

SOME BOUNDEDNESS RESULTS FOR
FANO-LIKE MOISHEZON MANIFOLDS

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ABSTRACT. We prove finiteness of the number of smooth blow-downs on Fano manifolds and boundedness results for the geometry of non projective Fano-like manifolds. Our proofs use properness of Hilbert schemes and Mori theory.

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INTRODUCTION

In this Note, we say that a compact complex manifold X is a *Fano-like manifold* if it becomes Fano after a finite sequence of blow-ups along smooth connected centers, *i.e.* if there exist a Fano manifold \tilde{X} and a finite sequence of blow-ups along smooth connected centers $\pi : \tilde{X} \rightarrow X$. We say that a Fano-like manifold X is *simple* if there exists a smooth submanifold Y of X (Y may not be connected) such that the blow-up of X along Y is Fano. If Z is a projective manifold, we call *smooth blow-down of Z (with an s -dimensional center)* a map π and a manifold Z' such that $\pi : Z \rightarrow Z'$ is the blow-up of Z' along a smooth connected submanifold (of dimension s). We say that a smooth blow-down of Z is projective (resp. non projective) if Z' is projective (resp. non projective).

It is well-known that any Moishezon manifold becomes projective after a finite sequence of blow-ups along smooth centers. Our aim is to bound the geometry of *Moishezon manifolds becoming Fano after one blow-up along a smooth center*, *i.e.* the geometry of *simple non projective Fano-like manifolds*.

Our results in this direction are the following, the simple proof of Theorem 1 has been communicated to us by Daniel Huybrechts.

THEOREM 1. *Let Z be a Fano manifold of dimension n . Then, there is only a finite number of smooth blow-downs of Z .*

Let us recall here that the assumption Z Fano is essential : there are projective smooth surfaces with infinitely many -1 rational curves, hence with infinitely many smooth blow-downs.

Since there is only a finite number of deformation types of Fano manifolds of dimension n (see [KMM92] and also [Deb97] for a recent survey on Fano manifolds) and since smooth blow-downs are stable under deformations [Kod63], we get the following corollary (see section 1 for a detailed proof) :

COROLLARY 1. *There is only a finite number of deformation types of simple Fano-like manifolds of dimension n .*

The next result is essentially due to Wiśniewski ([Wis91], prop. (3.4) and (3.5)). Before stating it, let us define

$$d_n = \max\{(-K_Z)^n \mid Z \text{ is a Fano manifold of dimension } n\}$$

and

$$\rho_n = \max\{\rho(Z) := \text{rk}(\text{Pic}(Z)/\text{Pic}^0(Z)) \mid Z \text{ is a Fano manifold of dimension } n\}.$$

The number ρ_n is well defined since there is only a finite number of deformation types of Fano manifolds of dimension n and we refer to [Deb97] for an explicit bound for d_n .

THEOREM 2. *Let X be an n -dimensional simple non projective Fano-like manifold, Y a smooth submanifold such that the blow-up $\pi : \tilde{X} \rightarrow X$ of X along Y is Fano, and E the exceptional divisor of π . Then*

- (i) *if each component of Y has Picard number equal to one, then each component of Y has ample conormal bundle in X and is Fano. Moreover*

$$\deg_{-K_{\tilde{X}}}(E) := (-K_{\tilde{X}|E'})^{n-1} \leq (\rho_n - 1)d_{n-1};$$

- (ii) *if Y is a curve, then (each component of) Y is a smooth rational curve with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n-1}$.*

Finally, we prove here the following result :

THEOREM 3. *Let Z be a Fano manifold of dimension n and index r . Suppose there is a non projective smooth blow-down of Z with an s -dimensional center. Then*

$$r \leq (n-1)/2 \text{ and } s \geq r.$$

Moreover,

- (i) *if $r > (n-1)/3$, then $s = n-1-r$;*
 (ii) *if $r < (n-1)/2$ and $s = r$, then $Y \simeq \mathbb{P}^r$.*

Recall that the index of a Fano manifold Z is the largest integer m such that $-K_Z = mL$ for L in the Picard group of Z .

REMARKS.

