New Uncertainty Principles for the Continuous Gabor Transform and the Continuous Wavelet Transform

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Abstract. Gabor and wavelet methods are preferred to classical Fourier methods, whenever the time dependence of the analyzed signal is of the same importance as its frequency dependence. However, there exist strict limits to the maximal time-frequency resolution of these both transforms, similar to Heisenberg's uncertainty principle in Fourier analysis. Results of this type are the subject of the following article. Among else, the following will be shown: if ψ is a window function, $f \in L^2(\mathbf{R}) \setminus \{0\}$ an arbitrary signal and $G_\psi f(\omega, t)$ the continuous Gabor transform of f with respect to ψ, then the support of $G_\psi f(\omega, t)$ considered as a subset of the time-frequency-plane $\mathbf{R}^2$ cannot possess finite Lebesgue measure. The proof of this statement, as well as the proof of its wavelet counterpart, relies heavily on the well known fact that the ranges of the continuous transforms are reproducing kernel Hilbert spaces, showing some kind of shift-invariance. The last point prohibits the extension of results of this type to discrete theory.

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1 Introduction

One of the basic principles in classical Fourier analysis is the impossibility to find a function f being arbitrarily well localized together with its Fourier transform $\hat{f}$. There are many ways to get this statement precise. The most famous of them is the so called *Heisenberg uncertainty principle* [Heis27], a consequence of Cauchy-Schwarz's inequality (c.f. [Chan89], for example):

Given $f \in L^2(\mathbf{R}) \setminus \{0\}$ arbitrary, one has

$$\left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx\right)^{1/2} \left(\int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2 d\xi\right)^{1/2} \geq \frac{\|f\|^2_{L^2(\mathbf{R})}}{2}, \tag{1}$$

where equality holds if and only if there exist some constants $C \in \mathbf{C}$, $k > 0$ such that $f(x) = Ce^{-kx^2}$.

Completely different techniques lead to further restrictions of this type, e.g. methods of complex analysis to the theorems of Paley-Wiener and Hardy [Chan89, Hard33], and a study of the spectral properties of compact operators to the work of Slepian, Pollak and Landau [Slep65, LaWi80, Slep83]. The uncertainty principles of Lenard, Amrein, Berthier and Jauch [Lena72, BeJa76, AmBe77] are mainly consequences of the geometric properties of abstract Hilbert spaces. Additional considerations provide the articles of Cowling-Price [CoPr84] and Donoho-Stark [DoSt89]. And those are just a few aspects of uncertainty in harmonic analysis. Deeper insight can be won from the book of Havin and Jöricke [HaJo94].

The representation of f as a function of x is usually called its *time-representation*, while *frequency-representation* is another name for the Fourier transform $\hat{f}(\xi)$. For applications, one often needs information about the frequency-behaviour of a signal at a certain time (resp. the time-behaviour of a certain frequency-component of the signal). This lead to the construction of several *joint time-frequency representations*, among those the Gabor transform (3). The motivation for the wavelet transform (12) was of similar nature. However, the latter should preferably be called a joint time-*scale* representation, since the parameter a in (12) cannot completely be identified with an inverse frequency, as it is often done in the literature.

Bearing in mind the limits of classical Fourier transform, one cannot expect to achieve perfect phase-space resolution by using such joint representations. Even worse, additional perturbations of the original signal may be introduced by the window (resp. wavelet) function ψ. Precise estimates tackling exactly that point are rare in literature. Usually, the time-frequency-resolution of a Gabor (resp. wavelet) transform is identified with the time-frequency localization of the function ψ [Chui92]. This can be seen even more clearly from the discrete transforms: the famous uncertainty principles of Balian-Low for the discrete Gabor transform [Bali81, Daub90] and Battle for the discrete wavelet transform [Batt89, Batt97] just estimate the maximal time-frequency resolution of the window (resp. wavelet) function ψ under the restriction that the daughter functions of ψ span a frame (resp. an – in some suitable sense – orthogonal set). As for the continuous wavelet transform, Dahlke and Maaß [DaMa95] proved a Heisenberg-like inequality related to the affine group. It is not so obvious, however, what consequences for the phase-space localization of $W_\psi f$ follow from this result. Presumably, Daubechies [Daub88, Daub92] was the first to analyze the energy content of $G_\psi f$ (resp. $W_\psi f$) restricted to a proper subset M of phase-space. But she considered only very *special* functions ψ and subsets M of very *special* geometry, chosen in such a way that the arguments of Slepian, Pollak and Landau could widely be transferred.

In section 4 of this article, a similar investigation will be performed for quite *general* functions ψ and almost *arbitrary* subsets M of phase-space. By this, one cannot expect to get such precise results as Daubechies did. While she computed the whole spectrum of a suitably constructed compact operator, we just derive an upper bound for its eigenvalues. This suffices, however, to estimate the maximal energy content of $G_\psi f$ (resp. $W_\psi f$) in M. Before doing so, we show in section 3 that if M is a set of finite Lebesgue (resp. affine) measure, there is no $f \in L^2(\mathbf{R})$ such that $\operatorname{supp} G_\psi f \subseteq M$ (resp. $\operatorname{supp} W_\psi f \subseteq M$). Here, $\operatorname{supp} h$ denotes the support of a given function h. We finish this article with some conclusions following from Heisenberg's uncertainty principle.

The results presented here are part of the author's PhD thesis [Wilc97].

2 Prerequisites from the Theory of Gabor and Wavelet Transforms

This section shall serve as a reference. It provides some of the most important definitions and theorems from the theory of (continuous) Gabor and wavelet transforms. Further introductory information, and especially the proofs of the results presented here, can be found, e.g., in [Chui92, Daub92, Koel94].

In the following, we denote by $\lambda^{(n)}$ n-dimensional Lebesgue measure, by $\mathbf{R}^*$ the set of real numbers without zero and by χ_M the characteristic function of the set M. The Fourier-Plancherel transform of a function $f \in L^2(\mathbf{R})$ is normalized by

$$(\mathcal{F}f)(\xi) := \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \quad (\xi \in \mathbf{R}).$$

2.1 Basic Gabor Theory

Definition 2.1 (*Gabor transform)*
1. A *window function* is a function $\psi \in L^2(\mathbf{R}) \setminus \{0\}$.
2. Given a window function ψ and $(\omega, t) \in \mathbf{R}^2$, we define the *daughter function* $\psi_{\omega t}$ of ψ by

$$\psi_{\omega t}(x) := \frac{1}{\sqrt{2\pi}} \psi(x-t) e^{i\omega x}. \tag{2}$$

3. The *Gabor transform (GT)* of a function $f \in L^2(\mathbf{R})$ with respect to the window function ψ is defined by

$$G_\psi f : \mathbf{R}^2 \to \mathbf{C}, \quad \text{where}$$

$$G_\psi f(\omega, t) := \int_{-\infty}^{\infty} f(x)\overline{\psi_{\omega t}(x)}dx. \tag{3}$$

4. Given a window function ψ, we define an operator G_ψ acting on $L^2(\mathbf{R})$ by

$$G_\psi : \ f \mapsto G_\psi f.$$

G_ψ is called the *operator of the Gabor transform* or, shorter, *the Gabor transform* with respect to ψ.

