Remarks on the Darboux Transform of Isothermic Surfaces

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Abstract. We study Darboux and Christoffel transforms of isothermic surfaces in Euclidean space. Using quaternionic calculus we derive a Riccati type equation which characterizes all Darboux transforms of a given isothermic surface. Surfaces of constant mean curvature turn out to be special among all isothermic surfaces: their parallel surfaces of constant mean curvature are Christoffel and Darboux transforms at the same time. We prove — as a generalization of Bianchi's theorem on minimal Darboux transforms of minimal surfaces — that constant mean curvature surfaces in Euclidean space allow ∞^3 Darboux transforms into surfaces of constant mean curvature. We indicate the relation between these Darboux transforms and Bäcklund transforms of spherical surfaces.

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1 Introduction

Transformations play an important role connecting surface theory with the theory of integrable systems. A well known example is the Bäcklund transformation of pseudospherical (and spherical [1]) surfaces in Euclidean 3-space which "adds solitons" to a given surface. In case of isothermic surfaces the Darboux transformation takes the role of the Bäcklund transform for pseudospherical surfaces. Darboux transforms of isothermic surfaces naturally arise in 1-parameter families ("associated families")

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allowing to rewrite the underlying (system of) partial differential equation(s) as an (infinite dimensional) integrable system [6], [4]. It is mainly for this reason that Darboux transformations provoke new interest among contemporary geometers — even though the subject was well studied around the turn of the century [5], [7] and [2]. A key tool in the study of Darboux transforms of an isothermic surface in Euclidean space is a careful analysis of the Christoffel transform (or dual isothermic surface) of the surface — which may be considered as a certain limiting case of Darboux transforms. In the present paper, we develop classical results further using quaternionic calculus which makes definitions elegant and calculations more efficient. Characterizations thus obtained turned out to be necessary in the development of the corresponding discrete theory [10].

In the first part of the paper, we develop isothermic surface theory in codimension 2 — which is a more appropriate setting when using quaternionic calculus. When restricting to codimension 1, all notions become classical. Here, we rely on the characterizations of Darboux and Christoffel pairs in $\mathbb{H}P^1$ given in [9]. The consequent use of the quaternionic setup yields a new and unified description for these surface pairs in $\mathbb{R}^4 \cong \mathbb{H}$. Even though the quaternionic calculus (as developed in [9]) provides a setting to study the global geometry of surface pairs in Möbius geometry (cf.[11]) we will restrict to local geometry in this paper, for two reasons: first, there are a number of possible definitions of a "globally isothermic surface" whose consequences have not yet been worked out. For example, definition 1 may well be read as a global definition but it is far too general to provide any global results. Secondly, Christoffel and Darboux transforms of a (compact) surface generally do not exist globally. Moreover, around certain types of umbilics they may not even exist locally. However, up to the problem of closing periods, the results on constant mean curvature surfaces can well be read as global results: here, the Christoffel transform can be determined without integration which ensures its global existence (with branch points at the umbilics of the original surface).

A central result is obtained by carefully analyzing the relation between Darboux and Christoffel pairs: we derive a Riccati type equation describing all Darboux transforms of a given isothermic surface. This equation is crucial for the explicit calculation of Darboux transforms — in the smooth case (all the pictures shown in this paper are obtained from this equation) as well as in the theory of discrete isothermic nets [10]. Moreover, most of our remaining results are different applications of the Riccati equation: first, we extend Bianchi's permutability theorems for Darboux and Christoffel transforms for the codimension 2 setup. We then discuss constant mean curvature surfaces in 3-dimensional Euclidean space as "special" isothermic surfaces: they can be characterized by the fact that their Christoffel transforms arise as Darboux transforms[3]. Together with the Riccati equation, this provides more detailed knowledge about the ∞^3 constant mean curvature Darboux transforms of a constant mean curvature surface — whose existence is a classical result due to Bianchi [1]. Our new proof shows that any such Darboux transform has (pointwise) constant distance to the Christoffel transform. This fact provides a geometric definition for a

[3]While the notion of "Darboux pairs" is naturally a conformal notion (i.e. relates surfaces in Möbius space) the notion of "Christoffel pairs" is a Euclidean one. This might explain the (untypical) fact that constant mean curvature surfaces in *Euclidean* space have a special position, *not* constant mean curvature surfaces in *any* space of constant curvature.

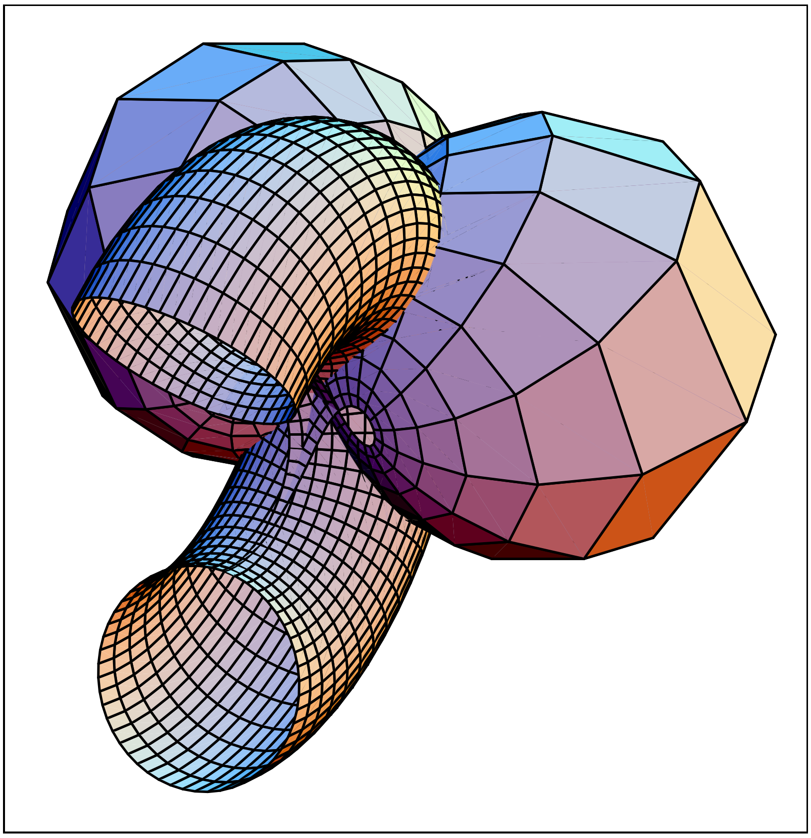

Figure 1: A Darboux transform of a torus of revolution

discrete analog of smooth constant mean curvature surfaces [10]. We conclude this paper relating this 3-dimensional family with the Bianchi-Bäcklund transformation for constant mean curvature surfaces discussed in [12] (cf.[1]).

2 DARBOUX PAIRS IN THE CONFORMAL 4-SPHERE

In 3-dimensional Möbius space (the conformal sphere S^3) an isothermic surface may be characterized by the existence of conformal curvature line coordinates around each (nonumbilic) point[4]. Note that the notion of principal curvature directions is conformally invariant — even though the second fundamental form is not. In higher codimensions the second fundamental form (with respect to any metric in the conformal class) takes values in the normal bundle. In order to diagonalize this vector valued second fundamental form, i.e. simultaneously diagonalize all components of a

[4] As mentioned earlier, there is a variety of possible definitions for isothermic surfaces which are all equivalent away from umbilics — for example, any of the characterizations of isothermic surfaces (cf.[9]) given in this paper could be used as (global) definitions instead of definition 1 (cf.[11]).

basis representation, the surface's normal bundle has to be flat[5]. This is an implicit prerequisite in the following

Definition 1 *A (2-dimensional) surface in (4-dimensional) Möbius space is called isothermic if around each (nonumbilic) point there exist conformal curvature line coordinates, i.e. conformal coordinates which diagonalize the (vector valued) second fundamental form taken with respect to any conformal metric of the ambient space.*

In order to understand the notion of a "Darboux pair of isothermic surfaces" we also have to learn what a "sphere congruence" is and what we will mean by "envelope of a sphere congruence":

Definition 2 *A congruence of 2-spheres in (4-dimensional) Möbius space is a 2-parameter family of 2-spheres.*

A 2-dimensional surface is said to envelope a congruence of 2-spheres if at each point it is tangent[6] to a corresponding 2-sphere.

