Some Properties of the Symmetric Enveloping Algebra of a Subfactor, with Applications to Amenability and Property T

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Abstract. We undertake here a more detailed study of the structure and basic properties of the symmetric enveloping algebra $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ associated to a subfactor $N \subset M$, as introduced in [Po5]. We prove a number of results relating the amenability properties of the standard invariant of $N \subset M, \mathcal{G}_{N,M}$, its graph $\Gamma_{N,M}$ and the inclusion $M \vee M^{\mathrm{op}} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$, notably showing that $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ is amenable relative to its subalgebra $M \vee M^{\mathrm{op}}$ iff $\Gamma_{N,M}$ (or equivalently $\mathcal{G}_{N,M}$) is amenable, i.e., $\|\Gamma_{N,M}\|^2 = [M : N]$. We then prove that the hyperfiniteness of $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ is equivalent to M being hyperfinite and $\Gamma_{N,M}$ being amenable. We derive from this a hereditarity property for the amenability of graphs of subfactors showing that if an inclusion of factors $Q \subset P$ is embedded into an inclusion of hyperfinite factors $N \subset M$ with amenable graph, then its graph $\Gamma_{Q,P}$ follows amenable as well. Finally, we use the symmetric enveloping algebra to introduce a notion of property T for inclusions $N \subset M$, by requiring $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ to have the property T relative to $M \vee M^{\mathrm{op}}$. We prove that this property doesn't in fact depend on the inclusion $N \subset M$ but only on its standard invariant $\mathcal{G}_{N,M}$, thus defining a notion of property T for abstract standard lattices $\mathcal{G}$.

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0. Introduction

Let $N \subset M$ be an inclusion of type II_1 factors with finite Jones index, $[M : N] < \infty$, and extremal. In short, its *symmetric enveloping* von Neumann algebra $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ is the unique (up to isomorphism) type II_1 factor S, generated by mutually commuting copies of M, M^{op} that satisfy $M' \cap S = M^{\mathrm{op}}$, $M^{\mathrm{op}\prime} \cap S = M$ and by a projection e_N which implements, at the same time, both the trace preserving expectation E_N of M onto N and the trace preserving expectation $E_{N^{\rm op}}$ of $M^{\rm op}$ onto $N^{\rm op}$.

One can construct this factor by first taking the C^*-algebra S_0 generated on the Hilbert space $L^2(M)$ by the operators of left and right multiplication by elements in M and by the orthogonal projection of $L^2(M)$ onto $L^2(N)$, then proving that there exists a unique trace τ on this C^*-algebra and then defining $M \underset{e_N}{\boxtimes} M^{\rm op}$ to be the type II_1 von Neumann factor obtained via the Gelfand-Naimark-Segal representation for (S_0, τ), i.e., $M \underset{e_N}{\boxtimes} M^{\rm op} \stackrel{def}{=} \overline{\pi_\tau(S_0)}$. This construction doesn't in fact depend on the (binormal) representation of the triple $(N \subset M, e_N, M^{\rm op} \supset N^{\rm op})$ that one starts with: any $M - M$ bimodule with an e_N-type projection on it, instead of $L^2(M)$, will do, provided certain obvious compatibility conditions for the commutants are satisfied.

The following exemple of symmetric enveloping algebras is quite relevant: if $N \subset M$ is an inclusion associated to a finitely generated discrete group G and an outer action σ of G on a type II_1 factor P (see e.g., 5.1.5 in [Po2]) then $(M \vee M^{\rm op} \subset M \underset{e_N}{\boxtimes} M^{\rm op})$ is isomorphic to $(P \bar{\otimes} P^{\rm op} \subset P \bar{\otimes} P^{\rm op} \rtimes_{\sigma \otimes \sigma^{\rm op}} G)$. In general, one has an interpretation of the *symmetric enveloping inclusion* $M \vee M^{\rm op} \subset M \underset{e_N}{\boxtimes} M^{\rm op}$ that is very much the same as this crossed product situation.

The symmetric enveloping algebra $M \underset{e_N}{\boxtimes} M^{\rm op}$ and the inclusions $M \vee M^{\rm op} \subset M \underset{e_N}{\boxtimes} M^{\rm op}$ were introduced in ([Po5]) in order to provide an additional tool for studying subfactors of finite index. It proved to be particularily useful for relating the analysis aspects of the theory of subfactors to its combinatorial features.

We undertake here a more detailed study of these objects and use them to get more insight into the structure of subfactors, notably proving a number of results on the amenability and the property T for subfactors $N \subset M$ and for their associated combinatorial invariants: the standard graph $\Gamma_{N,M}$ and the standard invariant $\mathcal{G}_{N,M}$.

Thus, we prove that $\mathcal{G}_{N,M}$ is amenable (by definition this means that its graph $\Gamma_{N,M}$ is amenable, i.e., it satisfies the Kesten-type condition $\|\Gamma_{N,M}\|^2 = [M : N]$) if and only if $M \underset{e_N}{\boxtimes} M^{\rm op}$ is amenable relative to $M \vee M^{\rm op}$ in the sense of [Po8]. In fact, we establish a few more additional equivalent characterizations of the amenability for $\Gamma_{N,M}$: a Følner type condition; a local Shannon-McMillan-Breiman type condition; a local bicommutation condition; a characterization in terms of the representations of $N \subset M$.

We then study the amenability in the special case of subfactors $N \subset M$ for which the algebras N, M involved are assumed amenable (or, equivalently, by Connes theorem [C1], hyperfinite) type II_1 factors. The key result along this line shows that the algebra $M \underset{e_N}{\boxtimes} M^{\rm op}$ is itself amenable if and only if both $\mathcal{G}_{N,M}$ and the single algebras N, M are amenable.

Again, some other characterizations of this situation are proved, notably an "injectivity"-type condition requiring $N \subset M$ to be the range of a norm one projection from its standard representation, or, equivalently, to be the range of a norm one projection from any of its (smooth) representations. Also, it is proved equivalent to an Effros-Lance type condition, requiring $S_0 = C^*(M, e_N, M') \subset \mathcal{B}(L^2(M))$ to be a simple C*-algebra . We call an inclusion of factors $N \subset M$ satisfying any of these conditions an *amenable inclusion*. While proving all these results we also show that if $N \subset M$ is amenable (i.e., N, M are hyperfinite and $\Gamma_{N,M}$ is amenable) then there exists a choice of a tunnel of subfactors $M \supset N \supset P_1 \supset P_2 ...$, obtained by taking downward basic constructions for certain induced-reduced algebras in the Jones tower (the choice being dictated by information contained in the standard invariant $\mathcal{G}_{N,M}$) such that the relative commutants $P'_k \cap N \subset P'_k \cap M$ exhaust $N \subset M$. This shows in particular that amenable subfactors are completely classified by their standard invariants $\mathcal{G}_{N,M}$ (see also [Po16]).

Next we derive the result that we regard as the most significant application of the methods developed in this paper, showing that the amenability in the category of inclusions of factors with finite Jones index $N \subset M$, with "morphisms" given by commuting square embeddings between such inclusions, is a hereditarity property. In the case one takes degenerate inclusions $N = M$ we recover Connes' hereditarity result for single hyperfinite type II$_1$ factors ([C1]). In terms of graphs, the result states that if an extremal inclusion of hyperfinite factors $N \subset M$ has amenable graph then any of its sub-inclusions $Q \subset P$ (i.e., $Q \subset P$ is embedded in $N \subset M$ as a commuting square) has amenable graph. It should be noted that the embedding of $Q \subset P$ into $N \subset M$ does not require $[P : Q] = [M : N]$, nor that $[M : P] < \infty$! This hereditarity property for the amenability of graphs is somewhat surprising and there is little that could appriorically predict it. It only holds true within the class of hyperfinite subfactors, as if we drop the amenability assumption on the ambient single algebras M involved it is no longer valid, in general.

Indeed, it is proved in ([Po7]) that given any abstract standard λ- lattice $\mathcal{G}$ and any of its sublattices $\mathcal{G}_0 \subset \mathcal{G}$, there exist subfactors $N \subset M$ and $N_0 \subset M_0$ and a commuting square inclusion of $N_0 \subset M_0$ into $N \subset M$, such that $\mathcal{G}_{N,M} = \mathcal{G}$ and $\mathcal{G}_{N_0,M_0} = \mathcal{G}_0$. But any standard λ-lattice $\mathcal{G}$ contains the Temperley-Lieb-Jones standard λ-lattice with graph A_∞, which is never amenable if $\lambda^{-1} > 4$. Thus if $\mathcal{G}$ is taken to be amenable, for instance to have finite depth, then $N \subset M$ has amenable graph while $N_0 \subset M_0$, which is embedded into it, doesn't. The reason is, of course, that in the examples of subfactors $N \subset M$ constructed in ([Po7]) the algebras N, M involved are not hyperfinite.

One consequence of the hereditarity result is that, for instance, one cannot embed subfactors $Q \subset P$ of index $\alpha > 4$ that are contructed by commuting squares of finite dimensional algebras like in ([Sc], [We]) and having graph $\Gamma_{Q,P}$ equal to A_∞ (note that by [H1] $\Gamma_{Q,P} = A_\infty$ if $\alpha < (5 + \sqrt{13})/2$) into hyperfinite subfactors of finite depth and index $> \alpha$. Also, by ([H2]) there exists a subfactor of index $\alpha = (5 + \sqrt{13})/2$, constructed from commuting squares of finite dimensional algebras and having graph A_∞, which thus, by our result, cannot be embedded into Haagerup's finite depth subfactor of same index $(5+\sqrt{13})/2$ ([H1]).

Our last application to the symmetric enveloping algebra approach is the consideration of a notion of property T for standard lattices. Thus, we prove that if a standard lattice $\mathcal{G}$ is given then $M \underset{e_N}{\boxtimes} M^{\rm op}$ has the property T relative to $M \vee M^{\rm op}$, in the sense of ([A-D],[Po8]), for some $N \subset M$ for which $\mathcal{G}_{N,M} = \mathcal{G}$, if and only if it has this property for any subfactor $N \subset M$ for which $\mathcal{G}_{N,M} = \mathcal{G}$. If $\mathcal{G}$ satisfies these conditions then we say that $\mathcal{G}$ has the *property* T. Note that this definition does not require the ambient factors involved to have the property T in the sense of Connes ([C4,5]). On the other hand, if $\mathcal{G}$ is a standard lattice coming from a discrete group G as described above, then $\mathcal{G}$ has the property T if and only if the group G has the property T in the classical sense of Kazhdan. Thus, our notion generalizes this notion, from discrete groups to the larger class of group-like objects $\mathcal{G}$. Our main result in this direction shows that if a sublattice $\mathcal{G}_0$ of a standard lattice $\mathcal{G}$ has the property T then $\mathcal{G}$ has the property T. As a consequence it follows that, generically, the Temperley-Lieb-Jones standard lattices with graph A_∞ do not have the property T.

Although we only work with type II_1 factors, many of the considerations in this paper can be suitably carried over to subfactors of type III (see the remarks 1.10.3°, 2.2.2°, 2.5.2°). The corresponding symmetric enveloping type III factors may prove to be a useful tool in the analysis of the Jones-Wassermann subfactors coming from representations of loop groups ([Wa], [Xu]). In a different direction, it would be interesting to relate the symmetric enveloping algebra associated to an extremal II_1 subfactor to Jones' affine Hecke algebra associated with that subfactor ([J3,4]). An explicit description of the symmetric enveloping algebras coming from certain special classes of subfactors ([BiH], [BiJ]) would certainly be most illuminating for getting some insight on this and other related problems.

The paper is organized in 9 Sections. In the first section we introduce the C^*-analogues of the symmetric enveloping algebras, needed in order to obtain the necessary universality properties and the functoriality of the von Neumann construction. A key ingredient for these considerations is the relative Dixmier property for subfactors of finite index, that we prove in the Appendix A.1.

In Sec. 2 we define the actual symmetric enveloping type II_1 factors (2.1, 2.2) and symmetric enveloping inclusions and prove their basic properties (2.6, 2.7, 2.9, 2.10). Also, we define a more general class of enveloping inclusions, in which to two given subfactors $N \subset M$ and $Q \subset P$ having the same higher relative commutant picture one associates their *concatenation* inclusion $M \vee P^{\rm op} \subset M \boxtimes P^{\rm op}$ (2.5.1°). We end that section by introducing a notion of index $[\mathcal{G} : \mathcal{G}_0]$ for sublattices $\mathcal{G}_0$ of standard lattices $\mathcal{G}$ (2.11, 2.12).

In Sec. 3 we discuss the example of symmetric enveloping algebras associated to subfactors coming from discrete groups acting outerly on factors, case in which it becomes an actual crossed-product algebra. In Sec. 4 we prove that even for general inclusions $N \subset M$ the corresponding symmetric enveloping algebras look very much like crossed products (4.5). Also, we prove some decomposition properties for such algebras, showing for instance that when N, M are hyperfinite then, regardless of whether $M \underset{e_N}{\boxtimes} M^{\text{op}}$ is hyperfinite or not, it is a *thin* factor, i.e., $M \underset{e_N}{\boxtimes} M^{\text{op}} = \overline{\text{sp}} R_1 R_2$, for some suitable hyperfinite subfactors R_1, R_2 (4.3). Also, we prove a general ergodicity property for the higher relative commutants of a subfactor which is quite useful in applications (4.8, 4.9).

In Sec. 5 and 6 we relate the amenability properties of $\mathcal{G}_{N,M}, \Gamma_{N,M}$ and $(M \vee M^{\text{op}} \subset M \underset{e_N}{\boxtimes} M^{\text{op}})$, obtaining a number of equivalent characterizations of the amenability for standard lattices and graphs (5.3, 6.1, 6.3, 6.4).

In Sec. 7 we discuss the case when $M \underset{e_N}{\boxtimes} M^{\text{op}}$ is hyperfinite, proving this equivalent to the amenability of $N \subset M$ and to various other properties of the representation theory of $N \subset M$ (7.1). For instance, we show that for hyperfinite subfactors it is enough that the universal graph $\Gamma^{u,rf}_{N,M}$ be amenable for $\Gamma_{N,M}$ to follow amenable (7.6). We also prove here the hereditarity property for amenable inclusions (7.5). The proof uses the characterization of the amenability for $N \subset M$ by the hyperfiniteness condition on $M \underset{e_N}{\boxtimes} M^{\text{op}}$, a fact that roughly reduces the argument to Connes' hereditarity of hyperfiniteness for single type II_1 factors. Sec.8 contains the proof of the Effros-Lance type characterization of amenability (8.1).

Finally, in Sec. 9 we introduce the property T for standard lattices and prove some results about this notion.

For most notations and general technical background we refer the reader to ([Po2,4,7]). More specific references are made in the text. For the reader's convenience we included an Appendix which, besides the already mentioned relative Dixmier property for subfactors of finite index, contains a generalized version of Connes' joint distribution trick needed in the proof of the Følner condition for graphs.

The results on amenability in this paper were presented by the author in lectures and seminars, during 1991-1997. A more formal announcement of these results, with sketches of proofs, appeared in [Po5], while a couple of statements on the equivalence of the definition of amenability with representations and the Kesten condition, respectively Følner condition, were already announced in [Po2], resp. [Po4]. A rather complete discussion of the role of amenability within the overall classification of subfactors, with a presentation of most of the results in this paper (including the ones on the property T) appeared in [Po11].

1. Symmetric Enveloping C^*-Algebras

In this and the next section we discuss the definition and basic properties of the symmetric enveloping algebras (C* in this section and von Neumann in the next) associated to an extremal inclusion of factors with finite index, as introduced in [Po5]. The statements below are similar to the ones in §1 of [Po5], but the proofs, that are only briefly sketched there, are given here in details.

So let $N \subset M$ be an inclusion of type II_1 factors with finite Jones index, $[M : N] < \infty$, which we assume to be extremal, i.e., $[pMp : Np] = [M : N]\tau(p)^2$, $\forall\, p \in \mathcal{P}(N' \cap M)$.

We denote by $M \subset M_1 = \langle M, e_N \rangle$ the (abstract) basic construction for $N \subset M$, e_N being the projection implementing the trace preserving conditional expectation E_N of M onto N.

We first construct the universal C^*-algebra generated by mutually commuting copies of M, M^{op} and an e_N-like projection implementing the expectations E_N, $E_{N^{\mathrm{op}}}$ on them.

A *representation* (π, π') of $(N \subset M, e_N, M^{\mathrm{op}} \supset N^{\mathrm{op}})$ is a pair of unital $*$-representations $\pi : M_1 \to \mathcal{B}(\mathcal{H})$, $\pi' : M_1^{\mathrm{op}} \to \mathcal{B}(\mathcal{H})$ such that $[\pi(M), \pi'(M^{\mathrm{op}})] = 0$, $\pi(e_N) = \pi'(e_N^{\mathrm{op}})$. Two such representations, (π_1, π_1') on $\mathcal{H}_1$ and (π_2, π_2') on $\mathcal{H}_2$, are equivalent if there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(x)U^* = \pi_2(x)$, $U\pi_1'(x)U^* = \pi_2'(x)$, $\forall\, x \in M_1$. A representation (π, π') on $\mathcal{H}$ is *cyclic* if $\exists \xi \in \mathcal{H}$ such that $\overline{\mathrm{Alg}(\pi(M_1), \pi'(M_1^{\mathrm{op}}))\xi} = \mathcal{H}$.

Note that if (π, π') is a representation on $\mathcal{H}$ then there exists a representation $(\bar\pi, \bar\pi')$ on the conjugate Hilbert space $\bar{\mathcal{H}}$ defined by $\bar\pi(x) = \overline{\pi'(x^{*\mathrm{op}})}$, $\bar\pi'(x^{\mathrm{op}}) = \overline{\pi(x^*)}$, $x \in M_1$, where $T \mapsto \bar T$ denotes the antiisomorphism from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\bar{\mathcal{H}})$ implemented by the conjugation $\mathcal{H} \ni \xi \mapsto \bar\xi \in \bar{\mathcal{H}}$.

We denote by $\hat{\mathcal{C}}$ the set of all equivalence classes of cyclic representants of $(N \subset M, e_N, M^{\mathrm{op}} \supset N^{\mathrm{op}})$ and by $\mathcal{C}$ a set of chosen representations for $\hat{\mathcal{C}}$ such that if $(\pi, \pi') \in \mathcal{C}$ then $(\bar\pi, \bar\pi') \in \mathcal{C}$.

1.1. Proposition. *There exists a unital C^*-algebra $\mathcal{U}$ with unital embeddings $j : M_1 \hookrightarrow \mathcal{U}$, $j' : M_1^{\mathrm{op}} \hookrightarrow \mathcal{U}$ such that*

a) $[j(M), j'(M^{\mathrm{op}})] = 0$,

b) $j(e_N) = j'(e_N^{\mathrm{op}})$.

and such that given any other unital C^*-algebra $\mathcal{U}_0$ with unital embeddings $j_0 : M_1 \hookrightarrow \mathcal{U}_0$, $j_0' : M_1^{\mathrm{op}} \hookrightarrow \mathcal{U}_0$ satisfying a), b) *(with (j', j_0') instead of (j, j')), there exists a unital $*$-algebra morphism $\pi : \mathcal{U} \to \mathcal{U}_0$ such that*

$$(*) \qquad \pi(j(x)) = j_0(x), \quad \pi(j'(x^{\mathrm{op}})) = j_0'(x^{\mathrm{op}}), \qquad \forall\, x \in M_1.$$

Moreover, $\mathcal{U}$ is unique (up to an isomorphism $(*)$) with these properties. Also, $\mathcal{U}$ is generated as a C^*-algebra by $j(M)$, $j'(M^{\mathrm{op}})$, $j(e_N)$ $(= j'(e_N^{\mathrm{op}}))$ and it has a unique antiautomorphism $^{\mathrm{op}}$ such that $j(x)^{\mathrm{op}} = j'(x^{\mathrm{op}})$, $(j'(x^{\mathrm{op}}))^{\mathrm{op}} = j(x)$, $\forall\, x \in M_1$ (so in particular $j(e_N)^{\mathrm{op}} = j'(e_N^{\mathrm{op}})^{\mathrm{op}} = j(e_N)$).

Proof. Put

$$\mathcal{U} \stackrel{def}{=} C^*\left(\left\{\bigoplus_{(\pi,\pi')\in\mathcal{C}} \pi(x), \bigoplus_{(\pi,\pi')\in\mathcal{C}} \pi'(y^{\text{op}}) \mid x, y \in M_1\right\}\right),$$

$$j(x) \stackrel{def}{=} \bigoplus_{(\pi,\pi')\in\mathcal{C}} \pi(x), \quad j'(x^{\text{op}}) \stackrel{def}{=} \bigoplus_{(\pi,\pi')\in\mathcal{C}} \pi'(x^{\text{op}}), \qquad x \in M_1.$$

$\mathcal{U}$, j, j' this way defined clearly satisfy a), b), and $(*)$ and the uniqueness is then trivial. Then we can define $^{\text{op}}$ on $\mathcal{U}$ by:

$$\mathcal{U} \ni \bigoplus_{(\pi,\pi')\in\mathcal{C}} \pi(x) \mapsto \bigoplus_{(\pi,\pi')\in\mathcal{C}} \bar{\pi}'(x^{*\text{op}}) \in \mathcal{U}$$

$$\mathcal{U} \ni \bigoplus_{(\pi,\pi')\in\mathcal{C}} \pi'(x^{\text{op}}) \mapsto \bigoplus_{(\pi,\pi')\in\mathcal{C}} \bar{\pi}(x^*) \in \mathcal{U}$$

Q.E.D.

1.2. Definition. We denote by $C^*_{u,\max}(M, e_N, M^{\text{op}})$ the C^*-algebra $\mathcal{U}$ constructed in 1.1 and call it the *universal symmetric enveloping C^*-algebra*. Also we denote by $C^*_{u,\text{bin}}(M, e_N, M^{\text{op}}) \stackrel{def}{=} C^*_{\max}(M, e_N, M^{\text{op}})/\cap \ker\pi$, where the intersection is over all representations π of $C^*_{\max}(M, e_N, M^{\text{op}})$ for which $\pi(M)$, $\pi(M^{\text{op}})$ are von Neumann algebras and call it the *universal binormal symmetric enveloping C^*-algebra* associated with $N \subset M$ (and the trace preserving expectation). We still denote by j, j' the embeddings of M_1, M_1^{op} into $C^*_{u,\text{bin}}(M, e_N, M^{\text{op}})$ resulting from the composition of the embeddings into $C^*_{u,\max}(M, e_N, M^{\text{op}})$ with the quotient map. Note that, with the notations in the proof of 1.1, if we let $\mathcal{C}^u_{\text{bin}} = \{(\pi, \pi') \in \mathcal{C} \mid \pi(M), \pi'(M^{\text{op}})$ are von Neumann algebras$\}$, then $C^*_{u,\text{bin}}(M, e_N, M^{\text{op}})$ can alternatively be defined as

$$C^*\left(\left\{\bigoplus_{\mathcal{C}^u_{\text{bin}}} \pi(x), \bigoplus_{\mathcal{C}^u_{\text{bin}}} \pi'(y^{\text{op}}) \mid x, y \in M_1\right\}\right)$$

with

$$j(x) = \bigoplus_{\mathcal{C}^u_{\text{bin}}} \pi(x), \quad j'(x^{\text{op}}) = \bigoplus_{\mathcal{C}^u_{\text{bin}}} \pi'(x^{\text{op}}).$$

Since $(\pi, \pi') \in \mathcal{C}^u_{\text{bin}}$ implies $(\bar{\pi}, \bar{\pi}') \in \mathcal{C}^u_{\text{bin}}$, it follows that $^{\text{op}}$ implements an antiautomorphism on $C^*_{u,\text{bin}}(M, e_N, M^{\text{op}})$, still denoted $^{\text{op}}$, satisfying $j'(x^{\text{op}}) = j(x)^{\text{op}}$, $j'(x^{\text{op}})^{\text{op}} = j(x)$.

In addition, $C^*_{u,\text{bin}}(M, e_N, M^{\text{op}})$ satisfies the following universality property:

1.3. Proposition. *Given any binormal representation (π_o, π_0') of $(N \subset M, e_N, M^{\text{op}} \supset N^{\text{op}})$ on a Hilbert space $\mathcal{H}_0$ there exists a unital $*$-representation $\pi : C^*_{u,\text{bin}}(M, e_N, M^{\text{op}}) \to \mathcal{B}(\mathcal{H}_0)$ such that $\pi(j(x)) = \pi_0(x)$, $\pi'(j'(x^{\text{op}})) = \pi_0'(x^{\text{op}})$, $\forall\ x \in M_1$. Moreover, $C^*_{u,\text{bin}}(M, e_n, M^{\text{op}})$ has a faithful representation $\tilde{\pi}$ such that $\tilde{\pi}(M)$, $\tilde{\pi}(M^{\text{op}})$ are von Neumann algebras. Also, $C^*_{u,\text{bin}}(M, e_N, M^{\text{op}})$ with the embeddings j, j' is unique (up to isomorphism) satisfying these properties.*

Proof. Trivial. Q.E.D.

1.4. LEMMA. *Let $N \subset M \overset{e_1}{\subset} M_1 \overset{e_2}{\subset} M_2 \subset \cdots$ be the Jones tower for $N \subset M$, with $e_1 = e_N$, and $M \overset{e_0}{\supset} N \overset{e_{-1}}{\supset} N_1 \supset \cdots$ be a choice of a tunnel. Let $\mathcal{S}_0$ be a unital C^*-algebra with unital $*$-embeddings $j_0 : M_1 \to \mathcal{S}_0$, $j_0' : M_1^{\mathrm{op}} \to \mathcal{S}_0$, such that $[j_0(M), j_0'(M^{\mathrm{op}})] = 0$, $j_0(e_N) = j_0'(e_N)$. Then j_0, j_0' extend uniquely to $*$-embeddings of $\bigcup_{n\geq 1} M_n$, $\bigcup_{n\geq 1} M_n^{\mathrm{op}}$ into $\mathcal{S}_0$, still denoted by j_0, j_0', such that $j_0(e_{n+2}) = j_0'(e_{-n}^{\mathrm{op}}), j_0'(e_{n+2}^{\mathrm{op}}) = j_0(e_{-n})$, $n \geq 0$.*

Proof. Trivial by the abstract characterization of the basic contruction in ([PiPo2]. [Po2]). Q.E.D.

1.5. LEMMA. *Let $\cdots \overset{e_{-1}}{\subset} N \overset{e_0}{\subset} M \overset{e_1}{\subset} M_1 \subset \cdots$, $\mathcal{S}_0$, j_0, j_0' be like in 1.4. Then we have*

$$\begin{aligned}\mathrm{Alg}(j_0(M_1), j_0'(M_1^{\mathrm{op}})) &= \mathrm{Alg}(j_0(M), j_0(e_N) = j_0'(e_N), j_0'(M^{\mathrm{op}}))\\ &= \bigcup_n \mathrm{sp}\, j_0'(M^{\mathrm{op}}) j_0(M_n) j_0'(M^{\mathrm{op}})\\ &= \bigcup_n \mathrm{sp}\, j_0(M) j_0'(M_n^{\mathrm{op}}) j_0(M)\\ &= \bigcup_n \mathrm{sp}\, j_0'(M^{\mathrm{op}}) j_0(M) j_0(f_{-n}^n) j_0(M) j_0'(M^{\mathrm{op}}),\end{aligned}$$

where f_{-n}^n is the Jones projection for $N_{n-1} \subset M \subset M_n$ obtained as a scalar multiple of the word of maximal length in $e_{-n+2}, \dots, e_0, e_1, \dots, e_n$ (cf. [PiPo2]) and it satisfies $j_0(f_{-n}^n) = j_0'((f_{-n}^n)^{\mathrm{op}})$. Similarly, for any $i \in \mathbb{Z}$ we have

$$\mathrm{Alg}(j_0(M), j_0(e_N), j_0'(M^{\mathrm{op}})) = \bigcup_n \mathrm{sp}\, j_0'(M_i^{\mathrm{op}}) j_0(M_n) j_0'(M_i^{\mathrm{op}}),$$

where $M_i = N_{-i-1}$ for $i \leq -1$, $M_0 = M$, $M_{-1} = N$.

Proof. It is sufficient to show that

$$\bigcup_n \mathrm{sp}\, j'(M^{\mathrm{op}}) j(M) j(f_{-n}^n) j(M) j(M^{\mathrm{op}})$$

is an algebra. If we denote by f_{-2n}^0 the Jones projection for $N_{2n-1} \subset N_{n-1} \subset M$ and by f_0^{2n} the one for $M \subset M_n \subset M_{2n}$, as in [PiPo2], then we have $M_n = \mathrm{sp}\, M f_{-n}^n M$, $M = \mathrm{sp}\, N_{n-1} f_{-2n}^0 N_{n-1}$ so that we get:

$$\begin{aligned}&j'(M^{\mathrm{op}}) j(M) j(f_{-n}^n) j(M) j'(M^{\mathrm{op}}) j(f_{-n}^n) j(M) j'(M^{\mathrm{op}})\\ &\quad\subset \mathrm{sp}\, j'(M^{\mathrm{op}}) j(M) j(f_{-n}^n)(j(f_{-2n}^0) j(N_{n-1}))(j'(N_{n-1}^{\mathrm{op}}) j'((f_{-2n}^0)^{\mathrm{op}}) j'(N_{n-1}^{\mathrm{op}}))\\ &\qquad \cdot j(f_{-n}^n) j(M) j'(M^{\mathrm{op}})\\ &\quad= \mathrm{sp}(j'(M^{\mathrm{op}}) j'(N_{n-1}^{\mathrm{op}}))(j(M) j(N_{n-1}))(j(f_{-n}^n) j(f_{-2n}^0) j'((f_{-2n}^0)^{\mathrm{op}}) j(f_{-n}^n))\\ &\qquad \cdot (j(N_{n-1}) j(M))(j'(N_{n-1}^{\mathrm{op}}) j'(M^{\mathrm{op}}))\\ &\quad= \mathrm{sp}\, j'(M^{\mathrm{op}}) J(M) j(f_{-2n}^{2n}) j(M) j'(M^{\mathrm{op}})\end{aligned}$$

in which we used that $[j(N_{n-1}), j(f_{-n}^n)] = 0$, $[j'(N_{n-1}^{\mathrm{op}}), j(f_{-n}^n)] = 0$ and $f_{-2n}^{2n} = \lambda^{-n} f_{-n}^n f_0^{2n} f_{-2n}^0 f_{-n}^n$, $j(f_0^{2n}) = j'((f_{-2n}^0)^{\mathrm{op}})$, $\lambda = [M : N]^{-1}$. Q.E.D.

1.6. Corollary. *Let* $\cdots \subset N_1 \overset{e_{-1}}{\subset} N \overset{e_0}{\subset} M \overset{e_1}{\subset} M_1 \subset \cdots$, f^n_{-n} *be as in* 1.5. *Then* $C^*_{u,\max}(M, e_N, M^{\rm op})$ *(respectively* $C^*_{u,\rm bin}(M, e_n, M^{\rm op})$*) is generated, as a* C^**-algebra, by* $j(M)$, $j(f^n_{-n}) = j'(f^n_{-n})$, $j'(M^{\rm op})$ *and there exists a natural isomorphism of* $C^*_{u,\max}(M, e_N, M^{\rm op})$ *(respectively* $C^*_{u,\rm bin}(M, e_N, M^{\rm op})$*) onto* $C^*(M, e_{N_{n-1}}, M^{\rm op})$ *(respectively* $C^*_{\rm bin}(M, e_{N_{n-1}}, M^{\rm op})$*), taking the canonical images of the elements in* M, $M^{\rm op}$ *in one algebra into the corresponding canonical images in the other algebra and* $j(f^n_{-n})$ *onto* $j(e_{N_{n-1}})$.

Proof. Trivial by definitions and 1.5. Q.E.D.

1.7. Lemma. *With the notations of* 1.4 *and* 1.5, *assume in addition that* $j_0(M)$, $j_0(e_N) = j_0'(e_N)$, $j_0'(M^{\rm op})$ *generate* $\mathcal{S}_0$ *as a* C^**-algebra, and that the following condition is satisfied:*

$$(*) \qquad j_0(M' \cap M_k) \subset j_0'(M^{\rm op}), \forall k \geq 1$$

Then $j_0(M_i)' \cap \mathcal{S}_0 = j_0'(M^{\rm op}_{-i})$, $(j_0'(M^{\rm op}_i))' \cap \mathcal{S}_0 = j_0(M_{-i})$, $\forall\, i \in \mathbb{Z}$, *and for all* k, i *in* $\mathbb{Z}$ *one has* $j_0(M_i' \cap M_k) = j_0(M_i)' \cap j_0(M_k) = j_0'(M^{\rm op}_{-i}) \cap (j_0'(M^{\rm op}_{-k}))' = j_0'((M^{\rm op}_{-k})' \cap M^{\rm op}_{-i})$. *Also, if* $x \in M'_{-n} \cap M_n$ *and* x' *denotes the canonical conjugate of* x $(= Jx^*J)$ *([Po2]), then* $j_0(x') = j'(x^{\rm op})$. *Moreover, for each* $i \in \mathbb{Z}$ *there exist unique conditional expectations* $\mathcal{E}^+_i : \mathcal{S}_0 \to j_0(M_i)' \cap \mathcal{S}_0 = j_0'(M^{\rm op}_{-i})$, $\mathcal{E}^-_i : \mathcal{S}_0 \to j_0'(M^{\rm op}_i)' \cap \mathcal{S}_0 = j_0(M_{-i})$ *such that* $\mathcal{E}^+_i(j_0(x)) = j_0(E_{M_i' \cap M_n}(x))$, $\mathcal{E}^-_i(j_0'(x^{\rm op})) = j_0'(E_{M_i' \cap M_n}(x)^{\rm op})$, $\forall\; x \in M_n$, $n \geq i$, *which satisfy* $\mathcal{E}^+_i = \mathcal{E}^+_i(j_0(u) \cdot j_0(u^*))$, $\mathcal{E}^-_i = \mathcal{E}^-_i(j_0'(u^{\rm op}) \cdot j_0'(u^{\rm op})^*)$, $\forall\; u \in \mathcal{U}(M_i)$.

Proof. Since $j_0(M' \cap M_k) \subset M^{\rm op}$ and $[j_0(M_k), j_0'(M^{\rm op}_{-k}] = 0$, it follows that $j_0(M' \cap M_k) \subset j_0'(M^{\rm op}_{-k})' \cap j_0'(M^{\rm op}) = j_0'(M^{\rm op}_{-k}{}' \cap M^{\rm op})$. But the two finite dimensional algebras involved in this inclusion have the same dimension, so they actually follow equal. By averaging over unitaries in M_i it then follows that $j_0(M_i' \cap M_k) \subset j_0'(M^{\rm op}), \forall i \geq 1$, giving in a similar way $j_0(M_i' \cap M_k) = j_0'(M^{\rm op}_{-k}{}' \cap M^{\rm op}_{-i})$. Then by duality isomorphisms these equalities follow for arbitrary $i, k \in \mathbb{Z}$.

By the relative Dixmier property for subfactors of finite index (see the Appendix, A.1), if for $x \in M_n$ we denote $C_{M_i}(x) = \overline{\rm co}^n\{uxu^* \mid u \in \mathcal{U}(M_i)\} \cap M_i' \cap M_n$ then $C_{M_i}(x) = \{E_{M_i' \cap M_n}(x)\}$ and $\forall\; x_1, \ldots, x_k \in M_n$, $\forall\; \varepsilon > 0$, $\exists\; u_1, \ldots, u_m \in \mathcal{U}(M_i)$ such that

$$\left\| \frac{1}{m} \sum_{l=1}^m u_l x_j u_l^* - E_{M_i' \cap M_n}(x_j) \right\| < \varepsilon, \qquad j = 1, 2, \ldots, k.$$

Since, by 1.5 we have

$$\mathrm{Alg}(j_0(M_1), j_0'(M^{\rm op}_1)) = \bigcup_n \mathrm{sp}\, j_0'(M^{\rm op}_{-i}) j_0(M_n) j_0'(M^{\rm op}_{-i}),$$

which is dense in $\mathcal{S}_0$, it follows that

$$\begin{aligned} j_0(M_i)' \cap \mathcal{S}_0 &= \bigcup_n j_0'(M_{-i}^{\rm op}) j_0(M_i' \cap M_n) j_0'(M_{-i}^{\rm op}) \\ &= \bigcup_n j_0'(M_{-i}^{\rm op}) j_0'((M_{-n}^{\rm op})' \cap (M_{-i}^{\rm op})) j_0'(M_{-i}^{\rm op}) \\ &= j_0'(M_{-i}^{\rm op}). \end{aligned}$$

Also, it follows that if $T = \sum_l j_0'(y_{1,l}^{\rm op}) j_0(x_l) j_0'(^{\rm op}_{2,l})$, for some $y_{1,l}, y_{2,l} \in M_{-i}$, $x_l \in M_n$, and we denote by $C_{i,\mathcal{S}_0}(T) = \overline{\rm co}^n \{ j_0(u) T j_0(u^*) \mid u \in \mathcal{U}(M_i)\} \cap (j_0(M_i))' \cap \mathcal{S}_0$, then

$$C_{i,\mathcal{S}_0}(T) = \left\{ \sum_l j_0'(y_{1,l}^{\rm op}) j_0(E_{M_i' \cap M_n}(x_l)) j_0'(y_{2,l}^{\rm op}) \right\}$$

is a single point set. Also, $C_{i,\mathcal{S}_0}(\alpha T_1 + \beta T_2) \subset \alpha C_{i,\mathcal{S}_0}(T_1) + \beta C_{i,\mathcal{S}_0}(T_2)$ and $1 \geq T \geq 0$ implies $1 \geq T' \geq 0$, $\forall\, T' \in C_{i,\mathcal{S}_0}(T)$. It follows that

$$\begin{aligned} \mathrm{Alg}\{j_0(M_1), j_0'(M_1^{\rm op})\} \ni T &= \sum_l j_0'(y_{1,l}^{\rm op}) j_0(x_l) j_o'(y_{2,l}^{\rm op}) \\ \mapsto \sum_l j_0'(y_{1,l}^{\rm op}) j_0(E_{M_i' \cap M_n}(x_l)) j_0'(y_{2,l}^{\rm op}) &\in (j_0(M_i))' \cap \mathcal{S}_0 = j_0'(M_{-i}^{\rm op}) \end{aligned}$$

is a well defined positive linear norm one projection onto $j_0'(M_{-i}^{\rm op})$ and the rest of the statement is then clear by continuity. Q.E.D.

1.8. DEFINITION. We denote $C^*_{\rm max}(M, e_N, M^{\rm op}) \overset{\rm def}{=} C^*_{u,\rm max}(M, e_N, M^{\rm op}) / \cap \ker \pi$, where the intersection is over all *smooth* representations π of $C^*_{u,\rm max}(M, e_N, M^{\rm op})$, i.e., representations satisfying the following smoothness condition (or axiom):

$$(*) \qquad \pi(j(M' \cap M_i)) \subset \pi(j'(M^{\rm op})), \quad i \in \mathbb{N}.$$

Note that by 1.7 this condition actually implies $\pi(j(M_k' \cap M_i)) = \pi(j'(M_{-i}^{\rm op\prime} \cap M_{-k}^{\rm op})), \forall i, k \in \mathbb{Z}$.

We call $C^*_{\rm max}(M, e_N, M^{\rm op})$ the *symmetric enveloping* C^**-algebra* associated with $N \subset M$. Similarly, we put $C^*_{\rm bin}(M, e_N, M^{\rm op}) \overset{\rm def}{=} C^*_{u,\rm bin}(M, e_N, M^{\rm op}) / \cap \ker \pi$, where the intersection is over all representations π of $C^*_{u,\rm bin}(M, e_N, M^{\rm op})$ such that $\pi(j(M))$, $\pi(j'(M^{\rm op}))$ are von Neumann algebras and such that axiom $(*)$ is satisfied. We call it the *binormal symmetric enveloping* C^**-algebra* associated with $N \subset M$. Note that, since $\mathcal{B}(L^2(M))$, with the representation of M and $M^{\rm op}$ as operators of left and right multiplication by elements in M and $e_N = \mathrm{proj}_{L^2(N)} \in \mathcal{B}(L^2(M))$, does satisfy the condition $(*)$, both these symmetric enveloping C*-algebras are non-degenerate.