Remark 2.2
1. Other names of the Gabor transform frequently used in the literature are *Weyl-Heisenberg transform, short time Fourier transform* and *windowed Fourier transform.*
2. If there is no danger of confusion, we drop the attribute *with respect to* ψ in the following.
3. From Plancherel's formula we get the *Fourier representations* of $G_\psi f$:

$$G_\psi f(\omega, t) = \mathcal{F}(f(x)\overline{\psi(x-t)})(\omega) = e^{-it\omega}\mathcal{F}(\hat{f}(\xi)\overline{\hat{\psi}(\xi-\omega)})(-t). \tag{4}$$

Denoting by $C_b(\mathbf{R}^2)$ the vector space of bounded continuous functions mapping $\mathbf{R}^2$ into $\mathbf{C}$, equipped with the maximum norm, we have

Theorem 2.3 (Covariance properties) *Let ψ be a window function. The Gabor transform G_ψ is a bounded linear operator from $L^2(\mathbf{R})$ to $C_b(\mathbf{R}^2)$ possessing the following covariance properties:*

for $f \in L^2(\mathbf{R})$ and $(\omega, t) \in \mathbf{R}^2$ arbitrary

$$[G_\psi f(\cdot - x_0)](\omega, t) = e^{-i\omega x_0} G_\psi f(\omega, t - x_0) \quad (x_0 \in \mathbf{R}), \tag{5}$$

$$\left[G_\psi(e^{i\omega_0 \cdot} f(\cdot))\right](\omega, t) = G_\psi f(\omega - \omega_0, t) \quad (\omega_0 \in \mathbf{R}). \tag{6}$$

Theorem 2.4 (Orthogonality relation) *Let ψ be a window function and $f, g \in L^2(\mathbf{R})$ arbitrary. Then we have*

$$\int_{-\infty}^{\infty} G_\psi f(\omega, t)\overline{G_\psi g(\omega, t)}d\omega dt = \|\psi\|^2_{L^2(\mathbf{R})}(f, g)_{L^2(\mathbf{R})}. \tag{7}$$

Corollary 2.5 (Isometry) *Let ψ be a window function. The normalized Gabor transform $\frac{1}{\|\psi\|_{L^2(\mathbf{R})}} G_\psi$ is an isometry from $L^2(\mathbf{R})$ into a subspace of $L^2(\mathbf{R}^2)$.*

Corollary 2.6 (Reproducing kernel) *Let ψ be a window function. Then $G_\psi(L^2(\mathbf{R}))$ is a reproducing kernel Hilbert space (r.k.H.s.) in $L^2(\mathbf{R}^2)$ with kernel function*

$$K_\psi(\omega', t'; \omega, t) := \frac{1}{\|\psi\|^2_{L^2(\mathbf{R})}} (\psi_{\omega t}, \psi_{\omega' t'})_{L^2(\mathbf{R})}. \tag{8}$$

The kernel is pointwise bounded:

$$|K_\psi(\omega', t'; \omega, t)| \leq 1 \qquad \forall\ (\omega', t'),\ (\omega, t) \in \mathbf{R}^2. \tag{9}$$

2.2 Basic Wavelet Theory

Definition 2.7 (*Wavelet transform)*
1. A function $\psi \in L^2(\mathbf{R}) \setminus \{0\}$ satisfying the *admissibility condition*

$$c_\psi := 2\pi \int_{-\infty}^{\infty} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|} < \infty \tag{10}$$

is called a *mother wavelet.*
2. Given a mother wavelet ψ and $(a, b) \in \mathbf{R}^* \times \mathbf{R}$, we define the *daughter wavelet* ψ_{ab} of ψ by

$$\psi_{ab}(x) := \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right). \tag{11}$$

3. The *wavelet transform (WT)* of a function $f \in L^2(\mathbf{R})$ with respect to the mother wavelet ψ is defined by

$$W_\psi f :\ \mathbf{R}^* \times \mathbf{R} \to \mathbf{C}, \quad \text{where}$$

$$W_\psi f(a, b) := \int_{-\infty}^{\infty} f(x) \overline{\psi_{ab}(x)} dx. \tag{12}$$

4. Given a mother wavelet ψ, we define an operator W_ψ acting on $L^2(\mathbf{R})$ *by*

$$W_\psi :\ f \mapsto W_\psi f.$$

W_ψ is called the *operator of the wavelet transform* or, shorter, *the wavelet transform* with respect to ψ.

Remark 2.8
From Plancherel's formula we get the *Fourier representation*

$$W_\psi f(a,b) = \mathcal{F}^{-1}(\sqrt{2\pi}\hat{f}(\xi)\sqrt{|a|}\overline{\hat{\psi}(a\xi)})(b). \tag{13}$$

Denoting by $C_b(\mathbf{R}^* \times \mathbf{R})$ the vector space of bounded continuous functions mapping $\mathbf{R}^2$ into $\mathbf{C}$, equipped with the maximum norm, we have

Theorem 2.9 (Covariance properties) *Let ψ be a mother wavelet. The wavelet transform W_ψ is a bounded linear operator from $L^2(\mathbf{R})$ to $C_b(\mathbf{R}^* \times \mathbf{R})$ possessing the following covariance properties:*

for $f \in L^2(\mathbf{R})$ and $(a,b) \in \mathbf{R}^* \times \mathbf{R}$ arbitrary

$$[W_\psi f(\cdot - x_0)](a,b) = W_\psi f(a, b - x_0) \quad (x_0 \in \mathbf{R}), \tag{14}$$

$$\left[W_\psi \left(\frac{1}{\sqrt{|c|}} f\left(\frac{\cdot}{c}\right)\right)\right](a,b) = W_\psi f\left(\frac{a}{c}, \frac{b}{c}\right) \quad (c \in \mathbf{R}^*). \tag{15}$$

Theorem 2.10 (Orthogonality relation) *Let ψ be a mother wavelet and $f, g \in L^2(\mathbf{R})$ arbitrary. Then we have*

$$\int_{-\infty}^{\infty} W_\psi f(a,b)\overline{W_\psi g(a,b)}\frac{dadb}{a^2} = c_\psi (f,g)_{L^2(\mathbf{R})}. \tag{16}$$

Corollary 2.11 (Isometry) *Let ψ be a mother wavelet. The normalized wavelet transform $\frac{1}{\sqrt{c_\psi}}W_\psi$ is an isometry from $L^2(\mathbf{R})$ into a subspace of $L^2(\mathbf{R}^* \times \mathbf{R}, d\mu_{aff})$, where $d\mu_{aff} := \frac{dadb}{a^2}$ denotes the so-called* affine measure.

Corollary 2.12 (Reproducing kernel) *Let ψ be a mother wavelet. Then $W_\psi(L^2(\mathbf{R}))$ is a r.k.H.s. in $L^2(\mathbf{R}^* \times \mathbf{R}, d\mu_{aff})$ with kernel function*

$$K_\psi(a',b';a,b) := \frac{1}{c_\psi}(\psi_{ab}, \psi_{a'b'})_{L^2(\mathbf{R})}. \tag{17}$$

The kernel is pointwise bounded:

$$|K_\psi(a',b';a,b)| \le \frac{\|\psi\|^2_{L^2(\mathbf{R})}}{c_\psi} \quad \forall\ (a',b'),\ (a,b) \in \mathbf{R}^* \times \mathbf{R}. \tag{18}$$

2.3 Group theoretical background

The parallel structures of the two foregoing sections suggest that Gabor and wavelet transform originate from a common root. As it is widely known, this root can be found in the theory of *unitary representations of locally compact groups*. Using the terminology of e.g. [GrMo85, HeWa89] we state one of the central results in that context:

Theorem 2.13 (Orthogonality relation) *Let G be a locally compact group with left Haar measure μ_L, $\mathcal{H}$ a complex Hilbert space and U a square integrable, irreducible, unitary representation of G on $\mathcal{H}$. Define*

$$\mathcal{A}_U := \{\psi \in \mathcal{H} : \ \psi \ is \ U - admissible\}, \tag{19}$$

where U-admissibility *of $\psi \in \mathcal{H}$ means*

$$0 < c_\psi^U := \int_G |(\psi, U(g)\psi)_{\mathcal{H}}|^2 d\mu_L(g) < \infty. \tag{20}$$

Then $\mathcal{A}_U$ is dense in $\mathcal{H}$, and there exists a unique positive operator $C_U : \mathcal{A}_U \to \mathcal{H}$ such that for all $\psi, \Psi \in \mathcal{A}_U$ and for all $f_1, f_2 \in \mathcal{H}$

$$\int_G (f_1, U(g)\psi)_{\mathcal{H}} \overline{(f_2, U(g)\Psi)_{\mathcal{H}}} d\mu_L(g) = (C_U\Psi, C_U\psi)_{\mathcal{H}} (f_1, f_2)_{\mathcal{H}}. \tag{21}$$

If G is unimodular, then C_U is a multiple of the identity operator.