Note that the requirements on a congruence of 2-spheres in 4-space to be enveloped by two surfaces are much more restrictive than on a hypersphere congruence [9]. Also, a congruence of 2-spheres in S^4 may have only *one* envelope — which generically does not occur in the hypersphere case. In the second half of the paper we will concentrate on the more familiar situation in 3-space.

If, however, we *have* two surfaces which envelope a congruence of 2-spheres the congruence will establish a point to point correspondence between its two envelopes by assigning the point of contact on one surface to the point of contact on the other surface. For a 3-dimensional ambient space it is well known [3] (cf. [7]) that two cases can occur if this correspondence preserves curvature lines[7] and is conformal: the congruence consists of planes in a certain space of constant curvature — in which case the two envelopes are Möbius equivalent — or, both envelopes are isothermic — in this case one surface is called a "Darboux transform" of the other (see [9], compare [3] or [4]). These remarks may motivate the following

Definition 3 *If a congruence of 2-spheres (which is not a plane congruence in a certain space of constant curvature) is enveloped by two isothermic surfaces, the correspondence between its two envelopes being conformal and curvature line preserving, the surfaces are said to form a Darboux pair. Each of the two surfaces is called a Darboux transform of the other.*

Before studying Darboux pairs in Euclidean space we will recall

3 A basic characterization for Darboux pairs

In order to discuss (Darboux) pairs of surfaces in 4- (or 3-) dimensional Möbius geometry we consider the conformal 4-sphere as the quaternionic projective line [9]:

$$S^4 \cong \mathbb{H}P^1 = \{x \cdot \mathbb{H} \mid x \in \mathbb{H}^2\}. \qquad (1)$$

[5]Since the principal directions of the (scalar) second fundamental forms with respect to any normal vector are conformally invariant, as in the codimension 1 case, the flatness of the normal bundle is a conformal invariant, too.

[6]As usually done in the 3-dimensional case, we also want to allow the surface to degenerate.

[7]This is what is called a "Ribaucour sphere congruence".

Note that we consider the space $\mathbb{H}^2$ of homogeneous coordinates of the quaternionic projective line as a *right* vector space over the quaternions $\mathbb{H}$.

Now, let $(f,\hat{f}) : M^2 \to \mathcal{P}$ be an immersion into the (symmetric) space of point pairs[8] in S^4,

$$\mathcal{P} := \{(x,y) \in S^4 \times S^4 \,|\, x \neq y\}. \tag{2}$$

We may write the derivatives of f and $\hat{f}$ as[9]

$$df = f\varphi + \hat{f}\omega, \;\; d\hat{f} = f\hat{\omega} + \hat{f}\hat{\varphi} \tag{3}$$

where $\varphi, \omega, \hat{\varphi}, \hat{\omega} : TM \to \mathbb{H}$ denote suitable quaternionic valued 1-forms. Then, the integrability conditions $d^2f = d^2\hat{f} = 0$ for f and $\hat{f}$ — the Maurer Cartan equations — read

$$\begin{array}{rcll} 0 &=& d\varphi + \varphi\wedge\varphi + \hat{\omega}\wedge\omega & \text{(Gauß equation for } f\text{)},\\ 0 &=& d\omega + \omega\wedge\varphi + \hat{\varphi}\wedge\omega & \text{(Codazzi equation for } f\text{)},\\ 0 &=& d\hat{\omega} + \hat{\omega}\wedge\hat{\varphi} + \varphi\wedge\hat{\omega} & \text{(Codazzi equation for } \hat{f}\text{)},\\ 0 &=& d\hat{\varphi} + \hat{\varphi}\wedge\hat{\varphi} + \omega\wedge\hat{\omega} & \text{(Gauß equation for } \hat{f}\text{)}. \end{array} \tag{4}$$

Since the quaternions are *not* commutative $\varphi\wedge\varphi \neq 0$ in general. Before continuing, let us list some useful identities for quaternionic 1-forms: let $\alpha, \beta : TM \to \mathbb{H}$ be quaternionic valued 1-forms and $g : M \to \mathbb{H}$ be a quaternionic valued function; then

$$\begin{array}{rcl} \alpha\wedge g\beta &=& \alpha g\wedge\beta,\\ \overline{\alpha\wedge\beta} &=& -\bar{\beta}\wedge\bar{\alpha},\\ d(g\alpha) &=& dg\wedge\alpha + g\cdot d\alpha,\\ d(\alpha g) &=& -\alpha\wedge dg + d\alpha\cdot g, \end{array} \tag{5}$$

where $(\alpha\wedge\beta)(x,y) := \alpha(x)\beta(y) - \alpha(y)\beta(x)$.

In this framework we are now able to state a basic characterization for Darboux pairs of isothermic surfaces (for more details[10] including a proof see [9]):

Proposition 1 *A pair of surfaces $(f,\hat{f}) : M^2 \to \mathcal{P}$ is a Darboux pair if and only if*

$$\omega\wedge\hat{\omega} = \hat{\omega}\wedge\omega = 0 \tag{6}$$

where $\omega, \hat{\omega} : TM \to \mathbb{H}$ are defined by

$$df = f\varphi + \hat{f}\omega, \;\; d\hat{f} = f\hat{\omega} + \hat{f}\hat{\varphi}. \tag{7}$$

It is easy to see that this characterization does not depend upon the choice of homogeneous coordinates for the two surfaces: given a change of homogeneous coordinates $(f,\hat{f}) \mapsto (fa, \hat{f}\hat{a})$, $a, \hat{a} : M \to \mathbb{H}$, we have

$$\begin{array}{rcl} d(fa) &=& (fa)\cdot(a^{-1}\varphi a + a^{-1}da) + (\hat{f}\hat{a})\cdot(\hat{a}^{-1}\omega a),\\ d(\hat{f}\hat{a}) &=& (fa)\cdot(a^{-1}\hat{\omega}\hat{a}) + (\hat{f}\hat{a})\cdot(\hat{a}^{-1}\hat{\varphi}\hat{a} + \hat{a}^{-1}d\hat{a}). \end{array} \tag{8}$$

[8] The homogeneous coordinates of a pair of (different) points in $\mathbb{H}P^1$ form a basis of $\mathbb{H}^2$. Thus, $\mathcal{P}$ can be identified with the symmetric space $\frac{Gl(2,\mathbb{H})}{\mathbb{H}_*\times\mathbb{H}_*}$. Sometimes it is more convenient to use suitably normalized coordinates: the group $Gl(2,\mathbb{H})$ may be replaced by a 15-dimensional subgroup $Sl(2,\mathbb{H})$ which is a double cover of the group of orientation preserving Möbius transformations of S^4 [9].

[9] We will use "f" and "$\hat{f}$" for the point maps into S^4 as well as for their homogeneous coordinates.

[10] In fact, this proposition states the connection between Darboux pairs and "curved flats" [8] in the symmetric space of point pairs.