We still denote by j, j' the canonical embeddings of M_1, M_1^{op} in $C^*_{\max}(M,e_N,M^{\text{op}})$ and $C^*_{\text{bin}}(M,e_N,M^{\text{op}})$. Note that the same argument as in 1.2 shows that the antiautomorphism ${}^{\text{op}}$ on $C^*_{u,\max}(M,e_N,M^{\text{op}})$ (respectively ${}^{\text{op}}$ on $C^*_{u,\text{bin}}(M,e_N,M^{\text{op}})$) implements an antiautomorphism, still denoted by ${}^{\text{op}}$, on $C^*_{\max}(M,e_N,M^{\text{op}})$ (resp. ${}^{\text{op}}$ on $C^*_{\text{bin}}(M,e_N,M^{\text{op}})$).
Also, by universality properties of $C^*_{u,\max}(M,e_N,M^{\text{op}})$ and $C^*_{u,\text{bin}}(M,e_N,M^{\text{op}})$ and the definitions, it follows that given any C^*-algebra $\mathcal{S}_0$ generated by copies of M, M^{op}, e_N satisfying 1.1 a), b), such that the corresponding tunnel-towers $\{M_i\}_i$, $\{M_j^{\text{op}}\}_j$ (cf. 1.4) satisfy the smoothness axiom 1.8 $(*)$, there exists a natural $*$-morphism of $C^*_{\max}(M,e_N,M^{\text{op}})$ onto $\mathcal{S}_0$ carrying $j(M_i)$, $j'(M_j^{\text{op}})$ onto the corresponding images of M_i, M_j^{op} $(\subset \mathcal{S}_0)$. If in addition $\mathcal{S}_0 \subset \mathcal{B}(\mathcal{H}_0)$ is so that the images of M, M^{op} are weakly closed, then this morphism factors to a $*$-morphism of $C^*_{\text{bin}}(M,e_N,M^{\text{op}})$.
The above can be regarded as the universality property satisfied by $C^*_{\max}(M,e_N,\ M^{\text{op}})$ and $C^*_{\text{bin}}(M,e_N,M^{\text{op}})$. Moreover, as a consequence of the prior results and definitions, if we denote by $\mathcal{S}$ either of these two algebras, then the following properties hold true:
1.9.1. $j(M_i)' \cap \mathcal{S} = j'(M_{-i}^{\text{op}})$, $j'(M_{-i}^{\text{op}})' \cap \mathcal{S} = j(M_i)$, $\forall\ i \in \mathbb{Z}$.
1.9.2. If $x \in M'_{-n} \cap M_n$ and x' denotes the canonical conjugate of x $(= Jx^*J)$ then $j(x') = j'(x^{\text{op}})$.
1.9.3. There exist unique conditional expectations $\mathcal{E}_i^- : \mathcal{S} \to j(M_{-i})$, $\mathcal{E}_i^+ : \mathcal{S} \to j'(M_{-i}^{\text{op}})$ such that $\mathcal{E}_i^-(j'(x^{\text{op}})) = j'(E_{M_i' \cap M_n}(x)^{\text{op}})$, $\mathcal{E}_i^+(j'(x)) = j(E_{M_i' \cap M_n}(x))$, $\forall\ x \in M_n$, $n \geq i$. Also, these expectations satisfy $\mathcal{E}_i^- = \mathcal{E}_i^-(j'(u^{\text{op}}) \cdot j'(u^{\text{op}*}))$, $\mathcal{E}_i^+ = \mathcal{E}_i^+(j(u) \cdot j(u^*))$, $\forall\ u \in \mathcal{U}(M_i)$.
1.9.4. $C^*_{\max}(M,e_{N_{n-1}},M^{\text{op}})$ (resp. $C^*_{\text{bin}}(M,e_{N_{n-1}},M^{\text{op}})$) naturally identifies with $C^*_{\max}(M,e_N,M^{\text{op}})$ (resp. $C^*_{\text{bin}}(M,e_N,M^{\text{op}})$), as in 1.6.

1.10. *Remarks.* 1°. Note that the smoothness condition 1.8 $(*)$ is redundant if $M' \cap M_n = Alg\{1, e_1, e_2, ..., e_n\}, \forall n$, i.e., in the case the graph of $N \subset M$ is of the form $\Gamma_{N,M} = A_n$ for some $n \leq \infty$.
2°. In the case $\mathcal{S}_0 \subset \mathcal{B}(\mathcal{H})$ is so that $j_0(M)$, $j_0'(M^{\text{op}})$ are von Neumann algebras (e.g., if $\mathcal{S}_0 = C^*_{\text{bin}}(M,e_N,M^{\text{op}})$) then one can give another proof to Lemma 1.7, which doesn't use the relative Dixmier property, as follows: if M is weakly separable (i.e., M has separable predual) then take $R \subset M$ to be a hyperfinite subfactor such that $R' \cap M_\infty = M' \cap M_\infty$ (cf. [Po2,9]), so in particular $R' \cap M_n = M' \cap M_n$, $\forall\ n$ (here $M_\infty = \overline{\cup M_n}^{\text{w}}$ as usual). Then denote by Φ the conditional expectation of $\mathcal{B}(\mathcal{H})$ onto $j_0(R)' \cap \mathcal{B}(\mathcal{H})$, obtained by averaging over a suitable amenable subgroup of $\mathcal{U}(R)$. Then clearly $\Phi|_{\mathcal{S}_0} = \mathcal{E}_0^+$ and the other expectations are obtained similarly. If M is not separable one can still apply [Po2,9] to get that $\forall\ B \subset \cup j(M_n)$ countably generated, $\exists\ R \subset M$ such that $E_{R' \cap M_n}(B) = E_{M' \cap M_n}(B)$, $\forall\ n$, and the rest of the proof is then similar.
3°. The considerations in this section are easily seen to cary over to the case when instead of an extremal inclusion of type II_1 factors $N \subset M$ (with trace preserving expectation) we take an extremal inclusion of factors of type III,

$\mathcal{N} \subset \mathcal{M}$ ([Po3]). However, in this more general case, some adjustements of the argument in 1.7 are needed, depending on the nature of the inclusion. Then, if $\mathcal{E}$ denotes the expectation of minimal index of $\mathcal{M}$ onto $\mathcal{N}$, an argument similar to 2° above can be used to prove the existence of a unique conditional expectation $\mathcal{E}^0$ from $C^*_{\text{bin}}(\mathcal{M}, e_{\mathcal{N}}, \mathcal{M}^{\text{op}})$ onto its C*-subalgebra generated by $\mathcal{M}$ and $\mathcal{M}^{\text{op}}$.

2. Symmetric Enveloping type II_1 Factors

2.1. THEOREM. *There exists a unique trace state* tr *on* $C^*_{\max}(M, e_N, M^{\text{op}})$ *and the corresponding ideal trace* $\mathcal{I}_{\text{tr}} = \{x \in C^*_{\max}(M, e_N, M^{\text{op}}) \mid \text{tr}(x^*x) = 0\}$ *is the unique maximal ideal in* $C^*_{\max}(M, e_N, M^{\text{op}})$*. In particular, there exists a unique state* τ_0 *on each quotient* C^**-algebra* $\mathcal{S}_0$ *of* $C^*_{\max}(M, e_N, M^{\text{op}})$ *(in particular on* $C^*_{\text{bin}}(M, e_N, M^{\text{op}})$*) and its ideal is the unique maximal ideal of* $\mathcal{S}_0$*.*

Proof. By the uniqueness properties of the expectations $\mathcal{E}_i^+$, $i \in \mathbb{Z}$, of a C^*-algebra $\mathcal{S}_0$ generated by $j_0(M_1)$, $j_0'(M_1^{\text{op}})$ onto $j_0'(M_{-i}^{\text{op}})$ like in 1.6, it follows that $\mathcal{E}_i^+ = E^{j_0'(M^{\text{op}})}_{j_0'(N_{i-1}^{\text{op}})} \circ \mathcal{E}_0^+$. Let τ be the trace on $j_0(M)$ and τ' the trace on $j_0'(M^{\text{op}})$ and define $\tau_0 = \tau' \circ \mathcal{E}_0^+$ on $\mathcal{S}_0$. Since $E^{j_0'(M^{\text{op}})}_{j_0'(N_{i-1}^{\text{op}})}$ is τ' preserving, we have for $i \geq 0$, $x \in \mathcal{S}_0$:

$$\tau_0(x) = \tau'(\mathcal{E}_0^+(x)) = \tau'\left(E^{j_0'(M^{\text{op}})}_{j_0'(N_{i-1}^{\text{op}})}(\mathcal{E}_0^+(x))\right) = \tau' \circ \mathcal{E}_i^+(x).$$

If $k \geq i$, $u \in \mathcal{U}(j_0(M_i))$, $x \in j_0(M_k)$, $y', y'' \in j_0'(N_{i-1}^{\text{op}})$ then we have:

$$\begin{aligned}\tau_0(uy'xy''u^*) &= \tau_0(y'uxu^*y'') = \tau'(\mathcal{E}_i^+(y'uxu^*y'')) = \tau'(y'\mathcal{E}_i^+(uxu^*)y'') \\ &= \tau'(y'E_{j_0(M_i' \cap M_k)}(uxu^*)y'') = \tau'(y'\mathcal{E}_i^+(x)y'') = \tau'(\mathcal{E}_i^+(y'xy'')) \\ &= \tau_0(y'xy'').\end{aligned}$$

Thus, by 1.6 it follows that $\tau_0(uTu^*) = \tau_0(T)$, $\forall\ T \in \mathcal{S}_0$, $\forall\ u \in \mathcal{U}(M_i)$. Also, if $u' \in j_0'(M^{\text{op}})$ is a unitary element and $x \in j_0(M_k)$, $y', y'' \in j_0'(M^{\text{op}})$ then we get:

$$\begin{aligned}\tau_0(u'y'xy''u'^*) &= \tau'(\mathcal{E}_0^+(u'^*y'xy''u')) = \tau'(u'y'\mathcal{E}_0^+(x)y''u'^*) \\ &= \tau'(y'\mathcal{E}_0^+(x)y'') = \tau'(\mathcal{E}_0'(y'xy'')) = \tau_0(y'xy'').\end{aligned}$$

This shows that $\tau_0(u'Tu'^*) = \tau_0(T)$, $\forall\ T \in \mathcal{S}_0$, $\forall\ u' \in \mathcal{U}(j_0'(M^{\text{op}}))$, by virtue of 1.6. Since the centralizer of τ_0 is an algebra and it contains both $\mathcal{U}(j_0(M_i))$, $\mathcal{U}(j_0'(M^{\text{op}}))$, with $i \geq 1$, τ_0 has all $\mathcal{S}_0 = C^*(j_0(M_i), j_0'(M^{\text{op}}))$ in its centralizer, thus, it is a trace.

If τ_1 is another trace on $\mathcal{S}_0$ and $(\pi_{\tau_1}, \mathcal{H}_{\tau_1}, \mathcal{E}_{\tau_1})$ is the corresponding GNS construction, then let $S_0 = \overline{\pi_{\tau_1}(\mathcal{S}_0)}^{\text{w}}$. Since the unit ball of $\pi_{\tau_1}(M_k)$ is complete in the norm given by $\|\pi_{\tau_1}(x)\xi_{\tau_1}\|$ (because the unit ball of $j_0(M_k)$ is complete in the norm $\tau_1(x^*x)^{1/2}$, by the uniqueness of the trace on the factor M_k) it follows that $\overline{\pi_{\tau_1}(M_k)} = \pi_{\tau_1}(M_k)$. Thus, for $x \in j_0(M_k)$, $y', y'' \in j_0'(M^{\rm op})$ we get:

$$\begin{aligned}\tau_1(y'xy'') &= \langle E_{\pi_{\tau_1}(j_0(M))' \cap S_0}(\pi_{\tau_1}(y'xy''))\xi_{\tau_1}, \xi_{\tau_1}\rangle \\ &= \langle(\pi_{\tau_1}(y')E_{\pi_{\tau_1}(j_0(M))' \cap S_0}(\pi_{\tau_1}(j_0(x))\pi_{\tau_1}(y'')\xi_{\tau_1}, \xi_{\tau_1}\rangle \\ &= \langle \pi_{\tau_1}(y'E_{j_0(M' \cap M_k)}(x)y'')\xi_{\tau_1}, \xi_{\tau_1}\rangle \\ &= \tau'(y'E_{j_0(M' \cap M_k)}(x)y'') \\ &= \tau_0(y'xy'').\end{aligned}$$

with the last part following from the uniqueness of the trace on $j_0'(M^{\rm op})$.
This shows that $\mathcal{S}_0$ has a unique trace τ_0 and also that if $\mathcal{I}_0 \subset \mathcal{S}_0$ is a two sided closed proper ideal then $\mathcal{S}_1 = \mathcal{S}_0/\mathcal{I}_0$ has a trace, which thus composed with the quotient map gives the trace on $\mathcal{S}_0$. Thus, $\mathcal{I}_0 \subset \mathcal{I}_{\tau_0}$, so $\mathcal{I}_{\tau_0}$ is the unique maximal ideal of $\mathcal{S}_0$. Q.E.D.

2.2. *Remarks.* 1°. Let

$$C^*_{\min}(M, e_N, M^{\rm op}) \stackrel{\rm def}{=} C^*_{\max}(M, e_N, M^{\rm op})/I_{\rm tr} \ (\simeq \pi_{\rm tr}(C^*_{\max}(M, e_N, M^{\rm op}))),$$

where $I_{\rm tr}$ is the trace ideal (= maximal ideal) of $C^*_{\max}(M, e_N, M^{\rm op})$ corresponding to the unique trace tr, as given by 2.1. From the previous theorem and its proof if follows that $C^*_{\min}(M, e_N, M^{\rm op})$ is simple, has a unique trace, still denoted tr, and has the Dixmier property, i.e., $\overline{\rm co}^n\{uxu^* \mid u \in \mathcal{U}(C^*_{\min}(M, e_N, M^{\rm op}))\} \cap \mathbb{C}1 = \{{\rm tr}(x)1\}$, $\forall\ x \in C^*_{\min}(M, e_N, M^{\rm op})$. In fact, by 2.1 any C^*-algebra $\mathcal{S}_0$ generated by mutually commuting copies of M, $M^{\rm op}$ and a projection e_N such that $N \subset M \stackrel{e_N}{\subset} {\rm Alg}(M, e_N)$ and $N^{\rm op} \subset M^{\rm op} \stackrel{e_N}{\subset} {\rm Alg}(M^{\rm op}, e_N)$ are basic constructions and such that the smoothness condition 1.8 $(*)$ is satisfied, has a unique trace tr, $I_{\rm tr}$ is its unique maximal ideal and $\mathcal{S}_0/I_{\rm tr} = C^*_{\min}(M, e_N, M^{\rm op})$.
Also it should be noted that if $N = M$ then $C^*_{\max}(M, e_N, M^{\rm op})$ coincides with $M \underset{\max}{\otimes} M^{\rm op}$, $C^*_{\rm bin}(M, e_N, M^{\rm op})$ with $M \underset{\rm bin}{\otimes} M^{\rm op}$ (as considered in [EL]) and $C^*_{\min}(M,\ e_N, M^{\rm op})$ with $M \underset{\min}{\otimes} M^{\rm op}$.
2°. Let $\mathcal{N} \subset \mathcal{M}$ be an extremal inclusion of von Neumann factors of type III, with the conditional expectation of minimal index $\mathcal{E}$, as in 1.10.3°. The construction analoguous to 2.1 is then as follows: one first considers the expectation $\mathcal{E}^0$ given by 1.10.3°; one takes a normal faithful state φ on $\mathcal{M}$ such that $\varphi \circ \mathcal{E} = \varphi$; instead of the trace tr, one defines a state ψ on $C^*_{\rm bin}(\mathcal{M}, e_{\mathcal{N}}, \mathcal{M}^{\rm op})$ by $\psi = (\varphi \otimes \varphi^{\rm op}) \circ \mathcal{E}^0$.

2.3. Corollary.

$$S = \overline{\pi_{\rm tr}(C^*_{\max}(M, e_N, M^{\rm op}))}^{\rm w} \simeq \overline{\pi_{\rm tr}(C^*_{\rm bin}(M, e_N, M^{\rm op}))}^{\rm w}$$

is a type II_1 *factor with embeddings* $\pi_{\mathrm{tr}} \circ j : M_1 \to S$, $\pi_{\mathrm{tr}} \circ j' : M_1^{\mathrm{op}} \to S$ *and an antisymmetry* $^{\mathrm{op}}$ *such that*
a) $[\pi_{\mathrm{tr}}(j(M)), \pi_{\mathrm{tr}}(j'(M^{\mathrm{op}}))] = 0$
b) $\pi_{\mathrm{tr}}(j(e_N)) = \pi_{\mathrm{tr}}(j'(e_N))$
c) $S = \mathrm{vN}(\pi_{\mathrm{tr}}(j(M)), \pi_{\mathrm{tr}}(j(e_N)), \pi_{\mathrm{tr}}(j'(M^{\mathrm{op}})))$
d) $\pi_{\mathrm{tr}}(j(x))^{\mathrm{op}} = \pi_{\mathrm{tr}}(j'(x^{\mathrm{op}}))$, $\forall\, x \in M$, $\pi_{\mathrm{tr}}(j(e_N))^{\mathrm{op}} = \pi_{\mathrm{tr}}(j(e_N))$.
Moreover, if S_0 *is another type* II_1 *factor with embeddings* $j_0 : M_1 \to S_0$, $j_0' : M_1^{\mathrm{op}} \to S_0$ *satisfying conditions a), b) (with* j_0 *instead of* $\pi_{\mathrm{tr}} \circ j$ *and* j_0' *instead of* $\pi_{\mathrm{tr}} \circ j'$*) and such that* $j_0(M' \cap M_n) \subset j_0'(M^{\mathrm{op}}), \forall n \geq 1$, *then there exists a unique isomorphism* σ *of* S *into* S_0 *such that* $j_0 = \sigma \circ \pi_{\mathrm{tr}} \circ j$, $j_0' = \sigma \circ \pi_{\mathrm{tr}} \circ j'$. *And if in addition* $S_0 = \mathrm{vN}(j_0(M), j_0(e_N), j_0'(M^{\mathrm{op}}))$, *then* σ *is onto.*

Proof. Trivial by 2.1. Q.E.D.

2.4. DEFINITION. We denote by $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ the type II_1 factor $S = \pi_{\mathrm{tr}}(\overline{C^*_{\max}(M, e_N, M^{\mathrm{op}})})$ in the previous corollary and call it the *symmetric enveloping type* II_1 *factor* associated with $N \subset M$. Also, we call $M \vee M^{\mathrm{op}} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ the *symmetric enveloping inclusion* associated with $N \subset M$. We will identify M, e_N, M^{op} with their corresponding canonical images in $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$, more generally we will identify M_n, M_n^{op}, e_n with their canonical images via $\pi_{\mathrm{tr}} \circ j$, $\pi_{\mathrm{tr}} \circ j'$ (cf 1.4), whenever some tunnel for $N \subset M$ is chosen. We've seen in 2.3 that $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ has an antisymmetry $^{\mathrm{op}}$ and that it satisfies a universality and uniqueness property. Also, from now on we will use the notation τ for the unique trace on the factor $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ (as in fact for any generic factor).

2.5. *Remarks.* 1°. As one can see from 2.1-2.3, the symmetric enveloping type II_1 factor $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ associated to an inclusion $N \subset M$ can be constructed out of any C*-algebra S_0 generated by copies of M and M^{op}, satisfying $M' \cap S_0 = M^{\mathrm{op}}$, and by a projection e_N, implementing the expectations E_N on M and $E_{N^{\mathrm{op}}}$ on M^{op}: just put $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ to be the completion of the algebra S_0/I_0 in the strong topology given by its unique trace, I_0 being the maximal ideal of S_0 or, alternatively, the ideal corresponding to the unique trace on S_0. In particular, $M \underset{e_N}{\boxtimes} M^{\mathrm{op}} = \overline{\mathrm{C}^*(M, e_N, J_M M J_M)/I_0}$. But one can also construct $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ by defining directly the Hilbert space of its standard representation. In order to show this, we will in fact consider a more general construction. Thus, let $N \subset M$ and $Q \subset P$ be extremal inclusions with the same extended higher relative commutant picture (or extended standard invariant), i.e., $\tilde{\to}\mathcal{G}_{N,M} = tilde \to \mathcal{G}_{Q,P} = \{A_{ij}\}_{i,j\in\mathbb{Z}}$. The *concatenation* algebra associated to these two inclusions is then the unique (up to isomorphism) type II_1 factor S generated by commuting copies of M, P^{op} and by a projection e, implementing both E_N and $E_{Q^{\mathrm{op}}}$, such that $M' \cap S = P^{\mathrm{op}}$. This algebra is denoted by $M \underset{e_N = e_Q}{\boxtimes} P^{\mathrm{op}}$

(or simply $M \boxtimes P^{\mathrm{op}}$, when no confusion is possible). Its uniqueness follows the same way as the uniqueness of $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ above. To prove its existence, we consider the following construction: Take $\{m_j\}_{j\in J}$ to be be an orthonormal basis of $A_{-\infty,\infty}$ over $A_{-\infty,0} \vee A_{0,\infty}$ and identify $A_{-\infty,0} \vee A_{0,\infty}$ with its image in $M \bar{\otimes} P^{\mathrm{op}}$ (through the choice of tunnels in M and P); note that the m_j's can be chosen of bounded norm and such that the set of indices J can be written as $\cup_n J_n$, where each J_n is finite and such that $\sum_{j\in J_n} m_j M \vee M^{\mathrm{op}}$ is a $M \vee M^{\mathrm{op}}$-bimodule of finite dimension (equivalently, $\sum_{j\in J_n} m_j P \vee P^{\mathrm{op}}$ is a $P \vee P^{\mathrm{op}}$-bimodule of finite dimension) See 4.5 below for how to get these m_j's. Then let $\mathcal{H}_n \stackrel{\mathrm{def}}{=} \sum_{j\in J_n} m_j \mathrm{L}^2(M \vee P^{\mathrm{op}})$ and $\mathcal{H} \stackrel{\mathrm{def}}{=} \vee_n \mathcal{H}_n$, the scalar product on $\mathcal{H}$ being defined by $\langle m_j \xi, m_i \eta\rangle = \langle E_{A_{-\infty,0}\vee A_{0,\infty}}(m_i^* m_j)\xi, \eta\rangle$, $\forall \xi, \eta \in \mathrm{L}^2(M \vee P^{\mathrm{op}})$. Finally, we let M, P^{op} and $e = e_N = e_{P^{\mathrm{op}}}$ act on $\mathcal{H}$ as follows: M (and similarly P^{op}) acts on each $\mathcal{H}_n$ by multiplication to the left, according to the relations $M m_i \subset \sum_{j\in J_n} m_j (M \vee A_{0,\infty})$, $\forall i \in J_n$, with the latter vector space being identified with a subset of $\mathcal{H}_n$; e acts also by multiplication to the left, by regarding $\mathcal{H}$ as a left $A_{-\infty,\infty}$ module in the obvious way and letting $e = e_1$. Then $M \underset{e_N = e_Q}{\boxtimes} P^{\mathrm{op}}$ is simply the von Neumann algebra generated by M, P^{op}, e_1 on $\mathcal{H}$.

It is easy to check that these actions of M, P^{op}, e on $\mathcal{H}$ are well defined, that they satisfy $M' \cap \mathrm{C}^*(M, e, P^{\mathrm{op}}) = P^{\mathrm{op}}$, $exe = E_N(x)e, eye = E_{P^{\mathrm{op}}}(y)e$, for $x \in M, y \in P^{\mathrm{op}}$, and that $\langle \cdot \hat{1}, \hat{1}\rangle$ defines a trace on $\mathrm{C}^*(M, e, P^{\mathrm{op}})$. This shows the existence of the concatenation algebra.

Note that, by using the same proofs as for $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$, it follows that the concatenation algebra has similar properties as the ones the symmetric enveloping algebras are shown to have in this section and in Sec. 4. Obviously, when $(Q \subset P) \simeq (N \subset M)$ this algebra coincides with the symmetric enveloping type II_1 factor associated with $N \subset M$.

Note that any extremal hyperfinite subfactor $N \subset R$ gives rise to a canonical non-separable concatenation algebra as follows: Let ω be a free ultrafilter on $\mathbb{N}$ and denote by R^ω the corresponding ultrapower algebras associated to the hyperfinite factor R. Then $(R' \cap R^\omega)' \cap R^\omega = R$ and more generally $(N_k' \cap R^\omega)' \cap R^\omega = N_k, \forall k$, where $R \supset N \supset N_1 \supset ...$ is a tunnel for $R \supset N$ (cf. [C1]). Thus, if we denote $P^0 = R' \cap R^\omega$ and Q^0 to be the downward basic construction for $P_1^0 = N' \cap R^\omega \supset R' \cap R^\omega = P^0$ and put $(Q \subset P \subset P_1) \simeq (Q^0 \subset P^0 \subset P_1^0)^{\mathrm{op}}$ then $N \subset M$ and $Q \subset P$ have the same higher relative commutant pictute (extended standard invariant) and the von Neumann algebra S generated by R and $P_1^{\mathrm{op}} = N' \cap R^\omega$ is isomorphic to the concatenation of $(N \subset R)$ and $(Q \subset P)$ (see also Remark 2.11, 1° in [Po3], with caution to the obvious typos there...).

2°. For an extremal inclusion of type III factors $\mathcal{N} \subset \mathcal{M}$ like in 1.10.3°, 2.2.2°, one defines its symmetric enveloping von Neumann algebra as $\pi_\psi(C^*_{\mathrm{bin}}(\mathcal{M}, e_{\mathcal{N}}, \overline{\mathcal{M}^{\mathrm{op}}}))$. It is easy to see that this algebra does not in fact depend on the normal faithful state φ, with $\varphi = \varphi \circ \mathcal{E}$, taken on $\mathcal{M}$.

The next proposition summarizes the main properties of the factor $M \underset{e_N}{\boxtimes} M^{\text{op}}$ and its canonical subalgebras:

2.6. PROPOSITION. *$M \underset{e_N}{\boxtimes} M^{\text{op}}$ with its subalgebras M_i, M_i^{op} projections e_k, $i, j, k \in \mathbb{Z}$, and antisymmetry ${}^{\text{op}}$ satisfy the conditions:*
a) $[M, M^{\text{op}}] = 0$;
b) $e_1^{\text{op}} = e_1 = e_N$, $e_n^{\text{op}} = e_{-n+2}$, $n \in \mathbb{Z}$, *and* $\cdots \subset N_1 \overset{e_{-1}}{\subset} N \overset{e_0}{\subset} M \overset{e_1}{\subset} M_1 \overset{e_2}{\subset} M_2 \subset \cdots$ *is a Jones tower-tunnel for* $N \subset M$, *where* $M_0 = M$, $M_{-1} = N$, $M_{-n} = N_{n-1}$, $n \geq 2$.
c) $\bigcup_{n\geq 1} M_i M_n^{\text{op}} M_i = \bigcup_{n \geq 1} M_j^{\text{op}} M_n M_j^{\text{op}} = \text{Alg}(M, e_N = e_1, M^{\text{op}})$, $\forall\, i, j \in \mathbb{Z}$ *and it is a dense $*$-subalgebra in* $M \underset{e_N}{\boxtimes} M^{\text{op}}$.
d) $M_j' \cap M \underset{e_N}{\boxtimes} M^{\text{op}} = M_{-j}^{\text{op}}$ *and* $(M_j^{\text{op}})' \cap M \underset{e_N}{\boxtimes} M^{\text{op}} = M_{-j}$, $\forall\, j \in \mathbb{Z}$.

Proof. Clear by 1.9 and the definition of $M \underset{e_N}{\boxtimes} M^{\text{op}}$. Q.E.D.

The bicommutant relations in d) above can actually be taken as an abstract characterization of the symmetric enveloping algebra:

2.7. PROPOSITION. *Let $N \subset M$ be an extremal inclusion and S be a type II_1 von Neumann algebra containing M. If $(M' \cap S \subset N' \cap S) \simeq (M^{\text{op}} \subset M_1^{\text{op}})$, and S is generated by M and $N' \cap S$ then $M \vee M' \cap S \subset S$ is naturally isomorphic to $M \vee M^{\text{op}} \subset M \underset{e_N}{\boxtimes} M^{\text{op}}$*

Proof. Let $e_0 \in M$ be a Jones projection for $N \subset M$ and $\{m_j\}_j$ an orthonormal basis of N over $N_1 = \{e_0\}' \cap N$ such that one of the m_j's equals 1. For $x \in S$ define $E(x) = \Sigma_j m_j e_0 x e_0 m_j^* \in S$. Note that if $x \in M' \cap S$ then m_j and e_0 commute with x so $E(x) = x$. Also, if $x \in N' \cap S$ then for each $y \in M$ we have

$$yE(x) = y \sum_j m_j e_0 x e_0 m_j^* = \lambda^{-1} \sum_{i,j} m_i e_0 E_N^M (e_0 m_i^* y m_j e_0) x e_0 m_j^*$$

$$= \lambda^{-1} \sum_{i,j} m_i e_0 x E_N^M (e_0 m_i^* y m_j e_0) e_0 m_j^* = \sum_i m_i e_0 x e_0 m_i^* y = E(x) y$$

showing that $[E(x), y] = 0$. Thus $E(x) \in M' \cap S$. This shows that E is a norm one projection of $N' \cap S$ onto $M' \cap S$ so by Tomiyama's theorem it is a conditional expectation. Also, if $x \in N' \cap S$ then we have

$$\tau(E(x)) = \tau(\Sigma_j m_j e_0 x e_0 m_j^*) = \tau(x e_0 \Sigma_j m_j^* m_j e_0)$$

$$= \Sigma_j \tau(x e_0 E_{N_1}^N (m_j^* m_j)) = \Sigma_j \tau(E_{N_1' \cap S}(x e_0 E_{N_1}^N (m_j^* m_j))) = \lambda^{-1} \tau(x e_0) = \tau(x)$$

Thus E is trace preserving as well, so it must coincide with the unique trace preserving expectation of $N' \cap S$ onto $M' \cap S$. Also, from the definition of $E(x)$, if $x \in N' \cap S$ then $e_0 x e_0 = E(x) e_0$. Thus, if $e_1 \in N' \cap S$ is a Jones projection for $(M' \cap S \subset N' \cap S) = (M^{\mathrm{op}} \subset M_1^{\mathrm{op}})$ then $e_0 e_1 e_0 = \lambda e_0$. But

$$\tau(e_0)^{-1} = [M : N] = [M^{\mathrm{op}} : N^{\mathrm{op}}] = [M_1^{\mathrm{op}} : M^{\mathrm{op}}] = \tau(e_1)^{-1},$$

so that $\tau(e_0) = \tau(e_1)$. Together with $e_0 e_1 e_0 = \lambda e_0$, this implies that $e_1 e_0 e_1 = \lambda e_1$. Thus, if $x, y \in N$ then $e_1 (x e_0 y) e_1 = \lambda x y e_1 = E_N^M (x e_0 y) e_1$, showing that e_1 implements the conditional expectation E_N^M. But, by its definition, e_1 also implements the conditional expectation of $M' \cap S$ onto $\{e_1\}' \cap (M' \cap S)$.
Since we also have the isomorphism $(M' \cap S \subset N' \cap S) \simeq (M^{\mathrm{op}} \subset M_1^{\mathrm{op}})$, which in turn implements an isomorphism $(\{e_1\}' \cap M' \cap S \subset M' \cap S \subset N' \cap S) \simeq (N^{\mathrm{op}} \subset M^{\mathrm{op}} \subset M_1^{\mathrm{op}})$, 2.3 applies to yield $(M \vee M' \cap S \subset S) \simeq (M \vee M^{\mathrm{op}} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}})$
. Q.E.D.

Note that from the above proposition and [Po2] it follows that if M is hyperfnite and the graph $\Gamma_{N,M}$ of $N \subset M$ is strongly amenable (see [Po2] for the definitions) then the inclusion $M \vee M^{\mathrm{op}} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ is isomorphic to the inclusion $M \vee M' \cap M_\infty \subset M_\infty$. The inclusions $M \vee M' \cap M_\infty \subset M_\infty$ for $N \subset M$ hyperfinite with finite depth, i.e., with finite (thus strongly amenable) graph, were considered and extensively studied by Ocneanu ([Oc], see also [EvKa]). Note that if M is an arbitrary type II_1 factor and $N \subset M$ is a subfactor of finite depth and we denote by $Q \subset P$ the standard model $N^{\mathrm{st}} \subset M^{\mathrm{st}}$ then $\mathcal{G}_{Q,P} = \mathcal{G}_{N,M}$ and $M \vee M' \cap M_\infty \subset M_\infty$ naturally identifies with the "concatenation" inclusion considered in 2.5.1°, i.e., with $M \vee P^{\mathrm{op}} \subset M \boxtimes P^{\mathrm{op}}$.
The next lemma provides some useful localization properties relating the Jones projections, the relative commutants and the antiisomorphism $^{\mathrm{op}}$. They are reminiscent of some well known facts (see e.g., [PiPo1] page 83, [Bi1] page 205).

2.8. Lemma. *Let $N \subset M$ be an extremal inclusion, $N \subset M \overset{e_N}{\subset} M_1$ its basic construction and $^{\mathrm{op}}$ the canonical antiautomorphism of $N' \cap M$ onto $M' \cap M_1$ (so $x^{\mathrm{op}} = J_M x^* J_M$, $x \in N' \cap M$).*
a) *If $x \in N' \cap M$ then $x e_N = x^{\mathrm{op}} e_N$ and x^{op} is the unique element $y' \in M' \cap M_1$ such that $y' e_N = x e_N$.*
b) *$e_N x y^{\mathrm{op}} e_N = \tau(xy) e_N$ and $\tau(x y^{\mathrm{op}} e_N) = \lambda \tau(xy)$, $\forall\ x, y \in N' \cap M$, where $\lambda = [M : N]^{-1}$.*
c) *If $q \in \mathcal{P}(N' \cap M)$, $q \neq 0$, then $N q q^{\mathrm{op}} \subset q M q q^{\mathrm{op}} \subset q q^{\mathrm{op}} M_1 q q^{\mathrm{op}}$ is a basic construction with Jones projections equal to*

$$\tau(q)^{-1} q q^{\mathrm{op}} e_N q q^{\mathrm{op}} = \tau(q)^{-1} q e_N q = \tau(q)^{-1} q^{\mathrm{op}} e_N q^{\mathrm{op}}.$$

d) *$E_{M \vee M' \cap M_1}(e_N) = \lambda \sum_{i,j,k} \tau(f_{jj}^k)^{-1} f_{ij}^k f_{ji}^{k\,\mathrm{op}}$, where $\{f_{ij}^k\}$ is a matrix unit for $N' \cap M$.*

Proof. a) If $y \in \hat{M}$ then $e_N(\hat{y}) = \widehat{E_N(y)}$ so that

$$x e_N(\hat{y}) = x \widehat{E_N(y)} = \widehat{E_N(y)} x = x^{\mathrm{op}} e_N(\hat{y}).$$

The uniqueness is clear because $y'e_N = 0$ implies $e_N y'^* y' e_N = 0$ so that $E_{N^{\rm op}}(y'^* y') = 0$, thus $y' = 0$.
b) By a) we have

$$e_N x y^{\rm op} e_N = e_N x y e_N = E_N(xy) e_N = E_{N' \cap N}(xy) = \tau(xy) e_N,$$

whenever $x, y \in N' \cap M$. The second part is then trivial.
c) Since

$$q q^{\rm op} \mathcal{B}(L^2(M)) q q^{\rm op} = \mathcal{B}(L^2(qMq))$$

and

$$(N q q^{\rm op}) \cap q q^{\rm op} \mathcal{B}(L^2(M)) q q^{\rm op} = q q^{\rm op} M_1 q q^{\rm op},$$

it follows that

$$N q q^{\rm op} \subset q M q q^{\rm op} \subset q q^{\rm op} M_1 q q^{\rm op}$$

is a basic construction. Also, if $e = \tau(q)^{-1} q q^{\rm op} e_N q q^{\rm op}$ then by a) we have

$$e = \tau(q)^{-1} q e_N q = \tau(q)^{-1} q^{\rm op} e_N q^{\rm op}.$$

Also, the range of $e = \tau(q)^{-1} q q^{\rm op} e_N q q^{\rm op}$ is clearly

$$L^2(Nq) = L^2(qNq) = q L^2(N) q = L^2(N) q$$

and so, since e is a projection we get $e = \text{proj}_{L^2(N)q} = \text{proj}_{L^2(Nqq^{\rm op})}$ as an element in $\mathcal{B}(L^2(qMq))$.
d) To prove this it is sufficient to show that $\tau(x y^{\rm op} e_N) = \tau(x y^{\rm op} a)$, $\forall\ x, y \in N' \cap M$, where $a = \lambda \sum_{i,j,k} \tau(f^k_{jj})^{-1} f^k_{ij} {f^k_{ji}}^{\rm op}$. It is then enough to check this for $x = f^{k'}_{rs}$, $y = f^{k''}_{s'r'}$. For the left hand side, by b) we have:

$$\tau(f^{k'}_{rs} {f^{k''}_{s'r'}}^{\rm op} e_N) = \lambda \delta_{k'k''} \delta_{ss'} \delta_{rr'} \tau(f^{k'}_{rr}).$$

For the right hand side we have:

$$\begin{aligned}
\tau(f^{k'}_{rs} {f^{k''}_{s'r'}}^{\rm op} a) &= \lambda \sum_{i,j,k} \tau(f^k_{jj})^{-1} \tau(f^k_{ij} {f^k_{ji}}^{\rm op} f^{k'}_{rs} {f^{k''}_{s'r'}}^{\rm op}) \\
&= \lambda \sum_{i,j,k} \tau(f^{k'}_{rr})^{-1} \delta_{kk'} \delta_{kk''} \delta_{jr} \delta_{jr'} \tau(f^k_{is}) \tau(f^k_{is'}) \\
&= \lambda \tau(f^{k'}_{rr})^{-1} \delta_{k'k''} \delta_{rr'} \delta_{ss'} \tau(f^{k'}_{ss})^2 \\
&= \lambda \delta_{k'k''} \delta_{ss'} \delta_{rr'} \tau(f^{k'}_{rr}).
\end{aligned}$$

Q.E.D.

2.9. Proposition. *Let $N \subset M$ be an extremal inclusion of type* II_1 *factors. Then we have*

a) $M \underset{e_{N_{n-1}}}{\boxtimes} M^{\mathrm{op}}$ *naturally identifies with* $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$*, by letting* $e_{N_{n-1}} \mapsto f^n_{-n}$.

b) *The inclusion* $M_1 \vee N^{\mathrm{op}} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ *naturally identifies with the reduced by* e_1^{op} *of the symmetric enveloping inclusion of* $M \subset M_1$, $M_1 \vee M_1^{\mathrm{op}} \subset M_1 \underset{e_M}{\boxtimes} M_1^{\mathrm{op}}$. *More generally,* $M_n \vee N^{\mathrm{op}}_{n-1} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ *is isomorphic to the reduced by* $(f^n_{-n})^{\mathrm{op}}$ *of* $M_n \vee M_n^{\mathrm{op}} \subset M_n \underset{e_M}{\boxtimes} M_n$.

c) *If* $p \in \mathcal{P}(N' \cap M)$ *and we denote by* $L \subset K$ *the inclusion* $Np \subset pMp$ *then* $K \underset{e_L}{\boxtimes} K^{\mathrm{op}}$ *is naturally embedded in* $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ *as the weakly closed* $*$*-subalgebra generated by* $pp^{\mathrm{op}}(M \vee M^{\mathrm{op}})pp^{\mathrm{op}}$ *and by*

$$e'_L \stackrel{\mathrm{def}}{=} \sigma(p)^{-1} pp^{\mathrm{op}} e_N pp^{\mathrm{op}}.$$

Also, the ientity of this algebra is pp^{op}.

d) *If* $T \subset S$ *denotes the symmetric enveloping inclusion associated with* $N \subset M$ *and* $T_0 \subset S_0$ *the symmetric enveloping inclusion associated with some other extremal inclusion of type* II_1 *factors* $N_0 \subset M_0$*, then the symmetric enveloping inclusion associated with* $N \bar{\otimes} N_0 \subset M \bar{\otimes} M_0$ *is naturally isomorphic to* $T \bar{\otimes} T_0 \subset S \bar{\otimes} S_0$.

Proof. a) Is clear by 1.6, 1.9.4 and 2.4.

b) follows then immediately, from 2.1, 2.3 and the fact that $(f^n_{-n})^{\mathrm{op}} M_n^{\mathrm{op}} (f^n_{-n})^{\mathrm{op}} = N^{\mathrm{op}}_{n-1} (f^n_{-n})^{\mathrm{op}} \simeq N^{\mathrm{op}}_{n-1}$.

To prove c) note that if π is the canonical representation of $C^*(M, e_N, JMJ)$ into $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ then the C^*-algebra generated by $pp^{\mathrm{op}}(M \cup M^{\mathrm{op}})pp^{\mathrm{op}}$ and e'_L is the image under π of $C^*(pJpJ(M \cup JMJ)pJpJ, pJpJe_N pJpJ)$ which naturally identifies with the C*-algebra generated by K, $J_K K J_K$ and e_L in $\mathcal{B}(L^2(K))$, where $L = Np \subset pMp = K$. Since this representation of $C^*_{u,\max}(K, e_L, K^{\mathrm{op}})$ is smooth, it follows that π implements a smooth representation of $C^*_{u,\max}(K, e_L, K^{\mathrm{op}})$ into $pp^{\mathrm{op}}(M \underset{e_N}{\boxtimes} M^{\mathrm{op}})pp^{\mathrm{op}}$. Since the latter has a trace, it follows by 2.1, 2.2 that $K \underset{e_L}{\boxtimes} K^{\mathrm{op}} = \left(C^*(pp^{\mathrm{op}} M pp^{\mathrm{op}}, e'_L, pp^{\mathrm{op}} M^{\mathrm{op}} \overline{pp^{\mathrm{op}}))}\right)^w \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$.

d) follows trivially from any of the characterizing universality properties of the symmetric enveloping algebras (e.g., from 2.7). Q.E.D.