Remark 2.14 Gabor transform is induced by a square-integrable, unitary, irreducible representation U_{WH} of the so called *Weyl-Heisenberg group* on $L^2(\mathbf{R})$. Here, U_{WH}-admissibility poses no additional restrictions: $A_{U_{WH}} = L^2(\mathbf{R})$.
Similarily, wavelet transform results from a representation U_{aff} of the *affine ("ax+b"-) group* on $L^2(\mathbf{R})$. In this case, U_{aff}-admissibility of a function $\psi \in L^2(\mathbf{R})$ corresponds to admissibility in the sense of *10.*
By this, covariance properties *5,6,14* and *15,* as well as the orthogonality relations *7,16* with corollaries are immediate consequences of group theory.
A helpful reference in the context of time-frequency distributions and group theory is the survey article of Miller *[Mill91].*

3 Restrictions on the Supports of Gabor and Wavelet Transforms

In 1977, Amrein and Berthier ([AmBe77], see also [HaJo94]) proved that the support of a function $f \in L^2(\mathbf{R}) \setminus \{0\}$ and the support of its Fourier transform $\hat{f}$ cannot both be sets of finite Lebesgue measure. Using the same techniques, we will show now that for any window function (resp. wavelet) ψ and any $f \in L^2(\mathbf{R}) \setminus \{0\}$ the support of the Gabor transform $G_\psi f$ (resp. wavelet transform $W_\psi f$) is a set of infinite Lebesgue (resp. affine) measure. As a preparation we need

Lemma 3.1 (Dimension of certain subspaces of a r.k.H.s.)
Let (Y, Σ_Y, μ_Y) be a σ-finite measure space, M a subset of Y with $\mu_Y(M) < \infty$, and $\mathcal{H} \subset L^2(Y, d\mu_Y)$ a r.k.H.s. with kernel K. Assuming that

$$\sup_{y', y \in Y} |K(y', y)| < \infty, \tag{22}$$

and defining

$$\mathcal{H}_M := \{F \in \mathcal{H} : \; F = \chi_M \cdot F\}, \tag{23}$$

the following estimate holds:

$$\dim \mathcal{H}_M \leq \left(\sup_{y',y \in Y} |K(y',y)| \right)^2 \mu_Y(M)^2 < \infty. \tag{24}$$

Proof: Using (22) and the finiteness of $\mu_Y(M)$ we get

$$\int_M \int_M |K(y',y)|^2 d\mu_Y(y') d\mu_Y(y) \leq \left(\sup_{y',y \in Y} |K(y',y)| \right)^2 \mu_Y(M)^2 < \infty, \tag{25}$$

hence, in particular, $K \in L^2(M \times M, d^2\mu_Y)$. Let $(e_n)_{n=1}^N$ ($N \in \overline{\mathbf{N}}$) be an arbitrary orthonormal family in $\mathcal{H}_M$, and define

$$\mathcal{E}_n(y',y) := e_n(y')\overline{e_n(y)} \quad (n \in \{1, \ldots, N\}).$$

Then for $m, n \in \{1, \ldots, N\}$

$$\begin{aligned} &\int_M \int_M \mathcal{E}_m(y',y)\overline{\mathcal{E}_n(y',y)} d\mu_Y(y') d\mu_Y(y) \\ = &\int_M \int_M e_m(y')\overline{e_m(y) e_n(y')} e_n(y) d\mu_Y(y') d\mu_Y(y) = \delta_{mn}, \end{aligned}$$

hence, $(\mathcal{E}_n)_{n=1}^N$ is an orthonormal family in $L^2(M \times M, d^2\mu_Y)$. Since we have shown that $K \in L^2(M \times M, d^2\mu_Y)$, Bessel's inequality, combined with the reproducing property of K, leads to

$$\begin{aligned} \|K\|^2_{L^2(M\times M, d^2\mu_Y)} &\geq \sum_{n=1}^N |(\mathcal{E}_n, K)_{L^2(M\times M, d^2\mu_Y)}|^2 \\ &= \sum_{n=1}^N |\int_M \int_M e_n(y')\overline{e_n(y) K(y',y)} d\mu_Y(y) d\mu_Y(y')|^2 \\ &= \sum_{n=1}^N |\int_M e_n(y')\overline{e_n(y')} d\mu_Y(y')|^2 = N. \end{aligned}$$

So, finally, (25) implies

$$N \leq \left(\sup_{y',y \in Y} |K(y',y)| \right)^2 \mu_Y(M)^2 < \infty,$$

and therefore each orthonormal set of $\mathcal{H}_M$ is finite. □

Now, choose M as a subset of $\mathbf{R}^2$ (resp. $\mathbf{R}^* \times \mathbf{R}$) and $\mathcal{H} = G_\psi(L^2(\mathbf{R})) \subset L^2(\mathbf{R}^2)$ (resp. $W_\psi(L^2(\mathbf{R})) \subset L^2\left(\mathbf{R}^* \times \mathbf{R}, \frac{da db}{a^2}\right)$). From section 2 we know that these two ranges are r.k.h.s with bounded kernels. Assuming, there exists at least one non-trivial function $F \in \mathcal{H}_M$, we will construct an infinite sequence of functions in $\mathcal{H}$ being linearly independent and supported in a set of finite measure. Since this is a contradicition to lemma 3.1, $\mathcal{H}_M$ must be zero space.

In the Gabor case, the construction is based on

Lemma 3.2 (Shifting lemma) *Let M, M_0 be two subsets of $\mathbf{R}^2$, $M_0 \subseteq M$, $\lambda^{(2)}(M_0) > 0$ and $\lambda^{(2)}(M) < \infty$. For $\omega_0 \in \mathbf{R}$ define*

$$M_0 - \omega_0 := \{(\omega, t) \in \mathbf{R}^2 : \ (\omega + \omega_0, t) \in M_0\}.$$

Then for each $\epsilon \in]0, \lambda^{(2)}(M_0)[$, there exists a real number $\omega^\epsilon \in \mathbf{R}$ such that

$$\lambda^{(2)}(M) < \lambda^{(2)}(M \cup (M_0 - \omega^\epsilon)) < \lambda^{(2)}(M) + \epsilon. \tag{26}$$

Proof: Consider the function

$$v: \ \mathbf{R} \to \mathbf{R}, \ \ \omega \mapsto \lambda^{(2)}(M \cup (M_0 - \omega)).$$

This function is continuous, since

$$\begin{aligned} v(\omega) &= \lambda^{(2)}(M) + \lambda^{(2)}(M_0) - \lambda^{(2)}(M \cap (M_0 - \omega)) \\ &= \lambda^{(2)}(M) + \lambda^{(2)}(M_0) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \chi_M(\tilde{\omega}, t) \cdot \chi_{M_0 - \omega}(\tilde{\omega}, t) d\tilde{\omega} dt \\ &= \lambda^{(2)}(M) + \lambda^{(2)}(M_0) - \int\int_M \chi_{M_0}(\tilde{\omega} + \omega, t) d\tilde{\omega} dt \\ &= \text{const.} - \|\chi_{M_0}(\cdot + \omega, \cdot)\|_{L^1(M)}, \end{aligned}$$

and $\lim_{|h| \to 0} \|f(\cdot + h, \cdot) - f(\cdot, \cdot)\|_{L^1(M)} = 0$ for every $f \in L^1(M)$ (cf. [Okik71], 3.6). Hence, evaluating v at two suitably chosen points and using the mean value theorem leads to assertion (26). Such points shall be constructed in the following.

From $M_0 \subseteq M$, one gets $v(0) = \lambda^{(2)}(M)$, and therefore the lower bound in relation (26).

Since $\lambda^{(2)}(M) < \infty$, given $\delta > 0$, there exists a bounded measurable subset M^δ of M such that $\lambda^{(2)}(M \setminus M^\delta) < \delta$ (cf. [EvGa92]). Choose $K^\delta > 0$ such that M^δ lies completely in the ball of radius K^δ centered at the origin. Put $\omega^\delta := 3K^\delta$. Then $M^\delta \cap (M^\delta + \omega^\delta) = \emptyset$, and

$$\begin{aligned} \int\int_M \chi_{M_0}(\omega + \omega^\delta, t) d\omega dt &\leq \int\int_{M^\delta} \chi_{M_0}(\omega + \omega^\delta, t) d\omega dt + \delta \\ = \int\int_{M^\delta + \omega^\delta} \chi_{M_0}(\tilde{\omega}, t) d\tilde{\omega} dt + \delta &\leq \int\int_{\mathbf{R}^n \setminus M^\delta} \chi_M(\tilde{\omega}, t) d\tilde{\omega} dt + \delta \leq 2\delta, \end{aligned}$$

hence, as before,

$$v(\omega^\delta) \geq \lambda^{(2)}(M) + \lambda^{(2)}(M_0) - 2\delta.$$

Now the mean value theorem shows that v takes all values between $\lambda^{(2)}(M)$ and $\lambda^{(2)}(M) + \lambda^{(2)}(M_0) - 2\delta$ with δ arbitrarily small. This proves the assertion. □