- a) For a Fano manifold X of dimension n and index r with second Betti number greater than or equal to 2, it is known that $2r \leq n+2$ [Wi91], with equality if and only if $X \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$.

- b) Fano manifolds of even dimension (resp. odd dimension n) and middle index (resp. index $(n+1)/2$) with $b_2 \geq 2$ have been intensively studied, see for example [Wis93]. Our Theorem 3 shows that there are no non projective smooth blow-down of such a Fano manifold, without using any explicit classification.
- c) The assumption that there is a *non projective* smooth blow-down of Z is essential in Theorem 3 (i) : the Fano manifold obtained by blowing-up \mathbb{P}^{2r-1} along a \mathbb{P}^{r-2} has index r .

1. PROOF OF THEOREM 1 AND COROLLARY 1. AN EXAMPLE.

1.1. PROOF OF THEOREM 1. *Thanks to D. Huybrechts for the following proof.* Let Z be a Fano manifold and $\pi : Z \rightarrow Z'$ a smooth blow-down of Z with an s -dimensional connected center. Let f be a line contained in a non trivial fiber of π . Then, the Hilbert polynomial $P_{-K_Z}(m) := \chi(f, m(-K_Z)|_f)$ is determined by s and n since $-K_Z \cdot f = n - s - 1$ and f is a smooth rational curve. Since $-K_Z$ is ample, the Hilbert scheme Hilb_{-K_Z} of curves in Z having P_{-K_Z} as Hilbert polynomial is a projective scheme, hence has a finite number of irreducible components. Since each curve being in the component \mathcal{H} of Hilb_{-K_Z} containing f is contracted by π , there is only a finite number of smooth blow-downs of Z with an s -dimensional center. \square

1.2. PROOF OF COROLLARY 1. Let us first recall ([Deb97] section 5.2) that there exists an integer $\delta(n)$ such that every Fano n -fold can be realized as a smooth submanifold of \mathbb{P}^{2n+1} of degree at most $\delta(n)$. Let us denote by T a closed irreducible subvariety of the disjoint union of Chow varieties of n -dimensional subvarieties of \mathbb{P}^{2n+1} of degree at most $\delta(n)$, and by $\pi : \mathcal{X}_T \rightarrow T$ the universal family.

Step 1 : Stability of smooth blow-downs. Fix t_0 in the smooth locus T_{smooth} of T and suppose that $X_{t_0} := \pi^{-1}(t_0)$ is a Fano n -fold and there exists a smooth blow-down of X_{t_0} (denote by E_{t_0} the exceptional divisor, P its Hilbert polynomial with respect to $\mathcal{O}_{\mathbb{P}^{2n+1}}(1)$). Let S be the component of the Hilbert scheme of $(n-1)$ -dimensional subschemes of \mathbb{P}^{2n+1} with Hilbert polynomial P and $u : \mathcal{E}_S \rightarrow S$ the universal family. Finally, let I be the following subscheme of $T \times S$:

$$I = \{(t, s) \mid u^{-1}(s) \subset X_t\}$$

and $p : I \rightarrow T$ the proper algebraic map induced by the first projection. Thanks to the analytic stability of smooth blow-downs due to Kodaira (see [Kod63], Theorem 5), the image $p(I)$ contains an analytic open neighbourhood of t_0 hence it also contains a Zariski neighbourhood of t_0 . Moreover, since exceptional divisors are rigid, the fiber $p^{-1}(t)$ is a single point for t in a Zariski neighbourhood of t_0 . Finally, we get algebraic stability of smooth blow-downs (the \mathbb{P}^r -fibered structure of exceptional divisor is also analytically stable - [Kod63], Theorem 4 - hence algebraically stable by the same kind of argument).