2.10. Proposition. *Let $N \subset M$ be an extremal inclusion of type* II_1 *factors.*

a) *If* $Q \subset N$ *is an extremal subfactor of* N *then* $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ *is unitally embedded as a subfactor of* $M \underset{e_Q}{\boxtimes} M^{\mathrm{op}}$*, by taking* $e_N \mapsto \sum_j m_j e_Q m_j^*$ $(= \sum_j m_j^{*\mathrm{op}} e_Q m_j^{\mathrm{op}})$*, where* $\{m_j\}_j$ *is a orthonormal basis of* N *over* Q*. Moreover, if there exists a tunnel* $M \supset N \supset N_1 \supset \cdots$ *for* $N \subset M$ *such that* $N_k \subset Q$ *for some* k*, then this unital embedding is in fact an equality.*

b) *If* $Q \subset P$ *is an extremal inclusion of factors embedded in* $N \subset M$ *as a commuting square, such that* $[P:Q]=[M:N]$ *and* $P'\cap P_n \subset M'\cap M_n$, $\forall\, n$ *then* $P \underset{e_Q}{\boxtimes} P^{\rm op}$ *is unitally embedded in* $M \underset{e_N}{\boxtimes} M^{\rm op}$, *by taking* $P \hookrightarrow M$, $P^{\rm op} \hookrightarrow M^{\rm op}$ *and* $e_Q \mapsto e_N$. *Also,* $P \underset{e_Q}{\boxtimes} P^{\rm op} \subset M \underset{e_N}{\boxtimes} M^{\rm op}$ *has finite index iff* $P \subset M$ *has finite index, with the estimate* $[M \underset{e_N}{\boxtimes} M^{\rm op} : P \underset{e_Q}{\boxtimes} P^{\rm op}] \leq [M:P]^2$. *Moreover, if* $P'\cap P_n = M'\cap M_n$, *then this embedding implements the nondegenerate commuting square:*

$$\begin{array}{ccc} M \vee M^{\rm op} & \subset & M \underset{e_N}{\boxtimes} M^{\rm op} \\ \cup & & \cup \\ P \vee P^{\rm op} & \subset & P \underset{e_Q}{\boxtimes} P^{\rm op}. \end{array}$$

Proof. a) It is easy to check by direct computation that

$$e_N = \sum_j m_j e_Q m_j^* = \sum_j (Jm_jJ)e_Q(Jm_j^*J)$$

in $\mathcal{B}(L^2(M))$, so a) follows from 2.2-2.6. The last part of a) then follows from 2.7 a) and the first part.

b) To prove the first part we only need to show that the representation of $C^*_{u,{\rm bin}}(P, e_Q, P^{\rm op})$ in $M \underset{e_N}{\boxtimes} M^{\rm op}$ satisfies the faithfulness condition 1.8 $(*)$, i.e., we need to show that $P'\cap P_n = {Q^{\rm op}_{n-1}}' \cap P^{\rm op}$, $\forall\, n$. By 2.1–2.4, it is sufficient to check this equality in a representation of $C^*_{\rm bin}(M, e_N, M^{\rm op})$, and we'll choose $C^*(M, e_N, JMJ) \subset \mathcal{B}(L^2(M))$ to do this. Let $e_P^M \in \mathcal{B}(L^2(M))$ be the orthogonal projection of $L^2(M)$ onto $L^2(P)$. Note that all the elements in $C^*(P, e_N, J_MPJ_M) \subset \mathcal{B}(L^2(M))$ commute with e_P^M and that if $x \in JMJ$ then $x \in JPJ$ iff $[x, e_P^M] = 0$. Now, if $x \in P'\cap P_n$ then $x \in M'\cap P_n$ by hypothesis, so $x \in JMJ \cap P_n$. Also, $[x, e_P^M]=0$, because $P_n \subset C^*(P, e_N, JPJ)$. Thus, $x \in JPJ \cap P_n$. But $P_n \subset M_n = JN_{n-1}J' \cap \mathcal{B}(L^2(M)) \subset JQ_{n-1}J'$. Thus, $x \in JPJ \cap (JQ_{n-1}J)'$. This proves the first part of b).

Further on, assume $[M:P] < \infty$ and take $\{m_j\}_j$ to be a finite orthonormal basis of M over P. Note that $M \underset{e_N}{\boxtimes} M^{\rm op} = \overline{\rm sp}(M \vee M^{\rm op}){\rm vN}\{e_j\}_{j\in\mathbb{Z}}$ and $P \underset{e_Q}{\boxtimes} P^{\rm op} = \overline{\rm sp}(P \vee P^{\rm op}){\rm vN}\{e_j\}_{j\in\mathbb{Z}}$, with $\{e_j\}_j \subset P_\infty$ being the Jones projections for a tower-tunnel for $Q \subset P$ (see 4.1, 4.2), and thus for $N \subset M$ as well. Since $M \vee M^{\rm op} = \Sigma_{i,j} m_i m_j^{*{\rm op}} P \vee P^{\rm op}$ it thus follows that $M \underset{e_N}{\boxtimes} M^{\rm op} = \Sigma_{i,j} m_i m_j^{*{\rm op}} P \underset{e_Q}{\boxtimes} P^{\rm op}$, showing that $M \underset{e_N}{\boxtimes} M^{\rm op}$ is a finitely generated left module over $P \underset{e_Q}{\boxtimes} P^{\rm op}$, with the estimate $[M \underset{e_N}{\boxtimes} M^{\rm op} : P \underset{e_Q}{\boxtimes} P^{\rm op}] \leq [M:P]^2$ as a bonus.

For the last part, we have that $\bigcup_n {\rm sp} P^{\rm op} P_n P^{\rm op}$ is so-dense in $P \underset{e_Q}{\boxtimes} P^{\rm op}$ and writing P_n as ${\rm sp} P f^n_{-n} P$ we get $E_{M\vee M^{\rm op}}(P \underset{e_Q}{\boxtimes} P^{\rm op}) = \overline{\rm sp} \bigcup_n ((P \vee P^{\rm op}) E_{M\vee M^{\rm op}}(f^n_{-n})(P \vee P^{\rm op}))$. But since $P'\cap P_n = M'\cap M_n$ and $Q'_{n-1} \cap P =$

$N'_{n-1} \cap M$, $\forall\ n$, it follows that $E_{M \vee M^{\rm op}}(f^n_{-n}) = E_{P \vee P^{\rm op}}(f^n_{-n})$, proving the desired commuting square condition. Q.E.D.

Let us end this section by considering a notion of index for sublattices of standard λ-lattices (see [Po7] for the definition of abstract standard lattices and for the notations and results used hereafter). We relate this notion with the content of this section by showing that the index of a sublattice coincides with the index of a certain canonically associated inclusion of symmetric enveloping type II_1 factors. This latter result will be used in Sections 5 and 8.

2.11. Definition. Let $\mathcal{G} = (A_{ij})_{0 \leq i \leq j}$ be a standard λ-lattice and $\mathcal{G}_0 = (A^0_{ij})_{0 \leq i \leq j}$ a sublattice. We define the index of $\mathcal{G}_0$ in $\mathcal{G}$ by $[\mathcal{G} : \mathcal{G}_0] \overset{def}{=} \lim_{n \to \infty} \mathrm{Ind} E^{A_{0n}}_{A^0_{0n}} = \mathrm{Ind} E^{A_{0,\infty}}_{A^0_{0,\infty}}$, where $\mathrm{Ind}(E)$ denotes as usual the index ([PiPo1]) of the conditional expectation E and $A_{i,\infty} = \overline{\cup_n A_{in}}$, $A^0_{i,\infty} = \overline{\cup_n A_{in}}$.

Let us make right away some comments on this definition. By (1.1.6 in [Po3]), if $\{m_j\}_j$ is an orthonormal basis of $A_{0,\infty}$ over $A^0_{0,\infty}$ (aprioricaly made up of square summable operators) then $\|\Sigma_j m_j m_j^*\| = \mathrm{Ind} E^{A_{0,\infty}}_{A^0_{0,\infty}}$. But both $A_{1,\infty} \subset A_{0,\infty}$ and $A^0_{1,\infty} \subset A^0_{0,\infty}$ are λ-Markov inclusions (see 1.1.5 in [Po2] for the definition), so the commuting square embedding of the latter into the former is nondegenerate (1.5,1.6 in [Po2]). Thus, by (1.6 in [Po2]) any orthonormal basis of $A_{1,\infty}$ over $A^0_{1,\infty}$ is an orthonormal basis of $A_{0,\infty}$ over $A^0_{0,\infty}$. Thus, in the above we may assume that $\{m_j\}_j$ lies in $A_{1,\infty}$. On the other hand, if bounded, $\Sigma_j m_j m_j^*$ belongs to the center of $A_{0,\infty}$ (see e.g. 1.1.5 in [Po3]), thus $\Sigma_j m_j m_j^* \in \mathcal{Z}(A_{0,\infty}) \cap A_{1,\infty} = \mathcal{Z}(A_{0,\infty}) \cap \mathcal{Z}(A_{1,\infty})$. But by (Corollary 1.4.2 in [Po2]) this latter intersection is in fact equal to the scalar multiples of the identity. Thus, $\Sigma_j m_j m_j^* \in \mathbb{C}1$. Altogether, this shows that we may as well take $[\mathcal{G} : \mathcal{G}_0] \overset{def}{=} \|\Sigma_j m_j m_j^*\| = \Sigma_j m_j m_j^*$, $\{m_j\}_j$ being an arbitrary orthonormal basis of $A_{i,\infty}$ over $A^0_{i,\infty}$, for some $i \geq 0$. The next proposition gives more ways to look at this index.

2.12. Proposition. *Let $\mathcal{G}$ be a standard λ-lattice with a sublattice $\mathcal{G}_0$. Let Q_0 be a non-atomic finite von Neumann algebra with a faithful trace and $N^{\mathcal{G}}(Q_0) \subset M^{\mathcal{G}}(Q_0)$, respectively $N^{\mathcal{G}_0}(Q_0) \subset M^{\mathcal{G}_0}(Q_0)$ be the associated extremal inclusions of type II_1 factors having $\mathcal{G}$, respectively $\mathcal{G}_0$ as standard invariants, given by the universal construction in ([Po7]). Let*

$$\begin{array}{ccc} N^{\mathcal{G}}(Q_0) & \subset & M^{\mathcal{G}}(Q_0) \\ \cup & & \cup \\ N^{\mathcal{G}_0}(Q_0) & \subset & M^{\mathcal{G}_0}(Q_0). \end{array}$$

be the corresponding commuting square like in ([Po7]). Then we have

$$[\mathcal{G} : \mathcal{G}_0] = [M^{\mathcal{G}}(Q_0) : M^{\mathcal{G}_0}(Q_0)] = [S : S_0],$$

where S and respectively S_0 denote the symmetric enveloping algebras of $N^{\mathcal{G}}(Q_0) \subset M^{\mathcal{G}}(Q_0)$ and respectively $N^{\mathcal{G}_0}(Q_0) \subset M^{\mathcal{G}_0}(Q_0)$.

Proof. Recall from [Po7] that $M^{\mathcal{G}}_\infty(Q_0)$ identifies with the free product with amalgamation $Q_0\bar{\otimes}A_{1,\infty} *_{A_{1,\infty}} A_{0,\infty}$, with $M^{\mathcal{G}_0}_\infty(Q_0)$ identifying with the subalgebra generated by Q_0 and $A^0_{0,\infty}$. By the resulting commuting square relations for these inclusions (see [Po7], pages 435 and 438), it follows that any orthonormal basis of $A_{1,\infty}$ over $A^0_{1,\infty}$ is an orthonormal basis of $Q_0\bar{\otimes}A_{1,\infty} *_{A_{1,\infty}} A_{0,\infty}$ over $Q_0\bar{\otimes}A^0_{1,\infty} *_{A^0_{1,\infty}} A^0_{0,\infty}$, thus of $M^{\mathcal{G}}_\infty(Q_0)$ over $M^{\mathcal{G}_0}_\infty(Q_0)$. But the commuting square embedding of $M^{\mathcal{G}_0}(Q_0) \subset M^{\mathcal{G}}(Q_0)$ into $M^{\mathcal{G}_0}_\infty(Q_0) \subset M^{\mathcal{G}}_\infty(Q_0)$ is nondegenarate (cf. [Po7]), so that in the end, if $\{m_j\}$ denotes an orthonormal basis of $A_{1,\infty}$ over $A^0_{1,\infty}$, we get $[M^{\mathcal{G}}(Q_0) : M^{\mathcal{G}_0}(Q_0)] = [M^{\mathcal{G}}_\infty(Q_0) : M^{\mathcal{G}_0}_\infty(Q_0)] = \Sigma_j m_j m_j^* = [\mathcal{G} : \mathcal{G}_0]$.
Finally, from the universality properties of the symmteric enveloping algebras and the definition of $N^{\mathcal{G}}(Q_0) \subset M^{\mathcal{G}}(Q_0)$ and $N^{\mathcal{G}_0}(Q_0) \subset M^{\mathcal{G}_0}(Q_0)$, we see that, if we denote by $N \subset M$ and $N_0 \subset M_0$ these two inclusions then $S_0 \subset S$ identifies with the inclusion $Q_0\bar{\otimes}N_0^{\rm op} *_{N_0^{\rm op}} M_0^{\rm op} \subset Q_0\bar{\otimes}N^{\rm op} *_{N^{\rm op}} M^{\rm op}$. But from the above we have that any orthonormal basis of $N^{\rm op}$ over $N_0^{\rm op}$ will be an orthonormal basis of S over S_0. Q.E.D.

3. A Class of Examples

Let Q be a type II$_1$ factor and $\sigma_1, \dots, \sigma_n$ a n-tuple of automorphisms of Q. Let $N \subset M$ be the locally trivial inclusion of factors associated with $\sigma_1, \dots, \sigma_n$ (see e.g. [Po2]), i.e., $M = Q \otimes M_{n+1}(\mathbb{C})$, $N = \{\sum_{i=0}^n \sigma_i(x) \otimes e_{ii} \mid x \in Q \simeq Q \otimes \mathbb{C}1\}$, where $\sigma_0 = \mathrm{id}_Q$ and $\{e_{ij}\}_{0\le i,j\le n}$ is a matrix unit for $M_{n+1}(\mathbb{C})$.
We still denote by σ_i the automorphism of $M = Q \otimes M_{n+1}(\mathbb{C})$ defined by $\sigma_i(x \otimes e_{kl}) = \sigma_i(x) \otimes e_{kl}$, $\forall\ x \in Q$, $0 \le k,l \le n$. Denote by G the discrete group generated by $\sigma_1, \dots, \sigma_n$ in $\mathrm{Aut}(M)/\mathrm{Int}(M)$. Also, we let $\sigma : G \to \mathrm{Aut}(M)/\mathrm{Int}(M)$ be the corresponding faithful G-kernel. Then note that the faithful G-kernel $\sigma\otimes\sigma^{\rm op}$ on $M\bar{\otimes}M^{\rm op}$ has vanishing $H^3(G, \mathbb{T})$ cohomology obstruction ([J5]), so that it can be viewed as a (properly outer) cocycle action of G on $M\bar{\otimes}M^{\rm op}$.
In this section we show that, with the above notations, we have

$$(M \vee M^{\rm op} \subset M \underset{e_Q}{\boxtimes} M^{\rm op}) \simeq (M\bar{\otimes}M^{\rm op} \subset (M\bar{\otimes}M^{\rm op}) \rtimes_{\sigma\otimes\sigma^{\rm op}} G),$$

in which the cross product is associated with the cocycle action $\sigma \otimes \sigma^{\rm op}$ as in (4.1 of [J5]). Since by the previous sections $M \vee M^{\rm op} \subset M \underset{e_Q}{\boxtimes} M^{\rm op}$ is the (weak closure of the) quotient of $C^*(M, JMJ) \subset C^*(M, e_N, JMJ)$, it will be sufficient to study this latter inclusion of algebras.
So let U_i be the unitary element acting on $L^2(M, \tau)$, defined on the dense subset $\hat{M} \subset L^2(M, \tau)$ by $U_i(\hat{x}) = \widehat{\sigma_i(x)}$, $x \in M$, $0 \le i \le n$. Note that $U_i x U_i^* = \sigma_i(x)$, $\forall\ x \in M$, $0 \le i \le n$, and $[J, U_i] = 0$. In particular, since $\sigma_i(e_{kl}) = e_{kl}$, $0 \le k,l \le n$, we also have $[U_i, e_{kl}] = 0$, $[U_i, Je_{kl}J] = 0$, $\forall\ i, k, l$.

3.1. LEMMA.
a) $e_N = \frac{1}{n+1} \sum_{i,j=0}^{n} U_j U_i^* e_{ji} J e_{ji} J.$

b) $U_j = (n+1) \sum_{k,l=0}^{n} J e_{lj} J e_{kj} e_N e_{0k} J e_{0l} J.$

Proof. a) If $x = \sum x_{ij} \otimes e_{ij} \in Q \otimes M_{n+1}(\mathbb{C}) = M$, then

$$\widehat{E_N(x)} = \frac{1}{n+1} \sum_{i,j} (\sigma_j \sigma_i^{-1}(x_{ii}) \otimes e_{jj})\widehat{\ } = \frac{1}{n+1} \sum_{i,j=0}^{n} U_j U_i^* e_{ji} J e_{ji} J(\hat{x})$$

proving the first formula.
b) By a) we have $e_{jj} e_N J e_{00} J = \frac{1}{n+1} U_j e_{j0} J e_{j0} J$, so that $U_j e_{00} J e_{00} J = (n+1) e_{0j} J e_{0j} J e_{jj} e_N J e_{00} J$. Thus we get

$$U_j = \sum_{k,l=0}^{n} e_{k0} J e_{l0} J (U_j e_{00} J e_{00} J) J e_{0l} J e_{0k} = (n+1) \sum_{k,l=0}^{n} e_{kj} J e_{lj} J e_N J e_{0l} J e_{0k}.$$

Q.E.D.

3.2. COROLLARY. *$C^*(M, e_N, JMJ) = C^*(M, \{U_i\}_{i \le n}, JMJ)$. In fact,*

$$C^*(M_{n+1}(\mathbb{C}), e_N, JM_{n+1}(\mathbb{C})J) = C^*(M_{n+1}(\mathbb{C}), \{U_i\}_{0 \le i \le n}, JM_{n+1}(\mathbb{C})J).$$

Proof. Trivial by the previous lemma. Q.E.D.

Describing $M \vee M^{\rm op} \subset M \underset{e_N}{\boxtimes} M^{\rm op}$ as a cross product is now an immediate consequence of the previous lemma and of 2.1, once we notice that $U_j(xJy^*J)U_j^* = \sigma_i(x) J \sigma_i(y^*) J$. To write the corresponding isomorphism in more specific terms, denote by u_i the image of U_i in $M \underset{e_N}{\boxtimes} M^{\rm op}$ (cf. 2.1) and by g_i the image of σ_i as an element of the group G.

3.3 THEOREM. *There exists a unique isomorphism γ, of $(M \vee M^{\rm op} \subset M \underset{e_N}{\boxtimes} M^{\rm op})$ onto $(M \bar{\otimes} M^{\rm op} \subset M \bar{\otimes} M^{\rm op} \rtimes_{\sigma \otimes \sigma^{\rm op}} G)$, satisfying:*
a) $\gamma(xy^{\rm op}) = x \otimes y^{\rm op}$, $x, y \in M$.
b) $u_{g_i} \overset{\rm def}{=} \gamma(u_i)$ *are unitary elements in the cross product* $M \bar{\otimes} M^{\rm op} \rtimes_{\sigma \otimes \sigma^{\rm op}} G$ *which implement the automorphism* $\sigma \otimes \sigma^{\rm op}(g_i)$, $0 \le i \le n$.
c) $\gamma(e_N) = \frac{1}{n+1} \sum_{i,j=0}^{n} u_{g_j} u_{g_i}^* e_{ji} \otimes e_{ij}^{\rm op}.$

d) $\gamma^{-1}(u_{g_j}) = u_j = (n+1) \sum_{k,l=0}^{n} e_{kj} e_{jl}^{\rm op} e_N e_{0k} e_{l0}^{\rm op}$, $0 \le j \le n$.

Proof. Trivial by 2.1 and 3.1. Q.E.D.

In the next section we will see that even for arbitrary extremal subfactors $N \subset M$ the resulting inclusion $M \vee M^{\rm op} \subset M \underset{e_N}{\boxtimes} M^{\rm op}$ can be interpreted as a 'cross-product'-type structure.

3.4 *Remarks.* 1°. As one knows (see e.g. 5.1.5 in [Po2]), the standard invariant $\mathcal{G}_{N,M}$ of the above locally trivial subfactor $N \subset M$ only depends on the cohomology obstruction in $H^3(G, \mathbb{T})$ ([J5]) of the corresponding G-kernel σ on Q. Thus, if we take another G-kernel σ', on another type II$_1$ factor Q' but with the same $H^3(G, \mathbb{T})$-obstruction as σ, and denote the similar locally trivial inclusion (corresponding to the same generators of G) by $N' \subset M'$, then $\mathcal{G}_{N',M'} = \mathcal{G}_{N,M}$ and we can thus consider the concatenation algebra 2.5.1° associated with these two inclusions. Then $M \vee M'^{\rm op} \subset M \boxtimes M'^{\rm op}$ is isomorphic to a cocycle cross product $M \bar{\otimes} M'^{\rm op} \subset (M \bar{\otimes} M'^{\rm op}) \rtimes_{\sigma \otimes \sigma'^{\rm op}} G$.
2°. Let $\mathcal{G} = (A_{ij})_{0 \leq i \leq j}$ be the standard λ-lattice associated to the locally trivial subfactor $N \subset M$, constructed from the automorphisms $\sigma_1, ..., \sigma_n$ acting on the factor Q as above, with G denoting the group generated by the σ_i's in $\mathrm{Aut}(Q)/\mathrm{Int}(Q)$ (and with the corresponding generators denoted hereafter by $g_1, ..., g_n$). Let $\mathcal{G}_0 = (A^0_{ij})_{i,j}$ be a sublattice of $\mathcal{G}$ with the property that A^0_{01} is a maximal abelian subalgebra of A_{01}. Note that this amounts to saying that $\mathcal{G}_0$ has same "generators" but possibly lesser "relations" than $\mathcal{G}$. Now take Q_0 to be an arbitrary finite von Neumann algebra without atoms. With Q_0 as "initial data", do the universal construction [Po7] of subfactors $N^{\mathcal{G}}(Q_0) \subset M^{\mathcal{G}}(Q_0)$ and $N^{\mathcal{G}_0}(Q_0) \subset M^{\mathcal{G}_0}(Q_0)$ with higher relative commutants picture given by $\mathcal{G}$ respectively $\mathcal{G}_0$, like at the end of Sec. 2, thus obtaining the non-degenerate commuting square of inclusions:

$$\begin{array}{ccc} N^{\mathcal{G}}(Q_0) & \subset & M^{\mathcal{G}}(Q_0) \\ \cup & & \cup \\ N^{\mathcal{G}_0}(Q_0) & \subset & M^{\mathcal{G}_0}(Q_0). \end{array}$$

One can then show that the above algebras and the inclusions involved can be alternatively described in terms of the following objects:
a). A type II$_1$ factor Q' with a faithful G kernel σ' on it such that if $N \subset M$ denotes the locally trivial subfactor constructed out of this G-kernel and the generators $g_1, ..., g_n$, like at the beginning of this section, then $(N \subset M) \simeq (N^{\mathcal{G}}(Q_0) \subset M^{\mathcal{G}}(Q_0))$;
b). An irreducible regular (in the sense of [D1]) subfactor $Q'_0 \subset Q'$, a group G_0 with generators $g'_1, ..., g'_n$ and a G_0-kernel σ'_0 on Q'_0 such that if $N_0 \subset M_0$ denotes the associated locally trivial subfactor, constructed from this G_0-kernel and the generators $g'_1, ..., g'_n$, like at the beginning of this section, then $(N_0 \subset M_0) \simeq (N^{\mathcal{G}_0}(Q_0) \subset M^{\mathcal{G}_0}(Q_0))$;
c). A group morphism ρ of G_0 onto G such that $\rho(g'_i) = g_i$ and such that if $H = \ker(\rho)$ denotes the corresponding kernel group then H is isomorphic to $\mathcal{N}_{Q'}(Q'_0)/\mathcal{U}(Q'_0)$ (so that Q' is a cocycle cross-product of Q'_0 by H), in such a way that if we denote by $\{u_h\}_{h \in H}$ a set of unitaries in $\mathcal{N}_{Q'}(Q'_0)$ that give

a cross-section for H then, modulo perturbations by inner automorphisms, σ' and σ'_0 are related as follows: $\sigma'(\rho(g'))(u_h a'_0) = u_{hgh^{-1}}\sigma'_0(g')(a'_0), \forall a'_0 \in Q'_0, h \in H, g' \in G_0$.
Moreover, through these identifications, $N_0 \subset M_0$ is embedded in $N \subset M$ by the inclusion $M_0 = Q'_0 \otimes M_{n+1}(\mathbb{C}) \subset Q' \otimes M_{n+1}(\mathbb{C}) = M$, and the corresponding commuting square is isomorphic to the above commuting square.
Thus, in this exemple the sublattice $\mathcal{G}_0$ of the lattice $\mathcal{G}$ (which was associated to the group G) corresponds to a "covering" group G_0 of the group G. Note that, with these identifications, we have that the index of $\mathcal{G}_0$ in $\mathcal{G}$ equals the order of the group H, $[\mathcal{G} : \mathcal{G}_0] = |H|$.
Finally, let us see what the symmetric enveloping algebras become in this case: if we extend the atomorphisms $\sigma'(g), \sigma'_0(g')$ to M, M_0 by putting them to act as the identity on $M_{n+1}(\mathbb{C})$, then the symmetric enveloping algebras S, S_0 of $N \subset M$ respectively $N_0 \subset M_0$, and the corresponding inclusion $S_0 \subset S$ (cf. 2.10, b)), are given by

$$S_0 = M_0 \bar{\otimes} M_0^{\mathrm{op}} \rtimes_{\sigma'_0 \otimes \sigma'^{\mathrm{op}}_0} G_0 \subset M \bar{\otimes} M^{\mathrm{op}} \rtimes_{\sigma' \otimes \sigma'^{\mathrm{op}}} G = S$$

with the inclusion being described similarily to c).

4. Thinness and Quasi-Regularity Properties

We've already seen that $\mathrm{sp} \bigcup_n M M_n^{\mathrm{op}} M = \mathrm{sp} \bigcup_n M^{\mathrm{op}} M_n M^{\mathrm{op}}$ is a $*$-subalgebra which is dense in $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ in the weak (or strong) operator topology. Let $\{e_n\}_{n\in\mathbb{Z}}$ be the Jones projections for the Jones tunnel-tower $\cdots N_1 \subset N \subset M \subset M_1 \subset \cdots$, with $e_N = e_1$, as in Sections 1–2, and denote by P the von Neumann algebra they generate in $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$. Fix $n \geq 0$ and choose an orthonormal basis $\{m_j\}_j$ of M over N_{n-1} that belongs to $\mathrm{vN}\{e_k\}_{k\leq 0} \subset P$ and an orthonormal basis $\{m_k^n\}_k$ of M_n over M that belongs to $\mathrm{vN}\{e_k\}_{k\leq n} \subset P$. Thus we have

$$\begin{aligned} M M_n^{\mathrm{op}} M &\subset M \left(\sum_k M^{\mathrm{op}} m_k^{n\,\mathrm{op}} \right) \left(\sum_j N_{n-1} m_j^* \right) = \sum_{j,k} M M^{\mathrm{op}} N_{n-1} m_k^{n\,\mathrm{op}} m_j^* \\ &= \sum_{j,k} M M^{\mathrm{op}} m_k^{n\,\mathrm{op}} m_j^* \subset \mathrm{sp}\, M M^{\mathrm{op}} P. \end{aligned}$$

Thus we obtain $\mathrm{sp} \bigcup_n M M_n^{\mathrm{op}} M \subset \mathrm{sp}(M \vee M^{\mathrm{op}})P$. Similarly, since $\sum_{j,k} M m_k^{n\,\mathrm{op}} m_j^* \subset M_\infty$, we get $\mathrm{sp} \bigcup_n M M_n^{\mathrm{op}} M \subset \mathrm{sp}\, M^{\mathrm{op}} M_\infty$, giving us the following:

4.1. Proposition. *With the above notations we have:*

$$\begin{aligned} S \overset{\mathrm{def}}{=} M \underset{e_N}{\boxtimes} M^{\mathrm{op}} &= \overline{\mathrm{sp}}(M \vee M^{\mathrm{op}})P = \overline{\mathrm{sp}}\, M_\infty M^{\mathrm{op}} \\ &= \overline{\mathrm{sp}}(M \vee M^{\mathrm{op}})(\mathrm{Alg}\{f_{-n}^n\}_n)(M \vee M^{\mathrm{op}}) \end{aligned}$$

the closure being taken in either of the wo*,* so *or* $\| \|_2$ *topologies in* S*.*

Proof. Since $\mathrm{sp}(M \vee M^{\mathrm{op}})P$, $M_\infty M^{\mathrm{op}}$ and $\mathrm{sp}(M \vee M^{\mathrm{op}})(Alg\{f^n_{-n}\}_n)(M \vee M^{\mathrm{op}})$ contain $\mathrm{sp} \bigcup_n M M_n^{\mathrm{op}} M$, which is a dense $*$-subalgebra in $S = M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$, we are done. Q.E.D.

Note that if M is hyperfinite then $M \vee M^{\mathrm{op}}$, M^{op}, M_∞ are all hyperfinite. Thus, in this case $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ can be written as a "product" of two hyperfinite subfactors. Recall from ([Po5]) that such situation is singled out by the following:

4.2. Definition. A type II_1 factor S for which there exist two hyperfinite type II_1 subfactors $R_1, R_2 \subset S$ such that $S = \overline{\mathrm{sp}}\, R_1 R_2$, the closure being taken in $\| \|_2$, is called a *thin* type II_1 factor.

With this terminology the above observation takes the form:

4.3. Corollary. *If* $N \subset M$ *is a extremal inclusion of hyperfinite type* II_1 *factors then* $S = M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ *is a thin type* II_1 *factor.*

From the above, the previous section and Connes' fundamental theorem ([C1]) we can already conclude:

4.4. Corollary. *If* $N \subset M$ *is an inclusion of factors associated to a faithful* G*-kernel* σ *on a hyperfinite type* II_1 *factor* R *like on Section* 3*, where* G *is a finitely generated discrete group, then* $M \underset{e_Q}{\boxtimes} M^{\mathrm{op}} \simeq R \otimes R^{\mathrm{op}} \rtimes_{\sigma \otimes \sigma^{\mathrm{op}}} G$ *is thin but it is hyperfinte iff* G *is amenable.*

More precise statements along these lines will be obtained in Sec.5 and 7. Let us note now that the Hilbert space $\mathcal{K}_n$ obtained as the closure of

$$(\mathrm{sp} M^{\mathrm{op}} M_n M^{\mathrm{op}})\hat{} = (\mathrm{sp} M M_n^{\mathrm{op}} M)\hat{}$$

in $L^2(M \underset{e_N}{\boxtimes} M^{\mathrm{op}}, \tau)$ is invariant to multiplication from left and right by both M and M^{op}, thus by $T = M \vee M^{\mathrm{op}}$. Thus $\mathcal{K}_n$ is a T-T bimodule.
Since $\mathrm{sp}\, M M_n^{\mathrm{op}} M = \mathrm{sp} \sum_{j,k} m_k m_j^{*\mathrm{op}} f^n_{-n}\; M M^{\mathrm{op}} = \mathrm{sp} \sum_{j,k} M M^{\mathrm{op}} f^n_{-n} m_j^{\mathrm{op}} m_k^*$, it follows that $\mathcal{K}_n$ has finite dimension both as a left and as a right T module. Thus, if p_n is the orthogonal projection of $L^2(S, \tau)$ onto $\mathcal{K}_n$ the p_n commutes with the operators of left and right multiplication by elements in T, i.e., $p_n \in T' \cap \langle S, T \rangle$. Also, since $\overline{\bigcup_n \mathcal{K}_n} = L^2(S, \tau)$, we have $p_n \nearrow 1$ and the above shows that $\mathrm{Tr}\, p_n < \infty$, $\forall\, n$, where $\mathrm{Tr} = \mathrm{Tr}_{\langle S, T \rangle}$ denotes the unique trace on $\langle S, T \rangle$ satisfying $\mathrm{Tr}(e_T) = 1$.
Thus, $T' \cap \langle S, T \rangle$ is generated by finite projections of $\langle S, T \rangle$ and the inclusion of factors $T p_n \subset p_n \langle S, T \rangle p_n$ has finite index for all n. Since $T' \cap S = \mathbb{C}1$ (cf. 2.3), by (1.8 in [PiPo1]) we can already conclude that $\mathrm{Tr}\, p \geq 1$, $\forall\, p \in T' \cap \langle S, T \rangle$ (so in particular $T' \cap \langle S, T \rangle$ is atomic) and that the multiplicity of any minimal projection p in $T' \cap \langle S, T \rangle$ is $\leq \mathrm{Tr}\, p$.
In fact we have the following more precise statement:

4.5. Theorem. *Let $N \subset M$ be an extremal inclusion of type* II_1 *factors and denote* $S = M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$, $T = M \vee M^{\mathrm{op}} \subset S$.

a) *If* $\{\mathcal{H}_k\}_{k\in K}$ *denotes the set of irreducible* M-M *bimodules corresponding to the set of even vertices of the standard graph* $\Gamma_{N,M}$ *of* $N \subset M$ *then* $L^2(S,\tau)$ *is isomorphic as a* T-T *bimodule with* $\bigoplus_{k\in K} \mathcal{H}_k \bar{\otimes} \bar{\mathcal{H}_k}^{\mathrm{op}}$ *and* $\mathcal{K}_n$ *with* $\bigoplus_{k\in K_n} \mathcal{H}_k \bar{\otimes} \bar{\mathcal{H}_k}^{\mathrm{op}}$ *(in which* $T \simeq M \bar{\otimes} M^{\mathrm{op}}$*).*

b) *If* $L^2(S,\tau)$ *is identified with* $\bigoplus_{k\in K} \mathcal{H}_k \bar{\otimes} \bar{\mathcal{H}_k}^{\mathrm{op}}$ *as in* a) *and* s_k *denotes the orthogonal projection of* $L^2(S,\tau)$ *onto its direct summand* $\mathcal{H}_k \bar{\otimes} \bar{\mathcal{H}_k}^{\mathrm{op}}$ *then* s_k *is a minimal projection in* $T' \cap \langle S,T\rangle$, $p_n = \sum_{k\in K_n} s_k$ *and* $T' \cap \langle S,T\rangle = \mathrm{vN}\{s_k\}_{k\in K} \simeq \ell^\infty(K)$. *Moreover,* $(\mathrm{Tr}\, s_k)^2 = [s_k \langle S,T\rangle s_k : T s_k] = v_k^4$, *where* $\vec{v} = (v_k)_{k\in K}$ *is the standard vector giving the weights at the even vertices of* $\Gamma_{N,M}$.

c) *The antiautomorphism* $^{\mathrm{op}}$ *on* S *leaves* T *invariant and thus implements an antiautomorphism on* $\langle S,T\rangle$, *still denoted by* $^{\mathrm{op}}$. *We have* $(T' \cap \langle S,T\rangle)^{\mathrm{op}} = T' \cap \langle S,T\rangle$, *the projection* s_k^{op} *coincides with* $J_S s_k J_S$ *and the corresponding bimodule is* $(\mathcal{H}_k \bar{\otimes} \bar{\mathcal{H}_k}^{\mathrm{op}})^- = \bar{\mathcal{H}}_k \bar{\otimes} \mathcal{H}_k^{\mathrm{op}}$.

Proof. Let $k \in K_1$ and choose $q = q_k \in N_1' \cap M$ to be a minimal projection in the direct summand labeled by k. Denote $v_q' = (\lambda\tau(q))^{1/2} q q^{\mathrm{op}} e_1 e_0 e_0^{\mathrm{op}}$ and $v_q = \lambda^{-2} E_{N'\cap M_1}(v_q')$. Note that $f = v_q' {v_q'}^*$ is the Jones projection for the irreducible inclusion $q^{\mathrm{op}} q N_1 \subset q^{\mathrm{op}} q M q \subset q^{\mathrm{op}} q M_2 q q^{\mathrm{op}}$ (cf. 2.8.b) and 2.8.c)). Note also that by applying twice the "push down lemma" (1.2 in [PiPo1]) and using the above definitions we get:

$$v_q e_0^{\mathrm{op}} e_0 = \lambda^{-2} E_{N'\cap M_1}(v_q') e_0^{\mathrm{op}} e_0$$

$$= \lambda^{-1} E_{N'\cap M_2}(\lambda^{-1} E_{N_1'\cap M_1}(v_q') e_0^{\mathrm{op}}) e_0 = \lambda^{-1} E_{N'\cap M_2}(v_q') e_0 = v_q',$$

implying that:

$$v_q e_0 e_0^{\mathrm{op}} v_q^* = v_q' e_0 e_0^{\mathrm{op}} {v_q'}^* = v_q' {v_q'}^* = f \le q^{\mathrm{op}} q.$$

Step I. We first prove that $L^2(\mathrm{sp} M v_q M) \simeq \mathcal{H}_k$ and that $L^2(\mathrm{sp} M^{\mathrm{op}} v_q M^{\mathrm{op}}) \simeq \bar{\mathcal{H}}_k^{\mathrm{op}}$. Indeed, since $e_0^{\mathrm{op}} = e_2$, by the definition of $\mathcal{H}_k$ we have $\mathcal{H}_k = L^2(\sum_j m_j M)$, where $\{m_j\}_j \subset M_1 = \langle M, e_1\rangle$ $(\subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}})$ are so that $\{m_j e_0^{\mathrm{op}} m_j^*\}_j$ are mutually orthogonal projections with $\sum_j m_j e^{\mathrm{op}} m_j^* = q^{\mathrm{op}} \in M' \cap M_2$. Since $v_q \in M_1$ and $v_q e_0^{\mathrm{op}} = q^{\mathrm{op}} v_q e_0^{\mathrm{op}}$, it follows that $v_q \in \sum m_j M$. Thus $M v_q M \subset \sum_j m_j M$, so that $L^2(\mathrm{sp} M v_q M) \subset \mathcal{H}_k$. Since $\mathcal{H}_k$ is irreducible and $L^2(\mathrm{sp} M v_q M)$ is a M-M bimodule, we actually have the equality $L^2(\mathrm{sp} M v_q M) = \mathcal{H}_k$.

To prove the second isomorphism, note that given any $T - T$ (resp. $M - M$) bimodule $\mathcal{H} \subset L^2(S,\tau)$, its conjugate $T - T$ (resp. $M - M$) bimodule $\bar{\mathcal{H}}$ can be identified with $(\mathcal{H})^* = \{\xi^* \mid \xi \in \mathcal{H}\}$ and its opposite $T^{\mathrm{op}} - T^{\mathrm{op}}$ (resp.

$M^{\rm op}$–$M^{\rm op}$) bimodule $\mathcal{H}^{\rm op}$ can be identified with $(\mathcal{H})^{\rm op} = \{\xi^{\rm op} \mid \xi \in \mathcal{H}\}$ (all this is trivial by the definitions). As a consequence, we also have $\bar{\mathcal{H}}^{\rm op} \simeq ((\mathcal{H})^*)^{\rm op}$. By taking into account that $(v_q^*)^{\rm op} = v_q$ and that $M^* = M$, from the isomorphism $L^2({\rm sp} M v_q M) \simeq \mathcal{H}_k$ and the above remark it thus follows that $L^2({\rm sp} M^{\rm op} v_q M^{\rm op}) \simeq \bar{\mathcal{H}}_k^{\rm op}$ as well.

Step II. We now prove that $\mathcal{H}_k \bar{\otimes} \bar{\mathcal{H}}_k^{\rm op} \simeq L^2({\rm sp} M M^{\rm op} v_q M M^{\rm op})$. To see this, by Step I it is sufficient to prove that there exists $\alpha \in \mathbb{C}$ such that:

$$\langle x_1 x_2^{\rm op} v_q y_1 y_2^{\rm op}, x_3 x_4^{\rm op} v_q y_3 y_4^{\rm op} \rangle = \alpha \langle x_1 v_q y_1, x_3 v_q y_3 \rangle \langle x_2^{\rm op} v_q y_2^{\rm op}, x_4^{\rm op} v_q y_4^{\rm op} \rangle,$$

$\forall\, x_i, y_j \in M$, $1 \leq i, j \leq 4$. By denoting $a = x_3^* x_1$, $b = y_1 y_3^*$, $c = x_2 x_4^*$, $d = y_4^* y_2$, it follows that it is sufficient to prove that

$$\langle a v_q b, c^{*{\rm op}} v_q d^{*{\rm op}} \rangle = \alpha \langle a v_q b, v_q \rangle \langle c^{\rm op} v_q d^{\rm op}, v_q \rangle,$$

$\forall\ a, b, c, d \in M$. Writing $b = b_1 e_0 b_2$, $d = d_1 e_0 d_2$ for $b_{1,2}, d_{1,2} \in N$ and using that

$$\begin{aligned}\langle a v_q b, c^{*{\rm op}} v_q d^{*{\rm op}} \rangle &= \langle b_2 a b_1 v_q e_0, d_2^{*{\rm op}} c^{*{\rm op}} d_1^{*{\rm op}} v_q e_0^{\rm op} \rangle,\\ \langle a v_q b, v_q \rangle &= \langle b_2 a b_1 v_q e_0, v_q \rangle,\\ \langle c^{\rm op} v_q d^{\rm op}, v_q \rangle &= \langle d_1^{\rm op} c^{\rm op} d_2^{\rm op} v_q e_0^{\rm op}, v_q \rangle,\end{aligned}$$

by putting a for $b_2 a b_1$ and c for $d_2 c d_1$, it follows that we only need to check that:

$$\langle a v_q e_0, c^{*{\rm op}} v_q e_0^{\rm op} \rangle = \alpha \langle a v_q e_0, v_q \rangle \langle c^{\rm op} v_q e_0^{\rm op}, v_q \rangle,$$

$\forall\ a, c \in M$. But

$$\langle a v_q e_0, c^{*{\rm op}} v_q e_0^{\rm op} \rangle = \tau(a c^{\rm op} v_q e_0 v_q^*) = \tau(a c^{\rm op} f)$$

and also

$$\begin{aligned}\langle a v_q e_0, v_q \rangle = \langle a v_q e_0, v_q e_0 \rangle = \tau(a v_q e_0 v_q^*) &= \lambda^{-1} \tau(a v_q e_0 e_0^{\rm op} v_q^*)\\ &= \lambda^{-1} \tau(a f),\end{aligned}$$

and similarly $\langle c^{\rm op} v_q e_0^{\rm op}, v_q \rangle = \lambda^{-1} \tau(c^{\rm op} f)$, where $f = v_q e_0 e_0^{\rm op} v_q^*$ is the Jones projection for the irreducible inclusion $N_1 q q^{\rm op} \subset q M q q^{\rm op} \subset q^{\rm op} q M_2 q q^{\rm op}$. Since $E_{M \vee M^{\rm op}}(f) = v_k^{-2} q q^{\rm op}$ (where $\vec{v} = (v_k)_{k \in K}$ is the standard vector as usual), we have $\tau(a c^{\rm op} f) = \alpha_0 \tau(a q) \tau(c^{\rm op} q^{\rm op})$, $\forall\ a, c \in M$, for some constant $\alpha_0 \in \mathbb{R}_+$. Also, we have $\tau(a f) = \alpha_1 \tau(a q)$, $\tau(c^{\rm op} f) = \alpha_1 \tau(c^{\rm op} q^{\rm op})$, $\forall\ a, c \in M$ for some constant $\alpha_1 \in \mathbb{R}_+$. This ends the proof.

