A Modular Compactification of the General Linear Group

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Abstract. We define a certain compactifiction of the general linear group and give a modular description for its points with values in arbitrary schemes. This is a first step in the construction of a higher rank generalization of Gieseker's degeneration of moduli spaces of vector bundles over a curve. We show that our compactification has similar properties as the "wonderful compactification" of algebraic groups of adjoint type as studied by de Concini and Procesi. As a byproduct we obtain a modular description of the points of the wonderful compactification of PGl_n.

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1. Introduction

In this paper we give a modular description of a certain compactification KGl_n of the general linear group Gl_n. The variety KGl_n is constructed as follows: First one embeds Gl_n in the obvious way in the projective space which contains the affine space of $n \times n$ matrices as a standard open set. Then one successively blows up the closed subschemes defined by the vanishing of the $r \times r$ subminors ($1 \leq r \leq n$), along with the intersection of these subschemes with the hyperplane at infinity.

We were led to the problem of finding a modular description of KGl_n in the course of our research on the degeneration of moduli spaces of vector bundles. Let me explain in some detail the relevance of compactifications of Gl_n in this context.

Let B be a regular integral one-dimensional base scheme and $b_0 \in B$ a closed point. Let $C \to B$ be a proper flat familly of curves over B which is smooth outside b_0 and whose fibre C_0 at b_0 is irreducible with one ordinary double point $p_0 \in C_0$. Let $\tilde{C}_0 \to C_0$ be the normalization of C_0 and let $p_1, p_2 \in \tilde{C}_0$

the two points lying above the singular point p_0. Thus the situation may be depicted as follows:

$$\begin{tikzcd}[column sep=small]
& (\tilde{C}_0, p_1, p_2) \arrow[dl] \arrow[dr] & & \\
\tilde{C}_0 & & C_0 \arrow[r] \arrow[d] & C \arrow[d] \\
& & b_0 \arrow[r] & B
\end{tikzcd}$$

where the left arrow means "forgetting the points p_1, p_2". There is a corresponding diagram of moduli-functors of vector bundles (v.b.) of rank n:

$$\begin{tikzcd}[column sep=small]
& \left\{ \begin{array}{c} \text{v.b. } E \text{ on } \tilde{C}_0 \\ \text{together with an} \\ \text{isomorphism } E[p_1] \xrightarrow{\sim} E[p_2] \end{array} \right\} \arrow[dl, "f_1"'] \arrow[dr, "f_2"', "\cong"] & & \\
\left\{ \begin{array}{c} \text{v.b. on} \\ \tilde{C}_0 \end{array} \right\} & & \left\{ \begin{array}{c} \text{v.b. on} \\ C_0 \end{array} \right\} \arrow[d] & \left\{ \begin{array}{c} \text{v.b. on} \\ C/B \end{array} \right\} \arrow[l] \arrow[d] \\
& & b_0 \arrow[r] & B
\end{tikzcd}$$

where $E[p_i]$ denotes the fibre of E at the point p_i (cf. section 3 below). The morphism f_1 is "forgetting the isomorphism between the fibres" and f_2 is "glueing together the fibres at p_1 and p_2 along the given isomorphism". The square on the right is the inclusion of the special fibre. It is clear that f is a locally trivial fibration with fibre Gl_n. Consequently, f_1 is not proper and thus $\{$v.b. on $C/B\}$ is not proper over B. It is desirable to have a diagram $(*)$:

$$\begin{tikzcd}[column sep=small]
& \left\{ \begin{array}{c} \text{generalized} \\ \text{v.b.-data on} \\ (\tilde{C}_0, p_1, p_2) \end{array} \right\} \arrow[dl, "\overline{f_1}"'] \arrow[dr, "\overline{f_2}"'] & & \\
\left\{ \begin{array}{c} \text{v.b. on} \\ \tilde{C}_0 \end{array} \right\} & & \left\{ \begin{array}{c} \text{generalized} \\ \text{v.b. on} \\ C_0 \end{array} \right\} \arrow[d] & \left\{ \begin{array}{c} \text{generalized} \\ \text{v.b. on} \\ C/B \end{array} \right\} \arrow[l] \arrow[d] \\
& & b_0 \arrow[r] & B
\end{tikzcd}$$

where the functors of "generalized" objects contain the original ones as open subfunctors and where $\{$generalized v.b. on $C/B\}$ is proper over B or at least satisfies the existence part of the valuative criterion for properness. The motivation is that such a diagram may help to calculate cohomological invariants of $\{$v.b. on $Y\}$ (Y a smooth projective curve) by induction on the genus of Y

(notice that the genus of $\tilde{C}_0$ is one less than the genus of the generic fibre of C/B).
In the current literature there exist two different approaches for the construction of diagram $(*)$. In the first approach the "generalized v.b." on C_0 are torsion-free sheaves (cf. [S1], [F], [NR], [Sun]). The second approach is by Gieseker who considered only the rank-two case (cf. [G]). Here the "generalized v.b." on C_0 are certain vector bundles on C_0, C_1 or C_2, where C_i is built from C_0 by inserting a chain of i copies of the projective line at p_0. (Cf. also [Tei] for a discussion of the two approaches). Of course, this is only a very rough picture of what is going on in these papers since I do not mention concepts of stability for the various objects nor the representability of the functors by varieties or by algebraic stacks.
In both approaches the morphism $\overline{f_2}$ is the normalization morphism (at least on the complement of a set of small dimension) and $\overline{f_1}$ is a locally trivial fibration with fibre a compactification of Gl_n. In the torsion-free sheaves approach this compactification is $\mathrm{Gr}(2n, n)$, the grassmanian of n-dimensional subspaces of a $2n$-dimensional vector space. In Gieseker's construction the relevant compactification of Gl_2 is KGl_2. An advantage of Gieseker's construction is that in contrast to the torsion-free sheaves approach, the space {generalized v.b. on C/B} is regular and its special fibre over b_0 is a divisor with normal crossings.
Very recently, Nagaraj and Seshadri have generalized Gieseker's construction of the right part of diagram $(*)$, i.e. the diagram

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{generalized}\\ \text{v.b. on}\\ C_0\end{array}\right\} & \longrightarrow & \left\{\begin{array}{c}\text{generalized}\\ \text{v.b. on}\\ C/B\end{array}\right\} \\ \downarrow & & \downarrow \\ b_0 & \longrightarrow & B \end{array}$$

to arbitrary rank n (cf. [NS], [S2]). Nagaraj's and Seshadri's "generalized vector bundles" on C_0 are certain equivalence classes of vector bundles on one of the curves $C_0, \ldots, C_n$, whose push-forward to C_0 are stable torsion free sheaves.
Without worrying about stability I have recently (and independently from Nagaraj and Seshadri) constructed the full diagram $(*)$ at least at the level of functors (details will appear in a forthcoming paper) and I have reasons to believe that the fibres of the corresponding morphism $\overline{f_1}$ should be represented by KGl_n. The present paper is the first step in the proof of this fact.
The compactification KGl_n of Gl_n has properties similar to those of the "wonderful compactification" of algebraic groups of adjoint type as studied by De Concini and Procesi (cf. [CP]). Namely:

1. The group $\mathrm{Gl}_n \times \mathrm{Gl}_n$ acts on KGl_n, extending the operation of $\mathrm{Gl}_n \times \mathrm{Gl}_n$ on Gl_n induced by right and left multiplication (cf. 5.6).
2. The complement of Gl_n in KGl_n is a divisor with normal crossings with irreducible components Y_i, Z_j $(i, j \in \{0, \ldots, n-1\})$ (cf. 4.2).

