a_0 which is not contained in the set of critical values. Then the normalization of the fibre product $X \times_C C'$ gives us a smooth fibration $\varphi : X' \to C'$ and a commutative diagram (cf. [Kod], Thm 6.3)

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \varphi \downarrow & & \downarrow \pi \\ C' & \xrightarrow{\sigma} & C \end{array} .$$

Here f is a finite covering which is unramified over $X - \pi^{-1}(a_0)$, because the multiplicities of π and σ compensate each other over a_i $(i \geq 1)$, and f has $\frac{d}{m_0}$ ramification divisors of order $m_0 - 1$ over $\pi^{-1}(a_0)$.
Assume that we knew $\Omega^1_{X'}$ is nef, then we would like to carry this over to Ω^1_X. However, it is not possible to apply Proposition 4 since f is ramified. But we have the following commutative diagram with exact rows which was already used in [Fu]

$$\begin{array}{ccccccc} 0 \longrightarrow & f^*(L) & \longrightarrow & f^*(\Omega^1_X) & \longrightarrow & \Omega^1_{X'/C'} & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \| & \\ 0 \longrightarrow & \varphi^*(K_{C'}) & \longrightarrow & \Omega^1_{X'} & \longrightarrow & \Omega^1_{X'/C'} & \longrightarrow 0. \end{array}$$

Let $D = \sum_{i=1}^{l} (m_i - 1) F_i$ then $L = \pi^*(K_C) \otimes \mathcal{O}_X(D)$ is the full subbundle of Ω^1_X associated to $\pi^*(K_C)$ (cf. [Re]). To prove the commutativity of this diagram one uses basically the fact that the restriction of f to a fibre of φ is unramified. For $i \geq 1$ we have $\pi^*(a_i) = m_i F_i$. So, defining $A := \sum_{i=1}^{l} \frac{(m_i - 1)}{m_i} \cdot a_i$ we get $L = \pi^*(K_C \otimes \mathcal{O}_C(A))$. Combining the diagram and Proposition 5, we obtain

COROLLARY 1 *Let X be a projective manifold of arbitrary dimension and let $\pi : X \to C$ be an almost smooth fibration, whose fibres are Abelian varieties. Assume furthermore that* (i) $g(C) \geq 1$ *or* (ii) $g(C) = 0$ *and* $\deg A \geq 2$. *Then Ω^1_X is nef.*

Proof: The process described above allows us to pass to a smooth Abelian fibration φ, for which $\Omega^1_{X'/C'}$ is nef by Proposition 5. Moreover the line bundle $L = \pi^*(K_C \otimes A)$ is nef, since our assumptions guarantee that $\deg(K_C \otimes A) = 2g(C) - 2 + \deg A \geq 0$. If L is nef, then $f^*(L)$ and $f^*(\Omega^1_X)$ are nef (Proposition 3). Since f is a finite surjective map, we finally deduce from Proposition 1 that Ω^1_X is nef.

Remark: (i) The corollary holds for arbitrary compact surfaces too, because Proposition 5 remains true in that case.
(ii) If S is a surface with $\kappa(S) = 1$ and $\pi : S \to \mathbb{P}_1$ is an almost smooth elliptic fibration, the condition that $\deg A \geq 2$ (resp. that L is nef) is automatically satisfied. We have $\deg(\pi_*(\omega_{S/\mathbb{P}_1})) = 0$ and therefore $\pi_*(\omega_{S/\mathbb{P}_1}) = \mathcal{O}_{\mathbb{P}_1}$ (cf. [BPV]). Now the formula for the canonical bundle of an elliptic fibration yields $K_S = \pi^*(K_{\mathbb{P}_1}) \otimes \mathcal{O}_S(D)$, so that $L = K_S$ is nef since $\kappa(S) = 1$.

Similarly we get

COROLLARY 2 *Let X be a projective 3-dimensional manifold with $\kappa(X) \geq 0$ and let $\pi : X \to C$ be an almost smooth fibration, whose fibres are hyperelliptic surfaces. Assume furthermore that* (i) $g(C) \geq 1$ *or* (ii) $g(C) = 0$ *and* $\deg A \geq 2$. *Then Ω^1_X is nef.*

Proof: To deduce from Proposition 6 that $\Omega^1_{X'/C'}$ is nef as a quotient of $\Omega^1_{X'}$, we have to assure that $g(C') \geq 1$. But $g(C') = 0$ leads to $-\infty = \kappa(X') \geq \kappa(X)$ which contradicts our assumptions.

In particular, these two corollaries yield the direction (ii)$\Rightarrow$ (i) in Theorem 3 which is now completely proved.