Step III. We next show that

$$L^2({\rm sp} M M^{\rm op} e_1 M M^{\rm op}) = \sum_{k \in K_1} L^2({\rm sp} M M^{\rm op} v_{q_k} M M^{\rm op}).$$

To see this let $q' \in N_1' \cap M$ be a minimal projection in the same simple direct summand as q and $u \in \mathcal{U}(N_1' \cap M)$ such that $u^* q u = q'$. Let $v'' = \lambda^{-1} E_{M_1}(u^{\rm op} v_q u^{*{\rm op}}) \in N' \cap M_1$ and note that $v'' e_0^{\rm op} v''^* = u^{\rm op}(v_q e_0^{\rm op} v_q^*) u^{*{\rm op}} \leq q'^{\rm op}$. By the same reasoning as in Step I, it follows that $L^2({\rm sp} M v'' M) = L^2(\sum_j m_j' M)$, where $\{m_j'\}_j \subset M_1$ is an orthonormal system such that $\sum m_j' e_0^{\rm op} m_j'^* = q'^{\rm op}$. But $v'' \in {\rm sp} M M^{\rm op} v_q M^{\rm op} M$, because $u^{\rm op} v_q u^{*{\rm op}} \in {\rm sp} M^{\rm op} v_q M^{\rm op}$ and $E_{M_1}(u^{\rm op} v_q u^{*{\rm op}}) = \lambda \sum_j b_j^{\rm op}(u^{\rm op} v_q u^{*{\rm op}}) b_j^{*{\rm op}} \in {\rm sp} M^{\rm op} v_q M^{\rm op}$ as well, where $\{b_j^*\}_j$ is an orthonormal basis of N over N_1.
Thus we have ${\rm sp} M M^{\rm op} v_q M^{\rm op} M \supset {\rm sp} M M^{\rm op} v_{q'} M M^{\rm op}$, $\forall\ q'$ chosen this way. Thus, if $\{m_j^k\}_j \subset M_1$ is a orthonormal system such that $\sum_j m_j^k e_0^{\rm op} m_j^k$ is the central support of $q^{\rm op}$ in $M' \cap M_2$ then ${\rm sp} M^{\rm op}(\sum_j m_j^k M) M^{\rm op} = {\rm sp} M M^{\rm op} v_q M^{\rm op} M$. Summing up over k and using that $\sum_k \sum_j m_j^k M = M_1 = {\rm sp} M e_1 M$, the statement follows.

Step IV. We now derive that

$$L^2({\rm sp} M M^{\rm op} f_{-n}^n M M^{\rm op}) \simeq \bigoplus_{k \in K_n} \mathcal{H}_k \bar{\otimes} \bar{\mathcal{H}}_k^{\rm op}$$

and then

$$L^2\left(M \underset{e_N}{\boxtimes} M^{\rm op}\right) = \bigoplus_{k \in K} \mathcal{H}_k \bar{\otimes} \bar{\mathcal{H}}_k^{\rm op}.$$

To see this, note first that $\mathcal{H}_k \bar{\otimes} \bar{\mathcal{H}}_k^{\rm op} \simeq \mathcal{H}_{k'} \bar{\otimes} \bar{\mathcal{H}}_{k'}^{\rm op}$ if and only if $\mathcal{H}_k \simeq \mathcal{H}_k'$. This fact follows immediately by interpreting $\mathcal{H}_k$ as irreducible representation of $M \otimes M^{\rm op}$, according to Connes' alternative view on correspondences (see [C4], [Po8]).
Since by Steps II and III we have $\vee_{k \in K_1} \mathcal{H}_k \bar{\otimes} \bar{\mathcal{H}}_k^{\rm op} = L^2({\rm sp} M M^{\rm op} e_1 M M^{\rm op})$, with $\mathcal{H}_k \bar{\otimes} \bar{\mathcal{H}}_k^{\rm op}$ mutually nonisomorphic, the first part of the statement follows for $n = 1$. By using this fact for $N_{n-1} \subset M \overset{f_{-n}^n}{\subset} M_n$, $n \geq 1$, we get it for any $n \geq 1$. The last part is now clear, since $\cup_n {\rm sp} M M^{\rm op} f_{-n}^n M M^{\rm op}$ is dense in $M \underset{e_N}{\boxtimes} M^{\rm op}$.

Step V. We finally show that if s_k denotes the minimal projection in $T' \cap \langle S, T\rangle$ labeled by $k \in K$ then ${\rm Tr}\, s_k = v_k^2$. This fact can be checked directly by using a similar strategy as in Step III. Instead, we will use the following more elegant argument: Since

$$v_k^4 = ({\rm Tr}_{\langle S,T\rangle} s_k)({\rm Tr}_{T'} s_k) = ({\rm Tr}_{\langle S,T\rangle} s_k)({\rm Tr}_{\langle S,T\rangle} J_S s_k J_S)$$

(cf. [J1]), we only need to show that $T' \cap \langle S, T\rangle \ni s_k \mapsto J_S s_k J_S \in T' \cap \langle S, T\rangle$ is ${\rm Tr}_{\langle S,T\rangle}$-preserving.
To see this note that since ${}^{\rm op}$ acts on S leaving T invariant, it implements a ${\rm Tr}_{\langle S,T\rangle}$-preserving anti-automorphism on $\langle S, T\rangle$, thus a ${\rm Tr}_{\langle S,T\rangle}$-preserving automorphism on the commutative algebra $T' \cap \langle S, T\rangle$. Moreover, if we put

$s_k L^2(S,\tau) = L^2(\text{sp}Tv_qT)$ as in Steps I and II and use that $v_q^{\text{op}} = v_q^*$, then we have

$$s_k^{\text{op}} L^2(S,\tau) = (L^2(\text{sp}Tv_qT))^{\text{op}}$$
$$= L^2(\text{sp}T^{\text{op}}v_q^{\text{op}}T^{\text{op}}) = L^2(\text{sp}Tv_q^{\text{op}}T) = L^2(\text{sp}Tv_q^*T)$$
$$= L^2(\text{sp}T^*v_q^*T^*) = L^2(\text{sp}Tv_qT)^* = J_S s_k J_S L^2(S,\tau)$$

Thus, $s_k^{\text{op}} = J_S s_k J_S$ so that $\text{Tr}(s_k) = \text{Tr}(s_k^{\text{op}}) = \text{Tr}(J_S s_k J_S)$. Q.E.D.

Note that the above theorem agrees with the exemples in Section 3. Indeed, if $\sigma \in \text{Aut}(P)$ is an automorphism of a type II_1 factor P and $\mathcal{H}_\sigma = L^2(\sigma)$ denotes the $P-P$ bimodule associated with σ as in [Po8] then an easy calculation shows that $\bar{\mathcal{H}_\sigma}^{\text{op}} = \mathcal{H}_{\sigma^{\text{op}}}$.

4.6. COROLLARY. *Let $N \subset M$ be an extremal inclusion. Then $N \subset M$ has finite depth if and only if $[M \underset{e_N}{\boxtimes} M^{\text{op}} : M \vee M^{\text{op}}] < \infty$. Moreover if these conditions are satisfied then $M \vee M^{\text{op}}$ has finite depth in $M \underset{e_N}{\boxtimes} M^{\text{op}}$.*

Proof. With the notations used in 4.5 and its proof, if we assume that $N \subset M$ has finite depth then K is finite so that by 4.5 we have $\dim(S' \cap \langle S,T\rangle) < \infty$ and each of the local indices is finite. But then, by Jones' formula ([J1]), it follows that $[S:T] < \infty$.
Conversely, if $[S:T] < \infty$ then $\dim(S' \cap \langle S,T\rangle) < \infty$, so that K follows finite, i.e., $N \subset M$ has finite depth.
Moreover, we see from 4.5 that if $[S:T] < \infty$ then the set of all $T-T$ irreducible bimodules generated by $L^2(S,\tau)$ under Connes' tensor product (fusion) are contained in the set of bimodules $\{\mathcal{H}_k \bar{\otimes} \bar{\mathcal{H}_{k'}}^{\text{op}}\}_{k,k' \in K}$ and is thus finite, i.e., $T \subset S$ has finite depth. Q.E.D.

4.7. *Remark.* As mentioned before, if M is hyperfinite and $N \subset M$ is a subfactor of finite depth then by ([Po15]) we have $M \vee M^{\text{op}} \subset M \underset{e_N}{\boxtimes} M^{\text{op}}$ is isomorphic to the inclusion $M \vee M' \cap M_\infty \subset M_\infty$ of [Oc]. This latter inclusion was already shown to have finite depth in [Oc] and in fact all its standard invariant (paragroup) has been calculated ([Oc], see also [EvKa]). In particular, for this class of symmetric enveloping inclusions, part b) of 4.5 can be recovered from ([Oc]). If $N \subset M$ is a finite depth subfactor with M not necessarily hyperfinite, then it is imediate to see that $M \vee M^{\text{op}} \subset M \underset{e_N}{\boxtimes} M^{\text{op}}$ has the same standard invariant (paragroup) as $P \vee P^{\text{op}} \subset P \underset{e_Q}{\boxtimes} P^{\text{op}}$ where $Q \subset P$ denotes the standard model for $N \subset M$, which is thus an inclusion of hyperfinite factors. Thus, for any $N \subset M$ with finite depth the standard invariant (paragroup) of $M \vee M^{\text{op}} \subset M \underset{e_N}{\boxtimes} M^{\text{op}}$ can be recovered from these results.
Theorem 4.5 shows that in the case $(T \subset S) = (M \vee M^{\text{op}} \subset M \underset{e_N}{\boxtimes} M^{\text{op}})$ then $L^2(S,\tau)$ is spanned by $T-T$ bimodules which are finitely generated both as left

and right T-modules. Equivalently, $T \subset S$ is such that $T' \cap \langle S, T\rangle$ is generated by finite projections of $\langle S, T\rangle$. Inclusions $T \subset S$ verifying this latter condition are called *discrete* in [ILP]. We'll introduce here a new terminology for such subfactors based on the former, more intrinsic characterization.

4.8. Definition. Let $T \subset S$ be an irreducible inclusion of type II_1 factors. We denote by $q\mathcal{N}_S(T) \stackrel{\text{def}}{=} \{x \in S \mid \exists\, x_1, x_2, \dots, x_n \in S$ such that $xT \subset \sum_{i=1}^n Tx_i$ and $Tx \subset \sum_{i=1}^n x_iT\}$. We call $q\mathcal{N}_S(T)$ the *quasi-normalizer* of T in S.

Note that the condition "$xT \subset \sum Tx_i$, $Tx \subset \sum x_iT$" is equivalent to "$TxT \subset (\sum_{i=1}^n Tx_i) \cap (\sum_{i=1}^n x_iT)$" and also to "$\mathrm{sp}TxT$ is finitely generated both as left and as a right T-module." It then follows readily that $\mathrm{sp}(q\mathcal{N}_S(T))$ is a $*$-algebra. Thus $P \stackrel{\text{def}}{=} \overline{\mathrm{sp}}(q\mathcal{N}_S(T)) = q\mathcal{N}_S(T)''$ is a subfactor of S containing T. Note also that $L^2(P) = \vee\{\mathcal{H} \mid \mathcal{H} \subset L^2(P)$, $\mathcal{H}$ is a T-T bimodule, $\dim({}_T\mathcal{H}) < \infty$, $\dim(\mathcal{H}_T) < \infty\}$ and that the orthogonal projection e_P, of $L^2(S)$ onto $L^2(P)$, satisfies $e_P = \vee\{f \in T' \cap \langle S, T\rangle \mid \mathrm{Tr} f < \infty,\ \mathrm{Tr} J_S f J_S < \infty\}$. All these facts are just reformulations of some results in [PiPo1] and [ILP], but can also be proved as exercises.
The terminology we wanted to introduce is then as follows:

4.9. Definition. Let $T \subset S$ be an irreducible inclusion. If $q\mathcal{N}_S(T)'' = S$, we say that T is *quasi-regular* in S. From the above remarks we see that an irreducible inclusion $T \subset S$ is discrete (as defined in [ILP]) iff T is quasi-regular in S.

Thus, from 4.5 it follows that if $N \subset M$ is an extremal inclusion of type II_1 factors then $M \vee M^{\mathrm{op}}$ is quasi-regular in $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$. Note that, even more, we showed that each irreducible T-T bimodule in $L^2(S)$ (where $T = M \vee M^{\mathrm{op}}$, $S = M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$) has multiplicity 1 and its (finite) dimension as a left T module coincides with its dimension as a right T-module. Thus, our symmetric enveloping inclusions have very similar properties to the inclusions given by cross-products of factors by outer actions of discrete groups.
We wanted to emphasize even more this aspect by choosing the terminology "quasi-normalizer", "quasi-regular" in analogy with Dixmier's notions of "normalizer" and "regularity" for an irreducible subfactor ([D1]). This is particularily justified by noticing that exemples of quasi-regular subfactors $T \subset S$ can be obtained by requiring S to be generated by unitary elements u such that uTu^* is included in T and has finite index in it (see the Appendix in [ILP] for a concrete exemple of such a situation).
Let us end this section with a result showing that the extended sequence of Jones projections in a tunnel-tower associated to a subfactor $N \subset M$ has a certain general ergodicity property with respect to the higher relative commutants that is very useful in applications (see e.g. 2.2 and 2.3 in [GePo]). We'll refer to this result as the *Ergodicity Theorem for Higher Relative Commutants*.

4.10. Theorem. *Let $N \subset M$ be a subfactor with finite index (but not necessarily extremal). Let $\{M_j\}_{j\in\mathbb{Z}}$ be a tunnel-tower for $N \subset M$, where*

$M_0 = M, M_{-1} = N$, and $\{e_j\}_{j\in\mathbb{Z}}$ be its corresponding Jones projections. Denote $A_{ij} = M_i' \cap M_j$ and $A_{-\infty,i} = \overline{\bigcup_{n\leq i} A_{ni}}$, $A_{-\infty,\infty} = \overline{\bigcup_i A_{-\infty,i}}$. Then we have:
a) $\{e_j\}'_{j\in\mathbb{Z}} \cap A_{-\infty,\infty} = \mathbb{C}$. *In particular, $A_{-\infty,\infty}$ is a factor.*
b) *If M has separable predual then the tunnel $\{M_j\}_{j\leq 0}$ can be chosen such that $\{e_j\}'_{j\leq k} \cap M_n \subset A_{-\infty,n}, \forall k \leq n$ in $\mathbb{Z}$.*
c) *If $N \subset M$ is extremal and its tunnel is chosen to satisfy condition* b) *then $\{e_j\}'_{j\in\mathbb{Z}} \cap M \underset{e_N}{\boxtimes} M^{\mathrm{op}} = \mathbb{C}$.*

Proof. a). Let θ be the trace preserving automorphism on $A_{-\infty,\infty}$ implemented by the duality isomorphism (1.5 of [PiPo1] or 1.3.3 of [Po2]), i.e., θ satisfies $\theta(A_{ij}) = A_{i+2,j+2}, \theta(e_k) = e_{k+2}, \forall i,j,k \in \mathbb{Z}$, with $\theta_{|A_{ij}}$ being defined as the restriction to $M_i' \cap M_j = M_i^{\alpha\prime} \cap M_j^{\alpha}$ of $\sigma'_{ij} : (M_i \subset M_{i+1} \subset ... \subset M_j)^\alpha \to (M_{i+2} \subset ... \subset M_{j+2})$, where $\sigma'_{ij}((x_{rs})_{r,s}) = \lambda^{i-j+1} \sum_{r,s} m_r e_{i+2} e_{i+3} ... e_{j+2} x_{rs} e_{j+2} ... e_{i+2} m_s^*$, in which $\{m_r\}_r$ is an orthonormal basis of $\mathrm{vN}\{e_n\}_{n\leq i+1}$ over $\mathrm{vN}\{e_n\}_{n\leq i}$ and $\lambda^{-1} = [M : N]$.
We first show that this automorphism satisfies the identity $\theta(z)e_{j+2} = ze_{j+2}$ for all $z \in \{e_k\}'_{k\leq j} \cap A_{-\infty,j}$. To this end let $\varepsilon > 0$ and $i \leq j$ be so that $\|E_{A_{ij}}(z) - z\|_2 < \varepsilon$. Put $z_0 = E_{A_{ij}}(z) \in \{e_{i+2}, ..., e_j\}' \cap A_{ij}$. From the above local formula for θ we have

$$e_{j+2}\theta(z_0)e_{j+2} = \lambda^{i-j+1} e_{j+2} (\sum_r m_r e_{i+2} ... e_{j+2} z_0 e_{j+2} ... e_{i+2} m_r^*) e_{j+2}$$

$$= \lambda^{i-j+3} \sum_r m_r e_{i+2} ... e_j (z_0 e_{j+2}) e_j ... e_{i+2} m_r^*$$

$$= \sum_r m_r (z_0 e_{i+2} e_{j+2}) m_r^* = (\sum_r m_r (z_0 e_{i+2}) m_r^*) e_{j+2}.$$

By taking into account that the orthonormal basis $\{m_r\}_r$ can be taken to be made up of no more than $[M : N] + 1$ elements, we thus get the estimates:

$$\|\theta(z)e_{j+2} - ze_{j+2}\|_2$$

$$\leq \|\theta(z) - \theta(z_0)\|_2 + \|z - z_0\|_2 + \|e_{J+2}\theta(z_0)e_{j+2} - z_0 e_{j+2}\|_2$$

$$\leq 2\varepsilon + \|\Sigma_r m_r (z_0 e_{i+2}) m_r^* e_{j+2} - z_0 e_{j+2}\|_2$$

$$\leq 2\varepsilon + \sum_r \|[m_r e_{i+2}, z_0]\|_2$$

$$\leq 2\varepsilon + ([M : N] + 1)^2 \varepsilon.$$

Letting ε tend to 0, we get the desired identity.
Now to prove part a) of the statement let $z \in \mathrm{vN}\{e_n\}'_{n\in\mathbb{Z}} \cap A_{-\infty,\infty}$ with $\tau(z) = 0$ and take $z_0 = E_{A_{-\infty,j}}(z)$ for some j. Note that $\tau(z_0) = 0$ as well. For such a z_0, and in fact for any z_0 in $\mathrm{vN}\{e_n\}'_{n\leq j} \cap A_{-\infty,j}$, we then have the estimates:

$$\|(z - z_0)e_{j+2}\|_2^2 = \tau((z - z_0)^*(z - z_0)e_{j+2})$$

$$= \tau(E_{\{e_l\}'_{l\geq j+2}\cap A_{-\infty,\infty}}((z-z_0)^*(z-z_0)e_{j+2}))$$

$$= \tau((z-z_0)^*(z-z_0)E_{\{e_l\}'_{l\geq j+2}\cap A_{-\infty,\infty}}(e_{j+2})) = \lambda\|z-z_0\|_2^2,$$

in which we used that by Jones ergodicity theorem we have $E_{\{e_l\}'_{l\geq j+2}\cap A_{-\infty,\infty}}(e_{j+2}) = \lambda 1$.
Since $z_0 e_{j+2} = \theta(z_0)e_{j+2}$, we get similarily:

$$\|(z-z_0)e_{j+2}\|_2^2 = \|(z-\theta(z_0))e_{j+2}\|_2^2$$

$$= \tau((z-\theta(z_0)^*(z-\theta(z_0))e_{j+2})$$

$$= \tau(E_{\{e_l\}'_{l\leq j+2}\cap A_{-\infty,\infty}}((z-\theta(z_0))^*(z-\theta(z_0))e_{j+2}))$$

$$= \tau((z-\theta(z_0))^*(z-\theta(z_0))E_{\{e_l\}'_{l\leq j+2}\cap A_{-\infty,\infty}}(e_{j+2}))$$

$$= \lambda\|z-\theta(z_0)\|_2^2,$$

in which we used the fact that $\theta(z_0)$ commutes with $\mathrm{vN}\{e_l\}_{l\leq j+2}$ and that by Jones ergodicity theorem we have $E_{\{e_l\}'_{l\leq j+2}\cap A_{-\infty,\infty}}(e_{j+2}) = \lambda 1$.
Altogether, the above shows that $\|z-z_0\|_2 = \|z-\theta(z_0)\|_2$ and by applying this recursively n times we get $\|z-z_0\|_2 = \|z-\theta^n(z_0)\|_2, \forall n \geq 1$.
On the other hand $\theta^n(A_{ij}) = A_{i+2n,j+2n}$ and so, if n is so that $2n > j-i$ then $\tau(z_1\theta^n(z_1)) = \tau(z_1)^2, \forall z_1 \in A_{ij}$, showing that θ is mixing on $A_{-\infty,\infty} = \overline{\bigcup_{i,j} A_{ij}}$. Thus, for $z_0 \in A_{-\infty,j}$ with $\tau(z_0) = 0$ we have

$$\lim_{n\to\infty}\|z_0-\theta^n(z_0)\|_2^2 = 2\|z_0\|_2^2.$$

Since $\|z_0-\theta^n(z_0)\|_2 \leq \|z_0-z\|_2 + \|z-\theta^n(z_0)\|_2$ and since for $z \in \mathrm{vN}\{e_n\}'_{n\in\mathbb{Z}} \cap A_{-\infty,\infty}$ we proved that $\|z-z_0\|_2 = \|z-\theta^n(z_0)\|_2, \forall n \geq 1$, in which $z_0 = E_{-\infty,j}(z)$, it follows that for each j we have the estimate:

$$2\|z_0\|_2^2 = \lim_{n\to\infty}\|z_0-\theta^n(z_0)\|_2^2 \leq 4\|z-z_0\|_2^2.$$

Now, letting j tend to infinity we get $\|z-z_0\|_2$ tend to 0 and $\|z_0\|_2$ tend to $\|z\|_2$, which from the above estimate forces $z=0$. This ends the proof of a).
b). Let $\{x_n\}_{n\geq 1} \subset M$ be a sequence of elements dense in the unit ball of M in the so-topology. We construct recursively a sequence of integers $0 < k_1 < k_2 < \ldots$ and a tunnel $M \supset N \supset N_1 \ldots \supset N_{k_1} \supset \ldots \supset N_{k_n} \supset \ldots$ for $N \subset M$ such that if $\{e_n\}_{n\leq 0}$ are the corresponding Jones projections and we denote by $B_n = \mathrm{Alg}\{e_j\}_{-k_n+1\leq j\leq -k_{n-1}-1}$ then we have:

$$\|E_{B_n'\cap M}(x_j) - E_{N'_{k_{n-1}}\cap M}(x_j)\|_2 < 2^{-n}, \forall j \leq n$$

Assume we have this up to some n. By ([Po1]) there exists a hyperfinite subfactor $R \subset N_{k_n}$ such that $E_{R'\cap M}(x) = E_{N'_{k_n}\cap M}(x), \forall x \in M$. On the other hand, by Jones ergodicity theorem, we can regard R as being generated by a sequence of Jones λ-projections e_j indexed over the integers $\leq -k_n - 1$. Thus, there will exist a sufficiently large k_{n+1} such that if we denote $B_{n+1} = \text{Alg}\{e_j\}_{-k_{n+1}+1 \leq j \leq -k_n - 1}$ then

$$\|E_{B'_{n+1}\cap M}(x_j) - E_{N'_{k_n}\cap M}(x_j)\|_2 < 2^{-n-1}, \forall j \leq n+1.$$

Now choose a Jones projection e_{-k_n} for $N_{k_n} \subset N_{k_n - 1}$ such that it commutes with $e_j \in B_{n+1}$ for $j \leq -k_n - 2$ and such that it satisfies the Jones-Temperley-Lieb relation for $j = -k_n - 1$ (see the proof of 4.4 on page 33 of [Po15]), i.e., $e_{-k_n} e_{-k_n-1} e_{-k_n} = \lambda e_{-k_n}$, and then simply define the corresponding tunnel $N_{k_n} \supset N_{k_n+1} \supset \ldots \supset N_{k_{n+1}}$ as given by these newly chosen Jones projections e_j with $-k_{n+1} + 1 \leq j \leq -k_n$.
Thus, if we take $A_n = \overline{\cup_{m\geq n} B_m} \subset \text{vN}\{e_j\}_{j\leq -k_n-1}$ then it follows from the above that $E_{A'_n\cap M}(x) \in \overline{\cup_k N'_k \cap M}$ for all $x \in \{x_j\}_j$ and thus by density for all $x \in M$. Thus even more so $\text{vN}\{e_j\}'_{j\leq -m} \cap M \subset \overline{\cup_k N'_k \cap M}$ for $m = k_n$ and thus in fact for all $m \geq 0$.
Finally, if $x \in M_n$ for some $n \geq 0$ then for any $\varepsilon > 0$ there exists $k \leq 0$ and $x' \in \text{sp}((\text{Alg}\{e_j\}_{k\leq j\leq n})M)$ such that $\|x - x'\|_2 < \varepsilon$. But then $x'' = E_{\{e_l\}'_{l\leq k-2}\cap M_n}(x')$ belongs to $\text{sp}((\text{Alg}(\{e_j\}_{k\leq j\leq n})\overline{\cup_i N'_i \cap M})$ which in turn is icluded into $\overline{\cup_i N'_i \cap M_n}$ and we have:

$$\|E_{\{e_l\}'_{l\leq k-2}\cap M_n}(x) - x''\|_2 = \|E_{\{e_l\}'_{l\leq k-2}\cap M_n}(x - x')\|_2 \leq \|x - x'\|_2 \leq \varepsilon.$$

Letting ε go to 0 and j to $-\infty$ we get

$$\lim_{j\to-\infty} E_{\{e_l\}'_{l\leq j-2}\cap M_n}(x) = E_{A_{-\infty,n}}(x), \forall x \in M_n$$

This ends the proof of b) and c) follows then imediately, by taking into account that $\bigcup_n \text{sp} M M_n^{\text{op}} M$ is dense in $M \underset{e_N}{\boxtimes} M^{\text{op}}$ and applying a) and b). Q.E.D.

4.11. Corollary. *Let $N \subset M$ be an extremal inclusion of type II_1 factors with separable preduals. There exists a choice of a tunnel $\{M_j\}_{j\leq 0}$ for $N \subset M$ such that if we denote $M_n = (M_{-n})^{\text{op}\prime} \cap M \underset{e_N}{\boxtimes} M^{\text{op}}$, $n \geq 1$, $M_\infty = \overline{\cup_n M_n}$, $M_\infty^{\text{op}} = \overline{\cup_n M_n^{\text{op}}}$ and $A_{-\infty,\infty} = \overline{\cup_n M'_{-n} \cap M_n}$ then $M_\infty, M_\infty^{\text{op}} \subset M \underset{e_N}{\boxtimes} M^{\text{op}}$ satisfy the conditions:*
a) $\overline{\text{sp}} M_\infty M_\infty^{\text{op}} = M \underset{e_N}{\boxtimes} M^{\text{op}}$.
b) $M_\infty \cap M_\infty^{\text{op}} = A_{-\infty,\infty}$ and $E_{M_\infty} E_{M_\infty^{\text{op}}} = E_{A_{-\infty,\infty}}$.
c) $A'_{-\infty,\infty} \cap M \underset{e_N}{\boxtimes} M^{\text{op}} = \mathbb{C}1$.

Proof.. Conditions a) and b) are actually valid for any choice of the tunnel while 4.9 clearly implies c). Q.E.D.

5. Relating the Amenability Properties of $\Gamma_{N,M}$, $\mathcal{G}_{N,M}$ and $(M \vee M^{\rm op} \subset M \underset{e_N}{\boxtimes} M^{\rm op})$

In [Po8] one considers a notion of relative amenability for inclusions of finite von Neumann algebras $T \subset S$ by requiring the existence of norm one projections from $\langle S, T\rangle$ onto S, equivalently of Connes-type S-hypertraces on $\langle S, T\rangle$. In the case $S = T \rtimes G$ for some discrete group G this condition on the inclusion $T \subset S$ is equivalent to the amenability of the group G.

As we have seen in the previous section, when $T = M \vee M^{\rm op} \subset M \underset{e_N}{\boxtimes} M^{\rm op} = S$, for $N \subset M$ a locally trivial subfactor associated to some faithful G-kernel σ, with G a finitely generated discrete group, then $(T \subset S) \simeq (T \subset T \rtimes_{\sigma\otimes\sigma^{\rm op}} G)$. Thus, the relative amenability of the inclusion $T \subset S$ is equivalent in this case to the amenability of G. On the other hand, one of the equivalent characterizations of the amenability of G is Kesten's condition requiring that the Cayley graph of G, Γ, corresponding to some finite, self-adjoint set of generators $g_0 = 1, g_1, \dots, g_n$, satisfies $\|\Gamma\| = n + 1$.

Recalling from [Po2,5] that the *standard graph* of a subfactor $\Gamma_{N,M}$ is called *amenable* if it satisfies the Kesten-type condition $\|\Gamma_{N,M}\|^2 = [M : N]$ and that its *standard invariant* $\mathcal{G}_{N,M}$ is called *amenable* if $\Gamma_{N,M}$ is amenable, and noticing that for the locally trivial subfactor $N \subset M$ corresponding to the above $(G; g_0, \dots, g_n; \sigma)$ the Cayley graph Γ coincides with the standard graph $\Gamma_{N,M}$, while $[M : N] = (n+1)^2$, it follows that in this case the amenability of G (thus, the relative amenability of $T \subset S$) is equivalent to the amenability of $\mathcal{G}_{N,M}$.

We prove in this section that in fact even for arbitrary extremal subfactors of finite index $N \subset M$ the relative amenability condition on $T = M \vee M^{\rm op} \subset M \underset{e_N}{\boxtimes} M^{\rm op} = S$ is equivalent to the amenability of the standard lattice $\mathcal{G}_{N,M}$. Along the lines, we will obtain some other related characterizations of the amenability of $\mathcal{G}_{N,M}$, thus of $\Gamma_{N,M}$.

Before stating the result, recall some terminology and notations from [Po2].

So let $(\mathcal{N} \overset{\mathcal{E}}{\subset} \mathcal{M}) = \bigoplus ((N \otimes P^{\rm op})^{**} \overset{(E\otimes {\rm id})^{**}}{\subset} (M \otimes P^{\rm op})^{**})$, the sum being taken over all isomorphism classes of type II_1 factors P that can be embedded with finite index in some amplification of M, i.e., factors P that are weakly stably equivalent to M in the sense of 1.4.3 in [Po8] (like for instance $P = M$). Then take first the atomic part of this inclusion, $(\mathcal{N} \overset{\mathcal{E}}{\subset} \mathcal{M})_{\rm at}$, and next the binormal part of the latter inclusion, $((\mathcal{N} \overset{\mathcal{E}}{\subset} \mathcal{M})_{\rm at})_{\rm bin}$ (i.e., the largest direct summand in which both M and $P^{\rm op}$ sit as von Neumann algebras), which we denote by $\mathcal{N}^u \overset{\mathcal{E}^u}{\subset} \mathcal{M}^u$, and call *the universal atomic (binormal) representation* of $N \subset M$. Also, the inclusion graph (or matrix) of $\mathcal{N}^u \subset \mathcal{M}^u$ is denoted by $\Gamma^u_{N,M}$ and called the *universal graph (or matrix)* of $N \subset M$.

Finally one defines $(\mathcal{N}^{\rm st} \overset{\mathcal{E}^{\rm st}}{\subset} \mathcal{M}^{\rm st})$ to be the minimal direct summand of $\mathcal{N}^u \subset$

$\mathcal{M}^u$ (or, equivalently, of $(\mathcal{N} \overset{\mathcal{E}}{\subset} \mathcal{M})_{\text{at}}$) containing the standard representation of $M \otimes M^{\text{op}}$, $\mathcal{B}(L^2(M))$ and call it *the standard representation* of $N \subset M$. It is easy to see (cf. e.g., [Po2]) that the commuting square embedding

$$\begin{array}{ccc} \mathcal{N}^{\text{st}} & \overset{\mathcal{E}^{\text{st}}}{\subset} & \mathcal{M}^{\text{st}} \\ \cup & & \cup \\ N \otimes M^{\text{op}} & \overset{E \otimes \text{id}}{\subset} & M \otimes M^{\text{op}} \end{array}$$

can be identified with the embedding

$$\begin{array}{ccc} \bigoplus_{\ell \in L} \mathcal{B}(\mathcal{K}_\ell) & \overset{\mathcal{E}^{\text{st}}}{\subset} & \bigoplus_{k \in K} \mathcal{B}(\mathcal{H}_k) \\ \cup & & \cup \\ N \otimes M^{\text{op}} & \overset{E \otimes \text{id}}{\subset} & M \otimes M^{\text{op}} \end{array}$$

in which $\{\mathcal{H}_k\}_{k \in K}$ (respectively, $\{\mathcal{K}_\ell\}_{\ell \in L}$) is the list of all irreducible M-M (resp. N-M) bimodules appearing as direct summands in $L^2(M_j)$, $j = 0, 1, 2, \ldots$, and $M \otimes M^{\text{op}}$ (resp, $N \otimes M^{\text{op}}$) is represented on each $\mathcal{H}_k$ (resp. $\mathcal{K}_\ell$) by operators of left and right multiplication by elements in M (respectively, right multiplication by elements in N and right multiplication by elements in M). Moreover, the inclusion matrix (or graph) for

$$\mathcal{N}^{\text{st}} \subset \bigoplus_{\ell \in L} \mathcal{B}(\mathcal{K}_\ell) \subset \bigoplus_{k \in K} \mathcal{B}(\mathcal{H}_k) = \mathcal{M}^{\text{st}}$$

(which is thus a direct summand of the universal graph $\Gamma^u_{N,M}$) is given by $(\Gamma_{N,M})^{\text{t}}$, while $\mathcal{E}^{\text{st}}$ is the unique expectation that preserves the trace Tr on $\mathcal{M}^{\text{st}} = \bigoplus_{k \in K} \mathcal{B}(\mathcal{H}_k)$ given by the weight vector $\vec{v} = (v_k)_{k \in K}$, with $v_k = \dim({}_M\mathcal{H}_{k\,M})^{1/2}$.

Finally, note that $N \subset M$ is in fact embedded in the smaller inclusion

$$\mathcal{N}^{\text{st,f}} \overset{\text{def}}{:=} (1 \otimes M^{\text{op}})' \cap \mathcal{N}^{\text{st}} \overset{\mathcal{E}^{\text{st,f}}}{\subset} (1 \otimes M^{\text{op}})' \cap \mathcal{M}^{\text{st}} \overset{\text{def}}{=:} \mathcal{M}^{\text{st,f}}$$

where $\mathcal{E}^{\text{st,f}}$ is the restriction of $\mathcal{E}^{\text{st}}$ to $\mathcal{M}^{\text{st,f}}$.

5.1. Definition. The commuting square embedding:

$$\begin{array}{ccc} \mathcal{N}^{\text{st},f} & \overset{\mathcal{E}^{\text{st,f}}}{\subset} & \mathcal{M}^{\text{st,f}} \\ \cup & & \cup \\ N & \overset{E_N}{\subset} & M \end{array}$$

is called the *finite (or reduced) standard representation* of $N \subset M$.

5.2. Lemma. *$\mathcal{N}^{\mathrm{st,f}}$, $\mathcal{M}^{\mathrm{st,f}}$ are finite type II_1 von Neumann algebras with atomic centers $\mathcal{Z}(\mathcal{N}^{\mathrm{st,f}}) = \mathcal{Z}(\mathcal{N}^{\mathrm{st}}) \simeq \ell^\infty(L)$, $\mathcal{Z}(\mathcal{M}^{\mathrm{st,f}}) = \mathcal{Z}(\mathcal{M}^{\mathrm{st}}) \simeq \ell^\infty(K)$. Moreover, the inclusion $\mathcal{N}^{\mathrm{st,f}} \subset \mathcal{M}^{\mathrm{st,f}}$ is a matricial inclusion having inclusion matrix (or graph) $(\Gamma_{N,M})^{\mathrm{t}}$. Also, $\mathcal{E}^{\mathrm{st,f}}$ is the unique conditional expectation of $\mathcal{M}^{\mathrm{st,f}}$ onto $\mathcal{N}^{\mathrm{st,f}}$ preserving the trace Tr on $\mathcal{M}^{\mathrm{st,f}}$ given by the weights $\{v_k^2\}_{k\in K}$ on the center $\mathcal{Z}(\mathcal{M}^{\mathrm{st,f}}) \simeq \ell^\infty(K)$.*

Proof. The first part is trivial, by the definition of $\mathcal{N}^{\mathrm{st,f}} \subset \mathcal{M}^{\mathrm{st,f}}$ and the properties of $\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}$. Then the last part is an immediate consequence of the first part and of 2.7 in [PiPo2]. Q.E.D.

5.3. Theorem. *Let $N \subset M$ be an extremal inclusion of type II_1 factors. The following conditions are equivalent:*
1) $\mathcal{G}_{N,M}$ is amenable.
1′) $\Gamma_{N,M}$ is amenable, i.e., $\Gamma_{N,M}$ satisfies the Kesten type condition $\|\Gamma_{N,M}\|^2 = [M : N]$.
2) $(\Gamma_{N,M}, \vec{v})$ satisfies the Følner-type condition: $\forall\, \varepsilon > 0$, $\exists F \subset K$ finite such that

$$\sum_{k\in\partial F} v_k^2 < \varepsilon \sum_{k\in F} v_k^2,$$

where

$$\partial F = \{k \in K \setminus F \mid \exists\, k_0 \in F \text{ such that } (\Gamma_{N,M}\Gamma_{N,M}^{\mathrm{t}})_{kk_0} \neq 0\}.$$

3) There exists a state ψ_0 on $\ell^\infty(K) \simeq T' \cap \langle S, T\rangle$ such that $\psi_0 \circ E$ has $S = M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ in its centralizer, where E is the unique Tr-preserving conditional expectation of $\langle S, T\rangle$ onto $T' \cap \langle S, T\rangle$.
4) $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ is amenable relative to $M \vee M^{\mathrm{op}}$.
5) There exists a norm one projection from $(\mathcal{N}^{\mathrm{st},f} \overset{\mathcal{E}^{\mathrm{st,f}}}{\subset} \mathcal{M}^{\mathrm{st,f}})$ onto $(N \overset{E_N}{\subset} M)$.
5′) There exists a $(N \subset M)$-hypertrace on $(\mathcal{N}^{\mathrm{st,f}} \overset{\mathcal{E}^{\mathrm{st,f}}}{\subset} \mathcal{M}^{\mathrm{st,f}})$.