3. The orbit closures of the operation of $\mathrm{Gl}_n \times \mathrm{Gl}_n$ on KGl_n are precisely the intersections $Y_I \cap Z_J$, where I, J are subsets of $\{0, \ldots, n-1\}$ with $\min(I) + \min(J) \geq n$ and where $Y_I := \cap_{i \in I} Y_i$, $Z_J := \cap_{j \in J} Z_j$ (cf. 9.4).
4. For each I, J as above there exists a natural mapping from $Y_I \cap Z_J$ to the product of two flag varieties. This mapping is a locally trivial fibration with standard fibre a product of copies of $\overline{\mathrm{PGl}}_{n_k}$ (the wonderful compactification of PGl_{n_k}) for some $n_k \geq 1$ and of one copy of KGl_m for some $m \geq 0$ (cf. 9.3).

Our main theorem 5.5 says that KGl_n parametrizes what we call "generalized isomorphisms" from the trivial bundle of rank n to itself. A generalized isomorphism between vector bundles E and F is by definition a diagram

$$\begin{tikzcd}[column sep=small]
E \arrow[r, phantom, "="] & E_0 \arrow[r, bend left, "\otimes"] & E_1 \arrow[l] \arrow[r, bend left, "\otimes"] & E_2 \arrow[l] & \cdots & E_{n-1} \arrow[r, bend left, "\otimes"] & E_n \arrow[l] \arrow[r, "\sim"] & F_n \arrow[r] & F_{n-1} \arrow[l, bend right, "\otimes"'] & \cdots & F_2 \arrow[r] & F_1 \arrow[r] \arrow[l, bend right, "\otimes"'] & F_0 \arrow[l, bend right, "\otimes"'] \arrow[r, phantom, "="] & F
\end{tikzcd}$$

with certain properties, where the E_i and F_j are vector bundles of the same rank as E and F and where the arrow $-\!\otimes\!\!\rightarrow$ indicates a morphims of the source into the target tensored with a line bundle to be specified. Cf. 5.2 for a precise definition.

The wonderful compactification $\overline{\mathrm{PGl}}_n$ of PGl_n is contained as an orbit closure in KGl_n, in fact $Y_0 \cong \overline{\mathrm{PGl}}_n$. Therefore theorem 5.5 implies a modular description of $\overline{\mathrm{PGl}}_n$. One of the reasons why I decided to publish the present paper separately from my investigations on the degeneration of moduli spaces of vector bundles on curves is the fact that $\overline{\mathrm{PGl}}_n$ has been quite extensively studied in the past (cf. [Lak1] for a historical overview and also the recent paper [Tha2]). Although some efford has been made to find a modular description for it, up to now only partial results in this direction have been obtained (cf. [V], [Lak2], [TK]). In section 8 we explain the connection of these results with ours. Recently Lafforgue has used $\overline{\mathrm{PGl}}_n$ to compactify the stack of Drinfeld's shtukas (cf. [Laf1], [Laf2]).

Sections 4 and 5 contain the main definitions: In section 4 we give the construction of KGl_n and in section 5 we define the notion of generalized isomorphisms. At the end of section 5 we state our main theorem 5.5. Its proof is given in sections 6 and 7. In section 8 we define complete collineations and compare our notion with the one given by previous authors, in section 9 we study the orbit closures of the operation of $\mathrm{Gl}_n \times \mathrm{Gl}_n$ on KGl_n and in section 10 we define an equivariant morphism of KGl_n onto the Grassmannian compactification of Gl_n and compute its fibres.

My interest in degeneration of moduli spaces of bundles on curves has been greatly stimulated by a workshop on conformal blocks and the Verlinde formula, organized in March 1997 by the physicists Jürgen Fuchs and Christoph Schweigert at the Mathematisches Forschungsinstitut in Oberwolfach. Part of this work has been prepared during a stay at the Mathematical Institute of the University of Oxford. Its hospitality is gratefully acknowledged. Thanks are due to Daniel Huybrechts for mentioning to me the work of Thaddeus, to M. Thaddeus himself for sending me a copy of part of his thesis and to M.

Rapoport for drawing my attention to the work of Laksov and Lafforgue. I would also like to thank Uwe Jannsen for his constant encouragement.

2. An elementary example

This section is not strictly necessary for the comprehension of what follows. But since the rest of the paper is a bit technical, I felt that a simple example might facilitate its understanding.

Let A be a discrete valuation ring, K its field of fractions, $\mathfrak{m}$ its maximal ideal, $t \in \mathfrak{m}$ a local parameter and $k := A/\mathfrak{m}$ the residue class field of A. Let E and F be two free A-modules of rank n and let $\varphi_K : E_K \xrightarrow{\sim} F_K$ be an isomorphism between the generic fibers $E_K := E \otimes_A K$ and $F_K := F \otimes_A K$ of E and F. We can choose A-bases of E and F such that φ_K has the matrix presentation $\mathrm{diag}(t^{m_1}, \ldots, t^{m_n})$ with respect to these bases, where $m_i \in \mathbb{Z}$ and $m_1 \leq \cdots \leq m_n$. Now let $a_0 := 0 =: b_0$ and for $1 \leq i \leq n$ set $a_i := -\min(0, m_{n+1-i})$ and $b_i := \max(0, m_i)$. Note that we have

$$\begin{aligned} & 0 = a_0 = \cdots = a_{n-l} \leq a_{n-l+1} \leq \cdots \leq a_n \\ \text{and} \quad & 0 = b_0 = \cdots = b_l \leq b_{l+1} \leq \cdots \leq b_n \end{aligned}$$

for some $l \in \{0, \ldots, n\}$. Let

$$E_n \subseteq \cdots \subseteq E_1 \subseteq E_0 := E \quad \text{and} \quad F_n \subseteq \cdots \subseteq F_1 \subseteq F_0 := F$$

be the A-submodules defined by

$$E_{i+1} := \begin{bmatrix} t^{a_{i+1}-a_i}\mathbb{I}_{n-i} & 0 \\ & \\ 0 & \mathbb{I}_i \end{bmatrix} E_i \quad , \quad F_{i+1} := \begin{bmatrix} \mathbb{I}_i & 0 \\ & \\ 0 & t^{b_{i+1}-b_i}\mathbb{I}_{n-i} \end{bmatrix} F_i \quad ,$$

where $\mathbb{I}_i$ denotes the $i \times i$ unit matrix. Then φ_K induces an isomorphism $\varphi : E_n \xrightarrow{\sim} F_n$ and we have the natural injections

$$\begin{aligned} E_i \hookrightarrow \mathfrak{m}^{a_i - a_{i+1}} E_{i+1} \quad &, \quad E_i \hookleftarrow E_{i+1} \\ F_{i+1} \hookrightarrow F_i \quad &, \quad \mathfrak{m}^{b_i - b_{i+1}} F_{i+1} \hookleftarrow F_i \quad . \end{aligned}$$