3 QUOTIENTS WITH NEF COTANGENT BUNDLE

The goal of this section is to prove that compact quotients of a manifold which is hyperbolic with respect to the Carathéodory-Reiffen pseudometric have a nef cotangent bundle. We will use the notion of singular hermitian metrics as introduced in [De1]:

DEFINITION 2 *Let L be a holomorphic line bundle over a compact complex manifold X and let $\theta_\alpha : L|_{U_\alpha} \xrightarrow{\simeq} U_\alpha \times \mathbb{C}$ be a local trivialization of L. Then a singular hermitian metric on L is given by*

$$\|\xi\| = |\theta_\alpha(\xi)| \cdot e^{-\varphi_\alpha(x)}, \qquad x \in U_\alpha, \quad \xi \in L_x,$$

where $\varphi_\alpha \in L^1_{loc}(U_\alpha)$ is an arbitrary real valued function, called the weight function of the metric with respect to the trivialization θ_α.

The curvature form of the singular metric on L is locally given by the closed $(1,1)$-current $c(L) = \frac{i}{\pi}\partial\bar{\partial}\varphi_\alpha$. We will write $c(L) \geq 0$, if $c(L)$ is a positive current in the sense of distribution theory, i.e. if the weight functions φ_α are plurisubharmonic.
Remark: We will say that a singular metric is continuous (or simply that it is a continuous metric), if the weight functions φ_α are continuous on the trivialization sets.
The main ingredient for the following arguments will be the next proposition which is independently due to Demailly, Shiffman and Tsuji (see e.g. [De2])

PROPOSITION 7 *Let L be a holomorphic line bundle on a compact complex manifold X. Then L is nef, if there exists a continuous metric with $c(L) \geq 0$.*

In fact the proposition is even true in the case where the Lelong numbers of the metric (which are zero everywhere for a continuous metric) are zero except for a countable set of points (cf. Thm. 4.2 in [JS]).
Let E be a holomorphic vector bundle over a compact complex manifold X. As in [Rei] and [Ko] we define

DEFINITION 3 *A Finsler structure on E is a continuous function $F : E \to \mathbb{R}_{\geq 0}$, so that for all $\eta \in E$:*
(i) *$F(\eta) > 0$ for $\eta \neq 0$,*
(ii) *$F(\lambda\eta) = |\lambda| F(\eta)$ for all $\lambda \in \mathbb{C}$.*
If we require in (i) *only $\geq$, F is said to be a Finsler pseudostructure.*

Let $P(E)$ denote the projective bundle of *lines* in the fibres of E, $p : P(E) \to X$ the projection and $\mathcal{O}_{P(E)}(-1)$ the subbundle of p^*E whose fibre over a point in $P(E)$ is given by the complex line represented by that point. Then we have a map $\tilde{p} : \mathcal{O}_{P(E)}(-1) \to E$ which is biholomorphic outside the zero sections of $\mathcal{O}_{P(E)}(-1)$ and E. The set of all plurisubharmonic functions on a complex manifold Y will be denoted by $PSH(Y)$.

PROPOSITION 8 (a) *Any Finsler structure F on E defines via*

$$||\xi|| := F \circ \tilde{p}(\xi), \qquad \xi \in \mathcal{O}_{P(E)}(-1).$$

a continuous metric on $\mathcal{O}_{P(E)}(-1)$.
(b) *If $\log F \in PSH(E\backslash\{0\})$, then $-\varphi_\alpha \in PSH(U_\alpha)$.*

Proof: (a) Let $\theta_\alpha : \mathcal{O}_{P(E)}(-1)\mid_{U_\alpha} \xrightarrow{\simeq} U_\alpha \times \mathbb{C}$ be a local trivialization and let s_α be a local holomorphic section of $\mathcal{O}_{P(E)}(-1)\mid_{U_\alpha}$ which describes the trivialization. Then the corresponding weight function is

$$-\varphi_\alpha(x) = \log\|s_\alpha(x)\| = \log F \circ \tilde{p}(s_\alpha(x)), \qquad x \in U_\alpha.$$

The map $\tilde{p} \circ s_\alpha : U_\alpha \to E$ is clearly holomorphic. Moreover for $x \in U_\alpha$ we have $s_\alpha(x) \neq 0$, so that property (i) in the definition of Finsler structures leads to $F \circ \tilde{p}(s_\alpha(x)) > 0$. From this we conclude $-\varphi_\alpha \in C^0(U_\alpha)$.
(b) If $f : Y \to Z$ is a holomorphic map between complex manifolds and the function $u \in PSH(Z)$, then $u \circ f \in PSH(Y)$ (cf. [JP], Appendix, PSH 7). So, since $\tilde{p} \circ s_\alpha$ is holomorphic, we have $-\varphi_\alpha \in PSH(U_\alpha)$.