4 Christoffel pairs of isothermic surfaces in Euclidean space

Another observation is that introducing a real parameter into the Maurer Cartan equations (4) we can obtain the Darboux pair equations (6) together with the original integrability conditions as integrability conditions of a 1-parameter family of Darboux pairs — the "associated family" of Darboux pairs[11]: writing

$$df_r = f_r\varphi + \hat{f}_r(r^2\omega),\ d\hat{f}_r = f_r(r^2\hat{\omega}) + \hat{f}_r\hat{\varphi} \tag{9}$$

with a parameter $r \in I\!\!R$ the Gauß equations for f_r and $\hat{f}_r$ become

$$\begin{array}{rcl} 0 & = & d\varphi + \varphi \wedge \varphi + r^4 \cdot \hat{\omega} \wedge \omega, \\ 0 & = & d\hat{\varphi} + \hat{\varphi} \wedge \hat{\varphi} + r^4 \cdot \omega \wedge \hat{\omega} \end{array} \tag{10}$$

while the Codazzi equations remain unchanged. This shows that if there exist surface pairs — not necessarily Darboux — $(f_r, \hat{f}_r)$ for more than one value of $r > 0$, then, we have a whole 1-parameter family of Darboux pairs.

Assuming we have such a 1-parameter family $(f_r, \hat{f}_r)$ of Darboux pairs a special situation will occur when $r \to 0$. To discuss this, we assume $\varphi = \hat{\varphi} = 0$ without loss of generality: we have $0 = d\varphi + \varphi \wedge \varphi$ and $0 = d\hat{\varphi} + \hat{\varphi} \wedge \hat{\varphi}$ and thus at least locally $\varphi = -da\,a^{-1}$ and $\hat{\varphi} = -d\hat{a}\,\hat{a}^{-1}$ with suitable functions $a, \hat{a} : M \to I\!\!H$. Rescaling by those and applying (8) gives $\varphi = \hat{\varphi} = 0$. Thus,

$$df_r = \hat{f}_r(r^2\omega),\ d\hat{f}_r = f_r(r^2\hat{\omega}), \tag{11}$$

and after the rescaling $(f, \hat{f}) \mapsto (f\frac{1}{r}, \hat{f}r)$ (or $(f, \hat{f}) \mapsto (fr, \hat{f}\frac{1}{r})$, respectively) we see that $\hat{f}$ (or f) becomes a fixed point in the conformal 4-sphere — which should be interpreted as a point at infinity. Thus, the other limit surfaces, f_0 and $\hat{f}_0$, naturally lie in (different) Euclidean spaces. Identifying these two Euclidean spaces "correctly" we obtain $df_0 = \bar{\omega}$ and $d\hat{f}_0 = \hat{\omega}$ [9].

These two limit surfaces $\hat{f}_0^c := f_0$ and $f_0^c := \hat{f}_0$ usually do *not* form a Darboux pair — in general they do *not* even envelope a congruence of 2-spheres[12]. But they do form what is called a Christoffel pair:

Definition 4 *Two surfaces $f_0, \hat{f}_0 : M^2 \to I\!\!R^4 \cong I\!\!H$ in Euclidean 4-space are said to form a Christoffel pair if they induce conformally equivalent metrics on M and have parallel tangent planes with opposite orientations. Each of the surfaces of a Christoffel pair is called a Christoffel transform or dual of the other.*

Note that the two surfaces of a Christoffel pair are automatically isothermic; in fact, isothermic surfaces can be characterized by the (local) existence of a Christoffel transform [9]. The Christoffel transform of an isothermic surface is unique[13] up to a

[11] As we mentioned in a previous footnote (10) Darboux pairs are actually curved flats in the symmetric space of point pairs — and curved flats arise in associated families.

[12] This might seem more natural if we remember that f_0 and $\hat{f}_0$ take values in "different" Euclidean spaces (cf. [4]). — However, one of these surfaces and the point at infinity (which are the remains of the other surface) *do* form a (degenerate) Darboux pair.

[13] Except in one case: Christoffel transforms of the 2-sphere appear in 1-parameter families. We will discuss this case later (see page 324).

homothety and a translation — so that in the sequel we will denote the Christoffel transform of an isothermic surface f by f^c.

Finally, let us state a characterization of Christoffel pairs similar to that for Darboux pairs:

PROPOSITION 2 *Two surfaces $f_0, \hat{f}_0 : M^2 \to \mathbb{R}^4 \cong \mathbb{H}$ form a Christoffel pair if and only if*

$$d\bar{f}_0 \wedge d\hat{f}_0 = d\hat{f}_0 \wedge d\bar{f}_0 = 0. \tag{12}$$

Both surfaces of a Christoffel pair are isothermic.

As for the characterization of Darboux pairs (page 317) a proof may be found in [9]. However, in case of 3-dimensional ambient space we will present an easy proof later (page 323) using some of the calculus we are going to develop.

Now we are prepared to study

5 Darboux pairs in $\mathbb{R}^4$

Let $(f, \hat{f}) : M^2 \to \mathcal{P}$ denote a pair of surfaces with

$$df = f\varphi + \hat{f}\omega, \ d\hat{f} = f\hat{\omega} + \hat{f}\hat{\varphi}, \tag{13}$$

as before. Assuming that $f, \hat{f} : M \to \mathbb{H} \times \{1\} \cong \mathbb{H}$ take values in Euclidean 4-space we see that $\varphi = -\omega$ and $\hat{\varphi} = -\hat{\omega}$, and hence

$$df = (\hat{f} - f) \cdot \omega, \ d\hat{f} = (f - \hat{f}) \cdot \hat{\omega}. \tag{14}$$

This allows us to rewrite condition (6) on f and $\hat{f}$ to form a Darboux pair[14] as

$$0 = df \wedge (f - \hat{f})^{-1} d\hat{f} = d\hat{f} \wedge (\hat{f} - f)^{-1} df. \tag{15}$$

As a first consequence of these equations we derive the equations

$$\begin{array}{rcccl} 0 & = & df \wedge (\hat{f} - f)^{-1} d\hat{f}(\hat{f} - f)^{-1} & = & (\hat{f} - f)^{-1} d\hat{f}(\hat{f} - f)^{-1} \wedge df, \\ 0 & = & d\hat{f} \wedge (f - \hat{f})^{-1} df(f - \hat{f})^{-1} & = & (f - \hat{f})^{-1} df(f - \hat{f})^{-1} \wedge d\hat{f} \end{array} \tag{16}$$

for any Darboux pair $(f, \hat{f})$. Since (15) also implies

$$0 = d[(\hat{f} - f)^{-1} d\hat{f}(\hat{f} - f)^{-1}] = d[(f - \hat{f})^{-1} df(f - \hat{f})^{-1}] \tag{17}$$

we conclude that the Christoffel transforms f^c and $\hat{f}^c$ of f and $\hat{f}$ are given by

$$\begin{array}{rcl} df^c & = & \overline{(\hat{f} - f)^{-1} d\hat{f}(\hat{f} - f)^{-1}}, \\ d\hat{f}^c & = & \overline{(f - \hat{f})^{-1} df(f - \hat{f})^{-1}}. \end{array} \tag{18}$$

Finally, if we fix the translations of f^c and $\hat{f}^c$ such that

$$\overline{(f^c - \hat{f}^c)} = (f - \hat{f})^{-1} \tag{19}$$

— note that $d\overline{(f - \hat{f})}^{-1} = d(f^c - \hat{f}^c)$ — we learn from the above characterization (15) of Darboux pairs that f^c and $\hat{f}^c$ also form a Darboux pair (cf. [2]):

[14] Hopefully, the reader will forgive our context dependent notation: f and $\hat{f}$ denote points in $\mathbb{H}P^1 \cong S^4$, vectors in $\mathbb{H}^2$ or numbers in $\mathbb{H} \cong \mathbb{R}^4$.