Observe that the compositions $E_{i+1} \hookrightarrow E_i \hookrightarrow \mathfrak{m}^{a_i - a_{i+1}} E_{i+1}$ and $E_i \hookrightarrow \mathfrak{m}^{a_i - a_{i+1}} E_{i+1} \hookrightarrow \mathfrak{m}^{a_i - a_{i+1}} E_i$ are both the injections induced by the inclusion $A \hookrightarrow \mathfrak{m}^{a_i - a_{i+1}}$. Furthermore, if $a_i - a_{i+1} < 0$ then the morphism of k-vectorspaces $E_{i+1} \otimes k \to E_i \otimes k$ is of rank i and the sequence

$$E_{i+1} \otimes k \to E_i \otimes k \to (\mathfrak{m}^{a_i - a_{i+1}} E_{i+1}) \otimes k \to (\mathfrak{m}^{a_i - a_{i+1}} E_i) \otimes k$$

is exact. This shows that the tupel

$$(\mathfrak{m}^{a_i - a_{i+1}},\ 1 \in \mathfrak{m}^{a_i - a_{i+1}}\ ,\ \ E_{i+1} \hookrightarrow E_i\ ,\ \ \mathfrak{m}^{a_i - a_{i+1}} E_{i+1} \hookleftarrow E_i\ ,\ \ i)$$

is what we call a "bf-morphism" of rank i (cf. definition 5.1). Observe now that if $a_i - a_{i+1} < 0$ and (f, g) is one of the following two pairs of morphisms:

$$E \otimes k \xrightarrow{f} (\mathfrak{m}^{-a_i} E_i) \otimes k \xrightarrow{g} (\mathfrak{m}^{-a_{i+1}} E_{i+1}) \otimes k \quad ,$$
$$E_i \otimes k \xleftarrow{g} E_{i+1} \otimes k \xleftarrow{f} E_n \otimes k \quad ,$$

then $\mathrm{im}(g \circ f) = \mathrm{im}(g)$. The above statements hold true also if we replace the E_i-s by the F_i-s and the a_i-s by the b_i-s. Observe finally that in the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \ker(E_n \otimes k \to E_0 \otimes k) \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & \ker(F_n \otimes k \to F_0 \otimes k) \arrow[r] \arrow[dr] & E_n \otimes k \cong F_n \otimes k \arrow[r] \arrow[d] & \mathrm{im}(F_n \otimes k \to F_0 \otimes k) \arrow[r] & 0 \\
 & & \mathrm{im}(E_n \otimes k \to E_0 \otimes k) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

the oblique arrows are injections.
All these properties are summed up in the statement that the tupel

$$\begin{aligned} \Phi \quad := \quad & ((\mathfrak{m}^{b_i - b_{i+1}},\ 1),\ (\mathfrak{m}^{a_i - a_{i+1}},\ 1),\ E_i \hookrightarrow \mathfrak{m}^{a_i - a_{i+1}} E_{i+1},\ E_i \hookleftarrow E_{i+1}, \\ & F_{i+1} \hookrightarrow F_i,\ \mathfrak{m}^{b_i - b_{i+1}} F_{i+1} \hookleftarrow F_i\ (0 \le i \le n-1),\ \varphi : E_n \xrightarrow{\sim} F_n) \end{aligned}$$

is a generalized isomorphism from E to F in the sense of definition 5.2, where for $a \le 0$ we consider $\mathfrak{m}^a$ as an invertible A-module with global section $1 \in \mathfrak{m}^a$. Observe that Φ does not depend on our choice of the bases for E and F. Indeed, it is well-known that the sequence $(m_1, \dots, m_n)$ is independent of such a choice and it is easy to see that $E_n = \varphi_K^{-1}(F) \cap E$, $\quad F_n = \varphi_K(E_n)$ and

$$E_i = E_n + \mathfrak{m}^{a_i} E \quad , \quad F_i = F_n + \mathfrak{m}^{b_i} F$$

for $1 \le i \le n-1$, where the $+$-sign means generation in E_K and F_K respectively. Observe furthermore that by pull-back the generalized isomorphism Φ induces a generalized isomorphism $f^*\Phi$ on a scheme S for every morphism $f : S \to \mathrm{Spec}\,(A)$. Of course the morphisms $f^*E_{i+1} \to f^*E_i$ etc. will be in general no longer injective, but this is not required in the definition.

3. Notations

We collect some less common notations, which we will use freely in this paper:

- For two integers $a \le b$ we sometimes denote by $[a, b]$ the set $\{c \in \mathbb{Z} \mid a \le c \le b\}$.
- For a $n \times n$-matrix with entries a_{ij} in some ring, and for two subsets A and B of cardinality r of $\{1, \dots, n\}$, we will denote by $\det_{AB}(a_{ij})$ the determinant of the $r \times r$-matrix $(a_{ij})_{i \in A, j \in B}$.
- For a scheme X we will denote by $\mathcal{K}_X$ the sheaf of total quotient rings of $\mathcal{O}_X$.

- For a scheme X, a coherent sheaf $\mathcal{E}$ on X and a point $x \in X$, we denote by $\mathcal{E}[x]$ the fibre $\mathcal{E} \otimes_{\mathcal{O}_X} \kappa(x)$ of $\mathcal{E}$ at x.
- For $n \in \mathbb{N}$, the symbol S_n denotes the symmetric group of permutations of the set $\{1, \ldots, n\}$.

4. Construction of the compactification

Let $X^{(0)} := \operatorname{Proj} \mathbb{Z}[x_{00},\, x_{ij}\ (1 \le i, j \le n)]$. We define closed subschemes

$$\begin{tikzcd}
Y_0^{(0)} \arrow[r, hook] & Y_1^{(0)} \arrow[r, hook] & \cdots \arrow[r, hook] & Y_{n-1}^{(0)} & \\
& Z_{n-1}^{(0)} \arrow[u, hook] \arrow[r, hook] & \cdots \arrow[r, hook] & Z_1^{(0)} \arrow[u, hook] \arrow[r, hook] & Z_0^{(0)}
\end{tikzcd}$$

of $X^{(0)}$, by setting $Y_r^{(0)} := V^+(\mathcal{I}_r^{(0)})$, $Z_r^{(0)} := V^+(\mathcal{J}_r^{(0)})$, where $\mathcal{I}_r^{(0)}$ is the homogenous ideal in $\mathbb{Z}[x_{00},\, x_{ij}\ (1 \le i, j \le n)]$, generated by all $(r+1)\times(r+1)$-subdeterminants of the matrix $(x_{ij})_{1 \le i,j \le n}$, and where $\mathcal{J}_r^{(0)} = (x_{00}) + \mathcal{I}_{n-r}^{(0)}$ for $0 \le r \le n-1$. For $1 \le k \le n$ let the scheme $X^{(k)}$ together with closed subschemes $Y_r^{(k)}$, $Z_r^{(k)} \subset X^{(k)}$ $(0 \le r \le n-1)$ be inductively defined as follows:

$X^{(k)} \to X^{(k-1)}$ is the blowing up of $X^{(k-1)}$ along the closed subscheme $Y_{k-1}^{(k-1)} \cup Z_{n-k}^{(k-1)}$. The subscheme $Y_{k-1}^{(k)} \subset X^{(k)}$ (respectively $Z_{n-k}^{(k)} \subset X^{(k)}$) is the inverse image of $Y_{k-1}^{(k-1)}$ (respectively of $Z_{n-k}^{(k-1)}$) under the morphism $X^{(k)} \to X^{(k-1)}$, and for $r \ne k-1$ (respectively $r \ne n-k$) the subscheme $Y_r^{(k)} \subset X^{(k)}$ (respectively of $Z_r^{(k)} \subset X^{(k)}$) is the complete transform of $Y_r^{(k-1)} \subset X^{(k-1)}$ (respectively $Z_r^{(k-1)} \subset X^{(k-1)}$). We set

$$\mathrm{KGl}_n := X^{(n)} \qquad \text{and} \qquad Y_r := Y_r^{(n)}\ ,\ Z_r := Z_r^{(n)} \quad (0 \le r \le n-1)\ .$$

We are interested in finding a modular description for the compactification KGl_n of $\mathrm{Gl}_n = \operatorname{Spec} \mathbb{Z}[x_{ij}/x_{00}\ (1 \le i, j \le n),\, \det(x_{ij}/x_{00})^{-1}]$.
Let $(\alpha, \beta) \in S_n \times S_n$ and set

$$x_{ij}^{(0)}(\alpha, \beta) := \frac{x_{\alpha(i),\beta(j)}}{x_{00}} \qquad (1 \le i, j \le n) \quad .$$