Theorem 3.3 *For any window function ψ and any set $M \subset \mathbf{R}^2$ of finite Lebesgue measure, we have*

$$G_\psi(L^2(\mathbf{R})) \cap \{F \in L^2(\mathbf{R}^2) : \ F = \chi_M \cdot F\} = \{0\}. \tag{27}$$

Proof: Let us assume, there exists a non-trivial function F_0 satisfying

$$F_0 \in G_\psi(L^2(\mathbf{R})) \cap \{F \in L^2(\mathbf{R}^2) : \ F = \chi_M \cdot F\}. \tag{28}$$

Let $M_0 \subseteq M$ denote the support of F_0, and choose $\epsilon \in]0, 2\lambda^{(2)}(M_0)[$ arbitrary. Using the notation of lemma 3.2 we define

$$\begin{aligned} M_1 &:= M, \\ M_2 &:= M_1 \cup (M_0 - \omega_1), \end{aligned}$$

where $\omega_1 \in \mathbf{R}$ is chosen such that

$$\lambda^{(2)}(M_1) < \lambda^{(2)}(M_2) < \lambda^{(2)}(M_1) + \epsilon \cdot 2^{-1},$$

and correspondingly for $k > 2$

$$M_k := M_{k-1} \cup (M_0 - \omega_1 - \cdots - \omega_{k-1}),$$

where $\omega_{k-1} \in \mathbf{R}$ satisfies

$$\lambda^{(2)}(M_{k-1}) < \lambda^{(2)}(M_k) < \lambda^{(2)}(M_{k-1}) + \epsilon \cdot 2^{-k+1}.$$

The existence of suitable translations $\omega_{k-1} \in \mathbf{R}$ is guaranteed by lemma 3.2, since $M_0 \subseteq M_1 \subset M_2 \subset \cdots \subset M_{k-2} \subset M_{k-1}$. Let $M^* := \bigcup_{k=1}^\infty M_k$. By construction

$$\lambda^{(2)}(M^*) \leq \lambda^{(2)}(M) + \epsilon \sum_{k=1}^\infty 2^{-k} = \lambda^{(2)}(M) + \epsilon.$$

Hence, $\lambda^{(2)}(M^*) < \infty$ for $\lambda^{(2)}(M) < \infty$. Let $F_1(\omega, t) := F_0(\omega, t)$, $F_k(\omega, t) := F_{k-1}(\omega + \omega_{k-1}, t)$ $(k \in \mathbf{N}, k > 1)$. Using the invariance property (6) of the Gabor transform, we see that $F_k \in G_\psi(L^2(\mathbf{R}))$ $(k \in \mathbf{N}, k > 1)$, and

$$\begin{aligned} \text{supp } F_k &= \text{supp } F_{k-1} - \omega_{k-1} \\ &= \text{supp } F_1 - \omega_1 - \cdots - \omega_{k-1} \\ &= M_0 - \omega_1 - \cdots - \omega_{k-1} \subseteq M_k \subset M^*. \end{aligned}$$

We now show the linear independence of the family $(F_k)_{k\geq 2}$. Let us assume, there exists a $k > 2$ such that

$$F_k = \sum_{\tilde{k}=2}^{k-1} a_{\tilde{k}} F_{\tilde{k}} \tag{29}$$

for some suitably chosen coefficients $a_2, a_3, \dots, a_{k-1} \in \mathbf{R}$. Then,

$$\text{supp } F_k \subseteq \bigcup_{\tilde{k}=2}^{k-1} \text{supp } F_{\tilde{k}},$$

and hence

$$\begin{aligned} &M_0 - \omega_1 - \cdots - \omega_{k-1} \\ &\quad \subseteq \{(M_0 - \omega_1) \cup (M_0 - \omega_1 - \omega_2) \cup \cdots \cup (M_0 - \omega_1 - \omega_2 - \cdots - \omega_{k-2})\} \\ &\quad \subseteq M_{k-1}. \end{aligned}$$

On the other hand, $\lambda^{(2)}(M_k) > \lambda^{(2)}(M_{k-1})$ implies that $M_k = M_{k-1} \cup (M_0 - \omega_1 - \cdots - \omega_{k-1})$ is a real superset of M_{k-1}. So, $M_0 - \omega_1 - \cdots - \omega_{k-1}$ cannot be a subset of M_{k-1}. Therefore, a linear combination of type (29) is not possible, and hence $(F_k)_{k\geq 2}$ is an infinite set of linearly independent functions with supp $F_k \subset M^*$, where $\lambda^{(2)}(M^*) < \infty$. From section 2 we know that $G_\psi(L^2(\mathbf{R}))$ is a r.k.H.s. with pointwise bounded kernel. Hence, following lemma 3.1, each subspace of $G_\psi(L^2(\mathbf{R}))$ consisting of functions supported on a set of finite measure must be of finite dimension. This shows that assumption (28) was wrong. □

From theorem 3.3 we get immediately

Corollary 3.4 (The support of a GT has infinite measure)
Let ψ be a window function. Then, for $f \in L^2(\mathbf{R}) \setminus \{0\}$ arbitrary, the support of $G_\psi f$ is a set of infinite Lebesgue measure.

Remark 3.5
Recalling the definition of the *cross-ambiguity function* of $f, g \in L^2(\mathbf{R})$

$$A(f,g)(\omega,t) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega\tilde{x}} f\left(\tilde{x} + \frac{t}{2}\right) \overline{g\left(\tilde{x} - \frac{t}{2}\right)} d\tilde{x}, \tag{30}$$

and rewriting (30) by

$$A(f,g)(\omega,t) = e^{-\frac{i\omega t}{2}} G_g f(-\omega, t), \tag{31}$$

we may conclude that supp $A(f,g)$ is of infinite measure, unless $f = 0$ or $g = 0$. This answers a question posed by Folland and Sitaram [FoSi97] which has been considered independently by Jaming [Jami98] and Janssen [Jans98]. Their proofs are based on the Fourier uncertainty principle of Benedicks [Bene85]. Using the same principle, Janssen disproved the existence of an *half-space* in $\mathbf{R}^2$ containing a finitely measured part of $\operatorname{supp} A(f,g)$ unless $f = 0$ or $g = 0$. Assuming f, g to be real-valued, this is a corollary to theorem 3.3, for $\lambda^{(2)}(\operatorname{supp} G_\psi f(\omega,t)|_{\omega<0}) < \infty$ implies $\lambda^{(2)}(\operatorname{supp} \overline{G_\psi f(\omega,t)}|_{\omega<0}) < \infty$, and therefore $\lambda^{(2)}(\operatorname{supp} G_\psi f(\omega,t)|_{\omega>0}) < \infty$, hence $\lambda^{(2)}(\operatorname{supp} G_\psi f(\omega,t)) < \infty$, where $\{(\omega,t) \in \mathbf{R}^2 \ : \omega < 0\}$ is representative for any subspace of $\mathbf{R}^2$ (cf. [Jans98]). In case $f = g$, complex values are admissible, as well.

Looking more closely at the proof of theorem 3.3 we find as its main ingredients

- a r.k.H.s. in an L^2-space with a pointwise bounded reproducing kernel,
- translation invariance in at least one fixed direction.

Consequently, results of this type hold in a much wider sense:

Theorem 3.6 (Abstract version) *Let $\mathcal{H}$ be a r.k.H.s. consisting of functions on $\mathbf{R}^n$ which are square-integrable with respect to Lebesgue measure. Assume, the reproducing kernel K of $\mathcal{H}$ is bounded. Let $U \neq \{0\}$ be a subspace of $\mathbf{R}^n$ such that $F \in \mathcal{H}$, $u \in U$ imply $F(\cdot - u) \in \mathcal{H}$. Then, for each $F \in \mathcal{H}$, one has $\lambda^{(n)}(\operatorname{supp} F) = \infty$.*

To obtain a corresponding result for the wavelet transform, we need an affine version of the shifting lemma 3.2. Using μ_{aff} instead of Lebesgue measure, we find analogously:

Lemma 3.7 (Affine shifting lemma) *Let M, M_0 be two subsets of $\mathbf{R}^* \times \mathbf{R}$, $M_0 \subseteq M$, $\mu_{aff}(M_0) > 0$ and $\mu_{aff}(M) < \infty$. For $b_0 \in \mathbf{R}$ define*

$$M_0 - b_0 := \{(a,b) \in \mathbf{R}^* \times \mathbf{R} : \ (a, b + b_0) \in M_0\}.$$

Then, for each $\epsilon \in]0, \mu_{aff}(M_0)[$, there exists a number $b^\epsilon \in \mathbf{R}$ such that

$$\mu_{aff}(M) < \mu_{aff}(M \cup (M_0 - b^\epsilon)) < \mu_{aff}(M) + \epsilon. \tag{32}$$

Hence, using (14) we can conclude as before

Theorem 3.8 *For any wavelet ψ and any set $M \subset \mathbf{R}^* \times \mathbf{R}$ of finite affine measure, we have*

$$W_\psi(L^2(\mathbf{R})) \cap \{F \in L^2(\mathbf{R}^* \times \mathbf{R}, d\mu_{aff}) : \ F = \chi_M \cdot F\} = \{0\}. \tag{33}$$

Corollary 3.9 (The support of a WT has infinite measure)
Let ψ be a wavelet. Then, for $f \in L^2(\mathbf{R}) \setminus \{0\}$ arbitrary, the support of $W_\psi f$ is a set of infinite affine measure.

