On a Conjecture of Izhboldin on Similarity of Quadratic Forms

Detlev W. Hoffmann[1]

Received: January 4, 1999

Communicated by Ulf Rehmann

Abstract. In his paper *Motivic equivalence of quadratic forms*, Izhboldin modifies a conjecture of Lam and asks whether two quadratic forms, each of which isomorphic to the product of an Albert form and a k-fold Pfister form, are similar provided they are equivalent modulo I^{k+3}. We relate this conjecture to another conjecture on the dimensions of anisotropic forms in I^{k+3}. As a consequence, we obtain that Izhboldin's conjecture is true for $k \leq 1$.

1991 Mathematics Subject Classification: Primary 11E81; Secondary 11E04.
Keywords and Phrases: Quadratic form, Pfister form, Albert form, similarity of quadratic forms.

In what follows, we will adhere to the same terminology and notations used in Izhboldin's article [I] mentioned in the abstract. In particular, if two quadratic forms ϕ and ψ are similar, we will write $\phi \sim \psi$.
Let F be a field of characteristic $\neq 2$. Recall that an Albert form α over F is a 6-dimensional quadratic form over F with signed discriminant $1 \in F^*/F^{*2}$ (i.e. $\alpha \in I^2F$), and an n-fold Pfister form over F is a form of type $\langle\langle a_1, \cdots, a_n\rangle\rangle := \langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$, $a_i \in F^*$. In his paper [I], Izhboldin states the following conjecture:

Conjecture 1 (Cf. Conjecture 5.1 in [I].) *Let q_1 and q_2 be Albert forms over F and let π_1 and π_2 be two k-fold Pfister forms over F ($k \geq 0$) such that $q_i \otimes \pi_i$, $i = 1, 2$ is anisotropic and $q_1 \otimes \pi_1 \equiv q_2 \otimes \pi_2 \bmod I^{k+3}F$. Then $q_1 \otimes \pi_1 \sim q_2 \otimes \pi_2$.*

[1]This research has been carried out in the framework of TMR Network ERB FMRX CT-97-0107 "Algebraic K-Theory, Linear Algebraic Groups and Related Structures."

In fact, this conjecture is a special case of a question asked by Lam [L, (6.6)]. Lam's original question was as follows. Suppose σ_i, $\rho_i \in P_nF$, $i = 1, 2$, and let $\phi_i = (\sigma_i \perp -\rho_i)_{\mathrm{an}}$ be the anisotropic part of $\sigma_i \perp -\rho_i$. If $\phi_1 \equiv \phi_2 \bmod I^{n+1}F$, does it then follow that $\phi_1 \sim \phi_2$? By a result of Elman and Lam [EL, Theorem 4.5], it is known that $\dim \phi_i \in \{2^{n+1} - 2^m, 1 \leq m \leq n+1\}$, and that if $\dim \phi_i = 2^{n+1} - 2^m$, then ρ_i and σ_i are $(m-1)$-linked, i.e. there exists an $(m-1)$-fold Pfister form which divides both ρ_i and σ_i. It is an easy exercise to show that Lam's question has a positive answer if $\dim \phi_1$ (or $\dim \phi_2$) equals 0 of 2^n (i.e. $m = n+1$ or $m = n$). In [I, Section 4], Izhboldin constructs counterexamples with $\dim \phi_1$ (or $\dim \phi_2$) equal to $2^{n+1} - 2^m$ with $1 \leq m \leq n-2$. The only remaining case $m = n-1$ boils down to Conjecture 1 above.
It turns out that this conjecture would have a positive answer if another well-known conjecture on quadratic forms were true, this other conjecture being

Conjecture 2 *Let $n \geq 2$ and let q be an anisotropic form in I^nF. If* $\dim q > 2^n$ *then* $\dim q \geq 2^n + 2^{n-1}$.

Proposition 1 *Conjecture 2 for $n = k+3$ implies Conjecture 1 for k.*

It was shown in [H 2] that Conjecture 2 holds for $n \leq 4$. As a consequence, we have

Corollary *Conjecture 1 holds for $k \leq 1$.*

Note that for $k = 0$ this is essentially Jacobson's theorem saying that two Albert forms are similar if and only if their associated biquaternion algebras are isomorphic (see [MS] for a quadratic form-theoretic proof of Jacobson's theorem).