Theorem 1 *If $f, \hat{f} : M^2 \to \mathbb{R}^4$ form a Darboux pair, then, their Christoffel transforms $f^c, \hat{f}^c : M^2 \to \mathbb{R}^4$ (if correctly scaled and positioned) form a Darboux pair, too.*

So far we learned how to derive the Christoffel transforms f^c and $\hat{f}^c$ of two surfaces f and $\hat{f}$ forming a Darboux pair. But usually it will be much easier to determine an isothermic surface's Christoffel transform than a Darboux transform. In the next section we will see that deriving Darboux transforms $\hat{f}$ and $\hat{f}^c$ of two surfaces f and f^c forming a Christoffel pair[15] comes down to solving

6 A Riccati type equation

Solving (18) for $d\hat{f}$ we obtain $d\hat{f} = (\hat{f} - f)d\bar{f}^c(\hat{f} - f)$. This yields the following Riccati type partial differential equation[16] for $g := (\hat{f} - f)$:

$$dg = g\, d\bar{f}^c\, g - df. \tag{20}$$

Using our characterization (12) of Christoffel pairs it is easily seen that this equation is "completely" (Frobenius) integrable. Note that — in agreement with our previous results — the common transform $g^c = \bar{g}^{-1}$ for Riccati equations yields

$$dg^c = g^c\, d\bar{f}\, g^c - df^c, \tag{21}$$

showing that $\hat{f}^c = f^c + g^c$ will provide a Darboux transform of f^c whenever $f + g$ is a Darboux transform of f coming from a solution g of (20).

Since every Darboux transform $\hat{f}$ of an isothermic surface f provides a Christoffel transform f^c of f via (18) *every* Darboux transform comes from a solution of (20) — if we do not fix the scaling of the Christoffel transform f^c. On the other hand *every* solution g of (20) defines a Darboux transform $\hat{f} = f + g$ of f since $df \wedge g^{-1}d(f+g) = d(f+g) \wedge g^{-1}df = 0$. This seems to be worth formulating as a

Theorem 2 *If $f, f^c : M^2 \to \mathbb{R}^4$ form a Christoffel pair of isothermic surfaces every solution of the integrable Riccati type partial differential equation*

$$dg = g\, d\bar{f}^c\, g - df \tag{22}$$

provides a Darboux transform $\hat{f} = f + g$ of f. On the other hand, every Darboux transform $\hat{f}$ of f is obtained this way — if we do not fix the scaling of f^c.

At this point, we should discuss the effect of a rescaling of the Christoffel transform f^c in the equation (20). For this purpose we examine the equations

$$dg = g\,(\pm r^4 d\bar{f}^c)\, g - df \tag{23}$$

[15] Note that the notation $\hat{f}^c$ for a Darboux transform of f^c makes sense because of our previous theorem: we have $\widehat{f^c} = \hat{f}^c$.

[16] The pictures in this paper were produced using Mathematica to numerically integrate this Riccati type equation.

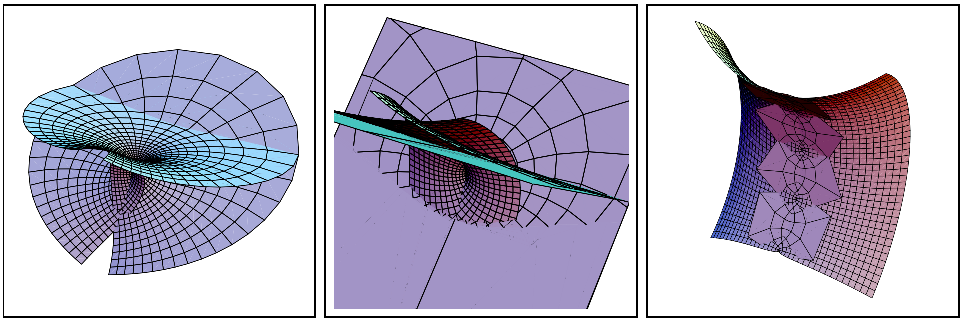

Figure 2: Darboux transforms of the Catenoid when $H^c \to \infty$

where $r \neq 0$ is a real parameter. For the derivatives of f and a Darboux transform $\hat{f} = f + g$ of f this yields

$$\begin{array}{rcccc} df &=& f \cdot [-g^{-1} df] &+& \hat{f} \cdot [g^{-1} df], \\ d\hat{f} &=& f \cdot [\mp r^4 d\bar{f}^c\, g] &+& \hat{f} \cdot [\pm r^4 d\bar{f}^c\, g]. \end{array} \tag{24}$$

Interpreting $f, \hat{f} : M^2 \to I\!H \cong I\!H \times \{1\}$ as homogeneous coordinates of the point pair map $(f, \hat{f}) : M^2 \to \mathcal{P}$ we may choose new homogeneous coordinates by performing a rescaling $(f, \hat{f}) \mapsto (fr, \hat{f}(rg)^{-1})$ to obtain[17]

$$\begin{array}{rcccc} d[fr] &=& [fr] \cdot [-g^{-1} df] &+& [\hat{f}(rg)^{-1}] \cdot [r^2 df], \\ d[\hat{f}(rg)^{-1}] &=& [fr] \cdot [\mp r^2 d\bar{f}^c] &+& [\hat{f}(rg)^{-1}] \cdot [df\, g^{-1}]. \end{array} \tag{25}$$

Even though this system resembles very much our original system (9) which describes the associated family of Darboux pairs, there is an essential difference: in (9) the forms φ, ω, $\hat{\varphi}$ and $\hat{\omega}$ are independent of the parameter r whereas the forms $g^{-1}df$ and $df\, g^{-1}$ in the system we just derived *do* depend on r. In fact, in the associated family $(f_r, \hat{f}_r)$ of Darboux pairs obtained from (9) both surfaces, f_r as well as $\hat{f}_r$, change with the parameter r whereas the parameter contained in the Riccati equation just effects the Darboux transform $\hat{f} = \hat{f}_r$ while the original surface f remains unchanged. However, the original system (9) appears in the linearization of our Riccati equation[18] which indicates a close relation of these two parameters.

As a first application of this parameter which occurs from rescalings of the Christoffel transform f^c in our Riccati equation we may prove an extension of Bianchi's permutability theorem [2] for Darboux transforms:

Theorem 3 *Let $\hat{f}_{1,2} : M^2 \to I\!H$ be two Darboux transforms of an isothermic surface $f : M^2 \to I\!H$,*

$$d\hat{f}_{1,2} = r_{1,2}(\hat{f}_{1,2} - f)\, d\bar{f}^c(\hat{f}_{1,2} - f), \tag{26}$$

where we fixed any scaling for the Christoffel transform f^c of f. Then, there exists an isothermic surface $\hat{f} : M^2 \to I\!H$ which is an r_1-Darboux transform of $\hat{f}_2$ and an

[17] Note that this rescaling provides an $Sl(2, I\!H)$ framing of the point pair map $(f, \hat{f})$ [9].

[18] Here, we would like to thank Fran Burstall for helpful discussions.

r_2-Darboux transform of $\hat f_1$ at the same time[19]*:*

$$d\hat f = r_{2,1}(\hat f - \hat f_{1,2})\, d\bar{\hat f}^{c}_{1,2}(\hat f - \hat f_{1,2}). \tag{27}$$

Moreover, the points of $\hat f$ lie on the circles determined by the corresponding points of f, $\hat f_1$ and $\hat f_2$, the four surfaces having a constant (real) cross ratio[20]

$$\frac{r_2}{r_1} \equiv (f - \hat f_1)(\hat f_1 - \hat f)^{-1}(\hat f - \hat f_2)(\hat f_2 - f)^{-1}. \tag{28}$$

To prove this theorem we simply *define* the surface $\hat f : M^2 \to \mathbb{H}$ by solving the cross ratio equation[21] (28) for $\hat f$:

$$\hat f := [r_2 \hat f_1(\hat f_1 - f)^{-1} - r_1 \hat f_2(\hat f_2 - f)^{-1}] \cdot [r_2(\hat f_1 - f)^{-1} - r_1(\hat f_2 - f)^{-1}]^{-1}. \tag{29}$$

Using this ansatz, it is a straightforward calculation to verify the Riccati equations (27) which proves the theorem.