Remark 3.10 There is no such result for *discrete* Gabor resp. wavelet transforms related to *orthonormal bases* [1]*:*
Let $(\psi_{jk})_{j,k\in\mathbf{Z}}$ *be an orthonormal wavelet basis in* $L^2(\mathbf{R})$ *and* $f = \psi = \psi_{00}$. *Then*

$$f = \sum_{j,k\in\mathbf{Z}} (f, \psi_{jk})_{L^2(\mathbf{R})}\psi_{jk} = \sum_{j.k\in\mathbf{Z}} \delta_{jk}\psi_{jk},$$

hence, there is just one *non-vanishing wavelet coefficient.*
This is a consequence of the fact that there is no translation invariance in the discrete setting.

4 Approximative Concentration of Gabor and Wavelet Transforms

From the foregoing section we know that the Gabor transform $G_\psi f$ of a function $f \in L^2(\mathbf{R}) \setminus \{0\}$ cannot possess a support of finite Lebesgue measure. In the following we will show that the portion of $G_\psi f$ lying outside some set M of finite Lebesgue measure cannot be arbitrarily small, either. For sufficiently *small* M, this can be seen immediately by estimating the Hilbert-Schmidt norm of a suitably defined operator. Taking into account some geometric properties of abstract Hilbert spaces, we find that restrictions of this kind hold for *arbitrary* sets of finite Lebesgue measure. More precise results going in that direction can be found by Daubechies [Daub88, Daub92], but only for *special* window functions ψ and *special* sets M.
The wavelet transform is treated in an analogous manner.

Theorem 4.1 (Concentration of $G_\psi f$ in small sets) *Let* ψ *be a window function and* $M \subset \mathbf{R}^2$ *with* $\lambda^{(2)}(M) < 1$. *Then, for* $f \in L^2(\mathbf{R})$ *arbitrary,*

$$\|G_\psi f - \chi_M \cdot G_\psi f\|_{L^2(\mathbf{R}^2)} \geq \|\psi\|_{L^2(\mathbf{R})}(1 - \lambda^{(2)}(M)^{1/2})\|f\|_{L^2(\mathbf{R})}. \tag{34}$$

Proof: Define $P_R : L^2(\mathbf{R}^2) \to L^2(\mathbf{R}^2)$ as the orthogonal projection from $L^2(\mathbf{R}^2)$ onto $G_\psi(L^2(\mathbf{R}))$, and $P_M : L^2(\mathbf{R}^2) \to L^2(\mathbf{R}^2)$ as the orthogonal projection from $L^2(\mathbf{R}^2)$ onto the subspace of functions supported in M. From corollary 2.5 we obtain

$$\begin{aligned} &\frac{1}{\|\psi\|_{L^2(\mathbf{R})}} \|G_\psi f - \chi_M \cdot G_\psi f\|_{L^2(\mathbf{R}^2)} \\ = \;& \frac{1}{\|\psi\|_{L^2(\mathbf{R})}} \|G_\psi f - P_M P_R(G_\psi f)\|_{L^2(\mathbf{R}^2)} \\ \geq \;& (1 - \|P_M P_R\|)\|f\|_{L^2(\mathbf{R})}, \end{aligned}$$

hence

$$\|G_\psi f - \chi_M \cdot G_\psi f\|_{L^2(\mathbf{R}^2)} \geq \|\psi\|_{L^2(\mathbf{R})}(1 - \|P_M P_R\|)\|f\|_{L^2(\mathbf{R})}. \tag{35}$$

[1] For definitions see e.g. [Daub92].

Being the projection onto a r.k.H.s., P_R can be represented by [Sait88]

$$P_R: \ F \mapsto P_R F(\omega, t) = (F(\omega', t'), K_\psi(\omega', t'; \omega, t))_{L^2(\mathbf{R}^2)}$$

with K_ψ defined by (8). Hence, for $F \in L^2(\mathbf{R}^2)$ arbitrary, we have

$$P_M P_R F(\omega, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_M(\omega, t) K_\psi(\omega', t'; \omega, t) F(\omega', t') d\omega' d' t$$

Therefore, the operator norm $\|P_M P_R\|$ can be estimated by the Hilbert-Schmidt norm $\|P_M P_R\|_{HS}$ (cf. [HaSu78]), using the fact that $\frac{1}{\|\psi\|_{L^2(\mathbf{R})}} G_\psi$ is an isometry:

$$\begin{aligned}
&\|P_M P_R\|_{HS}^2 \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\chi_M(\omega, t) K_\psi(\omega', t'; \omega, t)|^2 d\omega' dt' d\omega dt \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \chi_M(\omega, t) \frac{1}{\|\psi\|_{L^2(\mathbf{R})}^2} (\psi_{\omega t}, \psi_{\omega' t'})_{L^2(\mathbf{R})} \right|^2 d\omega' dt' d\omega dt \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| \chi_M(\omega, t) \frac{1}{\|\psi\|_{L^2(\mathbf{R})}^2} G_\psi \psi_{\omega t}(\omega', t') \right|^2 d\omega' dt' d\omega dt \\
=& \frac{1}{\|\psi\|_{L^2(\mathbf{R})}^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \iint_M \left| \frac{1}{\|\psi\|_{L^2(\mathbf{R})}} G_\psi \psi_{\omega t}(\omega', t') \right|^2 d\omega' dt' d\omega dt \\
=& \frac{1}{\|\psi\|_{L^2(\mathbf{R})}^2} \iint_M \left(\int_{-\infty}^{\infty} |\psi_{\omega t}(x)|^2 dx \right) d\omega dt \\
\leq& \frac{1}{\|\psi\|_{L^2(\mathbf{R})}^2} \|\psi\|_{L^2(\mathbf{R})}^2 \lambda^{(2)}(M) = \lambda^{(2)}(M).
\end{aligned}$$

Putting this into (35) proves the assertion. □

REMARK 4.2 Notice that the lower bound for $\|G_\psi f - \chi_M \cdot G_\psi f\|_{L^2(\mathbf{R}^2)}$ in (34) is the bigger the smaller $\lambda^{(2)}(M)$ is. This is in accordance with the philosophy of uncertainty.