Proof. 1) $\Longleftrightarrow$ 1′) is clear by the definitions.
To prove 1′) $\Longrightarrow$ 2) let $\Phi = \lambda V^{-1}\Gamma\Gamma^{\mathrm{t}}V$, where V is the diagonal matrix over K with entries $(v_k)_{k\in K}$. Note that Φ defines a bounded positive linear operator from $P \overset{\mathrm{def}}{=} T' \cap \langle S, T\rangle \simeq \ell^\infty(K)$ into itself such that $\Phi(1) = 1$. Note also that the trace Tr on P inherited from $\langle S, T\rangle$ has weights $(v_k^2)_{k\in K}$ as a measure on K, i.e., if $b \in P \simeq \ell^\infty(K)$ then

$$\|b\|_{1,\mathrm{Tr}} = \sum_{k\in K} |b_k| v_k^2.$$

For $a, b : K \to \mathbb{C}$, at least one of which has finite support, we denote $\langle a, b\rangle = \sum_{k\in K} a_k \bar{b}_k$. For each $b \in P \simeq \ell^\infty(K)$ with finite support we then have:

$$\begin{aligned}\mathrm{Tr}(\Phi(b)) &= \langle \Phi(b), V^2(1)\rangle = \langle b, \lambda V\Gamma\Gamma^{\mathrm{t}}V^{-1}V^2(1)\rangle \\ &= \langle b, \lambda V\Gamma\Gamma^{\mathrm{t}}V(1)\rangle = \langle b, V^2(1)\rangle = \mathrm{Tr}(b).\end{aligned}$$

Thus $\text{Tr} \circ \Phi = \text{Tr}$. In particular, by Kadison's inequality, this implies $\|\Phi(a)\|_{2,\text{Tr}} \leq \|a\|_{2,\text{Tr}}, \forall\, a \in L^2(T, \text{Tr})$.
Since $\|\lambda\Gamma\Gamma^{\text{t}}\| = 1$, it follows that $\forall\, \delta > 0\; \exists\, F_0 \subset K$ finite such that $T_0 = {}_{F_0}(\lambda\Gamma\Gamma^{\text{t}})_{F_0}$ satisfies $1 \geq \|T_0\| \geq 1 - \delta^2/2$. By the classical Perron-Frobenius theorem applied to T_0 (which is a finite symmetric matrix with nonnegative entries) it follows that there exists $b_0 \in \ell^\infty(K) \simeq P$, supported in the set F_0, with $b_0(k) \geq 0$, $\forall\, k$, and $\langle b_0, b_0 \rangle = 1$, such that $T_0 b_0 \geq (1 - \delta^2/2) b_0$. Thus, $\lambda\Gamma\Gamma^{\text{t}} b_0 \geq (1 - \delta^2/2) b_0$.
Let then $b \overset{\text{def}}{=} V^{-1}(b_0) \in \ell^\infty(K) \simeq P$ and note that

$$\|b\|_{2,\text{tr}}^2 = \langle V^{-1}(b_0), V^2 V^{-1}(b_0) \rangle = \langle b_0, b_0 \rangle = 1.$$

Moreover, we have:

$$\begin{aligned}\|\Phi(b) - b\|_{2,\text{Tr}}^2 &\leq 2 - 2\text{Tr}(\Phi(b)b) \\ &= 2 - 2\langle \lambda V^{-1}\Gamma\Gamma^{\text{t}}(b_0), V(b_0) \rangle \\ &= 2 - 2\langle \lambda\Gamma\Gamma^{\text{t}}(b_0), b_0 \rangle \\ &\leq 2 - 2(1 - \delta^2/2) = 2\delta^2/2 = \delta^2.\end{aligned}$$

Thus $\|b - \Phi(b)\|_{2,\text{Tr}} < \delta$ and $\|\Phi(b)\|_{2,\text{Tr}} \geq 1 - \delta$, while $\|b\|_{2,\text{Tr}} = 1$.
By Theorem A.2 it follows that if $\delta < 10^{-4}$ then there exists a finite spectral projection e of b such that $\|\Phi(e) - e\|_{2,\text{Tr}} < \delta^{1/4}\|e\|_{2,\text{Tr}}$.
In particular we have:

$$\begin{aligned}\|(1 - e)\Phi(e)\|_{2,\text{Tr}}^2 &\leq \|(1 - e)\Phi(e)\|_{2,\text{Tr}}^2 + \|e - e\Phi(e)\|_{2,\text{Tr}}^2 \\ &= \|e - \Phi(e)\|_{2,\text{Tr}}^2 < \delta^{1/4}\|e\|_{2,\text{Tr}}^2.\end{aligned}$$

Let $F \subset K$ be the support set of $e \in \ell^\infty(K) \simeq P$. By the first 3 lines of the proof of Lemma 3.2 on page 281 of [Po3], we have $v_k^{-1} v_{k_0} \geq \lambda$ for all $k_0, k \in K$ for which $(\Gamma\Gamma^{\text{t}})_{kk_0} \neq 0$. Thus we have

$$(\Phi)_{kk_0} = \lambda v_k^{-1} v_{k_0} \sum_{l \in L} a_{kl} a_{k_0 l} \geq \lambda^2,$$

forall $k, k_0 \in K$ for which the entry (k, k_0) of Φ is nonzero. In particular, this shows that $\Phi(e)(1 - e) \geq \lambda^2 \chi_{\partial F}$, where $\chi_{\partial F} \in \ell^\infty(K)$ is the characteristic function of $\partial F \subset K$. Thus we have

$$\begin{aligned}\lambda^4 \sum_{k \in \partial F} v_k^2 &= \|\lambda^2 \chi_F\|_{2,\text{Tr}}^2 \\ &\leq \|(1 - e)\Phi(e)\|_{2,\text{Tr}}^2 < \delta^{1/4}\|e\|_{2,\text{Tr}}^2 \\ &= \delta^{1/4} \sum_{k \in F} v_k^2.\end{aligned}$$

Thus, if $\varepsilon > 0$ was given and we take $\delta = (\lambda^4\varepsilon)^4$ then

$$\sum_{k\in\partial F} v_k^2 < \varepsilon \sum_{k\in F} v_k^2$$

thus proving $1') \Longrightarrow 2)$.
Proof of $2) \Longrightarrow 3)$. By 2), for each $\varepsilon = 2^{-n}$ there exisits a finite subset $F_n \subset K$ such that

$$\sum_{k\in\partial F_n} v_k^2 < 2^{-n} \sum_{k\in F_n} v_k^2$$

or, equivalently,

$$\left(\sum_{k\in\Gamma\Gamma^t F_n} v_k^2 - \sum_{k\in F_n} v_k^2\right) < 2^{-n} \sum_{k\in F_n} v_k^2.$$

Let $f_n \in T' \cap \langle S, T\rangle \simeq \ell^\infty(K)$ be the support projection of F_n. Let ω be a free ultrafilter on $\mathbb{N} \simeq K$ and define ψ_0 on $\ell^\infty(K) \simeq T' \cap \langle S, T\rangle$ by

$$\psi_0 = \lim_{n\to\omega} \mathrm{Tr}(\cdot f_n)/\mathrm{Tr} f_n.$$

Let $\psi \stackrel{\text{def}}{=} \psi_0 \circ E$ and note that $\psi = \lim_{n\to\omega} \mathrm{Tr}(\cdot f_n)/\mathrm{Tr} f_n$ on $\langle S, T\rangle$ as well. Note that for each n we have that $\mathrm{Tr}(\cdot f_n)/\mathrm{Tr} f_n$ has T in its centralizer and it is a normal state on $\langle S, T\rangle$. Since $T' \cap S = \mathbb{C}$ this implies that $\mathrm{Tr}(\cdot f_n)/\mathrm{Tr} f_n$ coincides with the trace τ when restricted to S. Thus, $\psi|_S = \tau$ and ψ has T in its centralizer. Let us show that ψ also has e_N in its centralizer. To do this, it is sufficient to prove that

$$\lim_{n\to\infty} (\|f_n e_N - e_N f_n\|_{1,\mathrm{Tr}}/\mathrm{Tr} f_n) = 0.$$

Let $f_n' \in T' \cap \langle S, T\rangle \simeq \ell^\infty(K)$ be the support projection of $F_n \cup \partial F_n$ and note that we have

$$\lim_{n\to\infty} (\|f_n' - f_n\|_{1,\mathrm{Tr}}/\mathrm{Tr} f_n) = \lim_{n\to\infty} \left(\sum_{k\in\partial F_n} v_k^2 / \sum_{k\in F_n} v_k^2\right) = 0.$$

Also, we have

$$\|f_n e_N - e_N f_n\|_{1,\mathrm{Tr}} \le \|e_N f_n - f_n' e_N f_n\| + 2\|f_n' - f_n\|_{1,\mathrm{Tr}}.$$

So, to prove that $[e_N, \psi] = 0$, it is in fact sufficient to prove that $f_n' e_N f_n = e_N f_n$, $\forall\ n$. We will show that, more generally, we have $s_{F'} e_N s_F = e_N s_F$, $\forall\ F \subset K$, where $F' = F \cup \partial F$ and $s_F = \sum_{k\in F} s_k$, $s_{F'} = \sum_{k\in F'} s_k$. To this end, it is clearly sufficient to do it for single element sets $F = \{k_1\}$. It then amounts to show that if $k_2 \in K \setminus F'$, then $s_{k_2} e_N s_{k_1} = 0$. By the proof of 4.5

we thus need to show that if $k_1, k_2 \in K_{n-1}$ for some n, with $k_2 \notin \{k_1\} \cup \partial\{k_1\}$ and we take a minimal projection q_i in the direct summand labeled by k_i in $N'_{2n-1} \cap M$, for each $i = 1, 2$, then we have $MM^{\mathrm{op}} v_{q_2} \perp e_1 MM^{\mathrm{op}} v_{q_1} MM^{\mathrm{op}}$, where $e_1 = e_N$ and $v_{q_i} = E_{N'_{n-1} \cap M_n}(q_i q_i^{\mathrm{op}} f_{-n}^n f_{-2n}^0 f_0^{2n})$, $i = 1, 2$.
Before proving this, note that for such q_1, q_2 we have $q_2 e_N q_1 = 0$ and in fact $q_2(N'_{2n-1} \cap M_1)q_1 = 0$. Now, if we take $x_{1,2} \in M$, $y_{1,2} \in M^{\mathrm{op}}$, $x, x_0 \in N_{n-1}$, $y, y_0 \in N_{n-1}^{\mathrm{op}}$, then we get

$$\begin{aligned}
&\tau(v_{q_2}^* y_0^{\mathrm{op}} y_2^{\mathrm{op}} x x_2 e_1 x_1 y_1^{\mathrm{op}} v_{q_1} x_0 f_{-2n}^0 x y^{\mathrm{op}} f_0^{2n} y_0^{\mathrm{op}}) \\
&\quad = \tau(E_{N'_{n-1} \cap M_n}(f_0^{2n} f_{-2n}^0 f_{-n}^n q_2 q_2^{\mathrm{op}}) y_0^{\mathrm{op}} y_2^{\mathrm{op}} x x_2 e_1 x_1 x_0 y_1^{\mathrm{op}} y^{\mathrm{op}} \\
&\quad\quad \cdot E_{N'_{n-1} \cap M_n}(q_1 q_1^{\mathrm{op}} f_{-n}^n f_0^{2n} f_{-2n}^0) f_{-2n}^0 f_0^{2n}).
\end{aligned}$$

Taking the conditional expectation onto $N'_{2n-1} \cap (M \underset{e_N}{\boxtimes} M^{\mathrm{op}})$ and denoting $Y_1^{\mathrm{op}} = y_0^{\mathrm{op}} y_2^{\mathrm{op}} \in M^{\mathrm{op}}$, $Y_2^{\mathrm{op}} = y_1^{\mathrm{op}} y^{\mathrm{op}} \in M^{\mathrm{op}}$, $X' = E_{N'_{2n-1}}(x x_2 e_1 x_1 x_0) \in N'_{2n-1} \cap M_1$, we thus obtain that the above is equal to:

$$\begin{aligned}
&\tau(f_0^{2n} f_{-2n}^0 f_{-n}^n q_2^{\mathrm{op}} q_2 Y^{\mathrm{op}} X' Y_2^{\mathrm{op}} q_1 q_1^{\mathrm{op}} f_{-n}^n f_0^{2n} f_{-2n}^0) \\
&= \tau(f_0^{2n} f_{-2n}^0 f_{-n}^n q_2^{\mathrm{op}} Y_1^{\mathrm{op}} (q_2 X' q_1) Y_2^{\mathrm{op}} q_1^{\mathrm{op}} f_{-n}^n) \\
&= 0
\end{aligned}$$

in which we first used that $v_{q_i} f_{-2n}^0 f_0^{2n} = q_i q_i^{\mathrm{op}} f_{-n}^n f_0^{2n} f_{-2n}^0$ and then we used that $q_2 X' q_1 = 0$.
Since the elements of the form $x_0 f_{-2n}^0 x$ with $x, x_0 \in N_{n-1}$ span all M, this finishes the proof of the fact that e_N is in the centralizer of ψ. Since ψ is equal to the trace on $S = M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ and has in its centralizer the weakly dense $*$-subalgebra generated by $T = M \vee M^{\mathrm{op}}$ and e_N in S, by [C3] it follows that ψ has all S in its centralizer. This ends the proof of 2) $\Longrightarrow$ 3).
The proof of 3) $\Longrightarrow$ 4) is then trivial, since the relative amenability of $T = M \vee M^{\mathrm{op}} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}} = S$ merely requires the existence of a state on $\langle S, T \rangle$ which has S in its centralizer, while condition 3) provides very special such states.
To prove 4) $\Longrightarrow$ 5) we need the following lemma.

5.4. Lemma. *Let $\mathcal{M}_0 = \mathrm{vN}(M \cup J_S M J_S)$, $\mathcal{N}_0 = \mathrm{vN}(N \cup J_S M J_S)$ and $\Phi_0 : \mathcal{B}(L^2(S)) \to \mathcal{B}(L^2(S))$ be defined by $\Phi_0(T) = \lambda \sum_j m_j^{\mathrm{op}} T m_j^{\mathrm{op}*}$, where $\{m_j^{\mathrm{op}}\}_j$ is an orthonormal basis of M_1^{op} over M^{op} and $\lambda = [M : N]^{-1}$ as usual. Then $\Phi_0(\mathcal{M}_0) = \mathcal{N}_0$, $\mathcal{E}_0 = \Phi_0|_{\mathcal{M}_0}$ is a conditional expectation and in fact*

$$\begin{array}{ccc}
\mathcal{N}_0 & \overset{\mathcal{E}_0}{\subset} & \mathcal{M}_0 \\
\cup & & \cup \\
N & \overset{E_N}{\subset} & M
\end{array}$$

is a commuting square embedding of $N \subset M$, which is isomorphic to the standard representation of $N \subset M$. Moreover, if $\mathcal{N}_0^{\mathrm{f}} = J_S M J_S' \cap \mathcal{N}_0 \subset J_S M J_S' \cap \mathcal{M}_0 = \mathcal{M}_0^{\mathrm{f}}$, then

$$\begin{array}{ccc} \mathcal{N}_0^{\mathrm{f}} & \overset{\mathcal{E}_0^{\mathrm{f}}}{\subset} & \mathcal{M}_0^{\mathrm{f}} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

is a commuting square embedding isomorphic to the finite standard representation of $N \subset M$.

Proof. By construction, we see that $\mathcal{N}_0 \subset \mathcal{M}_0$ is a direct summand of $(N \otimes M^{\mathrm{op}})^{**} \overset{(E\otimes \mathrm{id})^{**}}{\subset} (M \otimes M^{\mathrm{op}})^{**}$. Also, since $\mathcal{N}_0 \subset (M_1^{\mathrm{op}} \cup J_S M^{\mathrm{op}} J_S)' \cap \mathcal{B}(L^2(S))$, we have

$$\begin{aligned} \mathcal{M}_0' \cap \mathcal{N}_0 &\subset (M \cup J_S M J_S)' \cap (M_1^{\mathrm{op}} \cup J_S M^{\mathrm{op}} J_S)' \\ &= \mathrm{vN}(M \cup M_1^{\mathrm{op}} \cup J_S M J_S \cup J_S M^{\mathrm{op}} J_S)' \cap \mathcal{B}(L^2(S)) \\ &= (M \underset{e_N}{\boxtimes} M^{\mathrm{op}} \cup J_S(M \vee M^{\mathrm{op}}) J_S)' \\ &= J_S((M \vee M^{\mathrm{op}})' \cap M \underset{e_N}{\boxtimes} M^{\mathrm{op}}) J_S = \mathbb{C}1. \end{aligned}$$

Thus $\mathcal{Z}(\mathcal{M}_0) \cap \mathcal{Z}(\mathcal{N}_0) = \mathbb{C}$. But if p_0 denotes the projection of $L^2(S)$ onto $L^2(M)$ then clearly $p_0 \mathcal{M}_0 p_0 = \mathcal{M}_0 p_0$ is isomorphic to $\mathcal{B}(L^2(M))$ as a $M \otimes M^{\mathrm{op}}$ representation. Thus, $\mathcal{N}_0 \subset \mathcal{M}_0$ must in fact coincide with $\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}$. The last part is now clear, since this isomorphism sends $1 \otimes M^{\mathrm{op}}$ onto $J_S M J_S$. Q.E.D.

Proof of 4) $\Longrightarrow$ 5) $\Longleftrightarrow$ 5$'$). The equivalence of 5) and 5$'$) was proved in [Po2], the argument being identical to Connes' single algebra analogue statement. Let us then prove 4) $\Longrightarrow$ 5$'$). So let ψ be a S-hypertrace on $\langle S, T\rangle = J_S T J_S' \cap \mathcal{B}(L^2(S))$. Since $T = M \vee M^{\mathrm{op}}$ and $\mathcal{M}_0 \subset J_S M^{\mathrm{op}} J_S' \cap \mathcal{B}(L^2(S))$ (we've already noticed this in the above lemma) it follows that

$$\begin{aligned} \mathcal{M}_0^f = (J_S M J_S)' \cap \mathcal{M}_0 &\subset (J_S(\mathrm{vN}(M \cup M^{\mathrm{op}}))J_S)' \cap \mathcal{B}(L^2(S)) \\ &= (J_S T J_S)' \cap \mathcal{B}(L^2(S)) = \langle S, T\rangle. \end{aligned}$$

Thus ψ restricts to a state ϕ on $\mathcal{M}_0^{\mathrm{f}}$ which has M in its centralizer (since ψ has S in its centralizer and S contains M).
Note now that if $T \in \mathcal{M}_0^f$ then

$$\psi(e_N T) = \psi(u^{\mathrm{op}}(e_N T)u^{\mathrm{op}*}) = \psi((u^{\mathrm{op}} e_N u^{\mathrm{op}*})T), \qquad \forall\ u^{\mathrm{op}} \in \mathcal{U}(M^{\mathrm{op}}).$$

Averaging by unitaries in $\mathcal{U}(M^{\mathrm{op}})$ and using that $\overline{\mathrm{co}}^n\{u^{\mathrm{op}} e_N u^{\mathrm{op}*} \mid u^{\mathrm{op}} \in \mathcal{U}(M^{\mathrm{op}})\} \cap \mathbb{C}1 = \{\lambda 1\}$ (see the Appendix A.1), it follows that

$$\psi(e_N T) = \lambda \psi(T) = \lambda \phi(T).$$

But $e_N \in S$ is in the centralizer of ψ so

$$\psi(e_N T) = \psi(e_N T e_N) = \psi(\mathcal{E}_0(T) e_N).$$

By the same argument as above, the latter equals

$$\lambda\psi(\mathcal{E}_0(T)) = \lambda\phi(\mathcal{E}_0(T)).$$

Thus $\phi = \phi \circ \mathcal{E}_0^{\mathrm{f}}$ showing that ϕ is a $(N \subset M)$-hypertrace on $\mathcal{N}_0^f \overset{\mathcal{E}_0^{\mathrm{f}}}{\subset} \mathcal{M}_0^f$ thus on $\mathcal{N}_0^{\mathrm{st,f}} \overset{\mathcal{E}_0^{\mathrm{st,f}}}{\subset} \mathcal{M}_0^{\mathrm{st},f}$, proving 5′).

Proof of 5′) $\Longrightarrow$ 1). Since $\mathcal{N}_0^{\mathrm{st}f} \overset{\mathcal{E}_0^{\mathrm{st}f}}{\subset} \mathcal{M}_0^{\mathrm{st},f}$ has inclusion matrix $(\Gamma_{N,M})^{\mathrm{t}}$ and the trace Tr on $\mathcal{M}^{\mathrm{st},f}$ defined in 5.2 is preserved by $\mathcal{E}^{\mathrm{st},f}$, it follows by the general result in [Po13] that $\|\Gamma_{N,M}\|^2 = \|\Gamma_{N,M}^{\mathrm{t}}\|^2 = [M : N]$.
This ends the proof of the theorem. Q.E.D.

5.5 *Remarks.* 1°. Of all the equivalent characterisations of amenability for standard graphs, the Kesten-type amenability condition $\|\Gamma_{N,M}\|^2 = [M : N]$ seems to remain the easiest to check in practice. For instance, it immediately implies that if $[M : N] \leq 4$ then $\Gamma_{N,M}$ is amenable, and it is the condition that was used by Bisch and Haagerup to construct many examples of infinite depth subfactors with amenable graphs, by taking compositions between a fixed point algebra inclusion and a cross product inclusion, corresponding to actions of finite groups ([BiH]). Nevertheless, each of the other equivalent characterizations of amenability provided in [Po2-5] and in this paper has its own role in understanding various combinatorial and functional analytical aspects of this concept. The main interest in this notion of amenability comes from the fact that the hyperfinite subfactors having amenable graphs are precisely those that can be recovered from their standard invariants and are thus, in particular, completely classified by this invariant (see 7.1, 7.2 later in this paper, and also [Po16]).
2°. Note that in the proof of the Følner condition 5.3.2 for $\Gamma_{N,M}$, from the Kesten-type condition $\|\Gamma_{N,M}\|^2 = [M : N]$ (taken as the definition of the amenability for a graph) we do not actually use the fact that $\Gamma_{N,M}$ is standard, i.e., the fact that it comes from a subfactor. Indeed, the proof goes the same for any weighted bipartite graph (see [Po14] for more comments on this). However, by using the ergodicity property 4.8 of the standard invariant and of its subalgebra generated by the Jones projections, one can prove an interesting sharper Følner type condition for standard graphs. This will be discussed in a forthcoming paper.

5.6 Corollary. (a). *Let $\mathcal{G}$ be a standard λ-lattice and $\mathcal{G}_0$ a sublattice. If $\mathcal{G}_0$ is amenable then $\mathcal{G}$ is amenable. Conversely, if $[\mathcal{G} : \mathcal{G}_0] < \infty$ and $\mathcal{G}$ is amenable then $\mathcal{G}_0$ is amenable.*
(b). *Let $\mathcal{G}_k = \{A_{ij}^k\}_{i,j\geq 0}$ be standard λ_k-lattices with corresponding graphs $\Gamma_{\mathcal{G}_k} = \Gamma_k$, $k = 1, 2$. Let $\mathcal{G}$ denote the system of finite dimensional algebras*

$A_{ij} \stackrel{def}{=} A^1_{ij} \otimes A^2_{ij}$, $i, j \geq 0$, with the tensor product trace. Then $\mathcal{G}$ is a standard $\lambda_1\lambda_2$- lattice, its graph Γ is naturally identified with the tensor product of the graphs Γ_k (regarded as matrices) and we have that $\mathcal{G}$ is amenable if and only if both $\mathcal{G}_1$ and $\mathcal{G}_2$ are amenable.

Proof. (a). The first part follows trivially from the (Kesten-type) definition of amenability, since $\mathcal{G}_0 \subset \mathcal{G}$ implies $\|\Gamma_{\mathcal{G}}\| \geq \|\Gamma_{\mathcal{G}_0}\|$. The second part follows from 2.11, 5.3.4 and [Po8].
(b). The first part follows imediately from the axiomatization of standard lattices in [Po7]. The second part follows from the definition of the amenability, because we have $\|\Gamma\| = \|\Gamma_1\|\|\Gamma_2\|$, so that $(\lambda_1\lambda_2)^{-1} = \|\Gamma\|^2$ iff $\lambda_1^{-1} = \|\Gamma_1\|^2$ and $\lambda_2^{-1} = \|\Gamma_2\|^2$. Note that, by using 2.9.d) and [Po8], this part is an imediate consequence of 5.3.1 as well. Q.E.D.

6. Some More Characterizations of the Amenability for $\Gamma_{N,M}$ and $\mathcal{G}_{N,M}$

In this section we prove several more equivalent characterizations of the amenability for standard graphs and lattices, which clarify some of the results and ideas of the approach to amenability in [Po2,12]. We mention that, while related in spirit with the rest of the paper, the present section will not make explicit use of the symmetric enveloping algebras. So, in this respect, it can be regarded as a digression.
To state the first result, recall that if $B \subset A$ is an inclusion of von Neumann subalgebras of an ambient type II_1 factor then $H(A \mid B)$ denotes its Connes-Størmer relative entropy. By [PiPo1], if $N \subset M$ is an extremal inclusion of type II_1 factors then $H(M \mid N) = \ln([M : N])$. Also, if $N \subset M \subset M_1 \subset \cdots$ is the Jones tower associated to $N \subset M$ then

$$H(M' \cap M_{k+1} \mid M' \cap M_k) \leq H(M_{k+1} \mid M_k) = \ln([M_{k+1} : M_k]) = \ln([M : N]),$$

for all $k \geq 0$. More generally, if p is a projection in $M' \cap M_k$ then by [PiPo1] we have

$$\begin{aligned} H(p(M' \cap M_{k+1})p \,|p(M' \cap M_k)p) & \\ &\leq H(pM_{k+1}p \mid pM_kp) = \ln([pM_{k+1}p : pM_kp]) \\ &= \ln([M_{k+1} : M_k]) = \ln([M : N]) = H(M \mid N). \end{aligned}$$

Similarly, if $N^{\mathrm{st}} \subset M^{\mathrm{st}}$ denotes as usual the "model" inclusion generated by the higher relative commutants, as in [Po2], then the same remark as above shows that $H(pM^{\mathrm{st}}p \mid pN^{\mathrm{st}}p) \leq H(M \mid N)$, $\forall\, p \in \mathcal{P}(N^{\mathrm{st}})$.
The result that follows states that this "upper bound" for the "local relative entropies" is attained precisely when $\mathcal{G}_{N,M}$ (equivalently $\Gamma_{N,M}$) is amenable. Since $H(pM' \cap M_{k+1}p \mid pM' \cap M_kp)$ also represents the conditional entropy from step k to step $k+1$ of the restriction to the support set of p (in K or

L) of the random walk on the graph $\Gamma \circ \Gamma^{\mathrm{t}} \circ \Gamma \circ \Gamma^{\mathrm{t}} \cdots$, for $\Gamma = \Gamma_{N,M}$, with transition probabilities determined by $v = (v_k)_{k \in K}$, this maximality condition on the entropy can be interpreted as a *local Shanon-McMillan-Breimann type condition*, in the same spirit as 5.3.5 in [Po2].

6.1. Theorem. *Let $N \subset M$ be an extremal inclusion of type* II_1 *factors. The following conditions are equivalent.*
1) $\mathcal{G}_{N,M}$ *is amenable.*
2) $\forall\ \varepsilon > 0$, $\exists\ n \geq 1$ *and* $p \in \mathcal{P}(M' \cap M_n)$ *such that*

$$\|E_{(pM'\cap M_{n+1}p)' \cap (pM'\cap M_{n+2}p)}(e_{n+2}p) - \lambda p\|_2 < \varepsilon \|p\|_2.$$

3) $\forall\ \varepsilon > 0$, $\exists\ p \in \mathcal{P}(N_1^{\mathrm{st}})$ *such that*

$$\|E_{pN^{\mathrm{st}}p' \cap pM^{\mathrm{st}}p}(e_0 p) - \lambda p\|_2 < \varepsilon \|p\|_2.$$

4)

$$\lim_k \sup_{p \in \mathcal{P}(M' \cap M_k)} H(pM' \cap M_{k+1}p \mid pM' \cap M_k p) = H(M \mid N) = \ln([M:N]).$$

5)

$$\sup_{p \in \mathcal{P}(N^{\mathrm{st}})} H(pM^{\mathrm{st}}p \mid pN^{\mathrm{st}}p) = H(M \mid N).$$

Proof. First of all, note that since by [Po2] we have that $\Gamma_{N,M}$ is amenable if and only if Γ_{M,M_1} is amenable, it is suficient to prove the above equivalences in the case n is even in condition 2) and the k's are taken odd in condition 4).
1) $\Longrightarrow$ 2). If $\mathcal{G}_{N,M}$ is amenable then by 5.3 its graph $\Gamma_{N,M}$ verifies the Følner condition 5.3.2). Thus, $\forall\ \varepsilon > 0$, $\exists\ F \subset K$ finite non-empty such that

$$\sum_{k \in \partial F} v_k^2 < (\varepsilon/2) \sum v_k^2.$$

Let $n_0 \geq 1$ be such that $F' \overset{\mathrm{def}}{=} F \cup \partial F$ is included in K_n, $\forall\ n \geq n_0$.
For each $n \geq n_0$ let $\{p_k^n\}_{k \in K_n}$ be the list of minimal central projections of $M' \cap M_{2n}$. Note that $\forall\ k \in K$ we have

$$\lim_{n \to \infty} \dim(M' \cap M_{2n} p_k^n) = \infty.$$

Let $\delta > 0$. Let $\{m_k\}_{k \in F'}$ be positive integers such that

$$(*) \qquad \left| \frac{m_k}{m_{k'}} - \frac{v_k}{v_{k'}} \right| < \delta \min\{v_r/v'_r \mid r, r' \in F'\}, \qquad \forall\ k, k' \in F'.$$

Fix $n \geq n_0$ large enough such that

$$\dim(M' \cap M_{2n} p_k^n) \geq m_k^2, \qquad \forall\ k \in F'.$$

Then for each $k \in F'$ choose $q_k \in \mathcal{P}(M' \cap M_{2n}p_k^n)$ such that $\dim(q_k M' \cap M_{2n}q_k) = m_k^2$. Let $p \overset{\text{def}}{=} \sum_{k\in F'} q_k$. We will show that, for $\delta > 0$ small enough, p satisfies condition 2).

To this end denote by $G = \Gamma^{\rm t} F'$ the set of simple summands of $pM' \cap M_{2n+1}p$ and by $\{\bar{q}_l\}_{l\in G}$ the corresponding minimal central projections. Let also $\{s_k\}_{k\in F'}$, $\{t_l\}_{l\in G}$ denote the traces of the minimal projections in $pM' \cap M_{2n}p$ and respectively $pM' \cap M_{2n+1}p$.

Thus, if $\Gamma = (a_{kl})_{k\in K, l\in L}$ then for each $k \in F$, $l \in G$ with $a_{kl} \neq 0$ we have $t_l = \lambda \sum_{k'\in K} a_{k'l}s_{k'} = \lambda \sum_{k'\in F'} a_{k'l}s_{k'}$. Also, if we denote by $n_l^2 = \dim(\bar{q}_l M' \cap M_{2n+1}\bar{q}_l)$ and ${m'}_k^2 = \dim(q'_k M' \cap M_{2n+2}q'_k)$, where $\{q'_k\}_{k\in F''}$, $F'' = F' \cup \partial F'$, are the minimal central projections of $pM' \cap M_{2n+2}p$, then $n_l = \sum_{k'\in F'} a_{k'l}m_{k'}$ and $m'_k = \sum_{k''\in F'} b_{kk''}m_{k''}$, where $(b_{kk'})_{k,k'\in K} = \Gamma\Gamma^{\rm t}$.

From $(*)$ it follows that for $k \in F$ and $l \in G$ with $a_{kl} \neq 0$ we have the estimates:

$$\begin{aligned}
\left|\frac{t_l}{s_k} - \frac{n_l}{m'_k}\right| &= \left|\lambda \sum_{k'\in F'} a_{k'l}s_{k'}/s_k - \sum_{k'\in F'} a_{k'l}m_{k'} \Big/ \sum_{k''\in F'} b_{kk''}m_{k''}\right| \\
&\leq \left|\lambda \sum_{k'\in F'} a_{k'l}s_{k'}/s_k - \sum_{k'\in F'} a_{k'l}s_{k'} \Big/ \sum_{k''\in F'} b_{kk''}s_{k''}\right| + f(\delta) \\
&= \left|\lambda \sum_{k'\in F'} a_{k'l}s_{k'}/s_k - \sum_{k'\in F'} a_{k'l}s_{k'}/\lambda^{-1}s_k\right| + f(\delta) \\
&= f(\delta)
\end{aligned}$$

where $f(\delta) \to 0$ as $\delta \to 0$ and in which we used that for $k \in F$ we have

$$(\Gamma\Gamma^{\rm t}(s_{k''})_{k''\in F'})_k = \lambda^{-1}s_k.$$

With these estimates in mind recall that, with the above notations, we have (see e.g. Sec.6 in [PiPo1]):

$$\begin{aligned}
E_{(pM'\cap M_{2n+1}p)'\cap(pM'\cap M_{2n+2}p)}(e_{2n+2}p) &= \sum_{k\in F'',\ l\in G} (\tau(q'_k\bar{q}_l)^2/a_{kl}^2\tau(q'_k)\tau(\bar{q}_l))\bar{q}_l q'_k \\
&= \sum_{k\in F'',\ l\in G} (\lambda s_k n_l/m'_k t_l)\bar{q}_l q'_k.
\end{aligned}$$

But from the above estimates we see that for all $k \in F$ and $l \in G$ with $a_{kl} \neq 0$ we have:

$$|\lambda s_k n_l/m'_k t_l - \lambda| < f'(\delta)$$

where $f'(\delta) \to 0$ as $\delta \to 0$.

This would finish the proof if we could show that the trace of the sum of the projections $\bar{q}_l q'_k$ for $l \in G$ and $k \in F'' \setminus F$ is small with respect to the trace of p. To show this, it is sufficient to show that $\sum_{k\in F''-F}\tau(q'_k)$ is small with respect to $\tau(p)$. To this end, note first that we have

$$\begin{aligned}\sum_{k\in F}\tau(q'_k) &= \sum_{k\in F}\lambda m'_k t_k = \lambda\sum_{k\in F}\left(\sum_{k''\in F'} b_{kk''}m_{k''}\right)t_k\\ &\geq \lambda\sum_{k\in F}\sum_{k''\in F'} b_{kk''}m_k t_{k''} - \lambda\delta\sum_{k\in F,\ k''\in F'} b_{kk''}m_k t_k\\ &\geq \sum_{k\in F}\tau(q_k) - \lambda^{-2}\delta\sum_{k\in F}\tau(q_k)\end{aligned}$$

in which we first used $(*)$ and then the fact that $\sum_{k''\in K} b_{kk''}\leq\lambda^{-3}$, $\forall\ k\in K$ (see e.g., [Po3], page 281). Thus we get:

$$\begin{aligned}\sum_{k\in F''-F}\tau(q'_k) &= \sum_{k\in F''}\tau(q'_k) - \sum_{k\in F}\tau(q'_k)\\ = \tau(p) - \sum_{k\in F}\tau(q'_k) &\leq \tau(p) - \sum_{k\in F}\tau(q_k) + \lambda^{-2}\delta\sum_{k\in F}\tau(q_k)\\ &\leq \sum_{k'\in\partial F}\tau(q_{k'}) + \lambda^{-2}\delta\tau(p).\end{aligned}$$

But by applying $(*)$ again we also have

$$\begin{aligned}\sum_{k'\in\partial F}\tau(q_{k'})\Big/\tau(p) &= \sum_{k'\in\partial F}\tau(q_{k'})\Big/\sum_{k\in F'}\tau(q_k) \leq \sum_{k'\in\partial F}\tau(q_{k'})\Big/\sum_{k\in F}\tau(q_k)\\ &= \sum_{k'\in\partial F} m_{k'}s_{k'}\Big/\sum_{k\in F} m_k s_k = \sum_{k'\in\partial F}\left(s_{k'}\Big/\sum_{k\in F}\frac{m_k}{m_{k'}}s_k\right)\\ &\leq (1-\delta)^{-1}\sum_{k'\in\partial F}\left(s_{k'}\Big/\sum_{k\in F}\frac{v_k}{v_{k'}}\cdot s_k\right)\\ &= (1-\delta)^{-1}\sum_{k'\in\partial F} v_{k'}^2\Big/\sum_{k\in F} v_k^2 < (1-\delta)^{-1}\varepsilon/2.\end{aligned}$$

Altogether we get:

$$\begin{aligned}&\|E_{(pM'\cap M_{2n+1}p)'\cap(pM'\cap M_{2n+2}p)}(e_{2n+2}p) - \lambda p\|_2^2\\ &\qquad\leq f'(\delta)^2\tau(p) + \lambda^{-2}\delta\tau(p) + ((1-\delta)^{-1}\varepsilon^2/2)\tau(p)\\ &= (f''(\delta) + \varepsilon^2/2)\tau(p).\end{aligned}$$

Thus, if δ is chosen sufficiently small to make $f''(\delta) < \varepsilon^2/2$ then the above is majorized by $\varepsilon^2\tau(p)$, thus finishing the proof of 1) $\Longrightarrow$ 2).

Now, 2) $\Longleftrightarrow$ 3) is trivial by the definition of $N^{\mathrm{st}} \subset M^{\mathrm{st}}$. Also 4 $\Longleftrightarrow$ 5) is clear from the continuity properties of the relative entropy under commuting square conditions ([PiPo1]).
Then 2) $\Longrightarrow$ 4) follows from (4.2 in [PiPo1]).
Finally, to prove 4) $\Longrightarrow$ 1), recall from [PiPo2] that if $B \subset A$ is an inclusion of finite dimensional algebras with inclusion matrix T then $\mathrm{Ind}(E_A^B) \geq \|T\|^2 \geq \exp(H(B \mid A))$. Since the inclusion matrix T of $pM' \cap M_{2n+1}p \subset pM' \cap M_{2n+2}p$ is a restriction of $\Gamma_{N,M}$, we have

$$\|\Gamma_{N,M}\|^2 \geq \|T\|^2 \geq \exp(H(pM' \cap M_{2n+2}p \mid pM' \cap M_{2n+1}p)).$$

Thus, if the right hand side term can be made arbitrarily close to $\exp(H(M \mid N)) = [M : N]$ then we obtain $\|\Gamma_{N,M}\|^2 = [M : N]$, i.e., $\mathcal{G}_{N,M}$ follows amenable. Q.E.D.

6.2. Notation. We denote by $\tilde{M}$ the bicommutant of M in its enveloping algebra M_∞, i.e., $\tilde{M} = (M' \cap M_\infty)' \cap M_\infty$. Similarily we put $\tilde{N} = (N' \cap M_\infty)' \cap M_\infty$ and more generally $\tilde{M}_i = (M_i' \cap M_\infty)' \cap M_\infty$, $i \in \mathbb{Z}$, $\{M_i\}_{i \in \mathbb{Z}}$ being as usual a Jones tunnel-tower for $N \subset M$ and $M_0 = M, M_{-1} = N, M_{-n} = N_{n-1}, n \geq 2$. Note that there exists a unique conditional expectation $\tilde{E}$ from $\tilde{M}$ onto $\tilde{N}$ defined by $\tilde{E}(X) = \lambda \Sigma_j m_j X m_j^*$, for $X \in \tilde{M}$, $\{m_j\}_j$ being any orthonormal basis of $N' \cap M_\infty$ over $M' \cap M_\infty$ (e.g., an orthonormal basis of $\mathrm{vN}\{e_k\}_{k \geq 1}$ over $\mathrm{vN}\{e_k\}_{k \geq 2}$ will do, as the definition of $\tilde{E}$ is anyway easily seen to be independent of the choice of $\{m_j\}_j$) and that $\tilde{E}$ is implemented by e_1, i.e., $e_1 X e_1 = \tilde{E}(X) e_1$ (see Sec. 2.2 in [Po2] or 6.9 in [Po6]). The inclusion $\tilde{N} \overset{\tilde{E}}{\subset} \tilde{M}$ is in fact homogeneous λ-Markov in the sense of (1.2.3 and 1.2.11 of [Po3]) and we have a non-degenerate commuting square

$$\begin{array}{ccc} \tilde{N} & \overset{\tilde{E}}{\subset} & \tilde{M} \\ \cup & & \cup \\ N & \overset{E_N}{\subset} & M \end{array}$$

It should also be noted that, while $\tilde{E}(Y_1 e_0 Y_2) = \lambda Y_1 Y_2, \forall Y_{1,2} \in \tilde{N}$ (this relation can in fact be taken as the definition of $\tilde{E}$), in general $\tilde{E}$ is not trace preserving. In fact, one can easily show (see the proof of 6.4 hereafter) that it is trace preserving if and only if $\tilde{M} = M$, i.e., when the bicommutant relation holds true, $(M' \cap M_\infty)' \cap M_\infty = M$, equivalently when $\Gamma_{N,M}$ is strongly amenable (cf. 5.3.1 in [Po2]).
The Jones tower-tunnel of the above commuting square is obtained by defining the conditional expectations $\tilde{E}_i$ from $\tilde{M}_{i-1}$ onto $\tilde{M}_{i-2}$ in a similar manner with $\tilde{E}$.
Recalling from ([Po2]) that a representation $\mathcal{N} \overset{\mathcal{E}}{\subset} \mathcal{M}$ of $N \subset M$ is *smooth* if $N' \cap M_n \subset \mathcal{N}' \cap \mathcal{M}_n, \forall n$, note that by its construction, $\tilde{N} \subset \tilde{M}$ is obviously a smooth representation of $N \subset M$.