As indicated earlier, from now on we will concentrate on surfaces in 3-dimensional Euclidean space $\mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$:

7 Christoffel pairs in $\mathbb{R}^3$

In this situation, much of our previously developed calculus will simplify considerably. For example, we will be able to give an easy proof of our characterization of Christoffel pairs and to write down the Christoffel transform of an isothermic surface quite explicitly. First we note that our characterizations (15) and (12) of Darboux and Christoffel pairs of isothermic surfaces reduce to just *one* equation: if $f, \hat f : M^2 \to \mathrm{Im}\mathbb{H}$ both take values in the imaginary quaternions,

$$\begin{array}{rcl} d\hat f \wedge d\bar f &=& -\overline{d\bar f \wedge d\hat f}, \\ d\hat f \wedge (\hat f - f)^{-1} df &=& -\overline{df \wedge (f - \hat f)^{-1} d\hat f}. \end{array} \tag{30}$$

In order to continue we will collect some identities present in the codimension 1 case. We may orient an immersion $f : M^2 \to \mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$ by choosing a unit normal field $n : M^2 \to S^2$. This defines the complex structure J on M via

$$df \circ J = n\, df \tag{31}$$

— note that since f and n take values in the imaginary quaternions

$$n\, df = -\langle n, df\rangle + n \times df = n \times df = -df\, n. \tag{32}$$

The Hodge operator is then given as the dual of this complex structure:

$$*\eta = -\eta \circ J \tag{33}$$

[19]Note, that this claim makes no sense before we fix a scaling for the Christoffel transforms $\hat f^c_{1,2}$ of $\hat f_{1,2}$. But, according to our "permutability theorem" for Christoffel and Darboux transforms (theorem 1) there is a *canonical* scaling for $\hat f^c_{1,2}$ after we fixed the scaling of f^c.

[20]For a comprehensive discussion of the (complex) cross ratio in $\mathbb{R}^4 \cong \mathbb{H}$ see [10]. The idea for the proof given in this paper actually originated from the discrete version of this theorem.

[21]Note that the denominator does not vanish as long as $\hat f_1 \neq \hat f_2$. For $r_1 = r_2$ we get $\hat f = f$.

for any 1-form η on M.

With this notation we are now able to give a useful reformulation[22] of the equation arising in our characterizations of Darboux pairs and Christoffel pairs: if $\eta : TM \to \mathbb{H}$ is any quaternionic valued 1-form we have

$$(df \wedge \eta)(x, Jx) = df(x) \cdot (- * \eta(x) + n\,\eta(x)) \tag{34}$$

for any $x \in TM$. Consequently, $df \wedge \eta = 0$ if and only if

$$*\eta = n\,\eta. \tag{35}$$

This criterium shows that the space of imaginary solutions $\eta : TM \to \mathrm{Im}\mathbb{H}$ of the equation $0 = df \wedge \eta$ is pointwise 2-dimensional[23] — if η is an (injective) solution, then, every other solution $\tilde{\eta}$ is of the form

$$\tilde{\eta} = (a + b\,n) \cdot \eta \tag{36}$$

with suitable functions $a, b : M \to \mathbb{R}$. But *one* (imaginary) solution to the equation $0 = df \wedge \eta$ is easily found: it is well known that

$$d * df = -dn \wedge df = H\,df \wedge df \tag{37}$$

where H is the mean curvature of f. Thus

$$df \wedge (dn + H\,df) = 0 \tag{38}$$

which gives an injective solution $\eta = dn + H\,df$ away from umbilics of f.

At this point, we are ready to give the announced proof of our characterization of Christoffel pairs (12) in the 3-dimensional case:

THEOREM 4 *Two surfaces $f, f^c : M^2 \to \mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$ form a Christoffel pair if and only if*

$$df \wedge df^c = 0. \tag{39}$$

Generically, the Christoffel transform f^c of f is uniquely determined by f up to homotheties and translations of $\mathbb{R}^3$.

The fact that both surfaces of a Christoffel pair in 3-space are isothermic is classical (see for example [5]) — and thus we omit this calculation.

Now, in order to prove this theorem we note that from the above we know that $f^c : M^2 \to \mathrm{Im}\mathbb{H}$ satisfies (39) if and only if

$$*df^c = n\,df^c. \tag{40}$$

[22] At this point we would like to thank Ulrich Pinkall for many helpful discussions — this criterium is actually due to him.

[23] The space of solutions with values in the full quaternions is 4-dimensional as is easily seen: (36) becomes

$$\tilde{\eta} = (a + b\,n) \cdot \eta + (*\alpha + n\alpha)$$

with an arbitrary real 1-form $\alpha : TM \to \mathbb{R}$.

But this equation means that in corresponding points f and f^c have parallel tangent planes and that the almost complex structure induced by f^c with respect to $n^c := -n$ is just J — the same as that induced by f with respect to n. Thus,

$$df \wedge df^c = 0 \tag{41}$$

if and only if $f, f^c : M^2 \to \mathbb{R}^3$ have parallel tangent planes with opposite orientations and they induce conformally equivalent metrics, i.e. they form a Christoffel pair.

Now assume we have not just one but two Christoffel transforms f^c and $\tilde{f}^c$ of an isothermic surface $f : M^2 \to \mathbb{R}^3$. Then we know from (36) that

$$d\tilde{f}^c = (a + b\,n) \cdot df^c. \tag{42}$$

The integrability condition for $\tilde{f}^c$ reads

$$0 = da \wedge df^c + db \wedge *df^c + b\,H^c df^c \wedge df^c \tag{43}$$

showing that $a = const$ and $b = 0$ since $df^c \wedge df^c$ takes values in normal direction while all other components are tangential — provided that f^c is *not* a minimal surface[24]. This concludes the proof.

With (38) it also follows that

$$dn + H\,df = (a + b\,n)df^c \tag{44}$$

for suitable functions $a, b : M \to \mathbb{R}$. Similarly, we obtain

$$-dn + H^c\,df^c = (a^c + b^c n)df \tag{45}$$

by interchanging the roles of f and f^c. Adding these two equations yields $a = H^c$, $a^c = H$ and $b = b^c = 0$ since the forms df, $n\,df$, df^c and $n\,df^c$ are linearly independent (over the reals). As a consequence, we have a quite explicit formula relating the two surfaces of a Christoffel pair:

$$H^c df^c = dn + H\,df. \tag{46}$$

This equation shows that whenever one of the surfaces of a Christoffel pair is a minimal surface the other is totally umbilic (namely, a scaling of its Gauß map) and vice versa. This brings us back to our previous problem of the uniqueness of Christoffel transforms: assume we have a Christoffel pair (f, n) consisting of a minimal surface f and its Gauß map n. Then all the pairs

$$(\,a \int (\cos(t) + \sin(t)\,n) \cdot df\,,\ n\,) \tag{47}$$

with real constants a and t will also form Christoffel pairs. Up to homotheties (given by a) this will run us through the associated family of minimal surfaces (given by t) reflecting the fact that associated minimal surfaces have the same Gauß map[25].

Another fact that can be derived from (46) is that the (correctly scaled and positioned) Christoffel transform of a surface of constant mean curvature $H \neq 0$ is its

[24] The case of minimal Christoffel transforms will be discussed below.
[25] However, choosing "curvature lines" for the Gauß map will fix the minimal surface [9].