Step 2 : Stratification of T by the number of smooth blow-downs. For any integer $k \geq 0$, let us define

$$U_k(T) = \{t \in T_{smooth} \mid X_t \text{ is a Fano manifold and there exists at least } k \text{ smooth blow-downs of } X_t\};$$

and $U_{-1}(T) = T_{smooth}$. Thanks to Step 1, $U_k(T)$ is Zariski open in T , and thanks to Theorem 1,

$$\bigcap_{k \geq -1} U_k(T) = \emptyset.$$

Since $\{U_k(T)\}_{k \geq -1}$ is a decreasing sequence of Zariski open sets, by noetherian induction, we get that there exists an integer k such that $U_k(T) = \emptyset$ and we can thus define

$$k(T) := \max\{k \geq -1 \mid U_k(T) \neq \emptyset\}, \quad U(T) := U_{k(T)}(T).$$

Finally, we have proved that $U(T)$ is a non empty Zariski open set of T_{smooth} such that for every $t \in U(T)$, Z_t is a Fano n -fold with exactly $k(T)$ smooth blow-downs ($k(T) = -1$ means that for every $t \in T_{smooth}$, X_t is not a Fano manifold).

Now let $T_0 = T$, and T_1 be any closed irreducible component of $T_0 \setminus U(T_0)$. We get $U(T_1)$ as before and denote by T_2 any closed irreducible component of $T_1 \setminus U(T_1)$, and so-on. Again by noetherian induction, this process terminates after finitely many steps and we get a finite stratification of T such that each strata corresponds to an algebraic family of Fano n -folds with the same number of smooth blow-downs.

Step 3 : Conclusion. Since there is only a finite number of irreducible components in the Chow variety of Fano n -folds, each being finitely stratified by Step 2, we get a finite number of deformation types of simple Fano-like n -folds. \square

As it has been noticed by Kodaira, it is essential to consider only *smooth* blow-downs. A -2 rational smooth curve on a surface is, in general, not stable under deformations of the surface.

1.3. AN EXAMPLE. Before going further, let us recall the following well known example. Let Z be the projective 3-fold obtained by blowing-up \mathbb{P}^3 along a smooth curve of type $(3, 3)$ contained in a smooth quadric \mathcal{Q} of \mathbb{P}^3 . Let π denotes the blow-up $Z \rightarrow \mathbb{P}^3$. Then Z is a Fano manifold of index one and there are at least three smooth blow-downs of Z : π , which is projective, and two non projective smooth blow-downs consisting in contracting the strict transform \mathcal{Q}' of the quadric \mathcal{Q} along one of its two rulings (the normal bundle of \mathcal{Q}' in Z is $\mathcal{O}(-1, -1)$).

LEMMA 1. *There are exactly three smooth blow-downs of Z .*

PROOF : the Mori cone $NE(Z)$ is a 2-dimensional closed cone, one of its two extremal rays being generated by the class of a line f_π contained in a non trivial fiber of π , the other one, denoted by $[R]$, by the class of one of the two rulings

of \mathcal{Q}' (the two rulings are numerically equivalent, the corresponding extremal contraction consists in contracting \mathcal{Q}' to a singular point in a projective variety, hence is not a smooth blow-down). If E is the exceptional divisor of π , we have

$$E \cdot [f_\pi] = -1, E \cdot [R] = 3, \mathcal{Q}' \cdot [f_\pi] = 1, \mathcal{Q}' \cdot [R] = -1.$$

Now suppose there exists a smooth blow-down τ of Z with a 1-dimensional center, which is not one of the three previously described. Let L be a line contained in a non trivial fiber of τ , then since $-K_Z \cdot [L] = 1$, we have $[L] = a[f_\pi] + b[R]$ for some strictly positive numbers such that $a + b = 1$. Since we have moreover

$$\mathcal{Q}' \cdot [L] = a - b = 2a - 1 \in \mathbb{Z} \text{ and } E \cdot [L] = 3b - a = 3 - 4a,$$

we get $a = b = 1/2$. Therefore $\mathcal{Q}' \cdot [L] = 0$ hence L is disjoint from \mathcal{Q}' (it can not be contained in \mathcal{Q}' since $\mathcal{Q}'_{\mathcal{Q}'} = \mathcal{O}(-1, -1)$). It implies that there are two smooth blow-downs of Z with disjoint exceptional divisors, which is impossible since $\rho(Z) = 2$.