REMARK 4.3 Using mean value theorem and Cauchy-Schwarz's inequality, one gets immediately the related result

$$\begin{aligned}
\|\chi_M \cdot G_\psi f\|_{L^2(\mathbf{R}^2)} &\leq \lambda^{(2)}(M)^{1/2} \|G_\psi f\|_{L^\infty(\mathbf{R})} \\
&\leq \|\psi\|_{L^2(\mathbf{R})} \lambda^{(2)}(M)^{1/2} \|f\|_{L^2(\mathbf{R})}
\end{aligned}$$

(cf. [FoSi97]). The use of the projections P_R and P_M in the proof of theorem 4.1 leads to further conclusions, however:

Remark 4.4 (Stable reconstruction from incomplete noisy data)
Let ψ be a window function, $M \subset \mathbf{R}^2$ with $\lambda^{(2)}(M) < 1$ and P_M the orthogonal projection from $L^2(\mathbf{R}^2)$ onto the subspace of functions supported on M. Then there exists a linear operator $R_{\psi,M} : L^2(\mathbf{R}^2) \to L^2(\mathbf{R}^2)$, as well as a constant $K^G_{\psi,M} > 0$ such that for all $F \in G_\psi(L^2(\mathbf{R}))$, for all $n \in L^2(\mathbf{R}^2)$ and

$$\tilde{F} := (\mathbf{1} - P_M)F + n \tag{36}$$

we have

$$\|F - R_{\psi,M}\tilde{F}\|_{L^2(\mathbf{R}^2)} \leq K^G_{\psi,M}\|n\|_{L^2(\mathbf{R}^2)}. \tag{37}$$

Interpretation:
The *original signal* F can be *stably reconstructed* from the *measured signal* $\tilde{F}$ affected with *noise* n using exclusively data from the complement of M. Here, *stability* has to be understood in the sense that the reconstruction error is proportional to the $L^2(\mathbf{R}^2)$-norm of the noise. If there is no noise at all $(n = 0)$, perfect reconstruction of F from $\tilde{F} := (\mathbf{1} - P_M)F$ is possible.
An upper bound for the constant $K^G_{\psi,M}$ in (37) is given by

$$K^G_{\psi,M} \leq \frac{1}{1 - \lambda^{(2)}(M)^{1/2}}. \tag{38}$$

The connection between this result and Gerchberg-Papoulis' algorithm[ByWe85, DoSt89] for the reconstruction of incomplete Fourier data will be treated elsewhere.

Proof of (37):
Choose $R_{\psi,M} := (\mathbf{1} - P_M P_R)^{-1}$ with P_R defined as in the proof of theorem 4.1. From there we know that $\|P_M P_R\|^2 \leq \lambda^{(2)}(M) < 1$, showing that the Neumann series $\sum_{n=0}^{\infty}(P_M P_R)^n$ is convergent. Hence, $(\mathbf{1} - P_M P_R)^{-1}$ is well-defined. Now,

$$\begin{aligned} \|F - R_{\psi,M}\tilde{F}\|_{L^2(\mathbf{R}^2)} &= \|F - R_{\psi,M}(\mathbf{1} - P_M)F - R_{\psi,M}n\|_{L^2(\mathbf{R}^2)} \\ &= \|F - R_{\psi,M}(\mathbf{1} - P_M P_R)F - R_{\psi,M}n\|_{L^2(\mathbf{R}^2)} \\ &= \|F - F - R_{\psi,M}n\|_{L^2(\mathbf{R}^2)} \leq \|R_{\psi,M}\| \cdot \|n\|_{L^2(\mathbf{R}^2)}, \end{aligned}$$

where

$$\begin{aligned} \|R_{\psi,M}\| &= \|\mathbf{1} - P_M P_R\|^{-1} \\ &\leq (1 - \|P_M P_R\|)^{-1} \\ &\leq (1 - \lambda^{(2)}(M)^{1/2})^{-1}. \ \square \end{aligned}$$

Correspondingly, we obtain for the wavelet transform:

Theorem 4.5 (Concentration of $W_\psi f$ in small sets) *Let ψ be a mother wavelet and $M \subset \mathbf{R}^* \times \mathbf{R}$ with*

$$\frac{\|\psi\|_{L^2(\mathbf{R})}}{\sqrt{c_\psi}} \mu_{aff}(M)^{1/2} < 1.$$

Then for $f \in L^2(\mathbf{R})$ arbitrary,

$$\begin{aligned} &\|W_\psi f - \chi_M \cdot W_\psi f\|_{L^2(\mathbf{R}^* \times \mathbf{R}, \frac{da\,db}{a^2})} \\ &\qquad \geq \sqrt{c_\psi} \left(1 - \frac{\|\psi\|_{L^2(\mathbf{R})}}{\sqrt{c_\psi}} \mu_{aff}(M)^{1/2}\right) \|f\|_{L^2(\mathbf{R})}. \end{aligned} \tag{39}$$

Remark 4.6 Assuming $\|\psi\|_{L^2(\mathbf{R})} = 1$ we find (34) independent of ψ, while $\sqrt{c_\psi}$ cannot be eliminated from (39).

Analogously, the following abstract version of theorems 4.1, 4.5 can be proved:

Theorem 4.7 (Abstract concentration theorem for small sets) *Let G be a locally compact group with left Haar measure μ_L, $\mathcal{H}$ a complex Hilbert space, U a square integrable, irreducible, unitary representation of G on $\mathcal{H}$ and C_U the operator from theorem 21. For $\psi \in \mathcal{H}$ U-admissible we define an operator*

$$T_\psi : \mathcal{H} \to L^2(G, \mu_L), \quad f \mapsto T_\psi f,$$

setting

$$T_\psi f(g) := (f, U(g)\psi)_{\mathcal{H}} \quad (g \in G).$$

Then, for $M \subset G$ with $\frac{\|\psi\|_{\mathcal{H}}}{\|C_U \psi\|_{\mathcal{H}}} \mu_L(M)^{1/2} < 1$ and $f \in \mathcal{H}$ arbitrary,

$$\|T_\psi f - \chi_M \cdot T_\psi f\|_{L^2(G, d\mu_L)} \geq \|C_U \psi\|_{\mathcal{H}} \left(1 - \frac{\|\psi\|_{\mathcal{H}}}{\|C_U \psi\|_{\mathcal{H}}} \mu_L(M)^{1/2}\right) \|f\|_{\mathcal{H}}. \tag{40}$$

Question 4.8 Are there restrictions similar to (34) (resp. (39)) for 'bigger' sets, as well? More precisely: given an arbitrary set M of finite Lebesgue (resp. affine) measure – do there exist any constants $C^G_{\psi,M}$ (resp. $C^W_{\psi,M}$)> 0 such that for $f \in L^2(\mathbf{R})$ arbitrary

$$\|G_\psi f - \chi_M \cdot G_\psi f\|_{L^2(\mathbf{R}^2)} \geq C^G_{\psi,M} \|f\|_{L^2(\mathbf{R})} \tag{41}$$

$$\text{(resp. } \|W_\psi f - \chi_M \cdot W_\psi f\|_{L^2(\mathbf{R}^2)} \geq C^W_{\psi,M} \|f\|_{L^2(\mathbf{R})}) \text{ ?} \tag{42}$$

Using an abstract result of Havin and Jöricke [HaJo94] we will see that the answer to this question is 'yes'. We will not be able to give an estimate for $C^G_{\psi,M}$, $C^W_{\psi,M}$ by the measure of M, however.

Lemma 4.9 (Havin-Jöricke) *Let $\mathcal{H}_1, \mathcal{H}_2$ be two closed subspaces of a Hilbert space $\mathcal{H}$ satisfying*

$$\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}. \tag{43}$$

Let $P_{\mathcal{H}_1}$, $P_{\mathcal{H}_2}$ denote the corresponding orthogonal projections, and assume the product $P_{\mathcal{H}_1} P_{\mathcal{H}_2}$ to be a compact operator. Then, there exists a constant $C > 0$ such that for all $f \in \mathcal{H}$

$$\|P_{\mathcal{H}_1^\perp} f\|_{\mathcal{H}} + \|P_{\mathcal{H}_2^\perp} f\|_{\mathcal{H}} \geq C \|f\|_{\mathcal{H}}. \tag{44}$$

Proof: Cf. [HaJo94] I.3 §1.2 □

Remark 4.10 Subspaces $\mathcal{H}_1, \mathcal{H}_2$ satisfying (43) are said to form an *annihilating pair* or, shorter, an *a-pair.* Subspaces satisfying the harder condition (44) are said to form a *strongly annihilating pair* or, shorter*, strong a-pair,* cf. [HaJo94]. From the same reference we know that condition *(44)* is equivalent to

$$\alpha(\mathcal{H}_1, \mathcal{H}_2) > 0,$$

where $\alpha(\mathcal{H}_1, \mathcal{H}_2)$ denotes the *angle* [2] between $\mathcal{H}_1$ and $\mathcal{H}_2$, defined as the real number in $[0, \frac{\pi}{2}]$ satisfying

$$\cos(\alpha(\mathcal{H}_1, \mathcal{H}_2)) = \sup\{|(f,g)_{\mathcal{H}}| : \ f \in \mathcal{H}_1, \ \|f\|_{\mathcal{H}} \leq 1, \ g \in \mathcal{H}_2, \ \|g\|_{\mathcal{H}} \leq 1\}.$$

The angle $\alpha(\mathcal{H}_1, \mathcal{H}_2)$ is related to the projections $P_{\mathcal{H}_1}, \ P_{\mathcal{H}_2}$ according to:

$$\cos(\alpha(\mathcal{H}_1, \mathcal{H}_2)) = \|P_{\mathcal{H}_1} P_{\mathcal{H}_2}\|, \tag{45}$$

cf. [HaJo94], I.3 §1.1. The optimal constant C in (44) is as a function of $\alpha(\mathcal{H}_1, \mathcal{H}_2)$.