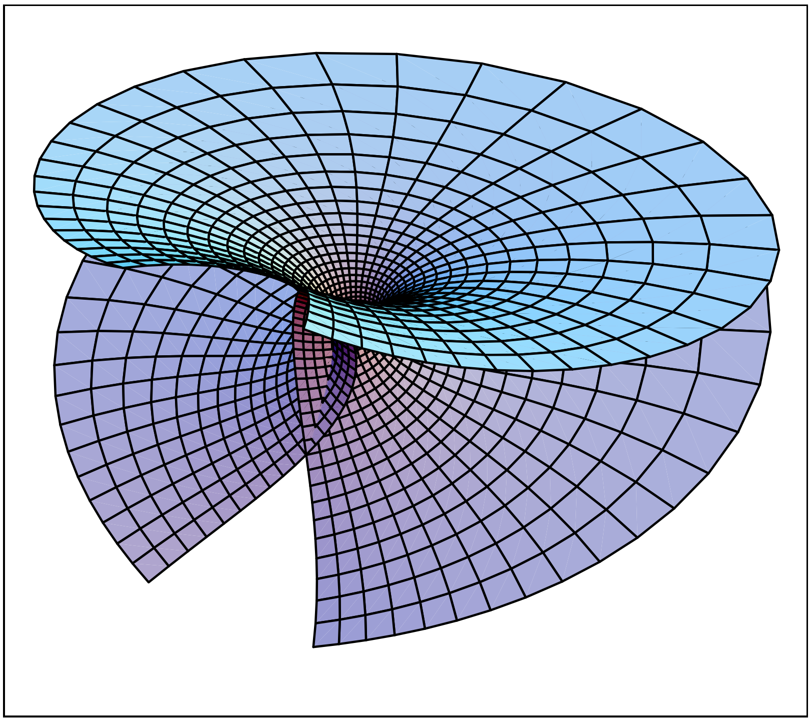

Figure 3: A Darboux transform of the Catenoid

parallel surface $f+\frac{1}{H}n$ of the same constant mean curvature $H^c = H$. Note that this parallel surface induces a conformally equivalent metric on the underlying manifold M^2 and consequently it is also a Darboux transform of the original surface[26] — the enveloped sphere congruence consisting of spheres with constant radius $\frac{1}{2H}$. Later, we will see that constant mean curvature surfaces in Euclidean space can be *characterized* by the fact that their Christoffel transforms are Darboux transforms too. Thus, in the remaining part of this paper we will study constant mean curvature ($H \neq 0$ or $H = 0$) Darboux transforms of

8 SURFACES OF CONSTANT MEAN CURVATURE

Using the reformulation (35) of our characterizing equation (15) of Darboux pairs we conclude that for any Darboux transform $\hat{f} = f + g$ of $f : M^2 \to \mathbb{R}^3$

$$*g\, d\hat{f} = n\, g\, d\hat{f} \tag{48}$$

where we used the fact that $g^{-1} = -\frac{1}{|g|^2} g$ for $g \in \mathrm{Im}\mathbb{H}$. Consequently, the normal field $\hat{n}$ of $\hat{f}$ is given by[27]

$$\hat{n} = \frac{gng}{|g|^2} = \frac{1}{|g|^2}(|g|^2 n - 2\langle n, g\rangle g) \tag{49}$$

since we must have $*d\hat{f} = -\hat{n} d\hat{f}$.

Thus, if the normal field of a Darboux transform $\hat{f}$ of an isothermic surface $f : M^2 \to \mathrm{Im}\mathbb{H}$ equals that of its Christoffel transform,

$$\hat{n} = n^c = -n, \tag{50}$$

then $g = a\, n$ for a suitable constant $a \in \mathbb{R}$ (remark that a has to be constant in order to obtain parallel tangent planes of $\hat{f}$ and f). With (46) we conclude

$$H df + dn = H^c df^c = H^c(df + dg) = H^c df + H^c a\, dn \tag{51}$$

which implies that either one of the surfaces is minimal and the other is totally umbilic, or, $H = H^c = \frac{1}{a}$ which means that f and $\hat{f} = f^c$ form a pair of parallel constant mean curvature surfaces.

Together with our previous remark (page 326) this leaves us with the following characterization of constant mean curvature surfaces:

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[26]Note that in order to obtain $g = \frac{n}{H}$ as a solution of our Riccati type equation (20) the Christoffel transform df^c of f has to be scaled such that $H^c = \frac{1}{H}$ — then, the Riccati equation is equivalent to (46). This means that the parallel constant mean curvature surface appears at a well defined location in the associated family.

[27]Note that with this formula we easily see that $\hat{f}$ is the second envelope of a sphere congruence enveloped by f:

$$2\langle g, n\rangle f + |g|^2 n = 2\langle g, n\rangle \hat{f} + |g|^2 \hat{n}.$$

The second fundamental form of $\hat{f}$ is quite complicated, but at least, when introducing frames it can be seen that it has the same principal directions as the second fundamental form of f^c. Since $\hat{f}$ also induces the conformally equivalent metric $|d\hat{f}|^2 = |g|^4 |df^c|^2$ we get half of a proof for our characterization (15) of Darboux pairs in the case of 3-dimensional ambient space.

THEOREM 5 *The (correctly scaled and positioned) Christoffel transform f^c of an isothermic surface $f : M^2 \to \mathbb{R}^3$ is also a Darboux transform $\hat{f}$ of f if and only if f is a surface of constant mean curvature $H \neq 0$. In this case $\hat{f} = f^c$ is the parallel surface of constant mean curvature.*

In order to study constant mean curvature Darboux transforms of constant mean curvature surfaces in general we have to calculate the mean curvature of a Darboux transform $\hat{f}$ of an isothermic surface. We will eventually derive the existence of a 3-parameter family of constant mean curvature Darboux transforms of a constant mean curvature surface, all of them having (pointwise) constant distance from the parallel constant mean curvature surface of the original surface. There are several ways to do so: we could calculate the second fundamental form of $\hat{f}$ — which is not convenient because this second fundamental form looks quite difficult — or, we could use (37) to directly calculate $\hat{H}$ with the help of our Riccati type equation (20). This second way is quite straightforward but not very interesting. So, we will present another way which grew out of discussions with Ulrich Pinkall[28]: observing that if $d\hat{f} = -\bar{g}df^c g$, the integrability condition for $\hat{f}$ becomes

$$0 = \bar{g}(\overline{dg\,g^{-1}} \wedge df^c - df^c \wedge dg\,g^{-1})g, \tag{52}$$

i.e. the reality of the form $df^c \wedge dg\,g^{-1}$. Since the volume form $\frac{1}{2}df^c \wedge *\, df^c$ induced by f^c is a basis of the real 2-forms on M this may be reformulated as

$$0 = df^c \wedge (dg\,g^{-1} - \frac{1}{2}U * df^c) \tag{53}$$

with a suitable function $U : M \to \mathbb{R}$. With (35) we obtain the equivalent equation

$$n^c dg - *dg = U\, df^c g \tag{54}$$

— the "Dirac equation" with reference immersion f^c.

Using this equation we may calculate the mean curvature $\hat{H}$ of $\hat{f}$ in terms of the function U via

$$d * d\hat{f} = \frac{1}{|g|^2}(U - H^c)\, d\hat{f} \wedge d\hat{f} \tag{55}$$

since

$$*\alpha \wedge *\beta = \alpha \wedge \beta \tag{56}$$

for any two 1-forms $\alpha, \beta : TM \to \mathbb{H}$ on a Riemann surface and hence

$$*df^c \wedge dg = \frac{1}{2}(*df^c \wedge dg - *df^c \wedge **\, dg) = \frac{1}{2}df^c \wedge (n^c dg - *dg). \tag{57}$$

Substituting our Riccati equation (20) into the Dirac equation yields $U = 2\langle n, g\rangle$ and consequently

$$\hat{H} = \frac{1}{|g|^2}(2\langle n, g\rangle - H^c). \tag{58}$$

[28]The Dirac equation (54) which we will discover on our way can be considered as a replacement for the Cauchy Riemann equations in a generalized "Weierstraß representation" for surfaces in $\mathbb{R}^3$. Given an immersion $f : M^2 \to \mathbb{R}^3$ this generalized "Weierstraß representation" will provide us with *any* immersion $\hat{f}$ which induces the same complex structure on M.