Proof of Proposition 1. Suppose that Conjecture 2 holds for $k+3$. Let q_1 and q_2 be Albert forms over F and let π_1 and π_2 be two k-fold Pfister forms over F ($k \geq 0$) such that $q_1 \otimes \pi_1 \equiv q_2 \otimes \pi_2 \bmod I^{k+3}F$ and such that $q_i \otimes \pi_i$ is anisotropic for $i = 1, 2$.
First, we note that we may assume $\pi_1 = \pi_2$ (cf. the remarks following Conjecture 5.1 in [I]). We denote this k-fold Pfister form by π. Since $q_i \otimes \pi \in I^{k+2}F$, we can scale q_i (and thus $q_i \otimes \pi$) without changing the equivalence $\bmod I^{k+3}F$, and we may thus assume that $q_i \cong \langle 1 \rangle \perp q_i'$, $\dim q_i' = 5$ for $i = 1, 2$. This yields $q_1' \otimes \pi \equiv q_2' \otimes \pi \bmod I^{k+3}F$.
In particular, $\pi \otimes (q_1' \perp -q_2')$ is a form of dimension $2^k(2^3 + 2) = 2^{k+3} + 2^{k+1}$ in $I^{k+3}F$. By Conjecture 2, $\pi \otimes (q_1' \perp -q_2')$ is isotropic. In particular, there exists $x \in F^*$ such that x is represented by both $\pi \otimes q_1'$ and $\pi \otimes q_2'$. Using the multiplicativity of Pfister forms (cf. [EL, Theorem 1.4]), there exist 4-dimensional forms q_i'', $i = 1, 2$, such that $\pi \otimes q_i' \cong \pi \otimes (\langle x \rangle \perp q_i'')$.
From this, it follows readily that $\pi \otimes q_1'' \equiv \pi \otimes q_2'' \bmod I^{k+3}F$. Note that $\dim(\pi \otimes q_i'') = 2^{k+2}$, so that $\pi \otimes q_1''$ and $\pi \otimes q_2''$ are (anisotropic) half-neighbors. As a consequence, $\pi \otimes q_1''$ becomes isotropic over the function field of $\pi \otimes q_2''$ (see, e.g., [H 3, Corollary 2.6] or [I, Lemma 3.3]). By [H 1, Theorem 1.4], this

implies that $\pi \otimes q_1''$ and $\pi \otimes q_2''$ are similar, so that there exists some $y \in F^*$ such that $\pi \otimes q_1'' \cong y\pi \otimes q_2''$. Thus, we obtain

$$\begin{array}{rcll} \pi \otimes q_1 & \equiv & \pi \otimes \langle 1, x\rangle \perp \pi \otimes q_1'' & \bmod I^{k+3}F \\ & \equiv & \pi \otimes q_2 & \bmod I^{k+3}F \\ & \equiv & y\pi \otimes q_2 & \bmod I^{k+3}F \\ & \equiv & y\pi \otimes \langle 1, x\rangle \perp y\pi \otimes q_2'' & \bmod I^{k+3}F \\ & \equiv & y\pi \otimes \langle 1, x\rangle \perp \pi \otimes q_1'' & \bmod I^{k+3}F \end{array}$$

and hence $\pi\otimes\langle 1,x\rangle \equiv y\pi\otimes\langle 1,x\rangle \bmod I^{k+3}F$. Now $\dim(\pi\otimes\langle 1,x\rangle) = 2^{k+1}$, and the Arason-Pfister Hauptsatz therefore implies that $\pi \otimes \langle 1,x\rangle \cong y\pi \otimes \langle 1,x\rangle$. We conclude that

$$\begin{array}{rcl} \pi \otimes q_1 & \cong & \pi \otimes \langle 1, x\rangle \perp \pi \otimes q_1'' \\ & \cong & y\pi \otimes \langle 1, x\rangle \perp y\pi \otimes q_2'' \\ & \cong & y\pi \otimes q_2 \ . \end{array}$$

□

Note that we didn't really make use of the fact that q_1 and q_2 are Albert forms. However, it is not difficult to show that if π is a k-fold Pfister form and $q = q' \perp \langle a\rangle \in IF$ such that $\pi \otimes q \in I^{k+2}F$, then if one chooses $b \in F^*$ such that $\tilde{q} = q' \perp \langle b\rangle \in I^2F$, one has $\pi \otimes q \cong \pi \otimes \tilde{q}$. So what is essential is the fact that $\pi \otimes q_i$ is in $I^{k+2}F$, in which case we may as well assume by what we just mentioned that q_i is an Albert form.
In the proof of Conjecture 2 for $n = 4$ in [H 2], one makes use of a certain property PD_2. It turns out that this property can be used to establish Conjecture 1 for $k = 1$ without invoking Conjecture 2 for $n = 4$. Let us recall the general definition of property PD_n.

