Publ. RIMS, Kyoto Univ.
8 (1972/73), 201-205

Abstract Potential Operators on Hilbert Space

(Dedicated to Professor Yasuo Akizuki on his 70th birthday)

By

Kôsaku YOSIDA*

Let X be a (real or complex) Hilbert space. A linear operator V with its domain $D(V)$ and range $R(V)$ both strongly dense in X is called an *abstract potential operator* (see K. Yosida [2], p. 412) if the inverse V^{-1} exists in such a way that

(1) $$A=-V^{-1}$$

is the infinitesimal generator of a one-parameter semi-group of class (C_0) of linear contraction operators on X into X. The purpose of the present note is to prove the following existence theorem. (Hereafter, we shall denote by S^a the strong closure of a subset S of X.)

Theorem. *Let U be a linear operator satisfying three conditions:*

(2) $$D(U)^a=X,$$

(3) $$R(U)^a=X,$$

(4) *U is accretive, that is,* $Re(Uf, f)\geqq 0$ *for every* $f\in D(U)$.

Then there exists at least one abstract potential operator V which is a closed linear accretive extension of U; V might coincide with U.

Proof. The proof is given in two steps. The first is to construct a maximal accretive extension V of U by virtue of R. S. Phillips' *theory of*

Received March 2, 1972.
* Dept. Math., Gakushuin Univ., 1-5-1, Mejiro, Toshima-ku, Tokyo 171, Japan.

Cayley transform (cf. B. Sz.-Nagy and C. Foias [1], p. 167). The second is to prove that this V is an abstract potential operator by making use of the *abelian ergodic theorem for pseudo-resolvents* (see K. Yosida [2], p. 215).

THE FIRST STEP. For every $\lambda>0$ and $f\in D(V)$, we have, by (4),

$$(5)\qquad \begin{aligned}\|\lambda Uf+f\|^2&=(\lambda Uf+f,\ \lambda Uf+f)=\|\lambda Uf\|^2+2Re(\lambda Uf, f)+\|f\|^2\\ &\geqq\|\lambda Uf\|^2+\|f\|^2\geqq\|\lambda Uf\|^2-2Re(\lambda Uf, f)+\|f\|^2=\|\lambda Uf-f\|^2.\end{aligned}$$

Hence the inverse $(\lambda U+I)^{-1}$ exists and moreover, the Cayley transform C defined through

$$(6)\qquad C\cdot(Uf+f)=(Uf-f)$$

is a contraction operator mapping $R(U+I)$ onto $R(U-I)$. Let us define a bounded linear extension $\hat{C}$ of C:

(7) through continuity on $R(U+I)^a$, and through putting $\hat{C}\cdot g=0$ on the orthogonal complement of $R(U+I)$.

This everywhere defined contraction operator $\hat{C}$ cannot admit eigenvalue one. Assume the contrary and let $\hat{C}\cdot f_0=f_0$ with $\|f_0\|=1$. Then its adjoint operator $\hat{C}^*$, which is also a contraction, must satisfy $\hat{C}^*\cdot f_0=f_0$ because

$$\begin{aligned}\|\hat{C}^*\cdot f_0-f_0\|^2&=\|\hat{C}^*\cdot f_0\|^2-2Re(\hat{C}^*\cdot f_0, f_0)+\|f_0\|^2\\ &\leqq\|f_0\|^2-2Re(f_0,\ \hat{C}\cdot f_0)+\|f_0\|^2=1-2+1=0.\end{aligned}$$

Thus we obtain, by (6) and (7),

$$(f_0, (U-I)f)=(f_0, \hat{C}\cdot(U+I)f)=(\hat{C}^*\cdot f_0, (U+I)f)=(f_0, (U+I)f),$$

hence $(f_0, f)=0$ and so $f_0=0$ by (2).

Therefore the inverse $(I-\hat{C})^{-1}$ exists and so we can define a linear operator V through

$$(8)\qquad V\cdot(I-\hat{C})f=(I+\hat{C})f.$$

V is an extension of U. In fact, we have, by (6), $(I-C)=I-(U-I)(U+I)^{-1}=2(U+I)^{-1}$, that is, $U=(I+C)(I-C)^{-1}$, proving that V is an extension of U. Here the existence of $(I-C)^{-1}$ is assured by that of $(I-\hat{C})^{-1}$. We can prove that V is accretive. For, by putting $f=(I-\hat{C})^{-1}g$ and observing (8) and the contraction property of $\hat{C}$, we obatin

$$Re(Vg,\, g)=Re((I+\hat{C})f,\, (I-\hat{C})f)=||f||^2-||\hat{C}\cdot f||^2\geqq 0.$$

We can also prove, by (8) and the boundedness of the operator $\hat{C}$, that V is a closed linear operator. Moreover, by (8), we have $(I+V)=I+(I+\hat{C})(I-\hat{C})^{-1}=2(I-\hat{C})^{-1}$, and so we obtain the existence theorem

(9) $R(V+I)=D(I-\hat{C})=X$ (and also $R(\lambda V+I)=X$ whenever $\lambda>0$).

Hence the accretive extension V is maximal as regards its range $R(\lambda V+I)$ for $\lambda>0$.

THE SECOND STEP. We will show that V is an abstract potential operator following after the proof of Theorem 2 on p. 414–415 in K. Yosida [2].