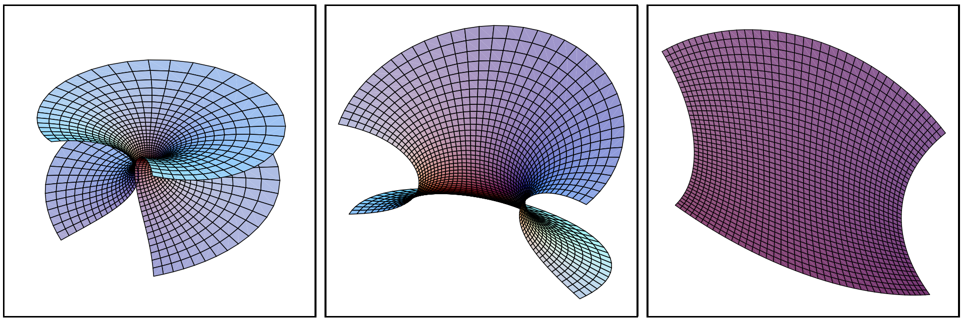

Figure 4: Darboux transforms of the Catenoid when $H^c \to 0$

Now we assume the mean curvature H of our original surface f to be constant — and consequently H^c is constant too — and rewrite this equation as

$$0 = h_{\hat{H}}(g) := \hat{H}\,|g|^2 - 2\langle n, g\rangle + H^c. \tag{59}$$

Taking the derivative of this function h_C where C denotes any constant and assuming H^c to be constant yields

$$dh_C(g) = -2\langle df^c, g\rangle \cdot h_C(g) - 2\langle df, g\rangle \cdot (C - H) \tag{60}$$

where we got rid of dn by using (46). This shows that whenever we choose an initial value $g(p_0) = g_0$ for a function $g : M^2 \to \mathrm{Im}\mathbb{H}$ such that $h_H(g_0) = 0$ the trivial solution $h_H \equiv 0$ will be the unique solution to the above (linear and homogeneous: $C = H$) differential equation. Thus our Riccati type equation (20) will produce a Darboux transform $\hat{f} = f + g$ of constant mean curvature $\hat{H} = H$ out of a surface of constant mean curvature ($H \neq 0$ or $H = 0$).

To conclude let us study the geometry of the condition $h_H(g) = 0$: for a minimal surface this simply says that the points $\hat{f}(p)$ of $\hat{f} = f + g$ always lie in distance $\frac{1}{2}H^c$ off the tangent planes $f(p) + d_p f(T_pM)$ of f. Since we also have the freedom of rescaling the Christoffel transform f^c of f we end up with a 3-parameter family of minimal Darboux transforms of a minimal surface (cf. [2]). A minimal Darboux transform of the Catenoid is shown in figure 3. Sending $H^c \to \pm\infty$ — note that in case of surfaces of constant mean curvature the associated family of Darboux pairs may be parameterized by H^c — the Darboux transforms look more and more like the original surface (Fig. 2) while sending $H^c \to 0$ the Darboux transforms approach a planar surface patch — the best compromise between the Catenoid's Christoffel transform and a minimal surface (Fig. 4).

In case of a surface of constant mean curvature $H \neq 0$ we may reformulate the condition $h_H(g) = 0$ as

$$|H\,g - n|^2 = 1 - H^c H \tag{61}$$

showing that the points $\hat{f}(p)$ lie on spheres centered on the parallel surface $f + \frac{1}{H}n$ and with constant radius $\frac{1}{H}\sqrt{1 - H^c H}$. Since the radius has to be real to provide real Darboux transforms we see that we have to have $H^c H \leq 1$ which restricts the range of the parameter H^c to a ray $H^c \leq \frac{1}{H}$ containing 0 (without loss of generality we assume

$H \geq 0$). As $H^c \to -\infty$ and $H^c \to 0$ we obtain the original surface and its Christoffel transform, respectively. But now, we obtain the Christoffel transform a second time — as a Darboux transform when $H^c = \frac{1}{H}$, i.e. when the spheres $h_H(g) = 0$ collapse to points. Figures 5 and 6 show constant mean curvature Darboux transforms of the cylinder.

To summarize the results we found in this section we formulate a theorem generalizing Bianchi's theorem on minimal Darboux transforms of minimal surfaces [2]:

THEOREM 6 *Any surface of constant mean curvature ($H \neq 0$ or $H = 0$) in Euclidean 3-space allows a 3-parameter family of Darboux transforms into surfaces of the same constant mean curvature.*

In case of a minimal surface all its minimal Darboux transforms have (pointwise) constant normal distance from the original surface while,

in case of a surface of constant mean curvature $H \neq 0$, all the constant mean curvature Darboux transforms have (pointwise) constant distance from the parallel constant mean curvature surface of the original surface.

Having a second look at the Darboux transform of the cylinder shown in figure 5 we recognize a strong similarity to Ivan Sterling's "doublebubbleton" [12]. This suggests a relation between our constant mean curvature Darboux transform and

9 THE BIANCHI-BÄCKLUND TRANSFORM OF CONSTANT MEAN CURVATURE SURFACES

We may supply any surface $f : M^2 \to \mathbb{R}^3$ of constant mean curvature $H = \frac{1}{2}$ with conformal coordinates $(x, y) : M^2 \to \mathbb{R}^2$ such that

$$\begin{array}{rcl} I & = & e^{2u}(dx^2 + dy^2), \\ II & = & e^u(\sinh(u)dx^2 + \cosh(u)dy^2) \end{array} \tag{62}$$

— reflecting the fact that every surface of constant mean curvature is isothermic. Then, a new surface of constant mean curvature — a "Bianchi-Bäcklund transform" of the original surface — can be obtained as $\hat{f} = f + g$ where

$$g = \frac{2}{\sinh(\beta)\cosh(\beta + \varphi)} \left(\cosh(\beta)e^{-u}[\cos\psi f_x - \sin\psi f_y] - \sinh\varphi n \right), \tag{63}$$

β denoting a real parameter and $\varphi + i\psi = \theta$ being given by the linear system

$$\begin{array}{rcl} \theta_x + iu_y & = & \sinh\beta \sinh\theta \cosh u + \cosh\beta \cosh\theta \sinh u \\ i\theta_y + u_x & = & -\sinh\beta \cosh\theta \sinh u - \cosh\beta \sinh\theta \cosh u. \end{array} \tag{64}$$

In fact, this transformation is obtained by applying two successive Bäcklund transforms to the surface of constant Gauß curvature [1] which is parallel to the original surface of constant mean curvature and then, taking the (correct) parallel surface of constant mean curvature [12]. In this construction, the second Bäcklund transform has to be matched to the first one such that the resulting surface of constant Gauß curvature is a real surface again.