PROPOSITION 9 *Let $E \to X$ be a holomorphic vector bundle over a compact complex manifold X. If there exists a Finsler structure $F : E \to \mathbb{R}_{\geq 0}$ such that $\log F \in PSH(E \setminus \{0\})$, then E^* is nef.*

Proof: To prove that E^* is nef, we have to show that $L := \mathcal{O}_{P(E)}(1) \cong \mathcal{O}_{\mathbb{P}(E^*)}(1)$ is nef. According to Proposition 8 the Finsler structure $F : E \to \mathbb{R}_{\geq 0}$ induces a continuous metric on $\mathcal{O}_{P(E)}(-1)$ so that $-\varphi_\alpha \in PSH(U_\alpha)$. For the dual bundle $L = \mathcal{O}_{P(E)}(1)$ equipped with the dual metric the weight functions are given by $\varphi_\alpha^* = -\varphi_\alpha$, hence we have a continuous metric on L whose current is positive and the assertion follows from Proposition 7.

Let X be a connected complex manifold. A Finsler (pseudo-) structure on the tangent bundle TX is called a differential (pseudo-) metric. Any such X admits a differential pseudometric: for $p \in X$ and $\eta \in TX_p$ we define

$$\gamma_X(p, \eta) := \sup\{|Dg(p).\eta| : g \in \mathcal{O}(X, \Delta), g(p) = 0\},$$

where Δ is the open unit disc in $\mathbb{C}$ and $\mathcal{O}(X, \Delta)$ the set of all holomorphic maps from X to Δ. Reiffen shows in [Rei]:

PROPOSITION 10 *The map $\gamma_X : TX \to \mathbb{R}_{\geq 0}$ is a differential pseudometric, which has the following invariance property. Let $f : X \to Y$ be a holomorphic map of connected complex manifolds, then*

$$\gamma_Y(f(p), Df(p).\eta) \leq \gamma_X(p, \eta),$$

in particular, for a biholomorphic map f the equality holds.

The function γ_X is called the Carathéodory-Reiffen pseudometric and X is said to be γ-hyperbolic, if γ_X is a differential metric.
Examples: (i) Any bounded domain $G \subset \mathbb{C}^n$ is γ-hyperbolic (cf. [JP], Chap. II, Prop. 2.3.2).
Proposition 10 immediately implies: let $i : X \to Y$ be a holomorphic immersion and let Y be γ-hyperbolic, then X is γ-hyperbolic too. This gives us
(ii) Let Y be a Stein manifold and let $\tilde{G}$ be a bounded domain in Y, i.e. there exists an embedding $Y \hookrightarrow \mathbb{C}^N$ and a bounded domain $G \subset \mathbb{C}^N$, such that $\tilde{G} = Y \cap G$ is connected. Then $\tilde{G}$ is γ-hyperbolic.

PROPOSITION 11 *Let X be a γ-hyperbolic manifold. Then the function*

$$\log \gamma_X : TX \setminus \{0\} \to (-\infty, +\infty)$$

is plurisubharmonic.

Proof: Since the logarithm is strictly increasing, we have

$$\log \gamma_X(p, \eta) = \sup\{\log |Dg(p).\eta| : g \in \mathcal{O}(X, \Delta), g(p) = 0\}.$$

The tangent map of a holomorphic map is again holomorphic, so that $\tilde{g}(p, \eta) := \log |Dg(p).\eta|$ is in $PSH(TX)$ (see [JP], Appendix, PSH 4). Hence $\log \gamma_X = \sup_g \{\tilde{g}\}$ is the supremum of plurisubharmonic functions. By assumption γ_X is a differential metric, i.e. γ_X is continuous and $\gamma_X : TX\setminus\{0\} \to \mathbb{R}_{>0}$, thus $\log\gamma_X : TX\setminus\{0\} \to (-\infty,\infty)$ is also continuous. Now we get our assertion from the following fact ([JP], Appendix, PSH 14). If a family $(u_\alpha)_{\alpha\in A}$ of plurisubharmonic functions is locally uniformly bounded from above, then the function

$$u_0 := (\sup_{\alpha\in A} u_\alpha)^*$$

is again plurisubharmonic, where "$*$" denotes the upper semicontinuous regularization. But we don't need to regularize $\log\gamma_X$, since it is already continuous and this assures also that the family $\{\tilde{g}\}$ is locally uniformly bounded from above.
Let $\mathcal{G}$ be a subgroup of the automorphism group $\mathrm{Aut}(X)$, which operates fixpointfree and properly discontinuously on X. Then the quotient $Q = X/\mathcal{G}$ is a Hausdorff space which admits a unique complex structure, such that the projection $\pi : X \to Q$ is a holomorphic and locally biholomorphic map. We can now prove