For $1 \le k \le n$ we define elements

$$\begin{array}{ll}
y_{ji}(\alpha,\beta)\ , \quad z_{ij}(\alpha,\beta) & (1 \le i \le k, \quad i < j \le n) \\
x_{ij}^{(k)}(\alpha,\beta) & (k+1 \le i, j \le n)
\end{array}$$

of the function field $\mathbb{Q}(X^{(0)}) = \mathbb{Q}(x_{ij}/x_{00}\ (1 \leq i,j \leq n))$ of $X^{(0)}$ inductively as follows:

$$y_{ik}(\alpha,\beta) := \frac{x_{ik}^{(k-1)}(\alpha,\beta)}{x_{kk}^{(k-1)}(\alpha,\beta)} \qquad (k+1 \leq i \leq n) \quad ,$$

$$z_{kj}(\alpha,\beta) := \frac{x_{kj}^{(k-1)}(\alpha,\beta)}{x_{kk}^{(k-1)}(\alpha,\beta)} \qquad (k+1 \leq j \leq n) \quad ,$$

$$x_{ij}^{(k)}(\alpha,\beta) := \frac{x_{ij}^{(k-1)}(\alpha,\beta)}{x_{kk}^{(k-1)}(\alpha,\beta)} - y_{ik}(\alpha,\beta)\ z_{kj}(\alpha,\beta) \qquad (k+1 \leq i,j \leq n) \quad .$$

Finally, we set $t_0(\alpha,\beta) := t_0 := x_{00}$ and

$$t_i(\alpha,\beta) := t_0 \cdot \prod_{j=1}^{i} x_{jj}^{(j-1)}(\alpha,\beta) \qquad (1 \leq i \leq n) \quad .$$

Observe, that for each $k \in \{0,\ldots,n\}$, we have the following decomposition of the matrix $[x_{ij}/x_{00}]$:

$$\left[\frac{x_{ij}}{x_{00}}\right] = n_\alpha \begin{bmatrix} \begin{matrix} 1 & & 0 \\ & \ddots & \\ y_{ij}(\alpha,\beta) & & 1 \end{matrix} & 0 \\ y_{ij}(\alpha,\beta) & \mathbb{I}_{n-k} \end{bmatrix} \begin{bmatrix} \frac{t_1(\alpha,\beta)}{t_0} & & & 0 \\ & \ddots & & \\ & & \frac{t_k(\alpha,\beta)}{t_0} & \\ 0 & & & \frac{t_k(\alpha,\beta)}{t_0}[x_{ij}^{(k)}(\alpha,\beta)] \end{bmatrix} \begin{bmatrix} \begin{matrix} 1 & & z_{ij}(\alpha,\beta) \\ & \ddots & \\ 0 & & 1 \end{matrix} & z_{ij}(\alpha,\beta) \\ 0 & \mathbb{I}_{n-k} \end{bmatrix} n_\beta^{-1}$$

Here, n_α is the permutation matrix associated to α, i.e. the matrix, whose entry in the i-th row and j-th column is $\delta_{i,\alpha(j)}$. For convenience, we define for each $l \in \{0,\ldots,n\}$ a bijection $\iota_l : \{1,\ldots,n+1\} \xrightarrow{\sim} \{0,\ldots,n\}$, by setting

$$\iota_l(i) = \begin{cases} i & \text{if} \quad 1 \leq i \leq l \\ 0 & \text{if} \quad i = l+1 \\ i-1 & \text{if} \quad l+2 \leq i \leq n+1 \end{cases}$$

for $1 \leq i \leq n+1$. With this notaton, we define for each triple $(\alpha,\beta,l) \in S_n \times S_n \times [0,n]$ polynomial subalgebras $R(\alpha,\beta,l)$ of $\mathbb{Q}(\mathrm{KGl}_n) = \mathbb{Q}(X^{(0)})$ together with ideals $\mathcal{I}_r(\alpha,\beta,l)$ and $\mathcal{J}_r(\alpha,\beta,l)$ $(0 \leq r \leq n-1)$ of $R(\alpha,\beta,l)$ as follows:

$$\begin{aligned} R(\alpha,\beta,l) &:= \mathbb{Z}\left[\frac{t_{\iota_l(i+1)}(\alpha,\beta)}{t_{\iota_l(i)}(\alpha,\beta)}\ (1 \leq i \leq n),\ y_{ji}(\alpha,\beta),\ z_{ij}(\alpha,\beta)\ (1 \leq i < j \leq n)\right], \\ \mathcal{I}_r(\alpha,\beta,l) &:= \left(\frac{t_{\iota_l(r+2)}(\alpha,\beta)}{t_{\iota_l(r+1)}(\alpha,\beta)}\right) \quad \text{if} \quad l \leq r \leq n-1 \quad \text{and} \quad \mathcal{I}_r(\alpha,\beta,l) := (1) \text{ else}, \\ \mathcal{J}_r(\alpha,\beta,l) &:= \left(\frac{t_{\iota_l(n-r+1)}(\alpha,\beta)}{t_{\iota_l(n-r)}(\alpha,\beta)}\right) \text{ if } n-l \leq r \leq n-1 \text{ and } \mathcal{J}_r(\alpha,\beta,l) := (1) \text{ else.} \end{aligned}$$

PROPOSITION 4.1. *There is a covering of KGl_n by open affine pieces $X(\alpha,\beta,l)$ $((\alpha,\beta,l) \in S_n \times S_n \times [0,n])$, such that $\Gamma(X(\alpha,\beta,l),\mathcal{O}) = R(\alpha,\beta,l)$ (equality as subrings of the function field $\mathbb{Q}(KGl_n)$). Furthermore, for $0 \leq r \leq n-1$ the ideals $\mathcal{I}_r(\alpha,\beta,l)$ and $\mathcal{J}_r(\alpha,\beta,l)$ of $R(\alpha,\beta,l)$ are the defining ideals for the* *closed subschemes* $Y_r(\alpha,\beta,l) := Y_r \cap X(\alpha,\beta,l)$ *and* $Z_r(\alpha,\beta,l) := Z_r \cap X(\alpha,\beta,l)$ *respectively.*

Proof. We make the blowing-up procedure explicit, in terms of open affine coverings. For each $k \in \{0,\dots,n\}$ we define a finite index set $\mathcal{P}_k$, consisting of all pairs

$$(p,q) = \left(\begin{bmatrix} p_0 \\ \vdots \\ p_k \end{bmatrix}, \begin{bmatrix} q_0 \\ \vdots \\ q_k \end{bmatrix} \right) \in \{0,\dots,n\}^{k+1} \times \{0,\dots,n\}^{k+1}$$

with the property that $p_i \neq p_j$ and $q_i \neq q_j$ for $i \neq j$ and that $p_i = 0$ for some i, if and only if $q_i = 0$. Observe that for each $k \in \{0,\dots,n\}$ there is a surjection $S_n \times S_n \times \{0,\dots,n\} \to \mathcal{P}_k$, which maps the triple (α,β,l) to the element

$$(p,q) = \left(\begin{bmatrix} \alpha(\iota_l(1)) \\ \vdots \\ \alpha(\iota_l(k+1)) \end{bmatrix}, \begin{bmatrix} \beta(\iota_l(1)) \\ \vdots \\ \beta(\iota_l(k+1)) \end{bmatrix} \right)$$

of $\mathcal{P}_k$. (Here we have used the convention that $\alpha(0) := 0$ for any permutation $\alpha \in S_n$). Furthermore, this surjection is in fact a bijection in the case of $k = n$. Let $(p,q) \in \mathcal{P}_k$ and chose an element (α,β,l) in its preimage under the surjection $S_n \times S_n \times \{0,\dots,n\} \to \mathcal{P}_k$. We define subrings $R^{(k)}(p,q)$ of $\mathbb{Q}(x_{ij}/x_{00}\ (1 \le i,j \le n))$ together with ideals $\mathcal{I}_r^{(k)}(p,q)$, $\mathcal{J}_r^{(k)}(p,q)$ $(0 \le r \le n)$, distinguishing three cases.

