Correction to the Paper "Classical Motivic Polylogarithm According to Beilinson and Deligne"

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Abstract. The Hodge theoretic appendix of our paper contains results (A.2.10, A.2.11) about the explicit shape of the groups of one-extensions in the category of Hodge structures. Regrettably, they are incorrect in general. Here, we give the right formulae. Since they coincide with the old ones for Hodge structures of strictly negative weights (e.g., Tate twists $A(n)$, $n \geq 1$), this correction has no effect on the main text of our paper.

In fact, the results from [B1] and [Jn3] quoted in the proof of A.2.10 refer to extensions in the category MHS'_A of mixed A-Hodge structures, while the statement of A.2.10 was about extensions in the category MHS_A of *graded-polarizable* Hodge structures. Using the notation of Appendix A, we have:

Theorem 1. *For any $H \in \mathrm{MHS}_A$, there is a canonical isomorphism*

$$W_{-1}H_{\mathbb{C}}/W_{-1}H_{\mathbb{C}} \cap (W_0 H_A + W_0 F^0 H_{\mathbb{C}}) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{MHS}_A}(A(0), H)$$

given by sending the class of $h \in W_{-1}H_{\mathbb{C}}$ to the extension described by the matrix

$$\begin{pmatrix} 1 & 0 \\ -h & \mathrm{id}_H \end{pmatrix} .$$

This means that we equip $\mathbb{C} \oplus H_{\mathbb{C}}$ with the diagonal weight and Hodge filtrations, and the A–rational structure extending the A–rational structure H_A of $H_{\mathbb{C}}$ by the vector

$$1 - h \in \mathbb{C} \oplus H_{\mathbb{C}} .$$

Proof. We may assume that H does not have strictly positive weights: look at the $\mathrm{Ext}^{\cdot}_{\mathrm{MHS}_A}(A(0),\ \ )$ sequence associated to the short exact sequence

$$0 \longrightarrow W_0 H \longrightarrow H \longrightarrow H/W_0 H \longrightarrow 0\,.$$

Our claim will be a consequence of a comparison of the $\mathrm{Ext}^{\cdot}_{\mathrm{MHS}'_A}(A(0),\ \ )$ and the $\mathrm{Ext}^{\cdot}_{\mathrm{MHS}_A}(A(0),\ \ )$ sequences associated to the short exact sequence

$$0 \longrightarrow W_{-1} H \longrightarrow H \longrightarrow \mathrm{Gr}^W_0 H \longrightarrow 0\,.$$

We use the formula from [B1], § 1 or [Jn3], 9.2, 9.3 for $\mathrm{Ext}^{\cdot}_{\mathrm{MHS}'_A}(A(0),\ \ )$, together with the following observations: 1) The Ext groups coincide if H has only strictly negative weights (since any extension in MHS'_A of $A(0)$ by H will then automatically be graded-polarizable). 2) $\mathrm{Ext}^1_{\mathrm{MHS}'_A}(A(0), H)$ is trivial if H is pure of weight 0 (any extension in MHS_A of pure objects of the same weight splits). 3) The canonical map from $\mathrm{Ext}^1_{\mathrm{MHS}_A}$ to $\mathrm{Ext}^1_{\mathrm{MHS}'_A}$ is injective.

It follows that $\mathrm{Ext}^1_{\mathrm{MHS}_A}(A(0), H)$ equals the image of

$$\mathrm{Ext}^1_{\mathrm{MHS}_A}(A(0), W_{-1}H) = W_{-1}H_{\mathbb{C}}/(W_{-1}H_A + W_{-1}F^0 H_{\mathbb{C}})$$

in

$$\mathrm{Ext}^1_{\mathrm{MHS}'_A}(A(0), H) = W_0 H_{\mathbb{C}}/(W_0 H_A + W_0 F^0 H_{\mathbb{C}})\,.$$

□

By replacing [B1], § 1 and [Jn3], 9.2, 9.3 by Theorem 1 in the original proofs of A.2.10 and A.2.11, one sees that the correct statements read as follows:

THEOREM A.2.10. *For any $H \in \mathrm{MHS}^+_A$, there is a canonical isomorphism*

$$\begin{aligned}\big(W_{-1}H_{\mathbb{C}}/W_{-1}H_{\mathbb{C}} \cap (W_0 H_A + W_0 F^0 H_{\mathbb{C}})\big)^+ &\xrightarrow{\ \sim\ } \mathrm{Ext}^1_{\mathrm{MHS}^+_A}(A(0), H) \\ &= H^1_{\mathfrak{H}^p}(\mathrm{Spec}(\mathbb{R})/\mathbb{R}, H)\,,\end{aligned}$$

where the superscript + on the left hand side denotes the fixed part of the de Rham–conjugation

$$\begin{aligned}W_{-1}H_{\mathbb{C}}/W_{-1}H_{\mathbb{C}} \cap (W_0 H_A + W_0 F^0 H_{\mathbb{C}}) &\xrightarrow{c_\infty} W_{-1}H_{\mathbb{C}}/W_{-1}H_{\mathbb{C}} \cap (W_0 H_A + W_0 \overline{F}^0 H_{\mathbb{C}}) \\ &= W_{-1}\,\iota^* H_{\mathbb{C}}/W_{-1}\,\iota^* H_{\mathbb{C}} \cap (W_0\,\iota^* H_A + W_0 F^0 \iota^* H_{\mathbb{C}}) \\ &\xrightarrow{F_\infty} W_{-1}H_{\mathbb{C}}/W_{-1}H_{\mathbb{C}} \cap (W_0 H_A + W_0 F^0 H_{\mathbb{C}})\,.\end{aligned}$$

The isomorphism is given by sending the class of $h \in W_{-1}H_{\mathbb{C}}$ to the extension described by the matrix

$$\begin{pmatrix} 1 & 0 \\ -h & \mathrm{id}_H \end{pmatrix}.$$

This means that we equip $\mathbb{C} \oplus H_{\mathbb{C}}$ with the diagonal weight and Hodge filtrations, and the A–rational structure extending the A–rational structure H_A of $H_{\mathbb{C}}$ by the vector

$$1 - h \in \mathbb{C} \oplus H_{\mathbb{C}} ,$$

thereby obtaining an extension E of $A(0)$ by H in the category MHS_A.

The conjugate extension $\iota^* E \in \mathrm{Ext}^1_{\mathrm{MHS}_A}(A(0), \iota^* H)$ is given, with the same notation, by the matrix

$$\begin{pmatrix} 1 & 0 \\ -F_\infty(h) & \mathrm{id}_{\iota^* H} \end{pmatrix} ,$$

and the extension of F_∞ to an isomorphism

$$F_\infty : E \xrightarrow{\sim} \iota^* E$$

sends $1 - h$ to $1 - F_\infty(h)$. Thus

$$(F_\infty)_{\mathbb{C}} = \mathrm{id} \oplus (F_\infty)_{\mathbb{C}} : \mathbb{C} \oplus H_{\mathbb{C}} \longrightarrow \mathbb{C} \oplus \iota^* H_{\mathbb{C}} .$$

COROLLARY A.2.11. *Let $X/\mathbb{R}$ be finite and reduced, and $M \in \mathrm{MHM}_A(X/\mathbb{R})$. Then there is a canonical isomorphism*

$$\begin{aligned} &\left(\bigoplus_{x \in X(\mathbb{C})} W_{-1} M_{x,\mathbb{C}} / W_{-1} M_{x,\mathbb{C}} \cap (W_0 M_{x,A} + W_0 F^0 M_{x,\mathbb{C}}) \right)^+ \\ \underset{\mathrm{A.2.10}}{\xrightarrow{\sim}} \; & \mathrm{Ext}^1_{\mathrm{MHS}_A^+} \left(A(0), \bigoplus_{x \in X(\mathbb{C})} M_x \right) \\ = \; & H^1_{\mathfrak{H}^p}(X/\mathbb{R}, M) . \end{aligned}$$

REFERENCES

[B1] A.A. Beilinson, "Notes on absolute Hodge cohomology", in "Applications of Algebraic K–theory to Algebraic Geometry and Number Theory", Proceedings of a Summer Research Conference held June 12–18, 1983, in Boulder, Colorado, Contemp. Math., vol. 55, Part I, AMS, Providence, pp. 35–68.

[Jn3] U. Jannsen, "Mixed Motives and Algebraic K– Theory", LNM 1400, Springer–Verlag 1990.

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