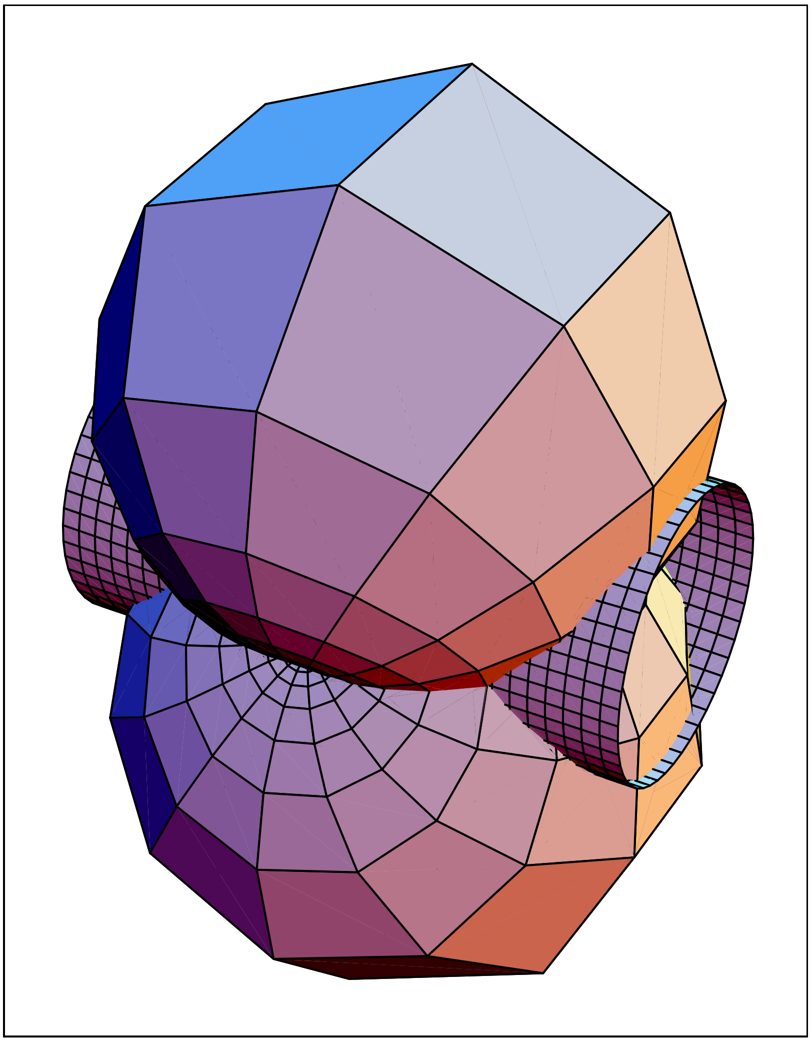

Figure 5: A Darboux transform of the cylinder

Fixing the scaling of the Christoffel transform f^c of f such that $H^c = H = \frac{1}{2}$, i.e. $f^c = f + 2n$, it is an unpleasant but straightforward calculation to see that our Riccati type equation

$$dg = g(\frac{\sinh^2(\beta)}{4} df^c)g - df \tag{65}$$

is equivalent to the above linear system (64) defining the function θ. Thus we have:

THEOREM 7 *Any Bianchi-Bäcklund transform of a surface of constant mean curvature is a Darboux transform.*

Analyzing the effect of the three parameters (β and initial values for φ and ψ) contained in the Bianchi-Bäcklund transform on the function $g : M \to \mathbb{R}^3$ at an initial point we find that any solution of our Riccati equation (20) with a *positive* multiple of the parallel constant mean curvature surface $f + 2n$ as Christoffel transform f^c can be obtained via a Bianchi-Bäcklund transform[29]. Those constant mean curvature Darboux transforms of a constant mean curvature surface where the Christoffel transform is taken a *negative* multiple of the parallel constant mean curvature surface (see Fig. 6) seem *not* to occur as Bianchi-Bäcklund transforms.

[29]Hereby, we also have to allow singularities $\varphi \to \infty$ to obtain vertical values of g too.

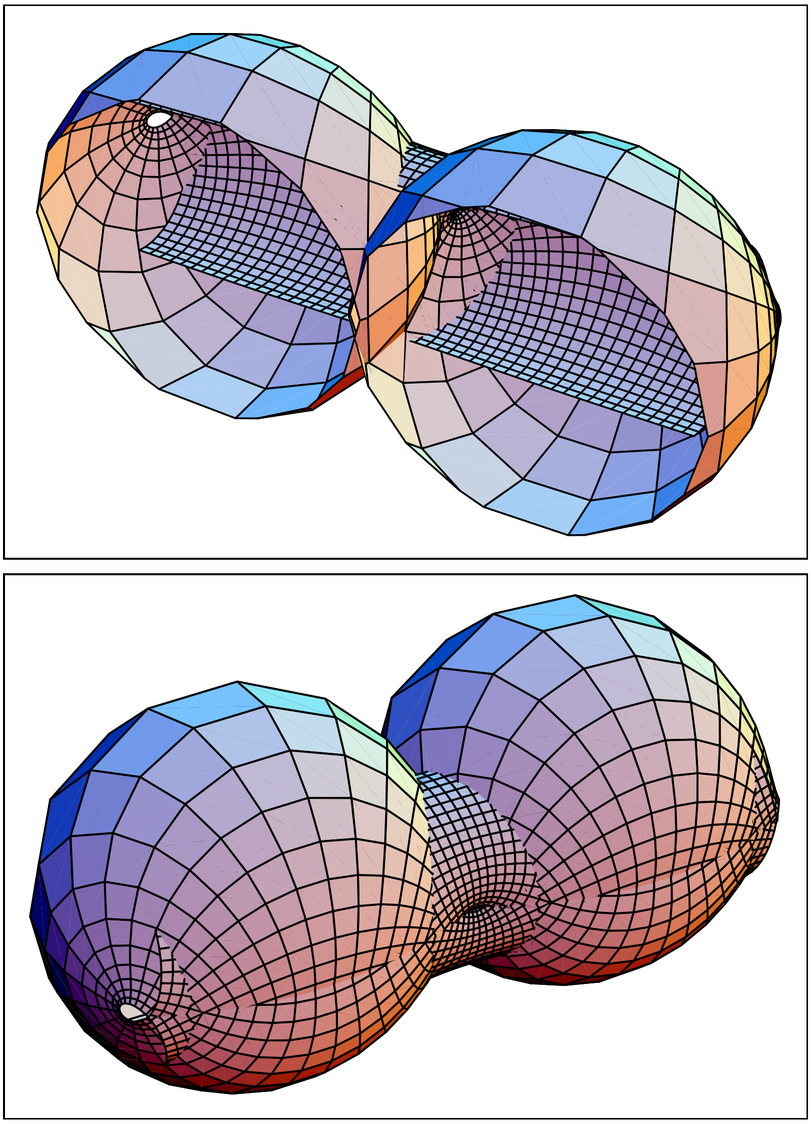

Figure 6: Another Darboux transform of the cylinder

References

[1] L. Bianchi: *Vorlesungen über Differentialgeometrie;* Teubner, Leipzig 1899 (Anhang zu Kapitel XVII: Zur Transformationstheorie der Flächen mit constantem positiven Krümmungsmass, pp.641-648)

[2] L. Bianchi: *Ricerche sulle superficie isoterme e sulla deformazione delle quadriche;* Annali Mat. 11 (1905) 93-157

[3] W. Blaschke: *Vorlesungen über Differentialgeometrie III;* Springer, Berlin 1929

[4] F. Burstall, U. Hertrich-Jeromin, F. Pedit and U. Pinkall: *Curved flats and Isothermic surfaces;* Math. Z. 225 199-209

[5] E. Christoffel: *Ueber einige allgemeine Eigenschaften der Minimumsflächen;* Crelle's J. 67 (1867) 218-228

[6] J. Cieśliński: *The Darboux-Bianchi transformation for isothermic surfaces;* Diff. Geom. Appl. 7 (1997) 1-28

[7] G. Darboux: *Sur les surfaces isothermiques;* Comptes Rendus 128 (1899) 1299-1305, 1538

[8] D. Ferus, F. Pedit: *Curved flats in symmetric spaces;* Manuscripta Math. 91 (1996) 445-454

[9] U. Hertrich-Jeromin: *Supplement on Curved flats in the space of point pairs and Isothermic surfaces: A quaternionic Calculus;* Doc. Math. J. DMV 2 (1997) 335-351

[10] U. Hertrich-Jeromin, T. Hoffmann, U. Pinkall: *A discrete version of the Darboux transform for isothermic surfaces;* to appear in A. Bobenko, R. Seiler, *Discrete integrable Geometry and Physics*, Oxford Univ. Press, Oxford 1997

[11] G. Kamberov: *Holomorphic quadratic differentials and conformal immersions;* Preprint (1997)

[12] I. Sterling, H.C. Wente: *Existence and Classification of Constant Mean Curvature Multibubbletons of Finite and Infinite Type;* Indiana Univ. Math. J. 42 (1993) 1239-1266

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