DEFINITION Let n be an integer ≥ 1. The field F is said to have the Pfister decomposition property for Pfister forms of fold $\leq n$, PD_n for short, if for each m $(1 \leq m \leq n)$, for each anisotropic $\pi \in P_{m-1}F$, for each $r \in \dot{F}$, and each anisotropic $\varphi \in \pi WF$, there exist forms σ and τ over F such that for $\rho := \pi \otimes \langle\langle r\rangle\rangle$ one has $\varphi \cong \pi \otimes \sigma \perp \rho \otimes \tau$ and $(\varphi_{F(\rho)})_{\mathrm{an}} \cong (\pi \otimes \sigma)_{F(\rho)}$.

PROPOSITION 2 *Suppose that F has PD_n for some $n \geq 1$. Then Conjecture 1 holds for $k = n - 1$.*

Proof. Suppose that F has PD_n for $n = k+1$. As in the previous proof, we may assume that we are in the situation where $\pi \otimes q_1 \equiv \pi \otimes q_2 \bmod I^{k+3}F$ with Albert forms q_i, $i = 1, 2$, a k-fold Pfister form π and with $\pi \otimes q_i$ being anisotropic for $i = 1, 2$. After scaling, we may assume that $q_1 \cong \langle 1, -r\rangle \perp q_1'$ for some $r \in F^*$. It follows that $\pi \otimes q_1$ contains the subform $\rho = \pi \otimes \langle\langle r\rangle\rangle$.
In particular, $\pi \otimes q_1$ becomes isotropic over the function field $F(\rho)$, and thus $\pi\otimes q_2$ also becomes isotropic over $F(\rho)$ (cf. [I, Theorem 4.3]). Property PD_{k+1} then implies that $\pi\otimes q_2$ contains a subform similar to ρ, and since we may scale

$\pi \otimes q_2 \in I^{k+2}F$ without changing the equivalence $\bmod I^{k+3}F$, we may assume that $\pi \otimes q_2 \cong \pi \otimes (\langle 1, -r\rangle \perp q_2')$ for some 4-dimensional form q_2'.
It follows that $\pi \otimes q_1' \equiv \pi \otimes q_2' \bmod I^{k+3}F$. As in the proof of Proposition 1, this implies that $\pi \otimes q_1'$ and $\pi \otimes q_2'$ are similar, and thus that $\pi \otimes q_1$ and $\pi \otimes q_2$ are also similar. □

It was proved by Rost that each field has property PD_2 (see [H 2, Lemma 2.6]). Again, we can conclude that Conjecture 1 holds for $k \leq 1$, this time by invoking PD_2.
In the case $n \geq 3$, we do not know whether PD_n holds for all fields nor whether PD_n for a field F implies that Conjecture 2 holds for F for $n+2$ (or vice versa).

References

[EL] Elman, R.; Lam, T.Y.: *Pfister forms and K-theory of fields.* J. Algebra 23 (1972), 181–213.

[H 1] Hoffmann, D.W.: *On quadratic forms of height* 2 *and a theorem of Wadsworth.* Trans. Amer. Math. Soc. 348 (1996), 3267–3281.

[H 2] Hoffmann, D.W.: *On the dimensions of anisotropic quadratic forms in I^4.* Invent. Math. 131 (1998), 185–198.

[H 3] Hoffmann, D.W.: *Similarity of quadratic forms and half-neighbors.* J. Algebra 204 (1998), 255–280.

[I] Izhboldin, O.T.: *Motivic equivalence of quadratic forms.* Doc. Math. J. DMV 3 (1998), 341–351.

[L] Lam, T.Y.: *Fields of u-invariant* 6 *after A. Merkurjev.* Israel Math. Conf. Proc. Vol. 1: Ring Theory 1989 (in honor of S.A. Amitsur) (ed. L. Rowen). pp.12–31. Jerusalem: Weizmann Science Press 1989.

[MS] Mammone, P.; Shapiro, D.B.: *The Albert quadratic form for an algebra of degree four.* Proc. Amer. Math. Soc. 105 (1989), 525–530.

Detlev W. Hoffmann
Équipe de Mathématiques
UMR 6623 du CNRS
Université de Franche-Comté
16, route de Gray
25030 Besançon Cedex, France
detlev@math.univ-fcomte.fr