Theorem 4.11 (Concentration of $G_\psi f$ in arbitrary sets of finite measure) *Let ψ be a window function and $M \subset \mathbf{R}^2$ with $\lambda^{(2)}(M) < \infty$. Then there exists a constant $C^G_{\psi,M} > 0$ such that for $f \in L^2(\mathbf{R})$ arbitrary (41) holds.*

Proof: Defining P_M,P_R as in the proof of theorem 4.1 and $\mathcal{H}_1, \mathcal{H}_2$ by

$$\mathcal{H}_1 \ := \ P_M(L^2(\mathbf{R}^2)), \tag{46}$$
$$\mathcal{H}_2 \ := \ P_R(L^2(\mathbf{R}^2)), \tag{47}$$

we conclude from theorem 3.3 that $\mathcal{H}_1$ and $\mathcal{H}_2$ form an a-pair. The proof of theorem 4.1 implies that for $M \subseteq \mathbf{R}^2$ arbitrary with $\lambda^{(2)}(M) < \infty$

$$\|P_M P_R\|_{HS} \leq (\lambda^{(2)}(M))^{1/2} < \infty.$$

[2] Cf. [Deut95] for more information on that subject.

Hence, $P_M P_R$ is a Hilbert-Schmidt operator and therefore compact, which means that $\mathcal{H}_1$, $\mathcal{H}_2$ form a strong a-pair. Now lemma 4.9 implies the existence of a constant $C > 0$ such that (44) holds for $P_{\mathcal{H}_1} := P_M$ and $P_{\mathcal{H}_2} := P_R$. Since $P_{\mathcal{H}_1^\perp}(G_\psi f) = (\mathbf{1} - P_R)G_\psi f = 0$, this leads to (41). □

Again, theorem 4.11 can be generalized to a wider class of transforms. Especially, we have the following wavelet counterpart:

Theorem 4.12 (Concentration of $W_\psi f$ in arbitrary sets of finite measure) *Let ψ be a mother wavelet and $M \subset \mathbf{R}^* \times \mathbf{R}$ with $\mu_{aff}(M) < \infty$. Then there exists a constant $C^W_{\psi,M} > 0$ such that for $f \in L^2(\mathbf{R})$ arbitrary (42) holds.*

The abstract version of theorem 4.11 is

Theorem 4.13 (Abstract concentration theorem for arbitrary sets) *Allowing $M \subset G$ with $\mu_L(M) < \infty$ arbitrary in the situation of theorem 4.7, there exists a constant $C^T_{\psi,M} > 0$ such that for all $f \in \mathcal{H}$*

$$\|T_\psi f - \chi_M \cdot T_\psi f\|_{L^2(G,d\mu_L)} \geq C^T_{\psi,M} \|f\|_{\mathcal{H}}. \tag{48}$$

5 Uncertainty Principles of Heisenberg Type

Up to now, we analyzed the concentration of $G_\psi f$ (resp. $W_\psi f$) as a function on two-dimensional phase-space. A different class of uncertainty principles results from comparing the localization of f (resp. $\hat{f}$) with the localization of its Gabor or wavelet transform regarded as function of *one*2 variable. Some results of that type, originating from an idea of Singer in the wavelet case [Sing92], will be presented in this final section.

Theorem 5.1 (UP of Heisenberg type for GT in ω) *Let ψ be a window function. Then, for $f \in L^2(\mathbf{R})$ arbitrary, the following inequality holds*

$$\left(\int_{-\infty}^{\infty} \omega^2 |G_\psi f(\omega,t)|^2 d\omega dt \right)^{1/2} \left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \right)^{1/2} \geq \frac{1}{2} \|\psi\|_{L^2(\mathbf{R})} \|f\|^2_{L^2(\mathbf{R})}. \tag{49}$$

Proof: Let us assume the non-trivial case that both integrals on the left hand side of (49) are finite. By translation invariance of Lebesgue integral we get

$$\begin{aligned} \|\psi\|^2_{L^2(\mathbf{R})} \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 |\psi(x-t)|^2 |f(x)|^2 dx dt \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 |\mathcal{F}^{-1}_\omega (G_\psi f(\omega,t))(x)|^2 dx dt, \end{aligned}$$

where $\mathcal{F}_\omega$ denotes Fourier transform with respect to the variable ω. Fixing $t \in \mathbf{R}$ arbitrary, Heisenberg's inequality implies

$$\left(\int_{-\infty}^{\infty} \omega^2 |G_\psi f(\omega,t)|^2 d\omega\right)^{1/2} \left(\int_{-\infty}^{\infty} x^2 |\mathcal{F}_\omega^{-1}(G_\psi f(\omega,t))(x)|^2 dx\right)^{1/2} \geq \frac{1}{2}\int_{-\infty}^{\infty} |G_\psi f(\omega,t)|^2 d\omega.$$

Integrating over t and using the inequality of Cauchy-Schwarz, as well as the isometry property of $\frac{1}{\|\psi\|_{L^2(\mathbf{R})}} G_\psi$, results in

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \omega^2 |G_\psi f(\omega,t)|^2 d\omega dt\right)^{1/2} \|\psi\|_{L^2(\mathbf{R})} \left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx\right)^{1/2} \\
&= \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \omega^2 |G_\psi f(\omega,t)|^2 d\omega dt\right)^{1/2} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 |\mathcal{F}_\omega^{-1}(G_\psi f(\omega,t))(x)|^2 dx dt\right)^{1/2} \\
&\geq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \omega^2 |G_\psi f(\omega,t)|^2 d\omega\right)^{1/2} \left(\int_{-\infty}^{\infty} x^2 |\mathcal{F}_\omega^{-1}(G_\psi f(\omega,t))(x)|^2 dx\right)^{1/2} dt \\
&\geq \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G_\psi f(\omega,t)|^2 d\omega dt = \frac{1}{2}\|\psi\|_{L^2(\mathbf{R})}^2 \|f\|_{L^2(\mathbf{R})}^2.
\end{aligned}$$

Dividing by $\|\psi\|_{L^2(\mathbf{R})}$ leads to (49). □

Remark 5.2 Note that the localization of ψ has no influence on (49).

Theorem 5.3 (UP of Heisenberg type for GT in t) *Let ψ be a window function. Then, for $f \in L^2(\mathbf{R})$ arbitrary, the following inequality holds*

$$\left(\int_{-\infty}^{\infty} t^2 |G_\psi f(\omega,t)|^2 d\omega dt\right)^{1/2} \left(\int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2 d\xi\right)^{1/2} \geq \frac{1}{2}\|\psi\|_{L^2(\mathbf{R})} \|f\|_{L^2(\mathbf{R})}^2. \tag{50}$$

Proof: Similiar to the proof of theorem 5.1 using the Fourier representation of $G_\psi f$. □

Corollary 5.4 (Phase space uncertainty of GT) *For ψ a window function, and $f \in L^2(\mathbf{R})$ arbitrary, we have*

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^2 |G_\psi f(\omega,t)|^2 d\omega dt\right)^{1/2} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \omega^2 |G_\psi f(\omega,t)|^2 d\omega dt\right)^{1/2} \cdot$$

$$\cdot \left(\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx\right)^{1/2} \left(\int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2 d\xi\right) \geq \frac{1}{4}\|\psi\|_{L^2(\mathbf{R})}\|f\|^4_{L^2(\mathbf{R})}.$$

Remark 5.5 Above corollary may be interpreted as follows: The better the phase space localization of the pair $(f, \hat{f})$, the worse is the phase space localization of the Gabor transform $G_\psi f(\omega, t)$.