Finally, if there is a smooth blow-down $\tau : Z \rightarrow Z'$ of Z with a 0-dimensional center, then Z' is projective and τ is a Mori extremal contraction, which is again impossible since we already met the two Mori extremal contractions on Z . \square

2. NON PROJECTIVE SMOOTH BLOW-DOWNS ON A CENTER WITH PICARD NUMBER 1. PROOF OF THEOREM 2.

The proof of Theorem 2 we will give is close to Wiśniewski's one but we give two intermediate results of independent interest.

2.1. ON THE NORMAL BUNDLE OF THE CENTER. Let us recall that a smooth submanifold A in a complex manifold W is contractible to a point (i.e. there exists a complex space W' and a map $\mu : W \rightarrow W'$ which is an isomorphism outside A and such that $\mu(A)$ is a point) if and only if $N_{A/W}^*$ is ample (Grauert's criterion [Gra62]).

The following proposition was proved by Campana [Cam89] in the case where Y is a curve and $\dim(X) = 3$.

PROPOSITION 1. *Let X be a non projective manifold, Y a smooth submanifold of X such that the blow-up $\pi : \tilde{X} \rightarrow X$ of X along Y is projective. Then, for each connected component Y' of Y with $\rho(Y') = 1$, the conormal bundle $N_{Y'/X}^*$ is ample.*

Before the proof, let us remark that Y is projective since the exceptional divisor of π is.

PROOF OF PROPOSITION 1 : (following Campana) we can suppose that Y is connected. Let E be the exceptional divisor of π and f a line contained in a non trivial fiber of π . Since $E \cdot f = -1$, there is an extremal ray R of the Mori cone $\overline{\text{NE}}(\tilde{X})$ such that $E \cdot R < 0$. Since $E \cdot R < 0$, R defines an extremal ray of the Mori cone $\overline{\text{NE}}(E)$ which we still denote by R (even if $\overline{\text{NE}}(E)$ is not a subcone

of $\overline{NE}(\tilde{X})$ in general !). Since $\rho(Y) = 1$, we have $\rho(E) = 2$, hence $\overline{NE}(E)$ is a 2-dimensional closed cone, one of its two extremal rays being generated by f . Then :

- either R is not generated by f and $E|_E$ is strictly negative on $\overline{NE}(E) \setminus \{0\}$. In that case, $-E|_E = \mathcal{O}_E(1)$ is ample by Kleiman's criterion, which means that $N_{Y/X}^*$ is ample.
- or, R is generated by f . In that case, the Mori contraction $\varphi_R : \tilde{X} \rightarrow Z$ factorizes through $\pi :$

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\varphi_R} & Z \\ \pi \downarrow & \nearrow \psi & \\ X & & \end{array}$$

where $\psi : X \rightarrow Z$ is an isomorphism outside Y . Since the variety Z is projective and X is not, ψ is not an isomorphism and since $\rho(Y) = 1$, Y is contracted to a point by ψ , hence $N_{Y/X}^*$ is ample by Grauert's criterion. \square

Let us prove the following consequence of Proposition 1:

PROPOSITION 2. *Let X be a non projective manifold, Y a smooth submanifold of X such that the blow-up $\pi : \tilde{X} \rightarrow X$ of X along Y is projective with $-K_{\tilde{X}}$ numerically effective (nef). Then, each connected component Y' of Y with $\rho(Y') = 1$ is a Fano manifold.*

PROOF : we can suppose that Y is connected. Let E be the exceptional divisor of π . Since $-E|_E$ is ample by Proposition 1, the adjunction formula $-K_E = -K_{\tilde{X}|_E} - E|_E$ shows that $-K_E$ is ample, hence E is Fano. By a result of Szurek and Wiśniewski [SZW90], Y is itself Fano. \square

2.2. PROOF OF THEOREM 2. For the first assertion, we only have to prove that

$$\text{deg}_{-K_{\tilde{X}}}(E) \leq (\rho_n - 1)d_{n-1}.$$

Let Y' be a connected component of Y and $E' = \pi^{-1}(Y')$. Then, since $-E|_{E'}$ is ample :

$$\text{deg}_{-K_{\tilde{X}}}(E') = (-K_{\tilde{X}|_{E'}})^{n-1} = (-K_{E'} + E|_{E'})^{n-1} \leq (-K_{E'})^{n-1} \leq d_{n-1}.$$

Now, if m is the number of connected components of Y , then

$$\rho_n \geq \rho(\tilde{X}) = m + \rho(X) \geq m + 1.$$

Putting all together, we get

$$\text{deg}_{-K_{\tilde{X}}}(E) \leq (\rho_n - 1)d_{n-1},$$

which ends the proof of the first point.