6.3. Theorem. *Let $N \subset M$ be an extremal inclusion of type* II_1 *factors. The following conditions are equivalent:*
1) $\Gamma_{N,M}$ *is amenable.*
2) *There exists a (possibly singular) trace* ψ *on* $\tilde{M}$ *such that* $\psi \circ \tilde{E} = \psi$.
3) *There exists a norm one projection of* $\tilde{N} \overset{\tilde{E}}{\subset} \tilde{M}$ *onto* $N \subset M$.
4) *If*

$$\begin{array}{ccc} \mathcal{N} & \subset & \mathcal{M} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

is a smooth representation of $N \subset M$ *such that there exists a norm one projection of* $\mathcal{M}$ *onto* M *(equivalently, a* M*-hypertrace on* $\mathcal{M}$*), then there exists a norm one projection of* $\mathcal{N} \overset{\mathcal{E}}{\subset} \mathcal{M}$ *onto* $N \subset M$ *(equivalently, a* $N \subset M$*-hypertrace on* $\mathcal{N} \overset{\mathcal{E}}{\subset} \mathcal{M}$*).*
5) *For any smooth representation of* $N \subset M$ *into an inclusion of type* II_1 *von Neumann algebras* $\mathcal{N} \overset{\mathcal{E}}{\subset} \mathcal{M}$*, there exists a norm one projection of* $\mathcal{N} \overset{\mathcal{E}}{\subset} \mathcal{M}$ *onto* $N \subset M$ *(equivalently, a* $N \subset M$*-hypertrace on* $\mathcal{N} \overset{\mathcal{E}}{\subset} \mathcal{M}$*).*

Proof. 1) $\Longrightarrow$ 2) By Theorem 6.1 (see condition 6.1.3) applied to $M \subset M_1$ and the anti-isomorphism between $N_1^{\mathrm{st}} \subset N^{\mathrm{st}} \overset{e_0}{\subset} M^{\mathrm{st}}$ and $M' \cap M_\infty \subset N' \cap M_\infty \overset{e_0}{\subset} N_1' \cap M_\infty$, it follows that there exist projections $p_n \in M' \cap M_\infty$ such that

$$\|E_{(p_n N' \cap M_\infty p_n)' \cap (p_n N_1' \cap M_\infty p_n)}(e_0 p_n) - \lambda p_n\|_2 / \|p_n\|_2 \leq 2^{-n}, \qquad \forall\, n.$$

We then define on M_∞ the state $\varphi \overset{\mathrm{def}}{=} \lim_{n\to\omega} \tau(p_n)^{-1}\tau(\,\cdot\, p_n)$. Note that, since $p_n \in M' \cap M_\infty$, we have $[p_n, (M' \cap M_\infty)' \cap M_\infty] = 0$, in other words $[p_n, \tilde{M}] = 0$. Thus, $[\varphi, \tilde{M}] = 0$, in particular $\varphi|_{\tilde{M}}$ is a trace. Moreover, by noting that $\tau(\,\cdot\, p_n) = \tau(E_{p_n B p_n}(\,\cdot\,) p_n)$ for any von Neumann subalgebra $B \subset M_\infty$ with $p_n \in B$, taking $B = (p_n N' \cap M_\infty p_n)' \cap p_n M_\infty p_n$ and using the above and the Cauchy-Schwartz inequality it follows that for all $x, y \in \tilde{N}$ we have:

$$\begin{aligned} &|\tau(x e_0 y p_n)/\tau(p_n) - \lambda \tau(x y p_n)/\tau(p_n)| \\ &\quad = |\tau(p_n y x e_0 p_n)/\tau(p_n) - \tau(p_n x y \lambda p_n)/\tau(p_n)| \\ &\quad = |\tau(E_{p_n B p_n}(p_n y x e_0 p_n)/\tau(p_n) - \tau(p_n x y \lambda p_n)/\tau(p_n)| \\ &\quad = \tau(p_n y x (E_{p_n B p_n}(e_0 p_n) - \lambda p_n))/\tau(p_n)) \\ &\quad \leq \|p\|_2 \|yx\|\ \|E_{p_n B p_n}(e_0) - \lambda p_n\|_2/\tau(p_n) \\ &\quad \leq 2^{-n}\|yx\|. \end{aligned}$$

Since $\tilde{E}(x e_0 y) = \lambda x y$, $\forall\, x, y \in \tilde{N}$, and $\mathrm{sp}\tilde{N} e_0 \tilde{N} = \tilde{M}$, it follows that

$$\lim_{n\to\infty} \|\tau(X p_n)/\tau(p_n) - \tau(\tilde{E}(X) p_n)/\tau(p_n)| = 0 \qquad \forall\, X \in \tilde{M}.$$

Thus, $\varphi(X) = \varphi(\tilde{E}(X))$, $\forall\ X \in \tilde{M}$. All this shows that $\psi \stackrel{\text{def}}{=} \varphi|_{\tilde{M}}$ is both a trace and satisfies $\psi = \psi \circ \tilde{E}$.
2) $\Longrightarrow$ 3). Since ψ is a trace on $\tilde{M}$, it is in particular a M-hypertrace and $\psi = \psi \circ \tilde{E}$ implies it is actually a $(N \subset M)$-hypertrace on $(\tilde{N} \stackrel{\tilde{E}}{\subset} \tilde{M})$; equivalently, there exists a conditional expectation of $\tilde{N} \subset \tilde{M}$ onto $N \subset M$.
3) $\Longrightarrow$ 4). If there exists a conditional expectation Φ of $\mathcal{M}$ onto M then by amplification it follows that there exist conditional expectations Φ_{2n} of $\mathcal{M}_{2n}$ onto M_{2n}, $\forall\ n \geq 0$. Let $\mathcal{F}_{2n} : \cup_k \mathcal{M}_k \to \mathcal{M}_{2n}$ be the conditional expectation implemented by $\cdots \circ \mathcal{E}_{2n+2} \circ \mathcal{E}_{2n+1}$ and denote $\Psi_{2n} : \cup_k \mathcal{M}_k \to M_\infty$ the aplications defined by $\Psi_{2n}(X) = \Phi_{2n} \circ \mathcal{F}_{2n}(X) \in M_{2n} \subset M_\infty$. Note that Ψ_{2n} is M_{2n}-M_{2n} linear. Finally, we put $\Psi(X) \stackrel{\text{def}}{=} \lim_{n\to\omega} \psi_{2n}(X)$, for $X \in \cup_k \mathcal{M}_k$, where ω is a free ultrafilter on $\mathbb{N}$. Thus, $\Psi(1) = 1$ and Ψ is M_{2n}-M_{2n} linear $\forall\ n$. Since the representation of $N \subset M$ into $\mathcal{N} \subset \mathcal{M}$ is smooth, $M' \cap M_j \subset \mathcal{M}' \cap \mathcal{M}_j$, $\forall\ j$. Thus, if $X \in \mathcal{M}$ then $[X, M' \cap M_j] = 0$ and by applying ψ we get $[\Psi(X), M' \cap M_j] = 0$. Thus, $\psi(X) \subset (\cup_j M' \cap M_j)' \cap M_\infty = \tilde{M}$. Similarly, we obtain that if $X \in \mathcal{N}$ then $\Psi(X) \in \tilde{N}$. But by 3) we have a conditional expectation of $\tilde{M}$ onto M, say Ψ_0, such that $\Psi_0(\tilde{N}) = N$.
We then define $\Psi_1 : \mathcal{M} \to M$ by $\Psi_1(X) = \Psi_0(\Psi(X))$, which is a conditional expectation and satisfies $\Psi_1(\mathcal{N}) = \Psi_0(\psi(\mathcal{N})) \subset \Psi_0(\tilde{N}) \subset N$.
4) $\Longrightarrow$ 5). Since $\mathcal{M}$ has projections $p \in \mathcal{Z}(\mathcal{M})$ such that $\mathcal{M}p$ is finite, it follows that there is a conditional expectation of $\mathcal{M}p$ onto $Mp \simeq M$, thus of $\mathcal{M}$ onto M and so 4) applies.
5) $\Longrightarrow$ 1) If 5) holds true then in particular there exists a norm one projection from the finite standard representation onto $N \subset M$, so by Theorem 5.3 we have 1). Q.E.D.

Recall from [Po2] that a standard λ-graph $(\Gamma, \vec{s})$ is called *ergodic* if $\vec{s}$ is the unique $\vec{s}$-bounded eigenvector for $\Gamma\Gamma^t$ corresponding to the eigenvalue λ^{-1}, equivalently, if $\mathcal{Z}(A_{0,\infty}) = \mathbb{C}$, where $A_{0,\infty}$ is the finite von Neumann algebra obtained as an inductive limit with the Bratteli diagram given by $\Gamma, \Gamma^t, \Gamma, ...$, starting from the even vertex $*$ of Γ, and having trace given by $\vec{s} = (s_k)_{k\in K}$. Note that if $N \subset M$ is a subfactor having standard graph $(\Gamma, \vec{s})$ then the algebra $A_{0,\infty}$ equals $M' \cap M_\infty$, where $N \subset M = M_0 \subset M_1 \subset ...$ is the Jones tower for $N \subset M$ and $M_\infty = (\cup_n M_n)^-$ as usual.
In what follows we'll call the standard λ-graph almost ergodic if $\dim\mathcal{Z}(A_{0,\infty}) < \infty$. This is equivalent to the fact that, up to scalar multiples, there are only finitely many $\vec{s}$-bounded eigenvectors for $\Gamma\Gamma^t$ corresponding to the eigenvalue λ^{-1} (see the proof of 1.4.2 in [Po2]). Note that Haagerup constructed extremal hyperfinite subfactors of index $\lambda^{-1} = 2 \cdot 4\cos^2\pi/5 = 3 + \sqrt{5}$ which have almost ergodic, but not ergodic, standard graph. The following consequence of 6.3 shows that this cannot happen if Γ is amenable.

6.4. Corollary. *If an amenable, extremal standard graph* $(\Gamma, \vec{s})$ *is almost ergodic then it is ergodic, and thus it is strongly amenable.*

Proof. Let $N \subset M$ be a subfactor having (weighted) standard graph equal to $(\Gamma, \vec{s})$. Denote like in 6.2 by $\tilde{N}_1 = (N_1' \cap M_\infty)' \cap M_\infty, \tilde{N} = (N' \cap M_\infty)' \cap M_\infty, \tilde{M} = (M' \cap M_\infty)' \cap M_\infty$ and by $\tilde{F}$ the expectation from $\tilde{M}$ onto $\tilde{N}_1$ defined as in 6.2 (so that in fact, with the notations there, we also have $\tilde{F} = \tilde{E}_{-1} \circ \tilde{E}_0$). Note that $\mathcal{Z}(\tilde{M}) = \mathcal{Z}(M' \cap M_\infty)$. Since $N_1 \subset M$ has amenable graph $(= \Gamma\Gamma^t)$, by 6.3 there exists a trace τ' on $\tilde{M}$ such that $\tau' \circ \tilde{F} = \tau'$. Since $\dim \mathcal{Z}(\tilde{M}) = \dim \mathcal{Z}(M' \cap M_\infty) < \infty$, it follows that there exists $a \in \mathcal{Z}(\tilde{M})_+$ such that $\tau'(X) = \tau(Xa), \forall X \in \tilde{M}$. Since $\tilde{E}$ is τ'-preserving, this implies that $a = \tilde{F}(a) \in \tilde{F}(\mathcal{Z}(\tilde{M})) = \mathcal{Z}(\tilde{N}_1)$. Thus $a \in \mathcal{Z}(\tilde{M}) \cap \mathcal{Z}(\tilde{N}_1) = \mathbb{C}1$, so $a = 1$ and $\tau' = \tau$.
Thus $\tilde{F}$ coincides with the trace preserving expectation F of $\tilde{M}$ onto $\tilde{N}$. In particular, this implies that $E_{(N_1' \cap M_\infty)' \cap M_\infty}(f) = F(f) = \lambda^2 1$, where $f \in M$ is the Jones projection for $N_1 \subset M$. By duality it follows that $E_{(M_{2j}' \cap M_\infty)' \cap M_\infty}(f_j) = \lambda^2 1$ for any $j \in \mathbb{Z}$, where f_j is the Jones projection for the inclusion $M_{2j} \subset M_{2j+2}$. By (5.3 in [Po2]) it follows that $M = (M' \cap M_\infty)' \cap M_\infty$, so in particular $M' \cap M_\infty$ is a factor, i.e., $(\Gamma, \vec{s})$ is ergodic. Q.E.D.

We now examine the effect of amenability on the universal graph $\Gamma^u_{N,M}$. To this end, let us denote, like in [Po2], by $\mathcal{N}^{u,f} \overset{\mathcal{E}^{u,f}}{\subset} \mathcal{M}^{u,f}$ the direct summand of $\mathcal{N}^u \subset \mathcal{M}^u$ given by all the irreducible representations $\mathcal{B}(\mathcal{H})$ of $M \otimes P^{\text{op}}$ which, when regarded as $M-P$ bimodules, have finite dimension, $\dim({}_M\mathcal{H}_P) < \infty$, P denoting here a generic "dummy" type II_1 factor weakly stably equivalent to M (in the sense of 1.4.3 in [Po8], i.e., P can be embedded with finite index in the amplification by some $\alpha > 0$ of M). Let $\Gamma^{u,f}_{N,M}$ denote its inclusion graph (or matrix). Recall from [Po2] that $\Gamma^{u,f}_{N,M}$ is in a natural way a weighted bipartite graph, the weights being given by the vector $((\dim_{M,P}\mathcal{H})^{1/2})$, which in fact also gives the weights of an $\mathcal{E}^{u,f}$-invariant trace on $\mathcal{M}^{u,f}$.

6.5. Theorem. *Let $N \subset M$ be an extremal inclusion of type II_1 factors. The following conditions are equivalent:*
1) *The standard graph $\Gamma_{N,M}$ is amenable, i.e., $\|\Gamma_{N,M}\|^2 = [M : N]$.*
2) *The graph $\Gamma^{u,f}_{N,M}$ is amenable, i.e., $\|\Gamma^{u,f}_{N,M}\|^2 = [M : N]$.*
3) *Each irreducible component Γ of $\Gamma^u_{N,M}$ satisfies $\|\Gamma\|^2 = [M : N]$.*
4) *For any $\varepsilon > 0$ there exists a subfactor $Q \subset N$, with $[N : Q] < \infty$, such that the inclusion matrix $T_0 = T_{Q' \cap N \subset Q' \cap M}$ satisfies $\|T_0\|^2 \geq [M : N] - \varepsilon$.*
4′) *For any $\varepsilon > 0$ there exists a factor P containing M with $[P : M] < \infty$, such that $\|T_{M' \cap P \subset N' \cap P}\|^2 \geq [M : N] - \varepsilon$.*

Proof. 3) $\Longrightarrow$ 2) is trivial.

2) $\Longrightarrow$ 1). For simplicity of notations, we let $(\mathcal{N} \overset{\mathcal{E}}{\subset} \mathcal{M}) = (\mathcal{N}^{u,f} \overset{\mathcal{E}^{u,f}}{\subset} \mathcal{M}^{u,f})$. Let K' be the set of simple summands of $\mathcal{M}$ and TT^{t} be the inclusion matrix of $\mathcal{M} \subset \mathcal{M}_2$. It follows that $\forall\, \varepsilon > 0$, $\exists\, k_0 \in K'$ such that

$$\lim_{n \to \infty} \|(TT^{\text{t}})^n \delta_{k_0}\|^{1/n} \geq \|TT^t\| - \varepsilon = [M : N] - \varepsilon.$$

But if for each $n \geq 0$ we denote by p_{2n} the minimal central projection in $\mathcal{M}_{2n}$ corresponding to $k_0 \in K'$, then

$$\|(TT^{\mathrm{t}})^n \delta_{k_0}\|^2 = \dim(\mathcal{M}p_0' \cap p_0\mathcal{M}_{2n}p_0) = \dim(\mathcal{M}p_{2n}' \cap \mathcal{M}_{2n}p_{2n}).$$

Moreover, by using that if $R \subset Q \subset P$ are inclusions of type II_1 factors with finite index then $\dim R' \cap P \leq ([P : Q]) \dim R' \cap Q$ (because the inclusion matrix of $R' \cap Q \subset R' \cap P$ has square norm $\leq [P : Q]$), it follows that, with the notations $M^0 = P' \cap \mathcal{M}, M_k^0 = P' \cap \mathcal{M}_k, k \geq 0$, we have:

$$\begin{aligned}\dim(\mathcal{M}p_{2n}' \cap \mathcal{M}_{2n}p_{2n}) &\leq \dim(Mp_{2n}' \cap M_{2n}^0 p_{2n}) \\ &\leq ([M_{2n}^0 p_{2n} : M_{2n}p_{2n}]) \dim(Mp_{2n}' \cap M_{2n}p_{2n}) \\ &= ([M^0 p_0 : Mp_0]) \dim(M' \cap M_{2n}) \\ &= ([M^0 p_0 : Mp_0]) \|(\Gamma_{N,M}\Gamma_{N,M}^{\mathrm{t}})^n \delta_*\|^2.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \|(TT^{\mathrm{t}})^n \delta_{k_0}\|^{1/n} \leq \lim_{n\to\infty} ([\mathcal{M}p_0 : Mp_0])^{1/2n} \|(\Gamma\Gamma^{\mathrm{t}})^n \delta_*\|^{1/n} = \|\Gamma\Gamma^{\mathrm{t}}\|$$

showing that $\|\Gamma\|^2 \geq [M : N] - \varepsilon$. Since ε was arbitrary, $\|\Gamma_{N,M}\|^2 = [M : N]$.
1) $\Longrightarrow$ 3). Let k' be any of the labels corresponding to an even vertex of Γ and let $q \in \mathcal{M}_{2n}^u$ be a minimal central projection corresponding to that same label. Then $\dim((\mathcal{M}^u q)' \cap \mathcal{M}_{2n}^u q) = \|(\Gamma\Gamma^t)^n \delta_{k'}\|^2$. But by smoothness, $(M' \cap M_{2n})q \subset (\mathcal{M}^u q)' \cap \mathcal{M}_{2n}^u q$, thus $\|\Gamma\Gamma^t\| \geq \|\Gamma_{N,M}\Gamma_{N,M}^t\|$.
1) $\Longrightarrow$ 4). This is clear, by simply taking $Q = N_k$, a subfactor in a Jones tunnel, with k large enough.
4) $\Longleftrightarrow$ 4′). This follows immediately by taking into account that if $Q \subset N(\subset M)$ is a subfactor of finite index in N and we denote $Q \subset N \subset M \subset M_1 \subset Q_1$ its basic construction, then $T_{Q' \cap N \subset Q' \cap M} = T_{M_1' \cap Q_1 \subset M' \cap Q_1}$.
4′) $\Longrightarrow$ 2). If $P \supset M$ is as in condition 4′) for some ε then let $p \in \mathcal{Z}(\mathcal{M}^{u,f})$ be the central projection supporting all the $N - M$ bimodules appearing as direct summands in ${}_N L^2(P)_M$. Then clearly $\|\Gamma\| \geq \|\Gamma_p\| \geq \|T_{M' \cap P \subset N' \cap P}\|$. Q.E.D.

6.6. Corollary. *Let $Q \subset N \subset M$ be inclusions of* II_1 *factors with finite index (not necessarily extremal).* (*i*). *If $Q \subset M$ has amenable graph then $Q \subset N$ and $N \subset M$ have amenable graphs.* (*ii*). *If $N \subset M$ has amenable graph and $p \in N' \cap M$ is a projection, then $Np \subset pMp$ has amenable graph.*

Proof. By [L], there exist extremal inclusions $Q_0 \subset N_0 \subset M_0$ such that: a). The higher relative commutants of $Q_0 \subset N_0$, $N_0 \subset M_0$ and respectively $Q_0 \subset M_0$ are algebraically isomorphic to those of $Q \subset N$, $N \subset M$ and respectively $Q \subset M$; so, in particular, the graphs of the induced-reduced algebras in the Jones towers of the corresponding subfactors are equa. b). $[N_0 : Q_0] = \mathrm{Ind}E_{\min}^{Q,N}$, $[M_0 : N_0] = \mathrm{Ind}E_{\min}^{N,M}$ and $[M_0 : Q_0] = \mathrm{Ind}E_{\min}^{Q,M}$ and the local indices in the Jones tower for the inclusions $Q_0 \subset N_0, N_0 \subset M_0$,

respectively $Q_0 \subset M_0$ are the same as for the initial incusions $Q \subset N, N \subset M$, respectively $Q \subset M$. With these in mind, let us prove (i) and (ii).
(i). Let $\Gamma^{u,f}_{Q,N}$, $\Gamma^{u,f}_{N,M}$ be as in [Po2] the inclusion matrices describing the inclusions $(Q \otimes M^{\text{op}})^{**}_{\text{at},f} \subset (N \otimes M^{\text{op}})^{**}_{\text{at},f} \subset (M \otimes M^{\text{op}})^{**}_{\text{at},f}$. Recall from ([Po2]) that $\Gamma^{u,f}_{Q,M} = \Gamma^{u,f}_{Q,N} \circ \Gamma^{u,f}_{N,M}$. Thus, if $\|\Gamma^{u,f}_{Q,M}\|^2 = \text{Ind}E^{Q,M}_{\min}$ then we get

$$\begin{aligned}\text{Ind}E^{Q,N}_{\min} \cdot \text{Ind}E^{N,M}_{\min} &= \text{Ind}E^{Q,M}_{\min} = \|\Gamma^{u,f}_{Q,M}\|^2 \leq \|\Gamma^{u,f}_{Q,N}\|^2 \|\Gamma^{u,f}_{N,M}\|^2 \\ &\leq \text{Ind}E^{Q,N}_{\min} \cdot \text{Ind}E^{N,M}_{\min},\end{aligned}$$

forcing the equalities $\|\Gamma^{u,f}_{Q,N}\|^2 = \text{Ind}E^{Q,N}_{\min}$, $\|\Gamma^{u,f}_{N,M}\|^2 = \text{Ind}E^{N,M}_{\min}$. But by the above considerations and 6.5 this implies $\Gamma_{Q,N}$ and $\Gamma_{N,M}$ are amenable.
(ii). This can be easily deduced from 6.5, by using the universal graphs as in the proof of (i) above. Instead, we'll use the following simpler argument: By the first part of the proof, we may assume $N \subset M$ is extremal. Then by 2.9.c) it follows that the finite standard representation of $Np \subset pMp$ is given by $\mathcal{N}^{\text{st},f} p \overset{\mathcal{E}}{\subset} p\mathcal{M}^{\text{st,f}} p$, where $\mathcal{E}$ is defined by $\mathcal{E}(pXp) = \tau(p)^{-1}\mathcal{E}^{\text{st,f}}(pXp)$, for $X \in \mathcal{M}^{\text{st,f}}$. But then, if Φ is a conditional expectation from $\mathcal{M}^{\text{st,f}}$ onto $\mathcal{N}^{\text{st,f}}$ sending M onto N then clearly Φ also sends $p\mathcal{M}^{\text{st,f}}p$ onto pMp and $\mathcal{N}^{\text{st,f}}p$ onto Np. By 5.3, this implies that $Np \subset pMp$ has amenable graph. Q.E.D.

We mention one last hereditarity property for the amenability of the graphs of subfactors, which has a self-contained and rather elementary proof.

6.7. Proposition. *Let*

$$\begin{matrix} N & \subset & M \\ \cup & & \cup \\ Q & \subset & P \end{matrix}$$

be a nondegenerate commuting square of inclusions of type II_1 *factors with finite index (thus,* $[M : N] = [P : Q] < \infty$, $[M : P] = [N : Q] < \infty$*). Then we have:*
a) $\|\Gamma_{N,M}\| = \|\Gamma_{Q,P}\|$, $H(M \mid N) = H(P \mid Q)$, $\text{Ind}E^{N,M}_{\min} = \text{Ind}E^{Q,P}_{\min}$ *and* $E_{Q'\cap P}(e_0) = E_{N'\cap M}(e_0)$, *where* $e_0 \in P$ *is a Jones projection for* $Q \subset P$ *(and thus for* $N \subset M$ *as well).*
b) $N^{\text{st}} \subset M^{\text{st}}$ *has atomic centers iff* $Q^{\text{st}} \subset P^{\text{st}}$ *has atomic centers.*
c) $\mathcal{G}_{N,M}$ *is amenable (resp. strongly amenable, resp. has finite depth) iff* $\mathcal{G}_{Q,P}$ *is amenable (resp. strongly amenable, resp. has finite depth).*

Proof. Let

$$\begin{matrix} \dots N_1 & \subset & N & \subset & M \\ \cup & & \cup & & \cup \\ \dots Q_1 & \subset & Q & \subset & P \end{matrix}$$

be a tunel for the given commuting square. Then $\dim N'_k \cap M \leq \dim Q'_k \cap M \leq [M : P] \dim Q'_k \cap P$, so that

$$\|\Gamma_{N,M}\|^2 = \lim_{k\to\infty} (\dim N'_k \cap M)^{1/k} \leq \lim_{k\to\infty} (\dim Q'_k \cap P)^{1/k} = \|\Gamma_{Q,P}\|^2.$$

Taking

$$\begin{array}{ccc} \langle N,Q\rangle & \subset & \langle M,P\rangle \\ \cup & & \cup \\ N & \subset & M \end{array}$$

and using that $\langle N,Q\rangle \subset \langle M,P\rangle$ is an amplified of $Q \subset P$ (so that $\Gamma_{\langle N,Q\rangle,\langle M,P\rangle} = \Gamma_{Q,P}$), by the first part we also get $\|\Gamma_{Q,P}\| \geq \|\Gamma_{N,M}\|$, thus $\|\Gamma_{N,M}\| = \|\Gamma_{Q,P}\|$.
Now remark that $E_P(Q_k' \cap M) = Q_k' \cap P$ and $E_P(\mathcal{Z}(Q_k' \cap M)) \subset \mathcal{Z}(Q_k' \cap P)$. Also, we have

$$\begin{aligned} &\mathrm{Ind}(E^{Q_k'\cap M}_{Q_k'\cap P}) \leq [M:P], \\ &\mathrm{Ind}(E^{Q_k'\cap M}_{N_k'\cap M}) \leq [Q_k' : N_k'] = [N_k : Q_k] = [M:P]. \end{aligned}$$

It follows that if we denote $R = \overline{\cup_k Q_k' \cap M}$ then $\mathrm{Ind}(E^R_{P^{\mathrm{st}}}) \leq [M:P]$, $\mathrm{Ind}(E^R_{M^{\mathrm{st}}}) \leq [M:P]$. Thus, P^{st} has atomic center iff R has atomic center iff M^{st} has atomic center.
Also, the above shows that

$$\begin{aligned} \sup_n \dim \mathcal{Z}(N_k' \cap M) < \infty &\Longleftrightarrow \sup_n \dim \mathcal{Z}(Q_k' \cap M) < \infty \\ &\Longleftrightarrow \sup_k \dim \mathcal{Z}(Q_k' \cap P) < \infty. \end{aligned}$$

Thus, $N \subset M$ has finite depth iff $Q \subset P$ has finite depth.
Since $Q \subset P$ is embedded as a commuting square in $N \subset M$, by the definition of relative entropy we have $H(P \mid Q) \leq H(M \mid N) \leq H(\langle M,P\rangle \mid \langle N,Q\rangle) = H(P \mid Q)$, thus, $H(M \mid N) = H(P \mid Q)$.
Next, if $e_0 \in P$ is a Jones projection then

$$E_{N'\cap M}(e_0) = E_{N'\cap M}(E_{Q'\cap M}(e_0)) = E_{N'\cap M}(E_{Q'\cap P}(e_0))$$

so that $\|E_{N'\cap M}(e_0)\|_2 \leq \|E_{Q'\cap P}(e_0)\|_2$ with equality iff $E_{N'\cap M}(e_0) = E_{Q'\cap P}(e_0)$. But $N \subset M$ is embedded as a commuting square in $\langle N,Q\rangle \subset \langle M,P\rangle$ which is an amplified of $Q \subset P$, so we get similarly $\|E_{Q'\cap P}(e_0)\|_2 \leq \|E_{N'\cap M}(e_0)\|_2$ giving $E_{N'\cap M}(e_0) = E_{Q'\cap P}(e_0)$.
To prove the statemnt about the minimal index, note from the formula of the Jones projection in ([PiPo1], page 83-84) that $E^{N,M}_{\min} = E^M_N(b^{1/2} \cdot b^{1/2})$ with $b \in \mathrm{Alg}\{E_{N'\cap M}(e_0)\} = \mathrm{Alg}\{E_{Q'\cap P}(e_0)\}$. Thus, $b \in P$ and $E^{N,M}_{\min}(P) = Q$, implying that $\mathrm{Ind}(E^{N,M}_{\min}) \geq \mathrm{Ind}(E^{Q,P}_{\min})$. Similarily, $\mathrm{Ind}(E^{N,M}_{\min}) \leq \mathrm{Ind}(E^{\langle N,Q\rangle,\langle M,P\rangle}_{\min})$ Thus, $\mathrm{Ind}E^{M,N}_{\min} = \mathrm{Ind}E^{P,Q}_{\min}$.
From the above, it follows in particular that $\mathrm{Ind}E^{N,M}_{\min} = \|\Gamma_{N,M}\|^2$ iff $\mathrm{Ind}E^{Q,P}_{\min} = \|\Gamma_{Q,P}\|^2$ so $\Gamma_{N,M}$ is amenable iff $\Gamma_{Q,P}$ is amenable (without the extremality assumtion required).

If $\Gamma_{N,M}$ is amenable and M^{st} is a factor (i.e., $\mathcal{G}_{N,M}$ is strongly amenable) then $\Gamma_{Q,P}$ is amenable and P^{st} has finite dimensional center. Thus 6.4 applies to get that $\Gamma_{Q,P}$ follows strongly amenable. Alternatively, and in order to keep the proof of this Proposition elementary and self-contained, note that the same proof as on (pages 235 and 183 of [Po2]) can be used to get the same conclusion, i.e., that P^{st} follows a factor and thus $\Gamma_{Q,P}$ strongly amenable. Q.E.D.

7. Hyperfiniteness of $M \underset{e_N}{\boxtimes} M^{\text{op}}$ and Hereditarity of the Amenability for Subfactors

Recall from [Po2] that an inclusion of factors $N \subset M$ is called *amenable* if it is the range of a norm one projection from any of its smooth representations, equivalently, if the algebras N, M are themselves amenable (i.e., hyperfinite by [C1]) and the graph $\Gamma_{N,M}$ is amenable ([Po2,3,4]), i.e., $\|\Gamma_{N,M}\|^2 = \text{Ind}E_{\min}^{N,M}$. In this section we will show that, in the case the inclusion $N \subset M$ is extremal, the amenability of $N \subset M$ is in fact equivalent to the hyperfiniteness of its symmetric enveloping algebra. We will then derive that the amenability of an inclusion is inherited by its "sub-inclusions"

7.1. Theorem. *Let $N \subset M$ be an extremal inclusion of type* II_1 *factors. The following conditions are equivalent:*
1) *$N \subset M$ is amenable.*
2) *$\Gamma_{N,M}$ is amenable and M is hyperfinite.*
3) *$\forall\ x_1, \dots, x_m \in M$, $\forall\ \varepsilon > 0$, $\exists\ n$, a projection f in $N' \cap M_n$, a subfactor $P \subset N$ such that $Pf \subset Nf \subset fM_nf$ is a basic construction and a finite dimensional subfactor $Q_0 \subset P$ such that*

$$x_i \in_\varepsilon Q_0 \vee (P' \cap M), \qquad i = 1, 2, \dots, m.$$

4) *$\forall\ x_1, \dots, x_m \in M$, $\forall\ \varepsilon > 0$, $\exists\ Q \subset N$ with $[N : Q] < \infty$ such that*

$$x_i \in_\varepsilon Q' \cap M, \qquad i = 1, 2, \dots, m.$$

5) *$M \underset{e_N}{\boxtimes} M^{\text{op}}$ is isomorphic to the hyperfinite type* II_1 *factor.*
6) *There exists a $M \underset{e_N}{\boxtimes} M^{\text{op}}$-hypertrace on $\mathcal{B}(L^2(M \underset{e_N}{\boxtimes} M^{\text{op}}))$.*
7) *There exists a $(N \subset M)$-hypertrace on $\mathcal{N}^{\text{st}} \subset \mathcal{M}^{\text{st}}$ (equivalently, a norm one projection of $\mathcal{N}^{\text{st}} \subset \mathcal{M}^{\text{st}}$ onto $N \subset M$).*

Proof. We will prove 1) $\Longrightarrow$ 7) $\Longrightarrow$ 2) $\Longrightarrow$ 3) $\Longrightarrow$ 4) $\Longrightarrow$ 5) $\Longrightarrow$ 6) $\Longrightarrow$ 7) and 2) $\Longrightarrow$ 1).
The implication 1) $\Longrightarrow$ 7) is trivial, as $\mathcal{N}^{\text{st}} \subset \mathcal{M}^{\text{st}}$ is just a particular case of a smooth representation.
If 7) is satisfied then by [Po13] we have $\|\Gamma_{N,M}\|^2 = [M : N]$ and N, M follow amenable (as ranges of norm one projections from the amenable von Neumann algebras $\mathcal{N}^{\text{st}}$, $\mathcal{M}^{\text{st}}$). Thus we have 7) $\Longrightarrow$ 2).

2) $\implies$ 3). This is essentially (4.4.1 in [Po2]), or the proof of (4.1 in [Po4], up to Step VI on page 291), with some changes and additional considerations that we explain below.

Like in the proof of 1) $\implies$ 2) in Theorem 6.1, we let F be an ε'-Følner set for $\Gamma_{N_1,N}$ (by 2) we have that $\Gamma_{N,M}$ is amenable, equivalently $\Gamma_{N_1,N}$ is amenable), then we choose a large n and some integers $m_k \leq (\dim N' \cap M_{2n+1} p_k^{n+1})^{1/2}$ such that

$$|\frac{m_k}{m_{k'}} - \frac{v_k}{v_{k'}}| < \delta \qquad \forall\ k, k' \in F, \tag{1}$$

where $\{p_k^{n+1}\}_k$ is now the list of minimal central projections in $N' \cap M_{2n+1}$ and $\vec{v} = (v_k)_{k \in K}$ is the standard vector of local indices (at even levels) for $\Gamma_{N_1,N}$. We then take $q_k \in \mathcal{P}(N' \cap M_{2n+1} p_k^{n+1})$ such that $\dim(q_k N' \cap M_{2n+1} q_k) = m_k^2$ and define $p = \sum_{k \in F} q_k$.

Let then $P_0 \subset N$ be a downward basic construction for $Np \subset pM_{2n+1}p$. By the choice of F (i.e., satisfying $\sum_{k \in F \cup \partial F} v_k^2 \leq (1+\varepsilon') \sum_{k \in F} v_k^2$) it follows that if $\{\bar{x}_j\}_j$ is an orthonormal basis of N over $P_0 \vee P_0' \cap N$ then $\{\bar{x}_j\}_j$ is almost an orthonormal basis of M over $P_0 \vee P_0' \cap M$ as well. Moreover, by the choice of integers $\{m_k\}_{k \in F}$ it follows that

$$\sum_j \tau(E_{P \vee P' \cap N}(\bar{x}_j^* \bar{x}_j) \bar{p}_{k_0}) / \tau(\bar{p}_{k_0}) \approx \sum_{k \in F} v_k^2, \qquad \forall\ k_0 \in F, \tag{2}$$

$\{\bar{p}_k\}_{k \in F}$ being the minimal central projections in $P_0' \cap N$. Also, since P_0 is a type II$_1$ factor, we may assume $E_{P_0 \vee P_0' \cap N}(\bar{x}_j^* \bar{x}_j) \bar{p}_k \in P_0 \bar{p}_k$, $\forall\ k \in F$. But then, by using first the approximate innerness of $Np \subset pM_{2n+1}p$ then the central freeness of $P_0 \subset M$, like in (Steps I, II, III in the proof of 4.1 in [Po4]), we obtain a conjugate of P_0 by a unitary element in N, say P_1, such that we have the type of estimates (a)–(f) on page 285 of [Po4] with P_1 instead of N_{m_0}. Then we go through Step IV on pages 286–288 of [Po4], noting that due to the condition (2) above, we don't need to take a further tunnel and that taking P_1 for N_m will do.

Then Step V on page 289 can be taken unchanged. Altogether, after doing all this we end up obtaining the following: $\forall\ x_1, \dots, x_l \in M$, $\forall\ \varepsilon > 0$, if $F \subset \text{Even}(\Gamma_{N_1,N})$ is a ε-Følner set, n is sufficiently large and $\{m_k\}_{k \in F}$ satisfy (1) with δ suficiently small, then there exists a choice of a downward basic construction $P_1 \subset N$ for $Np \subset pM_{2n+1}p$, where $p = \sum_{k \in F} q_k$ as above, and a projection $s_0 \in P_1$, such that for all $1 \leq i \leq l$ we have

$$\begin{gathered} \|[s_0, x_i]\|_2 < f(\varepsilon') \|s_0\|_2, \\ \|s_0 x_i s_0 - E_{s_0(P_1 \vee P_1' \cap M)s_0}(s_0 x_i s_0)\|_2 < f(\varepsilon') \|s_0\|_2 \end{gathered} \tag{3}$$

where $f(\varepsilon') \to 0$ as $\varepsilon' \to 0$.

Arguing like in Step VI on page 290 of [Po4] we obtain a family of such choices of downward basic constructions $(P_i)_{i \in I}$ with projections $(s_i)_{i \in I}$, $s_i \in P_i$, such that (P_i, s_i) satisfy (3) and $\sum_i s_i = 1$. But then there exists a downward basic construction $P \subset N$ for $Np \subset pM_{2n+1}p$ such that $s_i \in P$, $\forall\ i$, and $s_i P s_i = s_i P_i s_i$, $\forall\ i$. Thus P will satisfy

$$\|E_{P \vee P' \cap M}(x_i) - x_i\|_2 < f(\varepsilon'), \qquad 1 \leq i \leq n.$$

Since P is hyperfinite, by taking ε' so that $f(\varepsilon') < \varepsilon$ we get 7.1.3).
3) $\Longrightarrow$ 4) is trivial, by simply taking $Q = Q_0' \cap P$ in 3).
4) $\Longrightarrow$ 5). since $\mathrm{Alg}(M, e_N, M^{\mathrm{op}})$ is so-dense in $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ it is sufficient to prove that $\forall\ x_1, x_2, \dots, x_n \in M$, $\forall\ \varepsilon > 0$, $\exists\ B \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ finite dimensional such that

$$\|E_B(x_i) - x_i\|_2 < \varepsilon$$
$$\|E_B(e_N) - e_N\|_2 < \varepsilon$$
$$\|E_B(x_i^{\mathrm{op}}) - x_i^{\mathrm{op}}\|_2 < \varepsilon.$$

By 4) there exists $Q \subset N$ with $[N : Q] < \infty$ such that $x_i \in_\varepsilon Q' \cap M$. But

$$[Q^{\mathrm{op}\prime} \cap M \underset{e_N}{\boxtimes} M^{\mathrm{op}} : M^{\mathrm{op}\prime} \cap M \underset{e_N}{\boxtimes} M^{\mathrm{op}}] \leq [M^{\mathrm{op}} : Q^{\mathrm{op}}] < \infty$$

and thus

$$[Q^{\mathrm{op}\prime} \cap M \underset{e_N}{\boxtimes} M^{\mathrm{op}} : Q] \leq [M : Q]^2 < \infty,$$

implying that $B \overset{\mathrm{def}}{=} Q' \cap (Q^{\mathrm{op}\prime} \cap M \underset{e_N}{\boxtimes} M^{\mathrm{op}})$ has finite dimension. Since $e_N \in B$ and $Q' \cap M$, $Q^{\mathrm{op}\prime} \cap M^{\mathrm{op}} \subset B$, we are done.
5) $\Longrightarrow$ 6) is trivial, because hyperfinite algebras are amenable, so they have hypertraces.
6) $\Longrightarrow$ 7) By 5.2 we have $\mathcal{M}^{\mathrm{st}} = \mathrm{vN}(M, J_S M J_S) \subset \mathcal{B}(L^2(S))$, $\mathcal{N}^{\mathrm{st}} = \mathrm{vN}(N, J_S M J_S)$, where $S = M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$. Let then $\Phi : \mathcal{B}(L^2(S)) \to S$ be a conditional expectation. Since Φ is S-S linear and $[\mathcal{M}^{\mathrm{st}}, M^{\mathrm{op}}] = 0$ it follows that $\Phi(\mathcal{M}^{\mathrm{st}}) \subset M^{\mathrm{op}\prime} \cap S = M$. Similarly, since $[\mathcal{M}^{\mathrm{st}}, M_1^{\mathrm{op}}] = 0$, we get $\Phi(\mathcal{N}^{\mathrm{st}}) \subset M_1^{\mathrm{op}\prime} \cap S = N$.
2 $\Longrightarrow$ 1). If

$$\begin{array}{ccc} \mathcal{N} & \subset & \mathcal{M} \\ \cup & & \cup \\ N & \subset & M \end{array}$$

is an arbitrary smooth representation of $N \subset M$ then, M being hyperfinite, it follows that there exists a conditional expectation of $\mathcal{M}$ onto M. By Theorem 5.7 it then follows that there exists a conditional expectation of $\mathcal{N} \overset{\mathcal{E}}{\subset} \mathcal{M}$ onto $N \subset M$. Q.E.D.

7.2. *Remarks.* 1°. Note that by using condition 7.1.3 one can easily proceed to construct recursively a sequence of appropriate downward basic constructions for suitable local inclusions in the Jones tower-tunnel, say $M \supset N \supset P \supset P_1 \subset ...$, such that if we let $Q = \cap_n P_n$ and $(N^{0,\mathrm{st}} \subset M^{0,\mathrm{st}}) = (\overline{\cup_n P_n' \cap N} \subset \overline{\cup_n P_n' \cap M})$ then $(N \subset M) = (Q\bar{\otimes}N^{0,\mathrm{st}} \subset Q\bar{\otimes}M^{0,\mathrm{st}})$, with the isomorphism class of $N^{0,\mathrm{st}} \subset M^{0,\mathrm{st}}$ only depending on $\mathcal{G}_{N,M}$.

Indeed, from the proof of 7.1.2) $\Rightarrow$ 7.1.3) we see that, up to conjugacy by a unitary in N, the choice of the subfactor $P = P_1$ (equivalently, the choice of the projection $p \in N' \cap M_{2n+1}$) is determined by a choice of the ε-Følner set F and by a choice of the integers $\{m_k\}_{k\in F}$, which in turn both depend on ε. Similarily, each time one goes from step n to step $n+1$, one uses the Følner-type amenability condition for $P_n \subset M$ and some $\varepsilon = \varepsilon_{n+1}$ to get the next subfactor P_{n+1} (up to conjugation by a unitary element in P_n), from a downward basic construction that only depends on some choice of an ε_{n+1}-Følner set F_{n+1} and of some integers $\{m_j^{n+1}\}_{j\in F_{n+1}}$. Thus, if we let for instance $\varepsilon = 2^{-n}, \forall n$, and make the choice of the F_n's and m_j^n's this way, once for all, then the isomorphism class of $\{P_n' \cap N \subset P_n' \cap M\}_n$ will only depend on $\mathcal{G}_{N,M}$, as all the above choices can be "read" from this object through its amenability properties. In particular, the isomorphism class of $N^{0,\mathrm{st}} \subset M^{0,\mathrm{st}}$ will only depend on $\mathcal{G}_{N,M}$.

Thus, when complemented with this remark, we see that condition 7.1.3 in the above theorem shows that hyperfinite subfactors with amenable graphs are completely classified by their standard invariants (for more on this, see [Po16]).