First case: $0 \le l \le k-1$

$$\begin{aligned} R^{(k)}(p,q) \quad &:= \quad \mathbb{Z}\Bigg[\frac{t_{\iota_l(i+1)}(\alpha,\beta)}{t_{\iota_l(i)}(\alpha,\beta)}\ (1 \le i \le k),\ y_{ji}(\alpha,\beta),\ z_{ij}(\alpha,\beta) \left(\begin{array}{c} 1 \le i \le k, \\ i < j \le n \end{array} \right), \\ &\qquad\qquad x_{ij}^{(k)}(\alpha,\beta)\ (k+1 \le i,j \le n) \Bigg] \end{aligned}$$

$$\mathcal{I}_r^{(k)}(p,q) \quad := \quad \begin{cases} (1) & \text{if } r \in [0,l-1] \\ \left(\frac{t_{\iota_l(r+2)}(\alpha,\beta)}{t_{\iota_l(r+1)}(\alpha,\beta)} \right) & \text{if } r \in [l,k-1] \\ \left(\det_{AB}(x_{ij}^{(k)}(\alpha,\beta)) \left(\begin{array}{c} A,B \subseteq \{k+1,\dots,n\} \\ \sharp A = \sharp B = r+1-k \end{array} \right) \right) & \text{if } r \in [k,n-1] \end{cases}$$

$$\mathcal{J}_r^{(k)}(p,q) \quad := \quad \begin{cases} (1) & \text{if } r \in [0,n-l-1] \\ \left(\frac{t_{\iota_l(n-r+1)}(\alpha,\beta)}{t_{\iota_l(n-r)}(\alpha,\beta)} \right) & \text{if } r \in [n-l,n-1] \end{cases}$$

Second case: $l=k$

$$R^{(k)}(p,q) := \mathbb{Z}\left[\frac{t_{\iota_l(i+1)}(\alpha,\beta)}{t_{\iota_l(i)}(\alpha,\beta)}\ (1\le i\le k),\ y_{ji}(\alpha,\beta),\ z_{ij}(\alpha,\beta)\left(\begin{array}{c}1\le i\le k,\\ i<j\le n\end{array}\right),\right.$$
$$\left.\frac{t_k(\alpha,\beta)}{t_0}x_{ij}^{(k)}(\alpha,\beta)\ (k+1\le i,j\le n)\right]$$

$$\mathcal{I}_r^{(k)}(p,q) := \begin{cases}(1) & \text{if } r\in[0,l-1]\\ \left(\det_{AB}\left(\frac{t_k(\alpha,\beta)}{t_0}x_{ij}^{(k)}(\alpha,\beta)\right)\left(\begin{array}{c}A,B\subseteq\{k+1,\ldots,n\}\\ \sharp A=\sharp B=r+1-k\end{array}\right)\right) & \text{if } r\in[l,n-1]\end{cases}$$

$$\mathcal{J}_r^{(k)}(p,q) := \begin{cases}(1) & \text{if } r\in[0,n-l-1]\\ \left(\frac{t_{\iota_l(n-r+1)}(\alpha,\beta)}{t_{\iota_l(n-r)}(\alpha,\beta)}\right) & \text{if } r\in[n-l,n-1]\end{cases}$$

Third case: $k+1\le l\le n$

$$R^{(k)}(p,q) := \mathbb{Z}\left[\frac{t_{\iota_l(i+1)}(\alpha,\beta)}{t_{\iota_l(i)}(\alpha,\beta)}\ (1\le i\le k),\ \frac{t_0}{t_{k+1}(\alpha,\beta)},\right.$$
$$\left. y_{ji}(\alpha,\beta),\ z_{ij}(\alpha,\beta)\left(\begin{array}{c}1\le i\le k+1,\\ i<j\le n\end{array}\right),\ x_{ij}^{(k+1)}(\alpha,\beta)\ (k+2\le i,j\le n)\right]$$

$$\mathcal{I}_r^{(k)}(p,q) := \begin{cases}(1) & \text{if } r\in[0,k]\\ \left(\det_{AB}\left(x_{ij}^{(k+1)}(\alpha,\beta)\right)\left(\begin{array}{c}A,B\subseteq\{k+2,\ldots,n\}\\ \sharp A=\sharp B=r-k\end{array}\right)\right) & \text{if } r\in[k+1,n-1]\end{cases}$$

$$\mathcal{J}_r^{(k)}(p,q) := \begin{cases}\left(\frac{t_0}{t_{k+1}(\alpha,\beta)},\ \det_{AB}(x_{ij}^{(k+1)}(\alpha,\beta))\quad\left(\begin{array}{c}A,B\subseteq\{k+2,\ldots,n\}\\ \sharp A=\sharp B=n-r-k\end{array}\right)\right) & \text{if } r\in[0,n-k-1]\\ \left(\frac{t_{\iota_l(n-r+1)}(\alpha,\beta)}{t_{\iota_l(n-r)}(\alpha,\beta)}\right) & \text{if } r\in[n-k,n-1]\end{cases}$$

Observe that the objects $R^{(k)}(p,q)$, $\mathcal{I}_r^{(k)}(p,q)$, $\mathcal{J}_r^{(k)}(p,q)$ thus defined, depend indeed only on (p,q) and not on the chosen element (α,β,l). By induction on k one shows that $X^{(k)}$ is covered by open affine pieces $X^{(k)}(p,q)$ $((p,q)\in\mathcal{P}_k)$, such that $\Gamma(X^{(k)}(p,q),\mathcal{O})=R^{(k)}(p,q)$ (equality as subrings of the function field $\mathbb{Q}(X^{(k)})$), and such that the ideals $\mathcal{I}_r^{(k)}(\alpha,\beta)$ and $\mathcal{J}_r^{(k)}(\alpha,\beta)$ are the defining ideals of the closed subschemes $Y_r^{(k)}\cap X^{(k)}(p,q)$ and $Z_r^{(k)}\cap X^{(k)}(p,q)$ respectively. □

Corollary 4.2. *The scheme KGl_n is smooth and projective over Spec $\mathbb{Z}$ and contains Gl_n as a dense open subset. The complement of Gl_n in KGl_n is the union of the closed subschemes Y_i, Z_i $(0\le i\le n-1)$, which is a divisor with normal crossings. Furthermore, we have $Y_i\cap Z_j=\emptyset$ for $i+j<n$.*

Proof. This is immediate from the local description given in 4.1. □

We will now define a certain toric scheme, which will play an important role in the sequel. Let $M:=\mathbb{Z}^n$, with canonical basis $e_1,\ldots,e_n$. For $m\in M$ we denote by t^m the corresponding monomial in the ring $\mathbb{Z}[M]$. Furthermore, we write t_i/t_0 for the canonical generator t^{e_i} of $\mathbb{Z}[M]$. Let $N:=M^\vee$ be the dual of M with the dual basis $e_1^\vee,\ldots,e_n^\vee$. For $0\le l\le n$ let $\sigma_l\subset N_{\mathbb{Q}}:=N\otimes\mathbb{Q}$ be the cone generated by the elements $-\sum_{j=1}^{i} e_j^\vee$ $(1 \le i \le l)$ and the elements $\sum_{j=i}^{n} e_j^\vee$ $(l+1 \le i \le n)$. In other words:

$$\sigma_l = \sum_{i=1}^{l} \mathbb{Q}_+ \cdot \left(-\sum_{j=1}^{i} e_j^\vee \right) + \sum_{i=l+1}^{n} \mathbb{Q}_+ \cdot \left(\sum_{j=i}^{n} e_j^\vee \right) \quad .$$