We refer to [Wis91] prop. (3.5) for the second point. \square

3. ON THE DIMENSION OF THE CENTER OF NON PROJECTIVE SMOOTH BLOW-DOWNS. PROOF OF THEOREM 3.

Theorem 3 is a by-product of the more precise following statement and of Proposition 3 below :

THEOREM 4. *Let Z be a Fano manifold of dimension n and index r , $\pi : Z \rightarrow Z'$ be a non projective smooth blow-down of Z , $Y \subset Z'$ the center of π . Let f be a line contained in a non trivial fiber of π , then*

- (i) *if f generates an extremal ray of $\text{NE}(Z)$, then $\dim(Y) \geq (n - 1)/2$.*
- (ii) *if f does not generate an extremal ray of $\text{NE}(Z)$, then $\dim(Y) \geq r$. Moreover, if $\dim(Y) = r$, then Y is isomorphic to \mathbb{P}^r .*

In both cases (i) and (ii), Y contains a rational curve.

The proof relies on Wiśniewski's inequality (see [Wis91] and [AnW95]), which we recall now for the reader's convenience : let $\varphi : X \rightarrow Y$ be a Fano-Mori contraction (i.e $-K_X$ is φ -ample) on a projective manifold X , $\text{Exc}(\varphi)$ its exceptional locus and

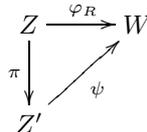
$$l(\varphi) := \min\{-K_X \cdot C ; C \text{ rational curve contained in } \text{Exc}(\varphi)\}$$

its length, then for every non trivial fiber F :

$$\dim \text{Exc}(\varphi) + \dim(F) \geq \dim(X) - 1 + l(\varphi).$$

PROOF OF THEOREM 4. The method of proof is taken from Andreatta's recent paper [And99] (see also [Bon96]).

First case : suppose that a line f contained in a non trivial fiber of π generates an extremal ray R of $\text{NE}(Z)$. Then the Mori contraction $\varphi_R : Z \rightarrow W$ factorizes through π :



where ψ is an isomorphism outside Y . In particular, the exceptional locus E of π is equal to the exceptional locus of the extremal contraction φ_R .

Let us now denote by ψ_Y the restriction of ψ to Y , $s = \dim(Y)$, π_E and $\varphi_{R,E}$ the restriction of π and φ_R to E . Since Z' is not projective, ψ_Y is not a finite map. Since φ_R is birational, W is \mathbb{Q} -Gorenstein, hence K_W is \mathbb{Q} -Cartier and $K_{Z'} = \psi^* K_W$. Therefore, $K_{Z'}$ is ψ -trivial, hence $K_Y + \det N_{Y/Z'}$ is ψ_Y -trivial. Moreover, $\mathcal{O}_E(1) = -E|_E$ is $\varphi_{R,E}$ -ample by Kleiman's criterion, hence $N_{Y/Z'}$ is ψ_Y -ample. Finally, ψ_Y is a Fano-Mori contraction, of length greater or equal to $n - s = \text{rk}(N_{Y/Z'})$. Together with Wiśniewski's inequality applied on Y , we get that for every non trivial fiber F of ψ_Y

$$2s \geq \dim(F) + \dim \text{Exc}(\psi_Y) \geq n - s + s - 1$$

hence $2s \geq n - 1$. Moreover, $\text{Exc}(\psi_Y)$ is covered by rational curves, hence Y contains a rational curve.