2°. Recently, F. Hiai and M. Izumi have further investigated our notion of amenability for standard lattices and weighted graphs coming from subfactors and obtained two more equivalent characterizations ([HiIz]): the first one requires the existence of invariant means on the (weighted) fusion algebra of all $M-M$ bimodules in the Jones tower of $N \subset M$; the second one is a ratio limit condition on the weight vector $\vec{v}$, stating that the (weighted) graph $(\Gamma_{N,M}, \vec{v})$ is amenable if and only if for every vertex $k \in K$ one has

$$\lim_{n\to\infty} \frac{\langle (\Gamma\Gamma^t)^n \delta_*, \delta_k\rangle}{\langle (\Gamma\Gamma^t)^n \delta_*, \delta_*\rangle} = v_k,$$

where $\Gamma = \Gamma_{N,M}$. This "ratio limit" result for group-like objects coming from subfactors, which generalizes in a non-trivial way a prior result of Avez for discrete groups ([Av]), shows that in fact the projections $q_k \in (N' \cap M_{2n+1})p_k^{n+1}$ in the proof of 2) $\Longrightarrow$ 3) of Theorem 7.1 can be taken equal to p_k^{n+1}. It also shows that the standard weight vector $\vec{v}$ of an amenable standard λ-lattice $\mathcal{G}$ can be completely recovered from its graph Γ.

It should be noted however that there exist no known examples of standard graphs Γ which admit two distinct standard weights, say $\vec{v}_1, \vec{v}_2$, for the same value of the index, i.e., such that $(\Gamma, \vec{v}_1) \not\simeq (\Gamma, \vec{v}_2)$. Whether such examples exist or not seems to be an interesting problem.

In order to prove the hereditarity result in its largest generality, namely without assuming that the inclusions involved are extremal, we'll need the following:

7.3. LEMMA. *Let $N \subset M$ be an inclusion of type* II_1 *factors, of finite index (but not necessarily extremal). Let B be a C^*-algebra containing M, such that $B = C^*(M, \bigcup_k N_k' \cap B)$ and such that B has a state ϕ with $[\varphi, M] = 0$. Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be the GNS representation for (B, φ). Then, as a M-M Hilbert bimodule, $\mathcal{H}_\varphi$ is a direct sum of irreducible bimodules $\mathcal{H}_\varphi = \bigoplus_j \mathcal{H}_j'$ with each $\mathcal{H}_j'$ isomorphic to a bimodule in the list $\{\mathcal{H}_k\}_{k\in K}$.*

Proof. Note first that if $\{\mathcal{K}_l^{\mathrm{op}}\}_{l\in L}$ denotes the list of all irreducible M-N bimodules contained in $\bigoplus_{k\in K} {}_M\mathcal{H}_{k\,N}$ (see the beginning of Section 5) and $\mathcal{H}_0' \simeq \bigoplus_j \mathcal{K}_{l_j}^{\mathrm{op}}$ is a M-N Hilbert bimodule contained in some $M-M$ bimodule $\mathcal{H}$, (i.e., $\mathcal{H}_0' \subset {}_M\mathcal{H}_N$) then $\overline{\mathrm{sp}} M\mathcal{H}_0' M \simeq \bigoplus_i \mathcal{H}_{k_i}$.
Then note that $\cup_k \mathrm{sp} M(N_k' \cap B)\xi_\varphi$ is dense in $\mathcal{H}_\varphi$. Indeed, we have

$$\begin{aligned}\mathrm{sp}M(N_k'\cap B)M(N_k'\cap B)M &= \mathrm{sp}M(N_k'\cap B)N_k f_{-k-1}^0 N_k\\ (N_k'\cap B)M &= \mathrm{sp}M(N_k'\cap B)f_{-k-1}^0(N_k'\cap B)M\\ &\subset \mathrm{sp}M(N_{2k+1}'\cap B)M.\end{aligned}$$

showing that

$$\mathrm{Alg}(M, \cup_k N_k' \cap B) = \cup_k \mathrm{sp}M(N_k'\cap B)M = \cup_k \mathrm{sp}M(N_k'\cap B)R,$$

where $R = \overline{\cup_k N_k' \cap M}$, the closure being taken in the norm $\|\cdot\|_2$ in M. But $\cup_k(N_k'\cap M)\xi_\varphi$ is dense in $R\xi_\varphi$ (because φ implements τ on M), so $\cup_k \mathrm{sp}M(N_k'\cap B)\xi_\varphi$ is dense in $\cup_k \mathrm{sp}M(N_k'\cap B)R\xi_\varphi = \cup_k \mathrm{sp}M(N_k'\cap B)M\xi_\varphi$ which is dense in $\mathcal{H}_\varphi$.
Let then $\mathcal{H}_\varphi' \overset{\mathrm{def}}{=} \vee\{\mathcal{H}' \subset \mathcal{H}_\varphi \mid \exists\, k \in K$ such that $\mathcal{H}' \simeq \mathcal{H}_k$ as M-M bimodules$\}$. Assume $\mathcal{H}_\varphi' \neq \mathcal{H}_\varphi$. Thus, there exists $\xi \in M(N_k'\cap B)\xi_\varphi$ such that $\xi \notin \mathcal{H}_\varphi'$. Let $\xi = X_0 Y_0' \xi_\varphi$, for some $X_0 \in M$, $Y_0' \in N_k' \cap B$. It follows that if $x \in M$, $y \in N_k$ then

$$\langle x\xi y, \xi\rangle = \langle xX_0Y_0'\xi_\varphi y, X_0Y_0'\xi_\varphi\rangle = \langle X_0^* x X_0 y Y_0'\xi_\varphi, Y_0'\xi_\varphi\rangle.$$

But the state on M defined by $\psi(X) = \langle XY_0'\xi_\varphi, Y_0'\xi_\varphi\rangle$, $X \in M$ has N_k in its centralizer so by A.1 it is automatically normal and of the form $\psi(X) = \tau(Xa)$ for some $a \in N_k' \cap M$. Thus we get

$$\langle x\xi y, \xi\rangle = \tau(X_0^* x X_0 y a) = \langle x(X_0 a^{1/2}\xi_\tau)y, (X_0 a^{1/2}\xi_\tau)\rangle$$

so if we define $\xi' = X_0 a^{1/2}\xi_\tau \in M\xi_\tau \subset L^2(M)$ then the above shows that $\mathcal{H}_0' = \overline{\mathrm{sp}} M\xi N_k$ is a M-N_k bimodule isomorphic to a sub-bimodule of ${}_M L^2(M)_{N_k}$. By the first part applied to $N = N_k$ it follows that $\overline{\mathrm{sp}} M\mathcal{H}_0' M$ is a sub-bimodule of $(\bigoplus_{k\in K}\mathcal{H}_k)^n$ for some multiplicity $n \leq \infty$, giving a contradiction. Q.E.D.

7.4. Corollary. *Let $N \subset M$ be an inclusion of type II_1 factors with finite index. Assume that for any $\varepsilon > 0$ there exists an amenable type II_1 von Neumann algebra B containing M such that*

$$\|E_{(N' \cap B)' \cap B}(e_0) - \lambda 1\|_2 < \varepsilon.$$

Then there exists a norm one projection from $\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}$ onto $N \subset M$.

Proof. If $B \supset M$ satisfies the condition in the hypothesis for some ε, then there exist some finite many unitary elements $u_1, \dots, u_n \in N' \cap B$ such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} u_i e_0 u_i^* - \lambda 1 \right\|_2 < \varepsilon.$$

Thus, by taking instead of B the von Neumann algebra generated by M and $\{u_1, \dots, u_n\}$, it follows that we may assume B is separable in the norm $\| \ \|_2$. Let then $\mathcal{H}$ be the M-M Hilbert bimodule obtained by summing up countably many copies of each $\mathcal{H}_k$, $k \in K$. By 7.3 we have $L^2(B) \subset \mathcal{H}$, for each B as in the hypothesis, where $L^2(B)$ has the M-M bimodule structure given by left-right multiplication by elements of M.
For each $\varepsilon = 1/n$ we choose an algebra B_n satisfying the hypothesis. We let $\Phi_n : \mathcal{B}(L^2(B_n)) \to B_n$ be norm one projections and define the state φ on $\mathcal{B}(\mathcal{H})$ by

$$\varphi(T) = \lim_{n \to \omega} \tau_n \circ \Phi_n(p_n T|_{L^2(B_n)})$$

where $p_n = \mathrm{proj}_{L^2(B_n)}$, τ_n is the trace on B_n and ω is a free ultrafilter on $\mathbb{N}$. Since each $\tau_n \circ \Phi_n$ is a M-hypertrace, φ follows a M-hypertrace. Moreover, if we identify $\mathcal{M}^{\mathrm{st}}$ with the von Neumann algebra generated in $\mathcal{B}(\mathcal{H})$ by the operators of left and right multiplication by M and $\mathcal{N}^{\mathrm{st}}$ with its von Neumann subalgebra generated by the operators of left multiplication by N and right multiplication by M, then $\mathcal{M}^{\mathrm{st}} = \mathrm{sp}\mathcal{N}^{\mathrm{st}} e_0 N$. Let $Y \in \mathcal{N}^{\mathrm{st}}$, $y \in N$. We want to show that $\varphi = \varphi \circ \mathcal{E}^{\mathrm{st}}$ on $\mathcal{M}^{\mathrm{st}}$, thus we need to show that $\varphi(Y e_0 y) = \lambda \varphi(Y y)$. But $[p_n, \mathcal{M}^{\mathrm{st}}] = 0$ and $[\mathcal{N}^{\mathrm{st}} p_n, N' \cap B_n] = 0$, so that $\Phi_n(\mathcal{N}^{\mathrm{st}} p_n) \subset (N' \cap B_n)' \cap B_n$. Thus $\Phi_n((Y e_0 y) p_n) = \Phi_n(Y p_n) e_0 y = y' e_0 y$, with $y' \in (N' \cap B_n)' \cap B_n$. Thus $\tau_n(y' e_0 y) = \tau_n(E_{(N' \cap B_n)' \cap B_n}(y' e_0 y)) = \tau_n(y' E_{(N' \cap B_n)' \cap B_n}(e_0) y)$. It follows that

$$\begin{aligned} |\tau_n \circ \Phi_n((Y e_0 y) p_n) - & \lambda \tau_n \circ \Phi_n((Y y) p_n)| \\ &= |\tau_n(y' E_{(N' \cap B_n)' \cap B_n}(e_0) y) - \lambda \tau_n(y' y)| \\ &\leq \|y'\| \, \|y\| \, \|E_{(N' \cap B_n) \cap B_n}(e_0) - \lambda 1\|_2 \\ &\leq \frac{1}{n} \|Y\| \, \|y\|. \end{aligned}$$

This proves that indeed $\varphi(Y e_0 y) = \lambda \varphi(Y y)$, so $\varphi = \varphi \circ \mathcal{E}^{\mathrm{st}}$ on $\mathcal{M}^{\mathrm{st}}$. Q.E.D.

We can now prove the announced hereditarity property for amenable inclusions.

7.5. THEOREM. *Let $N \subset M$ be an extremal inclusion of hyperfinite type II_1 factors with amenable graph (equivalently, with amenable standard invariant $\mathcal{G}_{N,M}$), i.e., $\|\Gamma_{N,M}\|^2 = [M : N]$. Assume $Q \subset P$ is an inclusion of factors embedded in $N \subset M$ as commuting squares (i.e., such that $E_N(P) = Q$), but without necessarily being extremal and not necessarily having the same index as $N \subset M$. Then $\Gamma_{Q,P}$ is amenable (equivalently, $\mathcal{G}_{Q,P}$ is amenable), i.e., $\|\Gamma_{Q,P}\|^2$ equals the minimal index of $Q \subset P$.*

Proof. By Theorem 7.1, $S = M \underset{e_N}{\boxtimes} M^{\text{op}}$ follows amenable so in particular the von Neumann algebra B generated in S by P and $Q' \cap S$ is also amenable. Let $e_0 \in P$ be a Jones projection for $Q \subset P$. Thus, $E_Q(e_0) = E_N(e_0) = \lambda_0 1 = [P : Q]^{-1}1$. Since $(N' \cap S)' \cap S = N$ it follows that $E_{(N'\cap S)'\cap S}(e_0) = \lambda_0 1$. Since $N' \cap S \subset Q' \cap S = Q' \cap B$, this implies that $E_{(Q'\cap B)'\cap B}(e_0) = \lambda_0 1$ as well.
Thus $Q \subset P$ satisfies the conditions in the hypothesis of 7.4, so there exists a norm one projection from the standard representation $\mathcal{Q}^{\text{st}} \subset \mathcal{P}^{\text{st}}$ onto $Q \subset P$. By [Po13] this implies $\|\Gamma_{Q,P}\|^2 = \|T_{\mathcal{Q}^{\text{st}}\subset\mathcal{P}^{\text{st}}}\|^2 = \text{Ind}E_{\min}^{Q,P}$. Q.E.D.

In Sec. 6 we've seen that for extremal inclusions of arbitrary type II_1 factors $N \subset M$ the condition $\|\Gamma^{u,f}_{N,M}\|^2 = [M : N]$ is sufficient to insure the amenability of the standard graph $\Gamma_{N,M}$. We now show that for inclusions of hyperfinite factors the weaker condition $\|\Gamma^{u,rf}_{N,M}\|^2 = [M : N]$ is enough, where $\Gamma^{u,rf}_{N,M}$ denotes the inclusion graph of the direct summand $\mathcal{N}^{u,rf} \subset \mathcal{M}^{u,rf}$ of $\mathcal{N}^u \subset \mathcal{M}^u$, in which $\mathcal{M}^{u,rf}$ consists of all irreducible representations $\mathcal{B}(\mathcal{H})$ of $M \otimes M^{\text{op}}$, with $\mathcal{H}$ having finite right dimension over M, i.e., $\dim(\mathcal{H}_M) < \infty$, but leaving the left dimensions $\dim({}_M\mathcal{H})$ arbitrary.

7.6. THEOREM. *Let $N \subset M$ be an extremal inclusion of hyperfinite type* II_1 *factors. The following conditions are equivalent:*
1) *$N \subset M$ has amenable graph, i.e., $\|\Gamma_{N,M}\|^2 = [M : N]$.*
2) *$\forall\ \varepsilon > 0$, $\exists\ P$ a hyperfinite factor containing M, such that* $\dim M' \cap P < \infty$ *and $\|T_{M'\cap P\subset N'\cap P}\|^2 \geq [M : N] - \varepsilon$.*
3) *$\|\Gamma^{u,rf}_{N,M}\|^2 = [M : N]$.*

Proof. 1) $\Longrightarrow$ 3) is trivial because $\Gamma^{u,rf}_{N,M} \supset \Gamma_{N,M}$.
3) $\Longrightarrow$ 2) By 3) there exists a direct summand $\mathcal{N} \subset \mathcal{M} = \oplus_{l\in L'}\mathcal{B}(\mathcal{K}'_l) \subset \oplus_{k\in K'}\mathcal{B}(\mathcal{H}'_k)$ of $\mathcal{N}^{u,rf} \subset \mathcal{M}^{u,rf}$ such that its inclusion graph Γ is connected and $\|\Gamma\|^2 > [M : N] - \varepsilon$. Take $K'_0 \subset K'$ finite and sufficiently large so that we still have $\|\Gamma^t_{K'_0}\|^2 > [M : N] - \varepsilon$.
By the definition of the universal representation $\mathcal{N}^{u,rf} \subset \mathcal{M}^{u,rf}$, if $Q = M' \cap \mathcal{N}$ then Q is a factor of type II_1, $\mathcal{N} = N \vee Q \subset M \vee Q = \mathcal{M}$ and Q has finite coupling constant in each direct summand $\mathcal{B}(\mathcal{H}'_k)$ of $\mathcal{M}$. But then, if one takes $P = Q' \cap \mathcal{B}(\oplus_{k\in K'_0}\mathcal{H}'_k)$ then $\|T_{M'\cap P\subset N'\cap P}\|^2 \geq \|\Gamma^t_{K'_0}\|^2 \geq [M : N] - \varepsilon$.
2) $\Longrightarrow$ 3) follows by noticing that $\mathcal{M}^{u,rf}$ contains the von Neumann algebra generated by the operators of left multiplication by M and right multiplication

by P on $L^2(P)$ as a direct summand (by taking $M^{\rm op} \simeq P^{\rm op}$, both being hyperfinite factors). Thus $T_{M'\cap P\subset N'\cap P}$ will be a restriction of the graph $\Gamma^{u,rf}_{N,M}$
2) $\Longrightarrow$ 1). Let $\varepsilon > 0$ and choose P a hyperfinite II_1 factor satisfying 2) for ε^{32}. Denote by T the bipartite graph describing the inclusions $M'\cap P \subset N'\cap P \subset N_1'\cap P \subset N_2'\cap P \subset \cdots$. Thus $\|T\|^2 \geq [M:N] - \varepsilon^{32}$ and there is a positive vector $w = (w_j)_{j\in J}$ such that $TT^{\rm t}w = \lambda^{-1}w$, giving the traces on $\{N_k'\cap P\}_{k\geq 1}$. But then A.2 applies the same way as in the proof of 1) $\Longrightarrow$ 2) in Theorem 5.3 to get a finite set $F \subset J$ such that $\sum_{j\in\partial F} w_j^2 < \varepsilon \sum_{j\in F} w_j^2$ (see 5.5.2° and [Po14]). Arguing like in the proof of 1) $\Longrightarrow$ 2) in Theorem 6.1 it then follows that there exist $k \geq 1$ and a projection $p \in N_{2k}'\cap P$ such that

$$\|E_{(pN'_{2k+1}\cap Pp)'\cap pPp}(e_{-2k-1}p) - \lambda p\|_2 < \varepsilon\|p\|_2.$$

By taking an $\alpha = [M:N]^{k+1}$-amplification of the inclusion $N_{2k+2} \subset N_{2k+1} \hookrightarrow pPp$ and using that $(N_{2k+2} \subset N_{2k+1})^\alpha = (N \subset M)$ it follows that there exists a hyperfinite type II_1 factor $P_0 \simeq (pPp)^\alpha$ such that $N \subset M \subset P_0$ and

$$\|E_{(N'\cap P_0)'\cap P_0}(e_0) - \lambda 1\|_2 < \varepsilon.$$

But then 7.3 applies to get that $\Gamma_{N,M}$ is amenable. Q.E.D

8. An Effros-Lance Type Characterization of Amenability

We will prove in this section yet another equivalent characterization for the amenability of a subfactor $N \subset M$, in terms of simplicity properties of the C^*-algebra $C^*_{\rm bin}(M, e_N, M^{\rm op})$. In the case $N = M$ our result reduces to the implication "$C^*_{\rm bin}(M, M^{\rm op})$ simple $\Longrightarrow \exists$ conditional expectations of $\mathcal{B}(L^2(M))$ onto M", which is one of the well known results of Effros and Lance in [EL], relating various amenability conditions for single von Neumann algebras (semidiscreteness, injectivity, etc).

8.1. Theorem. *Let $N \subset M$ be an extremal inclusion of type II_1 factors. The following conditions are equivalent:*
1°. *$N \subset M$ is amenable.*
2°. *$C^*_{\rm bin}(M, e_N, M^{\rm op})$ is simple.*
3°. *$C^*(M, e_N, JMJ)$ is simple.*
4°. *$C^*_{\rm bin}(M, e_N, M^{\rm op}) \simeq C^*(M, e_N, JMJ) \simeq C^*_{\rm min}(M, e_N, M^{\rm op})$, with the isomorphisms being given by the natural quotient maps.*

Proof. 1° $\Longrightarrow$ 2°. Let $C^*_{\rm bin}(M, e_N, M^{\rm op}) \hookrightarrow \mathcal{B}(\mathcal{H})$ be a faithful representation of $C^*_{\rm bin}(M, e_N, M^{\rm op})$ such that M and $M^{\rm op}$ are von Neumann algebras in $\mathcal{B}(\mathcal{H})$. It is sufficient to prove that if

$$x \in \text{Alg}(M, e_N, M^{\rm op}) = \bigcup_k \text{sp} M^{\rm op} M_k M^{\rm op} \subset C^*_{\rm bin}(M, e_N, M^{\rm op}) \subset \mathcal{B}(\mathcal{H}),$$

then $\|x\|_{\mathcal{B}(\mathcal{H})} \leq \|x\|_{\min}$, where $\|x\|_{\min}$ is the norm of (the image of) x in $C^*_{\min}(M, e_N, M^{\rm op})$. For such $x \in {\rm Alg}(M, e_N, M^{\rm op})$ let k be large enough such that

$$x = \sum_{i=n}^{n} y_i^{\rm op} z_i x_i^{\rm op} \in {\rm sp} M^{\rm op} M_k M^{\rm op},$$

for some $x_i, y_i \in M, z_i \in M^{\rm op}$, $1 \leq i \leq n$. We will prove that x can be approximated in the so topology on $\mathcal{B}(\mathcal{H})$ by elements $x' \in {\rm Alg}(M, e_N, M^{\rm op})$ such that $\|x'\|_{\mathcal{B}(\mathcal{H})} \leq \|x\|_{\min}$. By the inferior semicontinuity of the norm $\| \ \|_{\mathcal{B}(\mathcal{H})}$ with respect to the so-topology on $\mathcal{B}(\mathcal{H})$, this will show that $\|x\|_{\mathcal{B}(\mathcal{H})} \leq \|x\|_{\min}$ and will thus end the proof of $1^\circ \implies 2^\circ$.
To prove this approximation, let us first note that $\forall\ \xi_1, \dots, \xi_p \in \mathcal{H}$, $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ such that if $z_i' \in M_k$, with $\|z_i' - z_i\|_2 < \delta$, $\|z_i'\| \leq \|z_i\|$ then

$$x' \stackrel{\rm def}{=} \sum_{i=1}^{n} y_i^{\rm op} z_i' x_i^{\rm op}$$

satisfies $\|(x - x')\xi_j\| < \varepsilon$, $\forall\ 1 \leq j \leq p$. Indeed we have:

$$\|(x - x')\xi_j\| \leq \sum_{i=1}^{n} \|y_i^{\rm op}\| \ \|(z_i - z_i')x_i^{\rm op}\xi_j\|$$

and since the so-topology on the ball of radius $\|z_i\|$ (in the uniform norm) in M_k coincides with the topology given by the norm $\| \ \|_2$ on this ball, it follows that there exists $\delta > 0$ such that if $\|z_i - z_i'\|_2 < \delta$ then $\|(z_i - z_i')x_i^{\rm op}\xi_j\| < \varepsilon/n\|y_i\|$, $\forall\ i$. But then we have

$$\|(x - x')\xi_j\| < \sum_{i=1}^{n} \|y_i^{\rm op}\| \varepsilon/n \|y_i^{\rm op}\| = \varepsilon.$$

Now, if we assume $N \subset M$ is amenable then $M \subset M_k$ follows amenable and by [Po2] we get that $\forall\ \delta > 0$, $\exists$ finitely many tunnels $\{N_k^r\}_{1 \leq k \leq n_r}$, $r = 1, \dots, m$, and projections $p_r \in \mathcal{P}({N_{n_r}^r}' \cap M)$, $r = 1, \dots, m$, such that $\{p_r\}_r$ are mutually orthogonal, $\Sigma_r p_r = 1$ and

$$z_i' \stackrel{\rm def}{=} \sum_{r=1}^{m} p_r E^{M_k}_{{N^r}'_{n_r} \cap M_k}(z_i) p_r$$

satisfies $\|z_i' - z_i\|_2 < \delta$. Also, by its definition, z_i' checks $\|z_i'\| \leq \|z_i\|$. Furthermore, since $p_r \in M$ commute with $x_i^{\rm op}, y_i^{\rm op} \in M^{\rm op}$, $\forall\ r, i$, it follows that if we let $x' = \Sigma_i y_i^{\rm op} z_i' x_i^{\rm op}$ as above, then $\|x'\|_{\mathcal{B}(\mathcal{H})} = \sup_r \|x' p_r\|_{\mathcal{B}(\mathcal{H})}$. But since

$$x' p_r = \sum_i y_i^{\rm op} p_r E_{{N_{n_r}^r}' \cap M_k}(z_i) p_r x_i^{\rm op} = p_r \left(\sum_{i=1}^{n} y_i^{\rm op} E_{{N_{n_r}^r}' \cap M_k}(z_i) x_i^{\rm op} \right) p_r,$$

each $\|x'p_r\|_{\mathcal{B}(\mathcal{H})}$ is majorized by

$$\begin{aligned}\left\|\sum_{i=1}^n y_i^{\mathrm{op}} E_{N_{n_r}^r{}'\cap M_k}(z_i)x_i^{\mathrm{op}}\right\|_{\mathcal{B}(\mathcal{H})} &= \left\|\sum_{i=1}^n y_i^{\mathrm{op}} E_{N_{n_r}^r{}'\cap M_k}(z_i)x_i^{\mathrm{op}}\right\|_{M_{n_r}^{\mathrm{op}}} \\ &= \left\|\sum_{i=1}^n y_i^{\mathrm{op}} E_{N_{n_r}^r{}'\cap M_k}(z_i)x_i^{\mathrm{op}}\right\|_{C^*_{\min}(M,e_N,M^{\mathrm{op}})} \\ &\le \|x\|_{\min},\end{aligned}$$

with the last inequality following from the fact that, in the algebra $C^*_{\min}(M, e_N, M^{\mathrm{op}})$, the element $\Sigma_i y_i^{\mathrm{op}} E_{N_{n_r}^r{}'\cap M_k}(z_i)x_i^{\mathrm{op}}$ is the image of x under a conditional expectation.

Thus, from the the above remarks, if we take δ sufficiently small, we are done.

$2^\circ \implies 3^\circ$ is trivial, since by the definition of $C^*_{\mathrm{bin}}(M, e_N, M^{\mathrm{op}})$, $C^*(M, e_N, JMJ)$ is its quotient.

$3^\circ \implies 1^\circ$. If $C^*(M, e_N, JMJ)$ is simple then there exists an isomorphism

$$\varphi : S^0 \stackrel{\mathrm{def}}{=} C^*_{\min}(M, e_N, M^{\mathrm{op}}) \simeq C^*(M, e_N, JMJ) \subset \mathcal{B}(L^2(M)).$$

Since $S^0 \subset S \subset \mathcal{B}(L^2(S))$, where $S = M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ as usual, by Arveson's theorem φ can be extended to a completely positive map Φ from all $\mathcal{B}(L^2(S))$ to $\mathcal{B}(L^2(M))$. (Note that in fact we only use here a particular case of Arveson's theorem stating that if $B \subset A$ are unital C^*-algebras and $\pi_0 : B \to \mathcal{B}(\mathcal{H}_0)$ is a representation of B then there exists a Hilbert space $\mathcal{H} \supset \mathcal{H}_0$ and a representation $\pi : A \to \mathcal{B}(\mathcal{H})$ such that $\pi_0(b) = \mathrm{proj}_{\mathcal{H}_0}\pi(b)|_{\mathcal{H}_0}$, $\forall\, b \in B$. See 2.10.2 in [D2]). Since Φ is a unital $*$-homomorphism when restricted to S^0, it follows that it is a S^0-S^0 bimodule map. In particular, if $x_{1,2}^{\mathrm{op}} \in M^{\mathrm{op}} \subset S^0$ ($\subset \mathcal{B}(L^2(S))$) then $\Phi(x_1^{\mathrm{op}} T x_2^{\mathrm{op}}) = \varphi(x_1^{\mathrm{op}})\Phi(T)\varphi(x_2^{\mathrm{op}})$, $\forall\, T \in \mathcal{B}(L^2(S))$. Thus, if T satisfies $Tx^{\mathrm{op}} - x^{\mathrm{op}}T = 0$, $\forall\, x^{\mathrm{op}} \in M^{\mathrm{op}}$, then $\Phi(T)\varphi(x^{\mathrm{op}}) = \varphi(x^{\mathrm{op}})\Phi(T)$, $\forall\, x^{\mathrm{op}} \in M^{\mathrm{op}}$.

Thus we have $\Phi((M^{\mathrm{op}})' \cap \mathcal{B}(L^2(S)) = \varphi(M^{\mathrm{op}})' \cap \mathcal{B}(L^2(M))$. But $\varphi(M^{\mathrm{op}}) = JMJ$ and $JMJ' \cap \mathcal{B}(L^2(M)) = M$, so that $\Phi((M^{\mathrm{op}})' \cap \mathcal{B}(L^2(S))) = M$. Similarily we get

$$\begin{aligned}\Phi((M_1^{\mathrm{op}})' \cap \mathcal{B}(L^2(S))) &= \varphi(M_1^{\mathrm{op}})' \cap \mathcal{B}(L^2(M)) \\ &= JM_1J' \cap \mathcal{B}(L^2(M)) \\ &= N.\end{aligned}$$

But from 5.3 we have that $\mathcal{M}^{\mathrm{st}} \subset (M^{\mathrm{op}})' \cap \mathcal{B}(L^2(S))$ and $\mathcal{N}^{\mathrm{st}} \subset (M_1^{\mathrm{op}})' \cap \mathcal{B}(L^2(S))$, so Φ implements a positive unital M-M bimodule map from $\mathcal{M}^{\mathrm{st}}$ onto M carrying $\mathcal{N}^{\mathrm{st}}$ onto N. This shows that there exists a conditional expectation of ($\mathcal{N}^{\mathrm{st}} \subset \mathcal{M}^{\mathrm{st}}$) onto ($N \subset M$), so $N \subset M$ follows amenable.

All this shows that the conditions 1°–3° are equivalent. Since clearly 4° $\Longleftrightarrow$ 2°, all the conditions 1°–4° follow equivalent. Q.E.D.

8.2. *Remarks.* 1°. Note that when applied to the case $N = M$ the above proof of the implication 3° $\Rightarrow$ 1° in Theorem 8.1 reduces to a very short and elementary proof to one of the results in ([EL]).
2°. Recall from ([Bi3]) that $C^*(M, e_N, JMJ)$ contains the ideal $\mathcal{K}$ of compact operators over the Hilbert space $L^2(M, \tau)$ if and only if N contains no non-trivial central sequences of M. Thus, 8.1 implies that amenable inclusions always have non-trivial cental sequences contained in the subfactor (because if $C^*(M, e_N, JMJ)$ is simple then it cannot contain the ideal $\mathcal{K}$). In fact, 7.1.4 shows that there even exist non-commuting such central sequences, so that amenable inclusions split off the hyperfinite type II_1 factor (this is, of course, a consequence of the classification result 7.2.1° as well).

9. Property T for Subfactors and Standard Lattices

In this section we introduce a notion of property T for standard λ-lattices $\mathcal{G}$ (or, equivalently, for paragroups). When restricted to the class of standard lattices associated with subfactors coming from finitely generated discrete groups, our notion coincides with the classical property T of Kazhdan, which it thus generalizes, from discrete groups to the larger class of group-like objects $\mathcal{G}$. In order to define this notion, we will use a strategy similar to the approach to amenability in Section 5. Thus, the property T for a standard λ-lattice $\mathcal{G}$ will be defined by requiring $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ to have the property T relative to $M \vee M^{\mathrm{op}}$ in the sense of ([A-D], [Po8]), where $N \subset M$ is an extremal subfactor with $\mathcal{G}_{N,M} = \mathcal{G}$. This definition however depends on proving that such a property does not in fact depend on the subfactor $N \subset M$ one takes. We do prove this in the next few lemmas.
First of all, let us recall the definition of the relative property T, as introduced in ([A-D], [Po8]):

(∗). *Let U be a type II_1 factor and $B \subset U$ a von Neumann subalgebra of U. Then we say that U has the property* T *relative to B if there exists $\varepsilon > 0$ and $x_1, x_2, ..., x_n \in U$ such that whenever $\mathcal{H}$ is a given $U - U$ bimodule with a vector $\xi \in \mathcal{H}$ satisfying $\|\xi\| = 1, [\xi, B] = 0, \|[\xi, x_i]\| < \varepsilon$, it follows that $\mathcal{H}$ must contain a non-zero vector ξ_0 satisfying $[\xi_0, U] = 0$.*

Note that in the case $B = \mathbb{C}$ the above definition reduces to Connes' definition of property T for single type II_1 factors U. In general though, the definition (∗) does not require the ambient algebra U to have the property T. Instead, note that by ([A-D], [Po8]), if V is a type II_1 factor and G is a discrete group acting outerly on V, then $U = V \rtimes G$ has the property T relative to V if and only if the group G has Kazhdan's property T.
With this in mind, let us proceed with some technical results.

9.1. Lemma. *Let $V \subset U$ be an inclusion of type* II_1 *factors with $V' \cap U = \mathbb{C}1$. Then U has the property* T *relative to V if and only if $\forall\ \varepsilon > 0\ \exists\ \delta > 0$*

and $x_1, \dots, x_n \in U$ such that if $\varphi : U \to U$ is completely positive, unital, trace preserving, with $\varphi(v_1 x v_2) = v_1 \varphi(x) v_2$, $\forall\, v_1, v_2 \in V$, $x \in U$, and $\|\varphi(x_i) - x_i\|_2 < \delta$ then $\|\varphi(x) - x\|_2 < \varepsilon$, $\forall\, x \in U$, $\|x\| \leq 1$.

Proof. If U has the property T relative to V then the condition on completely positive maps holds true by 4.1.4 in [Po8].

Conversely, if this latter condition holds, then let $\mathcal{H}$ be a U-U bimodule with $\xi \in \mathcal{H}$, $\|\xi\| = 1$, $v\xi = \xi v$, $\forall\, v \in V$, $\|x_i \xi - \xi x_i\| < \delta' \stackrel{\text{def}}{=} \delta/2 \sum \|x_i\|_2$. Let $\varphi : U \to U$ be defined by $\tau(y\varphi(x)) = \langle x\xi y, \xi\rangle$, $x, y \in V$ as in [C4] (see [Po8]). Then φ is a well defined completely positive map and $\tau(\varphi(x)) = \langle x\xi, \xi\rangle$. Since $\xi = v\xi v^*$, $\forall\, v \in \mathcal{U}(V)$, one gets

$$\begin{aligned}\langle vxv^*\xi, \xi\rangle &= \langle vx\xi v^*, \xi\rangle = \langle vx\xi, \xi v\rangle = \langle vx\xi, v\xi\rangle \\ &= \langle v^* vx\xi, \xi\rangle = \langle x\xi, \xi\rangle\end{aligned}$$

for all $x \in U$. Averaging by unitaries $v \in \mathcal{U}(V)$ like in [Po1] and using that $V' \cap U = \mathbb{C}1$ and $E_{V'\cap U}(x) = \tau(x)1$, it follows that

$$\langle x\xi, \xi\rangle = \langle \tau(x)\xi, \xi\rangle = \tau(x), \qquad \forall\, x \in U.$$

Similarly, we obtain that

$$\tau(y\varphi(1)) = \langle \xi y, \xi\rangle = \tau(y), \qquad \forall\, y \in U.$$

Thus $\varphi(1) = 1$. Also, if $x, y \in U$, $v_1, v_2 \in V$ then

$$\begin{aligned}\tau(y\varphi(v_1 x v_2)) &= \langle v_1 x v_2 \xi y, \xi\rangle = \langle x\xi v_2 y, v_1^* \xi\rangle \langle x\xi v_2 y, \xi v_1^*\rangle \\ &= \langle x\xi v_2 y v_1, \xi\rangle = \tau(v_2 y v_1 \varphi(x)) = \tau(y(v_1\varphi(x)v_2)).\end{aligned}$$

This shows that $\varphi(v_1 x v_2) = v_1 \varphi(x) v_2$.

Finally, since $\|x_i\xi - \xi x_i\|_2 < \delta'$ we have

$$\begin{aligned}\|\varphi(x_i) - x_i\|_2^2 &= \tau(\varphi(x_i^*)\varphi(x_i)) = \tau(x_i^* x_i) - 2\mathrm{Re}\,\tau(x_i^*\varphi(x_i)) \\ &\leq \tau(\varphi(x_i^* x_i)) + \tau(x_i^* x_i) - 2\mathrm{Re}\,\tau(x_i^*\varphi(x_i)) \\ &= 2\tau(x_i^* x_i) - 2\mathrm{Re}\,\tau(x_i^*\varphi(x_i)) \\ &= 2\langle \xi x_i, \xi x_i\rangle - 2\mathrm{Re}\langle x_i\xi, \xi x_i\rangle \\ &\leq 2\|x_i\xi - \xi x_i\|\, \|\xi x_i\| \leq 2\delta'\|x_i\|_2 < \delta^2.\end{aligned}$$

Thus, φ this way defined satisfies the required condition, so $\|\varphi(x) - x\|_2 < \varepsilon$, $\forall\, x \in U$, $\|x\| \leq 1$. In particular, we have

$$\|\varphi(u) - u\|_2 < \varepsilon, \qquad \forall\, u \in \mathcal{U}(U).$$

Thus,

$$\begin{aligned}\|\xi u - u\xi\|^2 &= 2 - 2\mathrm{Re}\langle \xi u, \xi u\rangle = 2 - 2\mathrm{Re}\langle u^*\xi u, \xi\rangle \\ &= 2 - 2\mathrm{Re}\,\tau(\varphi(u)u^*) = 2\mathrm{Re}(\tau((u - \varphi(u))u^*)) \\ &\leq 2\|\varphi(u) - u\|_2 \leq 2\varepsilon.\end{aligned}$$

Thus, if $\varepsilon < 1/2$ then $\|u^*\xi u - \xi\| < 1$, $\forall\, u \in \mathcal{U}(U)$. But then $\exists\, \xi_0 \in \mathcal{H}$, $\|\xi_0 - \xi\| < 1$, such that $u\xi_0 = \xi_0 u$, $\forall\, u$ (see e.g., [Po1]). Thus $\mathcal{H}$ has a nonzero vector commuting with U, showing that U has the property T relative to V Q.E.D.

9.2. Lemma. *Let*

$$\begin{array}{ccc} V & \subset & U \\ \cup & & \cup \\ Q & \subset & P \end{array}$$

be a nondegenerate commuting square of type II_1 *von Neumann algebras with a countable set* $\mathcal{X} = \{f_n\}_n \subset P$ *such that* $\mathrm{sp}\, Q\mathcal{X}Q$ *is* $\| \, \|_2$*-dense in* P *and* $\mathrm{sp}\, V\mathcal{X}V$ *is* $\| \, \|_2$*-dense in* U. *Let* $\varphi : P \to P$ *be a unital,* τ*-preserving, completely positive map such that*

$$\varphi(q_1 x q_2) = q_1 \varphi(x) q_2, \qquad \forall\ q_1, q_2 \in Q,\ x \in P$$

and assume that $\forall\ n \geq 1$, $\exists\ \{m_j\}_j \subset L^2(V, \tau)$ *orthonormal basis of* V *over* Q *such that* $[m_j, f_n] = 0$, $[m_j, \varphi(f_n)] = 0$, $\forall\ j$. *Then there exists a unique unital,* τ*-preserving, completely positive map* $\tilde{\varphi} : U \to U$ *such that* $\tilde{\varphi}|_P = \varphi$ *and* $\tilde{\varphi}(v_1 x v_2) = v_1 \varphi(x) v_2$, $\forall\ v_1, v_2 \in V$, $x \in P$.

Proof. Let $e = e_P^U$. Let $\{m_j\}_j \subset L^2(V)$ be a fixed orthonormal basis of V over Q and note that any element in $\langle U, P\rangle$ can be written in the form $\Sigma_{i,j} m_i p_{ij} e m_j^*$, with $p_{ij} \in P$ (see Ch.1 in [Po2]). We first define an application $\tilde{\varphi} : \langle U, e\rangle \to \langle U, e\rangle$ by

$$\tilde{\varphi}\left(\sum_{i,j} m_i p_{ij} e m_j^*\right) = \sum_{i,j} m_i \varphi(p_{ij}) e m_j^*, \qquad p_{ij} \in P.$$

It is easy to see that $\tilde{\varphi}$ this way defined is completely positive and Tr-preserving and satisfies $\tilde{\varphi}(1) = 1, \tilde{\varphi}(Y_1 X Y_2) = Y_1 \tilde{\varphi}(X) Y_2, \forall\ Y_1, Y_2 \in \langle V, e\rangle,\ X \in \langle U, e\rangle$. Let us next show that $\tilde{\varphi}$ does not depend on the choice of the orthonormal basis $\{m_j\}$ of V over Q. So let $\{m'_j\}_j \subset L^2(V, \tau)$ be another such orthonormal basis. Then $m_i = \sum_k m'_k E_P^U(m_k^{'*} m_i)$ so that if $p \in P$ then $m_i p e m_j^* = \sum_{k,l} m'_k E_P^U(m_k^{'*} m_i) p E(m_j^* m'_l) e m_l^{'*}$ (note that the sums do make sense in $L^2(U, \tau)$, with convergence in $\| \, \|_2$, respectively so-topologies). By definition we thus have $\tilde{\varphi}(m_i p e m_j^*) = m_i \varphi(p) e m_j^*$ and since $E_P^U(m_k^{'*} m_i) \in Q$ and

$$\varphi(E_P^U(m_k'^* m_i) p E_P^U(m_j^* m'_l)) = E_P^U(m_k'^* m_i) \varphi(p) E_P^U(m_j^* m'_l),$$

we further get

$$m_i \varphi(p) e m_j^* = \sum_{k,l} m'_k (E_P^U(m_k'^* m_i) \varphi(p) E_P^U(m_j^* m'_l)) e m_l'^*$$

$$= \sum_{k,l} m'_k \varphi(E_P^U(m_k'^* m_i) p E_P^U(m_j^* m'_l)) e m_l'^*.$$

Taking linear combinations and limits, this shows that if

$$\sum_{i,j} m_i p_{ij} e m_j^* = \sum_{i,j} m'_i p'_{ij} e m_j'^*$$

then

$$\sum_{i,j} m_i\varphi(p_{ij})em_j^* = \sum_{i,j} m_i'\varphi(p_{ij}')em_j'^*,$$

showing that $\tilde{\varphi}$ does not depend on $\{m_j\}_j$.