Let Σ be the fan generated by all σ_l $(0 \le l \le n)$ and let $\widetilde{T} := X_\Sigma$ the associated toric scheme (over $\mathbb{Z}$). See e.g. [Da] for definitions. $\widetilde{T}$ is covered by the open sets $\widetilde{T}_l := X_{\sigma_l^\vee} = \operatorname{Spec} \mathbb{Z}[t^m \ (m \in \sigma_l^\vee \cap M)] = \operatorname{Spec} \mathbb{Z}[t_{\iota_l(i+1)}/t_{\iota_l(i)} \ (1 \le i \le n)]$. Observe that there are Cartier divisors $Y_{r,\widetilde{T}}$, $Z_{r,\widetilde{T}}$ $(0 \le r \le n-1)$ on $\widetilde{T}$, such that for each $l \in \{0, \ldots, n\}$ over the open part $\widetilde{T}_l \subset \widetilde{T}$,

$$Y_{r,\widetilde{T}} \quad \text{is given by the equation} \quad \begin{cases} 1 & \text{if } 0 \le r \le l-1 \\ t_{\iota_l(r+2)}/t_{\iota_l(r+1)} & \text{if } l \le r \le n-1 \end{cases}$$

$$Z_{r,\widetilde{T}} \quad \text{is given by the equation} \quad \begin{cases} 1 & \text{if } 0 \le r \le n-l-1 \\ t_{\iota_l(n-r+1)}/t_{\iota_l(n-r)} & \text{if } n-l \le r \le n-1 \end{cases}$$

Observe furthermore that $Y_{i,\widetilde{T}} \cap Z_{j,\widetilde{T}} = \emptyset$ for $i + j < n$ and that for each $r \in \{1, \ldots, n\}$, multiplication by t_r/t_0 establishes an isomorphism

$$\mathcal{O}_{\widetilde{T}}\left(-\sum_{i=0}^{n-r} Z_{i,\widetilde{T}} \right) \xrightarrow{\sim} \mathcal{O}_{\widetilde{T}}\left(-\sum_{i=0}^{r-1} Y_{i,\widetilde{T}} \right) \quad .$$

Lemma 4.3. *The toric scheme $\widetilde{T}$ together with the "universal" tupel*

$$(\mathcal{O}_{\widetilde{T}}(Y_{i,\widetilde{T}}),\ \mathbf{1}_{\mathcal{O}_{\widetilde{T}}(Y_{i,\widetilde{T}})},\ \mathcal{O}_{\widetilde{T}}(Z_{i,\widetilde{T}}),\ \mathbf{1}_{\mathcal{O}_{\widetilde{T}}(Z_{i,\widetilde{T}})}\ (0 \le i \le n-1),\ t_r/t_0\ (1 \le r \le n))$$

represents the functor, which to each scheme S associates the set of equivalence classes of tupels

$$(\mathcal{L}_i,\ \lambda_i,\ \mathcal{M}_i,\ \mu_i\ (0 \le i \le n-1), \varphi_r\ (1 \le r \le n)) \quad ,$$

where the $\mathcal{L}_i$ and $\mathcal{M}_i$ are invertible $\mathcal{O}_S$-modules with global sectons λ_i and μ_i respectively, such that for $i + j < n$ the zero-sets of λ_i and μ_j do not intersect, and where the φ_r are isomorphisms

$$\bigotimes_{i=0}^{n-r} \mathcal{M}_i^\vee \xrightarrow{\sim} \bigotimes_{i=0}^{r-1} \mathcal{L}_i^\vee \quad .$$

Here two tupels $(\mathcal{L}_i,\ \lambda_i,\ \mathcal{M}_i,\ \mu_i\ (0 \le i \le n-1), \varphi_r\ (1 \le r \le n))$ and $(\mathcal{L}_i',\ \lambda_i',\ \mathcal{M}_i',\ \mu_i'\ (0 \le i \le n-1), \varphi_r'\ (1 \le r \le n))$ are called equivalent, if there exist isomorphisms $\mathcal{L}_i \xrightarrow{\sim} \mathcal{L}_i'$ and $\mathcal{M}_i \xrightarrow{\sim} \mathcal{M}_i'$ for $0 \le i \le n-1$, such that all the obvious diagrams commute.

Proof. Let S be a scheme and $(\mathcal{L}_i,\ \lambda_i,\ \mathcal{M}_i,\ \mu_i\ (0 \le i \le n-1), \varphi_r\ (1 \le r \le n))$ a tupel defined over S, which has the properties stated in the lemma. Let us first consider the case, where all the sheaves $\mathcal{L}_i$, $\mathcal{M}_i$ are trivial and where there exists an $l \in \{0, \ldots, n\}$, such that λ_i and μ_j is nowhere vanishing for $0 \le i < l$ and $0 \le j < n-l$ respectively. Observe that under theses conditions there exists a unique set of trivializations $\mathcal{L}_i \xrightarrow{\sim} \mathcal{O}_S$, $\mathcal{M}_i \xrightarrow{\sim} \mathcal{O}_S$, $(0 \leq i \leq n)$ such that $\lambda_i \mapsto 1$ for $0 \leq i < l$, $\mu_j \mapsto 1$ for $0 \leq j < n-l$, and such that the diagrams

$$\begin{tikzcd}
\bigotimes_{i=0}^{n-r} \mathcal{M}_i^\vee \arrow[rr, "\varphi_r", "\sim"'] \arrow[dr, "\sim"] & & \bigotimes_{i=0}^{r-1} \mathcal{L}_i^\vee \arrow[dl, "\sim"'] \\
& \mathcal{O}_S &
\end{tikzcd}$$

commute for $1 \leq r \leq n$. Let $a_\nu \in \Gamma(S, \mathcal{O}_S)$ $(1 \leq \nu \leq n)$ be defined by $\lambda_i \mapsto a_{i+1}$ for $l \leq i \leq n-1$ and $\mu_j \mapsto a_{n-j}$ for $n-l \leq j \leq n-1$, and let $f_l : S \to \widetilde{T}_l$ be the morphism defined by $f_l^*(t_{\iota_l(\nu+1)}/t_{\iota_l(\nu)}) = a_\nu$ $(1 \leq \nu \leq n)$. It is straightforward to check that the induced morphism $f : S \to \widetilde{T}$ does not depend on the chosen number l and that it is unique with the property that the pull-back under f of the universal tupel is equivalent to the given one on S.

Returning to the general case, observe that there is an open covering $S = \cup_k U_k$, such that for each k there exists an l with the property that over U_k all the $\mathcal{L}_i$, $\mathcal{M}_i$ are trivial and that λ_i and μ_j is nowhere vanishing over U_k for $0 \leq i < l$ and $0 \leq j < n-l$. The above construction shows that there exists a unique morphism $f : S \to \widetilde{T}$ such that for each k the restriction to U_k of the pull-back under f of the universal tupel is equivalent to the restriction to U_k of the given one. Thus it remains only to show that the isomorphisms defining the equivalences over the U_k glue together to give a global equivalence of the pull-back of the universal tupel with the given one. However, this is clear, since it is easy to see that there exists at most one set of isomorphisms $\mathcal{L}_i \xrightarrow{\sim} \mathcal{L}_i'$, $\mathcal{M}_i \xrightarrow{\sim} \mathcal{M}_i'$ establishing an equvalence between two tuples $(\mathcal{L}_i,\ \lambda_i,\ \mathcal{M}_i,\ \mu_i,\ \varphi_r)$ and $(\mathcal{L}_i',\ \lambda_i',\ \mathcal{M}_i',\ \mu_i',\ \varphi_r')$. $\square$