Second case : suppose that a line f contained in a non trivial fiber of π does not generate an extremal ray R of $\text{NE}(Z)$. In that case, since $E \cdot f = -1$, there is an extremal ray R of $\text{NE}(Z)$ such that $E \cdot R < 0$. In particular, the exceptional locus $\text{Exc}(R)$ of the extremal contraction φ_R is contained in E , and since f is not on R , we get for any fiber F of φ_R :

$$\dim(F) \leq s = \dim(Y).$$

By the adjunction formula, $-K_E = -K_{Z|E} - E|_E$, the length $l_E(R)$ of R as an extremal ray of E satisfies

$$l_E(R) \geq r + 1,$$

where r is the index of Z . Together with Wiśniewski's inequality applied on E , we get :

$$r + 1 + (n - 1) - 1 \leq s + \dim(\text{Exc}(R)) \leq s + n - 1.$$

Finally, we get $r \leq s$, and since the fibers of φ_R are covered by rational curves, there is a rational curve in Y . Suppose now (up to the end) that $r = s$. Then E is the exceptional locus of the Mori extremal contraction φ_R . Moreover, $K_Z + r(-E)$ is a good supporting divisor for φ_R , and since every non trivial fiber of φ_R has dimension r , φ_R is a smooth projective blow-down. In particular, the restriction of π to a non trivial fiber $F \simeq \mathbb{P}^r$ induces a finite surjective map $\pi : F \simeq \mathbb{P}^r \rightarrow Y$ hence $Y \simeq \mathbb{P}^r$ by a result of Lazarsfeld [Laz83].

This ends the proof of Theorem 4. \square

The proof of Theorem 4 does not use the hypothesis Z Fano in the first case. We therefore have the following :

COROLLARY 2. *Let Z be a projective manifold of dimension n , $\pi : Z \rightarrow Z'$ be a non projective smooth blow-down of Z , $Y \subset Z'$ the center of π . Let f be a line contained in a non trivial fiber of π and suppose f generates an extremal ray of $\overline{\text{NE}}(Z)$. Then $\dim(Y) \geq (n - 1)/2$. Moreover, if $\dim(Y) = (n - 1)/2$, then Y is contractible on a point.*

We finish this section by the following easy proposition, which combined with Theorem 4 implies Theorem 3 of the Introduction :

PROPOSITION 3. *Let Z be a Fano manifold of dimension n and index r , $\pi : Z \rightarrow Z'$ be a smooth blow-down of Z , $Y \subset Z'$ the center of π . Then $n - 1 - \dim(Y)$ is a multiple of r .*

PROOF. Write

$$-K_Z = rL \quad \text{and} \quad -K_Z = -\pi^*K_{Z'} - (n - 1 - \dim(Y))E$$

where E is the exceptional divisor of π . Let f be a line contained in a fiber of π . Then $rL \cdot f = n - 1 - \dim(Y)$, which ends the proof. \square

PROOF OF THEOREM 3. Let Z be a Fano manifold of dimension n and index r and suppose there is a non projective smooth blow-down of Z with an s -dimensional center. By Proposition 3, there is a strictly positive integer k such that $n - 1 - kr = s$. By Theorem 4, either $n - 1 - kr \geq (n - 1)/2$ or

$n - 1 - kr \geq r$. In both cases, it implies that $r \leq (n - 1)/2$ and therefore $s \geq r$. If $r > (n - 1)/3$, since $n - 1 \geq (k + 1)r > (k + 1)(n - 1)/3$, we get $k = 1$ and $s = n - 1 - r$. \square

4. RATIONAL CURVES ON SIMPLE MOISHEZON MANIFOLDS.

The arguments of the previous section can be used to deal with the following well-known question : *does every non projective Moishezon manifold contain a rational curve ?* The answer is positive in dimension three (it is due to Peternell [Pet86], see also [CKM88] p. 49 for a proof using the completion of Mori's program in dimension three).

PROPOSITION 4. *Let Z be a projective manifold, $\pi : Z \rightarrow Z'$ be a non projective smooth blow-down of Z . Then Z' contains a rational curve.*

PROOF. With the notations of the previous section, it is clear in the first case where a line f contained in a non trivial fiber of π generates an extremal ray R of $\overline{NE}(Z)$ (in that case, the center of π contains a rational curve). In the second case, since f is not extremal and K_Z is not nef, there is a Mori contraction φ on Z such that any rational curve contained in a fiber of φ is mapped by π to a non constant rational curve in Z' . \square

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