We will now show that $\tilde{\varphi}(U) = U$ and that $\tilde{\varphi}|_P = \varphi$. To this end, let us first note that $\tilde{\varphi}(f_n) = \varphi(f_n)$, $\forall\ n$. Indeed, we have $f_n = f_n 1 = f_n \sum_i m_i e m_i^*$ in which we may assume $[m_j, f_n] = 0$, $\forall\ j$ (by the hypothesis and the above). Thus we get $f_n = \sum_i m_i f_n e m_i^*$.

According to the definition of $\tilde{\varphi}$ we get $\tilde{\varphi}(f_n) = \sum_i m_i\varphi(f_n)em_i^*$. But by the hypothesis we may also assume $[m_i, \varphi(f_n)] = 0$ so that we get $\sum_j m_j\varphi(f_n)em_j^* = \varphi(f_n)\sum_j m_j e m_j^* = \varphi(f_n)$.

Since $\tilde{\varphi}$ is V-bilinear (being $\langle Q, e\rangle$-bilinear) it follows that $\tilde{\varphi}(V\mathcal{X}V) = V\tilde{\varphi}(\mathcal{X})V = V\varphi(\mathcal{X})V \subset U$. In particular $\tilde{\varphi}_{|Q\mathcal{X}Q} = \varphi$.

The rest of the statement thus follows by continuity. Q.E.D.

9.3. Corollary. *Let*

$$\begin{array}{ccc} N & \subset & M \\ \cup & & \cup \\ N_0 & \subset & M_0 \end{array}$$

be a nondegenerate commuting square of type II_1 *factors with* $N \subset M$*,* $N_0 \subset M_0$ *extremal and* $N' \cap M_j = N_0' \cap M_{0j}$*,* $\forall\ j$*. Let* $T = M \vee M^{\mathrm{op}} \subset M \underset{e_N}{\boxtimes} M^{\mathrm{op}} = S$ *and* $T_0 = M_0 \vee M_0^{\mathrm{op}} \subset M_0 \underset{e_{N_0}}{\boxtimes} M_0^{\mathrm{op}} = S_0$*. If* S *has the property* T *relative to* T *then* S_0 *has the property* T *relative to* T_0*.*

Proof. By 2.5 we have $T_0' \cap S_0 = \mathbb{C}$, $T' \cap S = \mathbb{C}$. Also, by 2.8 S_0 is naturally included in S and we have a nondegenerate commuting square

$$\begin{array}{ccc} T & \subset & S \\ \cup & & \cup \\ T_0 & \subset & S_0. \end{array}$$

Let $\{N_{0,m}\}_m$ be some tunnel for $N_0 \subset M_0$ and N_m be the corresponding tunnel for $N \subset M$ and denote by $f_n = f_{-n}^n \in M_{0,n}$ the Jones projection for $N_{0,n-1} \subset M_0 \subset M_{0,n}$. By 4.1.4 in [Po8], since S has the property T relative to T and $\mathrm{sp}\cup_n T f_n T$ contains the dense $*$-subalgebra $\cup_n M M_n^{\mathrm{op}} M$ in S (cf. 4.1), it follows that $\forall\ \varepsilon > 0$ there exists n and δ such that if $\varphi : S \to S$ is unital, trace preserving, completely positive, T-T bimodule map with $\|\varphi(f_n) - f_n\|_2 < \delta$ then $\|\varphi(x) - x\|_2 < \varepsilon$, $\forall\ x \in S$, $\|x\| \leq 1$. Since

$$f_m \in (N_{0,m} \vee N_{0,m}^{\mathrm{op}})' \cap M_0 \underset{e_{N_0}}{\boxtimes} M_0^{\mathrm{op}} = (N_m \vee N_m^{\mathrm{op}})' \cap M \underset{e_N}{\boxtimes} M^{\mathrm{op}}, \forall\ m,$$

it follows that $\forall\ k$, $\exists\ \{m_j^k\}_j \subset N_k \vee N_k^{\mathrm{op}}$ an orthonormal basis of $N_k \vee N_k^{\mathrm{op}}$ over $N_{0,k} \vee N_{0,k}^{\mathrm{op}}$ (which will therefore be an orthonormal basis of T over T_0 as well). Thus $[m_j^k, f_k] = 0, \forall j$.

Let $\varphi_0 : S_0 \to S_0$ be a unital, trace preserving, completely positive, T_0-T_0 bimodule map satisfying $\|\varphi_0(f_n) - f_n\|_2 < \delta$. Since φ_0 is $T_0 - T_0$ bilinear and since $[f_k, N_{0,k} \vee N_{0,k}^{\text{op}}] = 0$, we get $[\varphi_0(f_k), N_{0,k} \vee N_{0,k}^{\text{op}}] = 0$. Thus we also have:

$$\varphi_0(f_m) \in (N_{0,m} \vee N_{0,m}^{\text{op}})' \cap M_0 \underset{e_{N_0}}{\boxtimes} M_0^{\text{op}} = (N_m \vee N_m^{\text{op}})' \cap M \underset{e_N}{\boxtimes} M^{\text{op}}, \forall\, m.$$

So we may apply Lemma 9.2 to get $\varphi : S \to S$ unital, τ-preserving, completely positive T-T bimodule map with $\varphi|_{S_0} = \varphi_0$. Thus $\|\varphi(f_n) - f_n\|_2 = \|\varphi_0(f_n) - f_n\|_2 < \delta$, implying that $\|\varphi(x) - x\|_2 < \varepsilon$, $\forall\, x \in S$, $\|x\| \leq 1$.
In particular, $\|\varphi_0(x) - x\|_2 < \varepsilon\ \forall\, x \in S_0$. By Lemma 9.1, this is suficient to ensure that S_0 has the property T relative to T_0. Q.E.D.

9.4. Proposition. *Let*

$$\begin{array}{ccc} N & \subset & M \\ \cup & & \cup \\ N_0 & \subset & M_0 \end{array}$$

be a nondegenerate commuting square of type II_1 *factors with* $N_0 \subset M_0$*,* $N \subset M$ *extremal and* $N_0' \cap M_{0,j} \subset N' \cap M_j$*,* $\forall\, j$ *(i.e.,* $N_0 \subset M_0$ *is smoothly embedded in* $N \subset M$*, in the sense of* [Po2]*). Let* $T_0 \subset S_0$*,* $T \subset S$ *be the corresponding symmetric enveloping inclusions. If* S_0 *has the property* T *relative to* T_0*, then* S *has the property* T *relative to* T*.*

Proof. By [Po8], $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ and $x_1, \dots, x_n \in S_0$ such that if $\mathcal{H}_0$ is a S_0-S_0 bimodule with a unit vector $\xi_0 \in \mathcal{H}_0$ satisfying $[y, \xi_0] = 0$, $\forall\, y \in T_0$, and $\|[x_i, \xi_0]\| < \delta$, $\forall\, i$, then there exists $\xi_1 \in \mathcal{H}_0$ satisfying $[x, \xi_1] = 0$, $\forall\, x \in S_0$, and $\|\xi_1 - \xi_0\| < \varepsilon$.
Let then $\mathcal{H}$ be a S-S bimodule with a unit vector $\xi_0 \in \mathcal{H}$ such that $[y, \xi_0] = 0$, $\forall\, y \in T$, and $\|[x_i, \xi_0]\| < \delta$, $\forall\, i$. Regarding $\mathcal{H}$ as a S_0-S_0 bimodule it follows that there exists $\xi_0' \in \mathcal{H}$ such that $[x, \xi_0'] = 0$, $\forall\, x \in S_0$, and $\|\xi_0' - \xi_0\| < \varepsilon$.
Denote

$$\begin{aligned} \mathcal{K} &= \{\xi \in \mathcal{H} \mid x\xi = \xi x,\ \forall\, x \in S_0\}, \\ \mathcal{K}_0 &= \{\eta_0 \in \mathcal{H} \mid [y, \eta_0] = 0,\ \forall\, y \in T = M \vee M^{\text{op}}\}, \\ \mathcal{K}_1 &= \{\eta_1 \in \mathcal{H} \mid [y, \eta_1] = 0,\ \forall\, y \in M_1 \vee N^{\text{op}}\}. \end{aligned}$$

With these notations, it follows that $\xi_0 \in \mathcal{K}_0$ and $\xi_0' \in \mathcal{K}$. We then need to construct some positive contractions $A, B \in \mathcal{B}(\mathcal{H})$ such that $0 \leq A, B \leq 1$, $A\xi = \xi = B\xi$, $\forall\, \xi \in \mathcal{K}$, $A\mathcal{K}_0 \subset \mathcal{K}_1$, $B\mathcal{K}_1 \subset \mathcal{K}_0$. For if we have such A and B, then

$$\|(BA)^n\xi_0 - \xi_0'\| = \|(BA)^n\xi_0 - (BA)^n\xi_0'\| \leq \|\xi_0 - \xi_0'\| < \varepsilon$$

so that if ξ_0'' is a weak limit point of $\{(1/n)\sum_{k=1}^n (BA)^k\xi_0\}_n$ then $BA\xi_0'' = \xi_0''$, $\xi_0'' \in \mathcal{K}_0$ (because all $(BA)^k\xi_0$ belong to $\mathcal{K}_0$) and $\|\xi_0'' - \xi_0'\| < \varepsilon$. But $0 \leq A \leq 1$, $0 \leq B \leq 1$, $BA\xi_0'' = \xi_0''$ implies that $A\xi_0'' = \xi_0''$, $B\xi_0'' = \xi_0''$, so that $\xi_0'' \in \mathcal{K}_0 \cap \mathcal{K}_1$. Thus $e_1\xi_0'' = \xi_0''e_1$, $y\xi_0'' = \xi_0''y$, $\forall\, y \in T$, and since T and e_1 generate S we get

$x\xi_0'' = \xi_0'' x$, $\forall\, x \in S$. This shows that $\mathcal{H}$ has a nonzero vector commuting with S.

Finally, in order to construct A, B with the required properties, let $\{p_j^n\}_{1\leq j\leq k_n} \subset M_{0,1} = \langle M_0, e_1\rangle$, $\{q_i^k\}_{1\leq i\leq m} \subset M_0^{\mathrm{op}}$ be partitions of the identity such that if p^n, respectively q^k denote the spectral projection of $|\sum_j p_j^n e_2 p_j^n - \lambda 1|$, respectively $|\sum_i q_i^k e_1 q_i^k - \lambda 1|$, corresponding to the interval $[\varepsilon, \infty]$, where $\lambda = [M_0 : N_0]^{-1} = [M : N]^{-1}$, then $\tau(p^n) < (1/n)\min_j \tau(p_j^n)$ and $\tau(q^k) < (1/k)\min_i \tau(q_i^k)$ (cf. the Appendix in [Po2], or [Po9]). We claim that if A is a weak limit of the sequence of operators $\{\sum_j p_j^n \cdot p_j^n\}_n \subset \mathcal{B}(\mathcal{H})$ and B is a weak limit of $\{\sum_i q_i^k \cdot q_i^k\}_k \subset \mathcal{B}(\mathcal{H})$ then A, B do satisfy the required conditions. Indeed, since $p_j^n, q_i^k \in S_0$ we have

$$\sum_j p_j^n \xi p_j^n = \xi, \quad \sum_i q_i^k \xi q_i^k = \xi, \qquad \forall\, \xi \in \mathcal{K},\ \forall\, k, n.$$

Thus, $A\xi = \xi = B\xi$, $\forall\, \xi \in \mathcal{K}$. Since $p_j^n \in \langle M_0, e_N\rangle \subset N^{\mathrm{op}\prime} \cap S$ it follows that if $[y, \eta_0] = 0$, $\forall\, y \in T = M \vee M^{\mathrm{op}}$, then

$$\left[x^{\mathrm{op}}, \sum_j p_j^n \eta_0 p_j^n\right] = 0, \qquad \forall\, x^{\mathrm{op}} \in N^{\mathrm{op}}.$$

Thus,

$$[x^{\mathrm{op}}, A\eta_0] = 0, \qquad \forall\, x^{\mathrm{op}} \in N^{\mathrm{op}},\ \forall\, \eta_0 \in \mathcal{K}_0.$$

Let $\eta_0 \in \mathcal{K}_0$ with $\|\eta_0\| = 1$ and note that, since η_0 commutes with T and $T' \cap S = \mathbb{C}$, η_0 follows a trace vector for S. Let also $\xi \in \mathcal{H}$, and $x \in M_1 = \langle M, e_1\rangle$ and note that

$$\begin{aligned}
&\left\|\lambda^{-1} \sum_j p_j^n E_M^{M_1}(p_j^n x p_i^n) - x p_i^n\right\|_2 \\
&\quad = \lambda^{-1/2} \left\|\lambda^{-1} \sum_j p_j^n e_2 p_j^n x p_i^n e_2 - x p_i^n e_2\right\|_2 \\
&\quad \leq \lambda^{-1/2} \left\|(1 - p^n)\left(\lambda^{-1} \sum_j p_j^n e_2 p_j^n - 1\right)\right\| \, \|x p_i^n e_2\|_2 + \lambda^{-1/2} \|p^n\|_2 \\
&\quad \leq 2\lambda^{-1/2} (\|x\|/n) \|p_i^n\|_2.
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\left\| x \sum_i p_i^n \eta_0 p_i^n - \lambda^{-1} \sum_{i,j} p_j^n E_M^{M_1}(p_j^n x p_i^n) \eta_0 p_i^n \right\|^2 \\
&\quad = \sum_i \left\| x p_i^n \eta_0 p_i^n - \lambda^{-1} \sum_j p_j^n E_M^{M_1}(p_j^n x p_i^n) \eta_0 p_i^n \right\|^2 \\
&\quad \leq \sum_i \left\| x p_i^n - \lambda^{-1} \sum_j p_j^n E_M^{M_1}(p_j^n x p_i^n) \right\|_2^2 \\
&\quad \leq 4\lambda^{-1}(\|x\|^2/n^2) \sum_j \|p_i^n\|_2^2 = 4\lambda^{-1}\|x\|^2/n^2.
\end{aligned}$$

Similarly we get

$$\left\| \sum_i p_i^n \eta_0 p_i^n x - \lambda^{-1} \sum_{i,j} p_j^n \eta_0 E_M^{M_1}(p_j^n x p_i^n) p_i^n \right\|^2 \leq 4\lambda^{-1}\|x\|^2/n^2$$

as well.

But since $y\eta_0 = \eta_0 y$, $\forall\ y \in M \subset M \vee M^{\mathrm{op}}$, we have

$$\sum_i \left(\sum_j p_j^n E_M^{M_1}(p_j^n x p_i^n) \right) \eta_0 p_i^n = \sum_j p_j^n \eta_0 \left(\sum_i E_M^{M_1}(p_j^n x p_i^n) p_i^n \right)$$

so by the above estimates we get

$$\left\| x \sum_i p_i^n \eta_0 p_i^n - \sum_j p_j^n \eta_0 p_j^n x \right\| < 8\lambda^{-1/2}\|x\|/n \to 0.$$

Since A is a weak limit of $\{\sum_i p_i^n \cdot p_i^n\}_n$ it follows that $[x, A\eta_0] = 0$, thus $A\mathcal{K}_0 \subset \mathcal{K}_1$. Similar calculations show that $B\mathcal{K}_1 \subset \mathcal{K}_0$ and A, B are thus constructed. As we have previously shown, this was sufficient to ensure that $\mathcal{H}$ has a nonzero vector commuting with S. Thus S has the property T relative to T. Q.E.D.

We can now conclude with the following:

9.5. THEOREM. *Let $N_0 \subset M_0$ be an extremal inclusion of type* II_1 *factors such that $M_0 \underset{e_{N_0}}{\boxtimes} M_0^{\mathrm{op}}$ has the property* T *relative to $M_0 \vee M_0^{\mathrm{op}}$. Then $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ has* *the property* T *relative to* $M \vee M^{\mathrm{op}}$ *for any extremal inclusion* $N \subset M$ *with* $\mathcal{G}_{N,M} = \mathcal{G}_{N_0,M_0}$

Proof. Since $N_0 \subset M_0$ is embedded smoothly in $N_0^\omega \subset M_0^\omega$ and the two subfactors have the same higher relative commutants, by 9.4 it follows that $M_0^\omega \underset{e_{N_0^\omega}}{\boxtimes} M_0^{\omega\mathrm{op}}$ has the property T relative to $M_0^\omega \vee M_0^{\omega\mathrm{op}}$. But by [Po9], the inclusion of factors $N^{\mathcal{G}}(R) \subset M^{\mathcal{G}}(R)$, where $\mathcal{G} = \mathcal{G}_{N_0,M_0}$ and R is the hyperfinite II_1 factors, is also embedded as a commuting square with same higher relative commutants in $N_0^\omega \subset M_0^\omega$. Thus, by 9.2 it follows that $M^{\mathcal{G}}(R) \boxtimes (M^{\mathcal{G}}(R))^{\mathrm{op}}$ has the property T relative to $M^{\mathcal{G}}(R) \vee (M^{\mathcal{G}}(R))^{\mathrm{op}}$. But $N^{\mathcal{G}}(R) \subset M^{\mathcal{G}}(R)$ is included in $N^\omega \subset M^\omega$ as well ([Po9]), so $M^\omega \underset{e_{N^\omega}}{\boxtimes} M^{\omega\mathrm{op}}$ has the property T relative to $M^\omega \vee M^{\omega\mathrm{op}}$, by 9.4. Then, again by 9.2 it follows that $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ has the property T relative to $M \vee M^{\mathrm{op}}$. Q.E.D.

9.6. Definition. We say that a standard λ-lattice $\mathcal{G}$ *has the property* T if $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ has the property T relative to $M \vee M^{\mathrm{op}}$ for some (and thus all!) subfactor $N \subset M$ with $\mathcal{G}_{N,M} = \mathcal{G}$.

The following class of examples shows that our notion of property T agrees with Kazhdan's classical notion for groups.

9.7. Proposition. *Let* $\mathcal{G}$ *be the standard* λ*-lattice of a locally trivial subfactor associated to some faithful* G*-kernel on some type* II_1 *factor. Then* $\mathcal{G}$ *has the property T if and only if the group* G *has the property* T.

Proof. Let P be the factor on which G acts and σ be the G-kernel on P. By Section 3 and 9.6, $\mathcal{G}$ has the property T iff $P \bar{\otimes} P^{\mathrm{op}} \rtimes_{\sigma \otimes \sigma^{\mathrm{op}}} G$ has the property T relative to $P \bar{\otimes} P^{\mathrm{op}}$. By ([A-D], [Po8]) this is equivalent to G having the property T. Q.E.D.

Let us next note some simple properties of this notion.

9.8. Proposition. (i) $\mathcal{G}$ *is both amenable and has the property* T *if and only if it has finite depth.*
(ii) *If* $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_1$ *(see part (b) in 5.6 for the definition) then* $\mathcal{G}$ *has the property* T *if and only if both* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *have the property* T.
(iii) $\mathcal{G}$ *has the property* T *if and only if* $\mathcal{G}^{\mathrm{op}}$ *has it.*
(iv) *If* $N \subset M$ *is an extremal inclusion* $\{M_i\}_i$ *is its tower, then* $\mathcal{G}_{N,M}$ *has the property* T *iff* $\mathcal{G}_{M_i,M_j}$ *has the property* T *for some* $i < j$ *iff* G_{M_i,M_j} *has the property* T *for all* $i < j$.

Proof. To prove (i), let $N \subset M$ be an extremal inclusion such that $\mathcal{G}_{N,M} = \mathcal{G}$. Then $\mathcal{G}$ is both amenable and has the property T iff $M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$ is both amenable and has the property T relative to $M \vee M^{\mathrm{op}}$. And by (4.1.4 in [Po8]) this is further equivalent to $[M \underset{e_N}{\boxtimes} M^{\mathrm{op}} : M \vee M^{\mathrm{op}}] < \infty$. But by 4.6, $[M \underset{e_N}{\boxtimes} M^{\mathrm{op}} : M \vee M^{\mathrm{op}}] < \infty$ is equivalent to $N \subset M$ having finite depth.

To prove (ii) let $N_j \subset M_j$, $j = 1, 2$, be such that $\mathcal{G}_{N_j,M_j} = \mathcal{G}_j$ and note that $\mathcal{G}_{N,M} = \mathcal{G}$ where $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$, $N = N_1 \bar{\otimes} N_2 \subset M_1 \bar{\otimes} M_2 = M$. Then $(T \subset S) = (T_1 \bar{\otimes} T_2 \subset S_1 \bar{\otimes} S_2)$, where $T \subset S$, $T_1 \subset S_1$, $T_2 \subset S_2$ are the symmetric enveloping inclusions associated with $N \subset M$, $N_1 \subset M_1$, respectively $N_2 \subset M_2$. If $T \subset S$ has the relative property T and $\{x_i\}_{1 \leq i \leq n} \subset S$ is its critical set for some $\varepsilon > 0$, then by [Po8] we may assume x_i are in the algebraic tensor product $S_1 \otimes S_2$, i.e., $x_i = \sum_j x^i_j \otimes y^i_j$, $x^i_j \in S_1$, $y^i_j \in S_2$. Let $\mathcal{H}_1$ be a S_1-S_1 bimodule with a unit vector $\xi_1 \in \mathcal{H}_1$ commuting with T_1 and δ_1-commuting with $\{x^i_j\}_{i,j}$. Denote by $\mathcal{H} = \mathcal{H}_1 \bar{\otimes} L^2(S_2)$, $\xi = \xi_1 \otimes \hat{1}$ and note that if δ_1 is sufficiently small then ξ ε-commutes with $\{x_i\}_i$. It follows that there exists $\xi' \in \mathcal{H}$, commuting with S at distance $K\varepsilon$ from ξ (see [Po8]), where K is a universal constant. But then, if $K\varepsilon < 1$, the projection ξ'' of ξ' onto $\mathcal{H}_1 \otimes \mathbb{C}1 \simeq \mathcal{H}_1$ is a nonzero vector commuting with S_1. This shows that S_1 has the property T relative to T_1. Similarly, S_2 has the property T relative to T_2.

Conversely, if S_i has property T relative to T_i for $i = 1, 2$ and $\mathcal{H}$ is a S-S bimodule with $\xi \in \mathcal{H}$ a unit vector commuting with $\{x_i \otimes y_j\}_{i,j}$, where $\{x_i\}_i \subset S_1$, $\{y_j\}_j \subset S_2$ are the critical sets for $T_1 \subset S_1$, respectively $T_2 \subset S_2$ then it follows that

$$\|u\xi - \xi u\| < \varepsilon, \qquad \forall\, u \in \mathcal{U}(S_1 \otimes 1) \cup \mathcal{U}(1 \otimes S_2).$$

Thus,

$$\|(u \otimes v)\xi - \xi(u \otimes v)\| < 2\varepsilon, \qquad \forall\, u \in \mathcal{U}(S_1),\ v \in \mathcal{U}(S_2).$$

A simple convexity argument in Hilbert space, or Ryll-Nardjewski's fixed point theorem then shows that there exists $\xi' \in \mathcal{H}$, $\|\xi' - \xi\| < 2\varepsilon$, commuting with all elements in the group $\mathcal{F} = \{u \otimes v \mid u \in \mathcal{U}(S_1),\ v \in \mathcal{U}(S_2)\}$. Since $\mathrm{sp}\, \mathcal{F} \supset S_1 \otimes S_2$ it follows that ξ' commutes with $S = S_1 \bar{\otimes} S_2$. Taking $\varepsilon < 1/2$, this shows that $\mathcal{H}$ has a nonzero vector commuting with S, so S has the property T relative to T.

To prove (iii) we only need to remark that the symmetric enveloping inclusions associated to $N \subset M$ and $N^{\mathrm{op}} \subset M^{\mathrm{op}}$ are identical, so that 9.5 applies to get that $\mathcal{G}_{N,M}$ has T iff $\mathcal{G}_{N^{\mathrm{op}},M^{\mathrm{op}}}$ $(= (\mathcal{G}_{N,M})^{\mathrm{op}})$ has this property.

Finally, to prove (iv) recall from [Po8] that if $V_0 \subset V \subset U$ are inclusions of factors and $[V : V_0] < \infty$ then U has the property T relative to V iff U has the property T relative to V_0. Thus, if $N \subset M$ is an extremal inclusion and we put $U = M \underset{e_N}{\boxtimes} M^{\mathrm{op}}$, $V = M \vee M^{\mathrm{op}}$, $V_0 = M \vee N^{\mathrm{op}}$, $V_1 = M_1 \vee N^{\mathrm{op}}$, it follows that U has the property T relative to V iff U has the property T relative V_1. But $V_1 \subset U$ is a reduced of the symmetric enveloping inclusion for $M \subset M_1$ (cf. 2.6) so, by [Po8] again, it has the relative property T iff $M_1 \vee M_1^{\mathrm{op}} \subset M_1 \underset{e_N}{\boxtimes} M_1^{\mathrm{op}}$ has relative propert T. Thus, $\mathcal{G}_{N,M}$ has T iff $\mathcal{G}_{M,M_1}$ has T. The rest follows from 2.6 a). Q.E.D.

We do not have more examples of property T standard λ-lattices other than the ones coming from groups (in 9.7) or the obvious ones that can be constructed by using jointly 9.7 and 9.8. For example, we do not know whether there exist standard lattices with the property T that come from irreducible subfactors. As for the minimal standard lattices generated by the Jones projections only, i.e., the so-called Temperley-Lieb-Jones standard lattices, we will prove below that generically they do not have the property T. This will in fact be an imediate corollary of the following more important consequence of 9.4:

9.9. Theorem. *Let $\mathcal{G}$ be a standard λ-lattice and $\mathcal{G}_0$ be a sublattice of $\mathcal{G}$. If $\mathcal{G}_0$ has the property* T *then $\mathcal{G}$ has the property* T. *Conversely, if $[\mathcal{G} : \mathcal{G}_0] < \infty$ and $\mathcal{G}$ has T then $\mathcal{G}_0$ has T.*

Proof. By [Po7] there exists a commuting square

$$\begin{array}{ccc} N & \subset & M \\ \cup & & \cup \\ N_0 & \subset & M_0 \end{array}$$

such that $\mathcal{G}_0 = \mathcal{G}_{N_0,M_0}$ and $\mathcal{G} = \mathcal{G}_{N,M}$. By 9.4 and the definition of property T for standard latices 9.6, it follows that if $\mathcal{G}_{N_0,M_0}$ has T then $\mathcal{G}_{N,M}$ has this property as well.
The last part is trivial, by [Po8], 2.7, 2.9 and 2.10. Q.E.D.

9.10. Corollary. *If a standard λ-lattice $\mathcal{G}_0$ is a sublattice of an amenable standard λ-lattice with infinite graph then $\mathcal{G}_0$ doesn't have the property* T. *In particular, if there exists an amenable subfactor of index λ^{-1} and infinite depth then the Temperley-Lieb-Jones standard lattice of graph A_∞ and index λ^{-1} does not have the property* T.

Proof. Trivial by 9.9. Q.E.D.

Let us end by mentioning a problem which at this point seems of interest:

9.11. *Problem.* Is it true that the property T for a standard lattice $\mathcal{G}$ only depends on its graph, i.e., if $\mathcal{G}, \mathcal{G}_0$ have the same (weighted) graph Γ and $\mathcal{G}$ has T, does it follow that $\mathcal{G}_0$ has T ? Note that in all the examples of property T standard lattices that we have in this paper (obtained by combining 9.7 with 9.8) this is indeed the case.
We strongly believe that this question has a positive answer. If this would be indeed the case, then one would have a notion of property T for standard graphs. We mention that in the combinatorial theory of groups there has been a steady interest towards generalizing the property T from groups to more general objects, in particular to (certain classes of) graphs. Since the standard lattices do generalize discrete groups and certain classes of Kac algebras and compact quantum groups ([Ba]), our definition of property T does provide a generalization along these lines.

Appendix

A.1. Relative Dixmier Property for Subfactors of Finite Index

We prove in this section a version for inclusions of type II_1 factors with finite Jones index of Dixmier's classical result on the norm closure of "averaging" elements by unitaries, as follows:

Theorem. *Let $N \subset M$ be an inclusion of factors of finite index. Then $N \subset M$ has the relative Dixmier property, i.e., for any $x \in M$, we have $\overline{\text{co}}^n\{uxu^* \mid u$ unitary element in $N\} \cap N' \cap M = \{E_{N'\cap M}(x)\}$.*

Proof. For $x \in M$ denote $C_N(x) = \overline{\text{co}}^n\{uxu^* \mid u \in \mathcal{U}(N)\}$. Since $E_{N'\cap M}(uxu^*) = E_{N'\cap M}(x), \forall u \in \mathcal{U}(N)$, it follows that $E_{N'\cap M}(y) = E_{N'\cap M}(x), \forall y \in C_N(x)$. Thus, if for some $x \in M$ we have $C_N(x) \cap N' \cap M \neq \emptyset$ then $C_N(x) \cap N' \cap M = \{E_{N'\cap M}(x)\}$.

By replacing if necessary x by $x - E_{N'\cap M}(x)$, it follows that it is sufficient to check that $0 \in C_N(x)$ for all $x \in M$ with $E_{N'\cap M}(x) = 0$. Moreover, by arguing like in the single algebra case ([D3]), it is sufficient to check this property for selfadjoint such elements x.

We will proceed by contradiction, assuming there exists an element $x = x^*$ in M, with $E_{N'\cap M}(x) = 0$, such that $0 \notin C_N(x)$. By the Hahn-Banach theorem there exists a functional $\Phi = \Phi^* \in M^*$ and $\varepsilon_0 > 0$ such that $\Phi(y) \geq \varepsilon_0,\ \forall y \in C_N(x)$. It follows that $\Psi(x) \geq \varepsilon_0,\ \forall \Psi \in \text{co}\{\Phi(u \cdot u^*) \mid u \in \mathcal{U}(N)\}$ so that $\Psi(x) \geq \varepsilon_0,\ \forall \Psi \in C_N(\Phi) \overset{\text{def}}{=} \overline{\text{co}}^{\sigma(M^*,M)}\{\Phi(u \cdot u^*) \mid u \in \mathcal{U}(N)\}$ and in fact $\Psi(y) \geq \varepsilon_0,\ \forall y \in C_N(x)$ as well.

To get to a contradiction we first show that there exists Ψ in $C_N(\Phi)$ which can be written as $\Psi = \Psi_1 - \Psi_2$, with $\Psi_{1,2}$ positive functionals on M which are scalar multiples of the trace τ when restricted to N. To this end let $\Phi = \Phi_1 - \Phi_2$ be the polar decomposition of Φ, into its positive and negative parts.

Let $\mathcal{V} = \{F \subset (N)_1 \mid F \text{ finite}\}$. By Dixmier's classical Theorem $\forall F \in \mathcal{V}, \exists u_F = (u_1^F, ..., u_{n_F}^F) \subset \mathcal{U}(N)$ such that $\|T_{u_F}(y) - \tau(y)1\| < 1/|F|, \forall y \in F$, where for $X \in M$ we denote $T_{u_F}(X) \overset{\text{def}}{=} (n_F)^{-1}\Sigma_j u_j^F X u_j^{F*}$. Then let ω be a free ultrafilter majorizing the filter $\mathcal{V}$ and for each $i = 1, 2$ define $\Psi_i(X) = \lim_{F\to\omega} \Phi_i(T_{u_F}(X))$, the limit being taken in the usual Banach sense. Then we clearly have $\Psi_{i|N} = c_i\tau_{|N}$, where $c_i = \Phi_i(1), i = 1, 2$. Also, if we let $\Psi = \Psi_1 - \Psi_2$ then $\Psi(X) = \lim_{F\to\omega} \Phi(T_{u_F}(X)), \forall X \in M$ and since $\Phi(T_{u_F}(\quad)) \in C_N(\Phi), \forall F$, it follows that Ψ belongs to $C_N(\Phi)$. Thus $\Psi = \Psi_1 - \Psi_2$ satisfies the desired conditions.

But by [PiPo1] we have $E_N(X) \geq \lambda X, \forall X \in M_+$, so by applying $\Psi_{1,2}$ to both sides we get $c_i\tau(X) = c_i\tau(E_N(X)) = \Psi_i(E_N(X)) \geq \lambda\Psi_i(X)$, implying that $\Psi_i \leq \lambda^{-1}c_i\tau,\ i = 1, 2$. Thus $\Psi_{1,2}$ actually follow normal on all M and so does Ψ. By Sakai's Radon-Nykodim type theorem there exists $a = a^* \in M$ such that $\Psi(X) = \tau(aX), \forall X \in M$. Putting this into the relation that Ψ satisfies gives $\tau(ya) \geq \varepsilon_0, \forall y \in C_N(x)$.

In particular, from the last relation and the trace property we get $\tau(xu^*au) = \tau(uxu^*a) \geq \varepsilon_0$. By taking convex combinations of elements of the form u^*au and weak limits and using that $\overline{\text{co}}^w\{u^*au \mid u \text{ unitary element in } N\} \cap N' \cap M = \{E_{N'\cap M}(a)\}$ (cf. [Po1]), we deduce that $\tau(xE_{N'\cap M}(a)) \geq \varepsilon_0$. But $\tau = \tau \circ E_{N'\cap M}$ and, since $E_{N'\cap M}(xE_{N'\cap M}(a)) = E_{N'\cap M}(x)E_{N'\cap M}(a)$ and x was assumed to satisfy $E_{N'\cap M}(x) = 0$, we obtain $0 \geq \varepsilon_0$ a contradiction which ends the proof of the theorem. Q.E.D.

A.2. A generalized version of Connes' perturbation theorem

In [C1] A. Connes proved a technical result about Hilbert norm perturbations of square integrable operators in semifinite von Neumann algebras.
We will use here a slight modification of his argument (essentially, of his "joint distribution trick") to derive the following version of his result, needed in the proof of Theorem 5.4:

A.2.1. Theorem. *Let P be a semifinite von Neumann algebra with a normal semifinite faithful trace denoted by* Tr. *Let Φ be a positive map on P satisfying the conditions:*
(1) $\Phi(1) = 1$, $\text{Tr} \circ \Phi \leq \text{Tr}$.
(2) $\sup\{\|\Phi(x)\|_{2,\text{Tr}} \mid x \in P,\ \|x\|_{2,\text{Tr}} \leq 1\} \leq 1$.
Let $\delta > 0$ be such that $\delta < (5)^{-4}$ and $b \in P_+$ satisfy the conditions:
(3) $\|b\|_{2,\text{Tr}} = 1$, $\|\Phi(b)\|_{2,\text{Tr}} \geq 1 - \delta$.
(4) $\|b - \Phi(b)\|_{2,\text{Tr}} < \delta$.
Then there exists $s > 0$ such that $\|e_s(b) - \Phi(e_s(b))\|_{2,\text{Tr}} < \delta^{1/4}\|e_s(b)\|$.

Proof. Like in [C1], let $X = \mathbb{R}_+^2 \setminus \{0\}$ and $H_0(x, y) = x$, $H_1(x, y) = y$. As on page 77 in [C1] it then follows that

$$\mu(A_0 \times A_1) \stackrel{\text{def}}{=} \text{Tr}(e_{A_0}(b)\Phi(e_{A_1}(b)))$$

for $A_i \subset \mathbb{R}_+$ Borel sets such that either $0 \neq \bar{A}_0$ or $0 \neq \bar{A}_1$, defines a Radon measure μ on X, which satisfies the properties:
a) $\|f(H_i)\|_{1,\mu} = \text{Tr}(\Phi_i(|f|(b)))$ (respectively $\|f(H_i)\|_{2,\mu}^2 = \text{Tr}(\Phi_i(|f|^2(b))) \leq \|f(b)\|_{2,\text{Tr}}^2$), for all $f : [0, \infty) \to \mathbb{C}$ Borel function with $f(0) = 0$ and $f(b) \in L^1(P, \text{Tr})$ (respectively $f(b) \in L^2(P, \text{Tr})$), $i = 0, 1$, where $\Phi_0 = id, \Phi_1 = \Phi$.
b) $\displaystyle\int_X f_0(H_0)\overline{f_1(H_1)}\text{d}\mu = \text{Tr}(f_0(b))\Phi(\bar{f}_1(b)))$, for all $f_i : [0, \infty) \to \mathbb{C}$ Borel with $f_i(0) = 0$ and $f_i(b) \in L^2(P, \text{Tr})$, $i = 0, 1$.
c) $\|f_0(H_0) - f_1(H_1)\|_{2,\mu} \geq \|f_0(b) - \Phi(f_1(b))\|_{2,\text{Tr}}$, $\forall\ f_i$ as in b).
d) $\|H_0 - H_1\|_{2,\mu}^2 = \text{Tr}(b^2) + \text{Tr}(\Phi(b^2)) - 2\text{Tr}(b\Phi(b)) \leq 6\delta$.
Indeed, a) and b) are clear by the proof of I.1 in [C1] and the definition of μ.

Further on, by a), b), (1), and Kadison's inequality we get:

$$\begin{aligned}\|f_0(H_0) - f_1(H_1)\|_{2,\mu}^2 &= \|f_0(H_0)\|_{2,\mu}^2 + \|f_1(H_1)\|_{2,\mu}^2 \\ &\quad - 2\mathrm{Re}\int_X f_0(H_0)\overline{f_1(H_1)}\mathrm{d}\mu \\ &= \mathrm{Tr}(f_0(b)^* f_0(b) + \mathrm{Tr}(\Phi(f_1(b)^* f_1(b))) \\ &\quad - 2\mathrm{Re}\ \mathrm{Tr}(f_0(b))\Phi(\overline{f_1}(b))) \\ &\geq \mathrm{Tr}(f_0(b)^* f_0(b_0)) \\ &\quad + \mathrm{Tr}(\Phi(f_1(b))^* \Phi(f_1(b))) \\ &\quad - 2\mathrm{Re}\ \mathrm{Tr}(f_0(b))\Phi(f_1(b))^*) \\ &= \|f_0(b_0) - \Phi(f_1(b))\|_{2,\mathrm{Tr}}^2.\end{aligned}$$

This proves c). Then d) is clear by noticing that the hypothesis and the Cauchy-Schwartz inequality imply:

$$\begin{aligned}&\mathrm{Tr}(b^2) + \mathrm{Tr}(\Phi(b^2)) - 2\mathrm{Tr}(b\Phi(b)) \\ &\quad \leq \mathrm{Tr}(b^2) + \mathrm{Tr}(b^2) - 2\mathrm{Tr}(b\Phi(b)) \\ &\quad = 2 - 2\mathrm{Tr}(b\Phi(b)) \\ &\quad \leq 2 - 2\mathrm{Tr}(b^2) + 2\delta \\ &\quad \leq 2(1 - (1-\delta)^2) + 2\delta \leq 6\delta\end{aligned}$$

Remark now that we have, like in proof of 1.2.6 in [C1], the estimate:

$$\begin{aligned}&\int_{\mathbb{R}_+^*} \|e_{t^{1/2}}(H_0) - e_{t^{1/2}}(H_1)\|_{2,\mu}^2 \mathrm{d}t \\ &\quad = \|H_0^2 - H_1^2\|_{1,\mu} \leq \|H_0 - H_1\|_{2,\mu}\|H_0 + H_1\|_{2,\mu}.\end{aligned}$$

But d) implies $\|H_0 - H_1\|_{2,\mu} \leq (6\delta)^{1/2}$ and a) implies $\|H_0 + H_1\|_{2,\mu} \leq \|H_0\|_{2,\mu} + \|H_1\|_{2,\mu} \leq \|b\|_{2,\mathrm{Tr}} + \|b\|_{2,\mathrm{Tr}} = 2$. Thus, by applying c) to the function $f = \chi_{[t^{1/2},\infty)}$, for each $t > 0$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}_+^*} \|e_{t^{1/2}}(b) - \Phi(e_{t^{1/2}}(b))\|_{2,\mathrm{Tr}}^2 \mathrm{d}t \\ (*) \qquad &\quad \leq 2(6\delta)^{1/2} = 2(6\delta)^{1/2} \int_{\mathbb{R}_+^*} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \mathrm{d}t.\end{aligned}$$

This implies that if we denote by D the set of all $t > 0$ for which

$$g(t) \stackrel{\mathrm{def}}{=} \|e_{t^{1/2}}(b_0) - \Phi(e_{t^{1/2}}(b))\|_{2,\mathrm{Tr}}^2 \mathrm{d}t < \delta^{1/4} \|e_{t^{1/2}}(b)\|_{2,\mathrm{Tr}}^2$$

then

$$\int_D \|e_{t^{1/}}(b)\|_{2,\mathrm{Tr}}^2 \mathrm{d}t \geq 1 - 5\delta^{1/4}.$$

Indeed, for if $\int_D \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \mathrm{d}t < 1 - 5\delta^{1/4}$, by taking into account that $g(t) \geq \delta^{1/4}\|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2$ for $t \in \mathbb{R}_+^* \setminus D$, we would get:

$$\begin{aligned}\int_{\mathbb{R}_+^*} g(t)\mathrm{d}t &\geq \int_{\mathbb{R}_+^*\setminus D} g(t)\mathrm{d}t \\ &\geq \delta^{1/4} \int_{\mathbb{R}_+^*\setminus D} \|e_{t^{1/2}}(b_0)\|_{2,\mathrm{Tr}}^2 \mathrm{d}t \\ &\geq 5\delta^{1/2} > 2(6\delta)^{1/2}.\end{aligned}$$

which is in contradiction with $(*)$.
In particular, since $\delta < 5^{-4}$, we have $1 - 5\delta^{1/4} > 0$ so that $D \neq \emptyset$. Thus, any $s > 0$ with $s^2 \in D$ will satisfy the condition in the conclusion. Q.E.D.

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