Remark 5.6 The symmetry between f and ψ in the definition of Gabor transform leads to similar relations between $G_\psi f$ and ψ (resp. $\hat{\psi}$):

$$\left(\int_{-\infty}^{\infty} \omega^2 |G_\psi f(\omega,t)|^2 d\omega dt\right)^{1/2} \left(\int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx\right)^{1/2} \geq \frac{1}{2}\|f\|_{L^2(\mathbf{R})}\|\psi\|^2_{L^2(\mathbf{R})},$$

$$\left(\int_{-\infty}^{\infty} t^2 |G_\psi f(\omega,t)|^2 d\omega dt\right)^{1/2} \left(\int_{-\infty}^{\infty} \xi^2 |\hat{\psi}(\xi)|^2 d\xi\right)^{1/2} \geq \frac{1}{2}\|f\|_{L^2(\mathbf{R})}\|\psi\|^2_{L^2(\mathbf{R})}.$$

Theorem 5.7 (UP of Heisenberg type for the WT in b) *Let ψ be a mother wavelet. Then, for $f \in L^2(\mathbf{R})$ arbitrary,*

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} b^2 |W_\psi f(a,b)|^2 \frac{dadb}{a^2}\right)^{1/2} \left(\int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2 d\xi\right)^{1/2} \geq \frac{\sqrt{c_\psi}}{2}\|f\|^2_{L^2(\mathbf{R})}. \quad (51)$$

Proof: Similar to the proof of theorem 5.1. Assuming the existence of both integrals on the left hand side of (51), we get from the admissibilty condition (10) for ψ

$$2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi^2 |\hat{\psi}(a\xi)|^2 |\hat{f}(\xi)|^2 \frac{da}{|a|} d\xi = c_\psi \int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2 d\xi.$$

Using the Fourier representation of the wavelet transform (13), this implies

$$\int_{-\infty}^{\infty} \xi^2 |\mathcal{F}_b(W_\psi f(a,b))(\xi)|^2 \frac{da}{a^2} d\xi = c_\psi \int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2 d\xi. \quad (52)$$

On the other hand, Heisenberg's inequality leads to

$$\left(\int_{-\infty}^{\infty} b^{|}W_\psi f(a,b)|^2 db\right)^{1/2} \left(\int_{-\infty}^{\infty} \xi^2 |\mathcal{F}_b(W_\psi f(a,b))(\xi)|^2 d\xi\right)^{1/2}$$
$$\geq \frac{1}{2} \int_{-\infty}^{\infty} |W_\psi f(a,b)|^2 db$$

for all $a \in \mathbf{R}^*$. Integrating with respect to $\frac{da}{a^2}$ gives

$$\int_{-\infty}^{\infty} \left[\left(\int_{-\infty}^{\infty} b^2 |W_\psi f(a,b)|^2 db\right)^{1/2} \left(\int_{-\infty}^{\infty} \xi^2 |\mathcal{F}_b(W_\psi f(a,b))(\xi)|^2 d\xi\right)^{1/2} \right] \frac{da}{a^2}$$
$$\geq \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |W_\psi f(a,b)|^2 \frac{da}{a^2} db.$$

The left hand side of this inequality may be estimated from above using Cauchy-Schwarz's inequality. The right hand side can be rewritten by the isometry of $\frac{1}{\sqrt{c_\psi}} W_\psi$. From (52) we therefore get

$$\left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b^2 |W_\psi f(a,b)|^2 db \frac{da}{a^2}\right)^{1/2} \left(\int_{-\infty}^{\infty} \xi^2 |\mathcal{F}_b(W_\psi f(a,b))(\xi)|^2 d\xi \frac{da}{a^2}\right)^{1/2}$$
$$= \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} b^2 |W_\psi f(a,b)|^2 db \frac{da}{a^2}\right)^{1/2} \sqrt{c_\psi} \left(\int_{-\infty}^{\infty} \xi^2 |\hat{f}(\xi)|^2 d\xi\right)^{1/2}$$
$$\geq \frac{1}{2} c_\psi \|f\|^2_{L^2(\mathbf{R})}. \quad \square$$

Remark 5.8 There is not so much symmetry between the parameters a and b of the wavelet transform as there is symmetry between ω and t in the Gabor case. An uncertainty relation between $W_\psi f$ as a function of a and f as a function of x will be derived in the following using a slightly *modified definition of wavelet transform.* Making use of Kaiser's observation [Kais95] that "frequency filters" $\hat{f}(\xi) \mapsto w_f(\xi)\hat{f}(\xi)$ often correspond to "scale filters" $W_\psi f(a,b) \mapsto w_S(a) W_\psi f(a,b)$. Here, w_F, w_S denote some suitable filter functions.

Theorem 5.9 (UP of Heisenberg type for WT in a) *Let ψ be a mother wavelet, $\hat{\psi}(\xi) = 0$ for $\xi < 0$ and $f \in L^2(\mathbf{R}) \setminus \{0\}$ arbitrary. Consider* *the following modified definition of wavelet transform:*

$$\tilde{W}_\psi : \ f \mapsto \tilde{W}_\psi f(a,b) := \int_{-\infty}^{\infty} f(x)\overline{\psi(ax-b)}dx \quad ((a,b) \in \mathbf{R}^+ \times \mathbf{R}). \tag{53}$$

Then

$$\int_0^\infty \int_{-\infty}^{\infty} a^2 |\tilde{W}_\psi f(a,b)|^2 da db \cdot \int_{-\infty}^{\infty} x^2 |f(x)|^2 dx \geq \pi(\mathcal{M}(|\hat{\psi}|^2))(2) \|f\|^2_{L^2(\mathbf{R})}. \tag{54}$$

Here, $\mathcal{M} : \ f \mapsto (\mathcal{M}f)(\sigma) := \int_0^\infty f(x) x^{-\sigma} \frac{dx}{x}$ *denotes classical Mellin transform. We have equality in (54), if there exist some constants* $C \in \mathbf{C}$ *and* $k > 0$ *such that* $f(x) = Ce^{-k\frac{x^2}{2}}$.

Proof: In the following we assume that $\int_0^\infty \int_{-\infty}^{\infty} a^2 |W_\psi f(a,b)|^2 da db < \infty$ and $\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx < \infty$. Otherwise, (54) is trivially satisfied. The Fourier representation of $\tilde{W}_\psi$ is given by

$$\tilde{W}_\psi f(a,b) = \sqrt{2\pi}\mathcal{F}^{-1}(\hat{f}(a\xi)\overline{\hat{\psi}(\xi)})(b),$$

what can be seen by replacing ψ by $\mathcal{F}^{-1}(\hat{\psi})$. Using Plancherel's identity, we get

$$\begin{aligned} \int_{-\infty}^{\infty} |\tilde{W}_\psi f(a,b)|^2 db &= 2\pi \int_{-\infty}^{\infty} |\hat{f}(a\xi)|^2 |\hat{\psi}(\xi)|^2 d\xi \\ &= 2\pi \int_{-\infty}^{\infty} |\hat{f}(u)|^2 \left|\hat{\psi}\left(\frac{u}{a}\right)\right|^2 \frac{du}{a}. \end{aligned}$$

Integrating by $a^2 da$ leads to

$$\begin{aligned} \int_0^\infty \int_{-\infty}^{\infty} a^2 |\tilde{W}_\psi f(a,b)|^2 da db &= \int_0^\infty a^2 2\pi \left(\int_{-\infty}^{\infty} |\hat{f}(u)|^2 \left|\hat{\psi}\left(\frac{u}{a}\right)\right|^2 \frac{du}{a} \right) da \\ &= \int_{-\infty}^{\infty} |\hat{f}(u)|^2 \left(2\pi \int_0^\infty a \left|\hat{\psi}\left(\frac{u}{a}\right)\right|^2 da \right) du \\ &= \int_0^\infty K(u) |\hat{f}(u)|^2 du \end{aligned}$$

with

$$K(u) := 2\pi \int_{-\infty}^{\infty} \left| \hat{\psi}\left(\frac{u}{a}\right) \right|^2 \frac{du}{a}. \tag{55}$$

(This is the previously mentioned correspondence between "scale" and "frequency filters".) Introducing Mellin transform, we see that $K(u)$ is just a function of u^2:

$$\begin{aligned} K(u) &= 2\pi \int_0^{\infty} \frac{u}{v} |\hat{\psi}(v)|^2 u \frac{dv}{v^2} \\ &= 2\pi u^2 \int_0^{\infty} |\hat{\psi}(v)|^2 v^{-2} \frac{dv}{v} \\ &= 2\pi u^2 \mathcal{M}(|\hat{\psi}|^2)(2). \end{aligned}$$

Now, the remainder follows from Heisenberg's uncertainty principle. □

Remark 5.10 Estimates for the variance of $W_\psi f(a,b)$ in both a and b were proved by Flandrin [Flan98].

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