V being accretive, we have, as in (5), $||\lambda Vf+f||\geqq||\lambda Vf||$ for every $f\in D(V)$ and $\lambda>0$. Hence, by (9), we can define a bounded linear operator

(10) $$J_\lambda=V(\lambda V+I)^{-1}$$

satisfying

(11) $$||\lambda J_\lambda||\leqq 1.$$

It is easy to see that J_λ is a *pseudo-resolvent*, i.e.,

(12) $$J_\lambda-J_\mu=(\mu-\lambda)J_\lambda J_\mu.$$

Therefore, by (11), we can apply the abelian ergodic theorem to the effect that

(13) $$R(J_\mu)^a=\{x\in X;\, s\text{-}\lim_{\lambda\uparrow\infty}\lambda J_\lambda x=x\}\qquad\text{for all } \mu>0,$$

(14) $$R(I-\mu J_{\mu})^a=\{x\in X;\ s\text{-}\lim_{\lambda\downarrow 0}\lambda J_{\lambda}\, x=0\}\qquad\text{for all } \mu>0.$$

By $R(V)^a=R(U)^a=X$, we have $R(J_{\mu})^a=X$ by (10) and so, by (11) and (12), the null space of J_{λ} consists of zero vector only, independently of $\lambda>0$. Hence J_{λ} is the resolvent of a linear operator, i. e.,

(15) $J_{\lambda}=(\lambda I-A)^{-1}$, where $A=\lambda I-J_{\lambda}^{-1}$ is independent of $\lambda>0$.

We have thus $D(A)^a=R(J_{\mu})^a=X$ and so, by (11), the operator A is the infinitesimal generator of a contraction semi-group of class (C_0). We can also prove that $R(A)^a=X$. For, we have, by (10) and (15),

$$(\lambda I-A)J_{\lambda}(\lambda Vf+f)=\lambda Vf+f=(\lambda I-A)Vf=\lambda Vf-AVf,$$

that is,

(16) $$-AVf=f\qquad\text{whenever } f\in D(V),$$

proving that $R(A)^a=D(V)^a=D(U)^a=X$. Thus, by (14) and $AJ_{\mu}= =(\mu J_{\mu}-I)$, we obtain $s\text{-}\lim_{\lambda\downarrow 0}\lambda J_{\lambda}f=0$ for all $f\in X$. This implies that the inverse A^{-1} exists. In fact, the condition $Af_0=0$ is equivalent to $\lambda(\lambda I-A)^{-1}f_0=f_0$ and hence $f_0=s\text{-}\lim_{\lambda\downarrow 0}\lambda J_{\lambda}f_0=0$.

Thus $-A^{-1}$ is an abstract potential operator. On the other hand, (16) shows that the inverse V^{-1} exists. Hence, by $(\lambda I-A)=J_{\lambda}^{-1}= =(\lambda V+I)V^{-1}=\lambda I+V^{-1}$, we obtain $-A=V^{-1}$, completing the proof of our Theorem.

Remark. We shall verify (2), (3) and (4) for Newtonian and logarithmic potentials

(17) $$(Uf)(y)=\int_{R^n}K_n(|y-z|)f(z)dz\qquad(n\geqq 2),$$

$$K_n(r)=r^{2-n}\quad\text{for } n\geqq 3,\qquad\text{and}\quad K_2(r)=\log r^{-1}.$$

The proof of $D(U)^a=R(U)^a=X=L^2(R^n)$ can be obtained by making use of the fact that, for $0<\delta_1<\delta_2$,

$$u_{x,\delta_1,\delta_2}(y)=(K_n(\delta_1)-K_2(\delta_2))^{-1}\int_{R^n}K_n(|y-z|)(d\nu_{x,\delta_1}(z)-d\nu_{x,\delta_2}(z))$$

is continuous in y satisfying

$$\begin{aligned} u_{x,\delta_1,\delta_2}(y)&=1 && \text{if } |y-x|\leqq\delta_1,\\ &=0 && \text{if } |y-x|\geqq\delta_2,\\ 0<u_{x,\delta_1,\delta_2}(y)&<1 && \text{if } \delta_1<|y-x|<\delta_2. \end{aligned}$$

Here $\nu_{x,\delta}$ is the unit measure uniformly distributed over the hypersurface of the sphere of centre x and radius δ in R^n.

The Gauss-Frostmann energy inequality

$$\int_{R^n}(Uf)(y)\cdot\overline{f(y)}\,dy\geqq 0 \qquad (n\geqq 2)$$

holds good whenever $f\in L^2(R^n)$ is of compact support satisfying $\int_{R^n}f(y)\,dy=0$. It is easy to prove that such f's constitute a strongly dense subset of $L^2(R^n)$.

ANOTHER TREATMENT OF THE SECOND STEP (*Added on* 20 *April*, 1972). As in the above proof of the non-existence of the eigenvalue 1 for the operator $\hat{C}$, we can show that $\hat{C}\cdot f_0=-f_0$ implies $\hat{C}^*\cdot f_0=-f_0$ and hence $(f_0, Uf)=0$, proving by (3) the non-existence of the eigenvalue -1 for $\hat{C}$. Thus $V=(I+\hat{C})(I-\hat{C})^{-1}$ given by (8) admits the inverse $V^{-1}=$ $=(I-\hat{C})(I+\hat{C})^{-1}$. Hence we can prove that V is an abstract potential operator without appealing to the abelian ergodic theorem.

Remark (added during the proof). On reading the pre-print, Prof. K. Sato gave interesting comments and extensions. See his paper to appear.

References

[1] Sz.-Nagy, B. and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*, North Holland Publ. Co., 1970.

[2] Yosida, K., *Functional Analysis*, the Third Ed., Springer-Verlag, 1971.