THEOREM 6 *Let X be a γ-hyperbolic manifold and let $Q = X/\mathcal{G}$ be a compact quotient as above. Then the cotangent bundle Ω^1_Q is nef.*

Proof: As local coordinates ψ for Q we can take π^{-1} restricted to appropriate open sets such that a coordinate change is described by $\psi_1 \circ \psi_0^{-1} = f$, where $f \in \mathcal{G}$ (cf. [W], Chap. V, Prop. 5.3.). Then we define for $q \in Q$ and $\xi \in TQ_q$

$$F(q,\xi) := \gamma_X(\psi(q), D\psi(q).\xi).$$

Since the Carathéodory-Reiffen metric γ_X is invariant under biholomorphic transformations (Proposition 10), this definition does not depend on the choice of the local coordinate and gives us a differential metric F on TQ. Moreover Proposition 11 implies that $\log F \in PSH(TQ\setminus\{0\})$. Now the assertion follows from Proposition 9.

In particular, compact quotients of a bounded domain in $\mathbb{C}^n$ or in a Stein manifold have nef cotangent bundles.

REFERENCES

[BPV] Barth, W., Peters, C., Van de Ven, A.: Compact Complex Surfaces. Erg. d. Math. Bd. 3, Berlin Heidelberg New York: Springer 1984

[Ca] Campana, F.: Réduction d'Albanese d'un morphisme propre et faiblement kählérien. I. Comp. Math. 54, 373-398 (1985)

[CP] Campana, F., Peternell, T.: Projective manifolds whose tangent bundles are numerically effective. Math. Ann. 289, 169-187 (1991)

[De1] Demailly, J.-P.: Singular hermitian metrics on positive line bundles. In: Hulek, K. et al. (eds.): Complex Algebraic Varieties (Lect. Notes Math., vol. 1507, pp. 87-104) Berlin Heidelberg New York: Springer 1992

[De2] Demailly, J.-P.: Regularization of closed positive currents and intersection theory. J. Alg. Geom. 1, 361-409 (1992)

[DPS] Demailly, J.-P., Peternell, T., Schneider, M.: Compact complex manifolds with numerically effective tangent bundles. J. Alg. Geom. 3, 295-345 (1994)

[Fu] Fujiwara, T.: Varieties of small Kodaira dimension whose cotangent bundles are semiample. Comp. Math. 84, No.1, 43-52 (1992)

[Gr] Griffiths, P.: Periods of integrals on algebraic manifolds, III. Publ. Math. I.H.E.S. No. 38, 125-180 (1970)

[Ha] Hartshorne, R.: Algebraic Geometry. Graduate Texts in Math. 52, Berlin Heidelberg New York: Springer 1977

[JP] Jarnicki, M., Pflug, P.: Invariant Distances and Metrics in Complex Analysis. De Gruyter Expositions in Math. 9, Berlin New York: De Gruyter 1993

[JS] Ji, K., Shiffman, B.: Properties of compact complex manifolds carrying closed positive currents. J. Geom. Anal. (1993)

[Ko] Kobayashi, S.: Negative vector bundles and complex Finsler structures. Nagoya Math. J., Vol. 57, 153-166 (1975)

[Kod] Kodaira, K.: On Compact Analytic Surfaces: II. Ann. Math. 77, 563-626 (1963)

[Mo] Mori, S.: Projective manifolds with ample tangent bundles. Ann. Math. (2) 110, 593-606 (1979)

[PW] Peternell, T., Wilson, P.M.H.: Threefolds with extremal Chern classes. Higher dimensional complex varieties, Trento Proceedings, 357-378 (1996)

[Re] Reid, M.: Bogomolov's theorem $c_1^2 \leq 4c_2$. Intl. Symp. on Algebraic Geometry Kyoto, 623-642 (1977)

[Rei] Reiffen, H.J.: Die Carathéodorysche Distanz und ihre zugehörige Differentialmetrik. Math. Ann. 161, 315-324 (1965)

[Ue] Ueno, K.: Classification Theory of Algebraic Varieties and Compact Complex Spaces. Lect. Notes Math., vol. 439, Berlin Heidelberg New York: Springer 1975

[W] Wells, R.O.: Differential Analysis on Complex Manifolds. Graduate Texts in Math. 65, Berlin Heidelberg New York: Springer 1980

[Zh] Zheng, F.: On semi-positive threefolds. Thesis, Harvard (1990)

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