For each pair $(\alpha, \beta) \in S_n \times S_n$ we define the open subset $X(\alpha, \beta) \subseteq \mathrm{KGl}_n$ as the union of the open affines $X(\alpha, \beta, l)$ $(0 \leq l \leq n)$. Let

$$\begin{aligned}
U^- &:= \operatorname{Spec} \mathbb{Z}[y_{ji}\ (1 \leq i < j \leq n)] \quad, \\
U^+ &:= \operatorname{Spec} \mathbb{Z}[z_{ij}\ (1 \leq i < j \leq n)] \quad.
\end{aligned}$$

Let $y : X(\alpha, \beta) \to U^-$ (respectively $z : X(\alpha, \beta) \to U^+$) be the morphism defined by the property that $y^*(y_{ji}) = y_{ji}(\alpha, \beta)$ (respectively $z^*(z_{ij}) = z_{ij}(\alpha, \beta)$) for $1 \leq i, j \leq n$. Observe that just as in the case of $\widetilde{T}$, multiplication by the rational function $t_r(\alpha, \beta)/t_0$ provides an isomorphism

$$\mathcal{O}_{X(\alpha,\beta)}\left(-\sum_{i=0}^{n-r} Z_i(\alpha, \beta)\right) \xrightarrow{\sim} \mathcal{O}_{X(\alpha,\beta)}\left(-\sum_{i=0}^{r-1} Y_i(\alpha, \beta)\right)$$

for $1 \leq r \leq n$, where $Y_i(\alpha, \beta)$ (respectively $Z_i(\alpha, \beta)$) denotes the restriction of Y_i (respectively Z_i) to the open set $X(\alpha, \beta)$. Thus, by lemma 4.3, the tupel

$$(\mathcal{O}(Y_i(\alpha, \beta)),\ \mathbf{1},\ \mathcal{O}(Z_i(\alpha, \beta)),\ \mathbf{1}\ (i \in [0, n-1]),\ t_r(\alpha, \beta)/t_0\ (r \in [1, n]))$$

defines a morphism $t : X(\alpha, \beta) \to \widetilde{T}$.

Lemma 4.4. *The morphism* $(y,t,z) : X(\alpha,\beta) \to U^- \times \widetilde{T} \times U^+$ *is an isomorphism.*

Proof. Let $\Omega(\alpha,\beta) \subset X(\alpha,\beta)$ be the preimage of Gl_n under the morphism $X(\alpha,\beta) \hookrightarrow \mathrm{KGl}_n \to X^{(0)}$. By definition of KGl_n, we have for all $l \in \{0,\dots,n\}$:

$$\begin{aligned}\Omega(\alpha,\beta) &= X(\alpha,\beta,l) \setminus \bigcup_{i=0}^{n-1} (Y_i(\alpha,\beta,l) \cup Z_i(\alpha,\beta,l)) \\ &= \operatorname{Spec}\mathbb{Z}[y_{ji}(\alpha,\beta),\ z_{ij}(\alpha,\beta)\ (1 \le i < j \le n), \\ &\qquad\qquad (t_i(\alpha,\beta)/t_0)^{\pm 1}\ (1 \le i \le n)] \quad .\end{aligned}$$

Let $T := \operatorname{Spec}\mathbb{Z}[(t_i/t_0)^{\pm 1}] \subset \widetilde{T}$ be the Torus in $\widetilde{T}$. We have an isomorphism $\Omega(\alpha,\beta) \xrightarrow{\sim} U^- \times T \times U^+$ defined by $y_{ji} \mapsto y_{ji}(\alpha,\beta)$, $z_{ij} \mapsto z_{ij}(\alpha,\beta)$, $t_i/t_0 \mapsto t_i(\alpha,\beta)/t_0$, and a commutative quadrangle

$$\begin{array}{ccc} X(\alpha,\beta) & \xrightarrow{(y,t,z)} & U^- \times \widetilde{T} \times U^+ \\ \uparrow & & \uparrow \\ \Omega(\alpha,\beta) & \xrightarrow{\sim} & U^- \times T \times U^+ \end{array} \quad ,$$

where the vertical arrows are the natural inclusions. Furthermore, the map (y,t,z) induces an isomorphism $X(\alpha,\beta,l) \xrightarrow{\sim} U^- \times \widetilde{T}_l \times U^+$ for $0 \le l \le n$. Using the fact that $X(\alpha,\beta)$ is separated and that $\Omega(\alpha,\beta)$ dense in $X(\alpha,\beta)$, the lemma now follows easily. □

5. bf-morphisms and generalized isomorpisms

Definition 5.1. Let S be a scheme, $\mathcal{E}$ and $\mathcal{F}$ two localy free $\mathcal{O}_S$-modules and r a nonnegative integer. A *bf-morphism of rank* r *from* $\mathcal{E}$ *to* $\mathcal{F}$ is a tupel

$$g = (\mathcal{M},\ \mu,\quad \mathcal{E} \to \mathcal{F},\quad \mathcal{M} \otimes \mathcal{E} \leftarrow \mathcal{F},\quad r)\ ,$$

where $\mathcal{M}$ is an invertible $\mathcal{O}_S$-module and μ a global section of $\mathcal{M}$ such that the following holds:

1. The composed morphisms $\mathcal{E} \to \mathcal{F} \to \mathcal{M} \otimes \mathcal{E}$ and $\mathcal{F} \to \mathcal{M} \otimes \mathcal{E} \to \mathcal{M} \otimes \mathcal{F}$ are both induced by the morphism $\mu : \mathcal{O}_S \to \mathcal{M}$.
2. For every point $x \in S$ with $\mu(x) = 0$, the complex

$$\mathcal{E}[x] \to \mathcal{F}[x] \to (\mathcal{M} \otimes \mathcal{E})[x] \to (\mathcal{M} \otimes \mathcal{F})[x]$$

   is exact and the rank of the morphism $\mathcal{E}[x] \to \mathcal{F}[x]$ equals r.

The letters "bf" stand for "back and forth". As a matter of notation, we will sometimes write $g^\sharp$ for the morphism $\mathcal{E} \to \mathcal{F}$ and $g^\flat$ for the morphism $\mathcal{F} \to \mathcal{M} \otimes \mathcal{E}$ occuring in the bf-morphism g. Note that in case μ is nowhere vanishing, the number $\operatorname{rk} g := r$ cannot be deduced from the other ingredients of g. Sometimes we will use the following more suggestive notation for the bf-morphism g:

$$g = \left(\begin{tikzcd} \mathcal{E} \arrow[r, "r", "(\mathcal{M},\mu)"'] & \mathcal{F} \arrow[l, bend right, "\otimes"'] \end{tikzcd} \right) .$$

In situations where it is clear, what $(\mathcal{M},\mu)$ and r are, we will sometimes omit these data from our notation:

$$g = \left(\begin{tikzcd} \mathcal{E} \arrow[r] & \mathcal{F} \arrow[l, bend right, "\otimes"'] \end{tikzcd} \right) .$$

DEFINITION 5.2. Let S be a scheme, $\mathcal{E}$ and $\mathcal{F}$ two locally free $\mathcal{O}_S$-modules of rank n. A *generalized isomorphism from $\mathcal{E}$ to $\mathcal{F}$* is a tupel

$$\begin{aligned} \Phi \quad = \quad & (\mathcal{L}_i,\, \lambda_i,\, \mathcal{M}_i,\, \mu_i, \quad \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \quad \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\ & \mathcal{F}_{i+1} \to \mathcal{F}_i, \quad \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \le i \le n-1), \quad \mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n) , \end{aligned}$$

where $\mathcal{E} = \mathcal{E}_0,\, \mathcal{E}_1, \ldots, \mathcal{E}_n,\, \mathcal{F}_n, \ldots, \mathcal{F}_1,\, \mathcal{F}_0 = \mathcal{F}$, are localy free $\mathcal{O}_S$-modules of rank n and the tupels

$$\begin{aligned} & (\mathcal{M}_i,\, \mu_i, \quad \mathcal{E}_{i+1} \to \mathcal{E}_i, \quad \mathcal{M}_i \otimes \mathcal{E}_{i+1} \leftarrow \mathcal{E}_i, \quad \mathrm{i}) \\ \text{and} \qquad & (\mathcal{L}_i,\, \lambda_i, \quad \mathcal{F}_{i+1} \to \mathcal{F}_i, \quad \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i, \quad \mathrm{i}) \end{aligned}$$

are bf-morphisms of rank i for $0 \le i \le n-1$, such that for each $x \in S$ the following holds:

1. If $\mu_i(x) = 0$ and (f,g) is one of the following two pairs of morphisms:
$$\mathcal{E}[x] \xrightarrow{f} ((\otimes_{j=0}^{i-1}\mathcal{M}_j) \otimes \mathcal{E}_i)[x] \xrightarrow{g} ((\otimes_{j=0}^{i}\mathcal{M}_j) \otimes \mathcal{E}_{i+1})[x] \ ,$$
$$\mathcal{E}_i[x] \xleftarrow{g} \mathcal{E}_{i+1}[x] \xleftarrow{f} \mathcal{E}_n[x] \ ,$$
then $\mathrm{im}(g \circ f) = \mathrm{im}(g)$. Likewise, if $\lambda_i(x) = 0$ and (f,g) is one of the following two pairs of morphisms:
$$\mathcal{F}_n[x] \xrightarrow{f} \mathcal{F}_{i+1}[x] \xrightarrow{g} \mathcal{F}_i[x] \ ,$$
$$((\otimes_{j=0}^{i}\mathcal{L}_j) \otimes \mathcal{F}_{i+1})[x] \xleftarrow{g} ((\otimes_{j=0}^{i-1}\mathcal{L}_j) \otimes \mathcal{F}_i)[x] \xleftarrow{f} \mathcal{F}[x] \ ,$$
then $\mathrm{im}(g \circ f) = \mathrm{im}(g)$.
2. In the diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \ker(\mathcal{E}_n[x] \to \mathcal{E}_0[x]) \arrow[d] \arrow[dr] & & \\
0 \arrow[r] & \ker(\mathcal{F}_n[x] \to \mathcal{F}_0[x]) \arrow[r] \arrow[dr] & \mathcal{E}_n[x] \cong \mathcal{F}_n[x] \arrow[r] \arrow[d] & \mathrm{im}(\mathcal{F}_n[x] \to \mathcal{F}_0[x]) \arrow[r] & 0 \\
 & & \mathrm{im}(\mathcal{E}_n[x] \to \mathcal{E}_0[x]) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

the oblique arrows are injections.

Definition 5.3. A *quasi-equivalence* between two generalized isomorphisms

$$\begin{aligned}
\Phi &= (\mathcal{L}_i,\, \lambda_i,\, \mathcal{M}_i,\, \mu_i, \quad \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \quad \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\
&\qquad \mathcal{F}_{i+1} \to \mathcal{F}_i, \quad \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \le i \le n-1), \quad \mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n)\ , \\
\Phi' &= (\mathcal{L}'_i,\, \lambda'_i,\, \mathcal{M}'_i,\, \mu'_i, \quad \mathcal{E}'_i \to \mathcal{M}'_i \otimes \mathcal{E}'_{i+1}, \quad \mathcal{E}'_i \leftarrow \mathcal{E}'_{i+1}, \\
&\qquad \mathcal{F}'_{i+1} \to \mathcal{F}'_i, \quad \mathcal{L}'_i \otimes \mathcal{F}'_{i+1} \leftarrow \mathcal{F}'_i \quad (0 \le i \le n-1), \quad \mathcal{E}'_n \xrightarrow{\sim} \mathcal{F}'_n)
\end{aligned}$$

from $\mathcal{E}$ to $\mathcal{F}$ consists in isomorphisms $\mathcal{L}_i \xrightarrow{\sim} \mathcal{L}'_i$ and $\mathcal{M}_i \xrightarrow{\sim} \mathcal{M}'_i$ for $0 \le i \le n-1$, and isomorphisms $\mathcal{E}_i \xrightarrow{\sim} \mathcal{E}'_i$ and $\mathcal{F}_i \xrightarrow{\sim} \mathcal{F}'_i$ for $0 \le i \le n$, such that all the obvious diagrams are commutative. A quasi-equivalence between Φ and Φ' is called an *equivalence*, if the isomorphisms $\mathcal{E}_0 \xrightarrow{\sim} \mathcal{E}'_0$ and $\mathcal{F}_0 \xrightarrow{\sim} \mathcal{F}'_0$ are in fact the identity on $\mathcal{E}$ and $\mathcal{F}$ respectively.

After these general definitions, we now return to our scheme KGl_n. The notations are as in the previous section.
From the matrix-decomposition on page 560 (for $k = n$) we see that the matrix $[x_{ij}/x_{00}]_{1\le i,j\le n}$ has entries in the subspace $\Gamma(\mathrm{KGl}_n, \mathcal{O}(\sum_{i=0}^{n-1} Z_i))$ of the function field $\mathbb{Q}(\mathrm{KGl}_n)$ of KGl_n. Therefore it defines a morphism

$$\boldsymbol{x} : E_0 \longrightarrow \mathcal{O}\left(\sum_{i=0}^{n-1} Z_i\right) \cdot F_0 \quad ,$$

where $E_0 = F_0 = \oplus^n \mathcal{O}_{\mathrm{KGl}_n}$.
Let $E_n \subset E_0$ be the preimage under $\boldsymbol{x}$ of $F_0 \subset \mathcal{O}(\sum_{i=0}^{n-1} Z_i)\cdot F_0$ and let $F_n \subset F_0$ be the image under $\boldsymbol{x}$ of E_n. Thus $\boldsymbol{x}$ induces a morphism

$$E_n \longrightarrow F_n \quad ,$$

which we again denote by $\boldsymbol{x}$. For $1 \le i \le n-1$ we define $\mathcal{O}_{\mathrm{KGl}_n}$-submodules E_i and F_i of $\oplus^n \mathcal{K}_{\mathrm{KGl}_n}$ as follows:

$$\begin{aligned}
E_i &:= E_n + \mathcal{O}\left(-\sum_{j=0}^{i-1} Z_j\right) \cdot E_0 \\
F_i &:= F_n + \mathcal{O}\left(-\sum_{j=0}^{i-1} Y_j\right) \cdot F_0
\end{aligned}$$

(the plus-sign means generation in $\oplus^n \mathcal{K}_{\mathrm{KGl}_n}$). Observe that for $0 \le i \le n-1$ we have the following natural injections:

$$\begin{gathered}
E_i \hookrightarrow \mathcal{O}(Z_i) \cdot E_{i+1} \quad , \quad E_i \hookleftarrow E_{i+1} \\
F_{i+1} \hookrightarrow F_i \quad , \quad \mathcal{O}(Y_i) \cdot F_{i+1} \hookleftarrow F_i
\end{gathered}$$

Proposition 5.4. *The tupel*

$$\begin{aligned}
\Phi_{univ} \;:=\; &(\mathcal{O}(Y_i),\, \mathbf{1}_{\mathcal{O}(Y_i)},\, \mathcal{O}(Z_i),\, \mathbf{1}_{\mathcal{O}(Z_i)},\, E_i \hookrightarrow \mathcal{O}(Z_i) \cdot E_{i+1}, \;\; E_i \hookleftarrow E_{i+1}, \\
&F_{i+1} \hookrightarrow F_i,\, \mathcal{O}(Y_i) \cdot F_{i+1} \hookleftarrow F_i \;(0 \le i \le n-1), \;\; \boldsymbol{x} : E_n \to F_n)
\end{aligned}$$

is a generalized isomorphism from $\oplus^n\mathcal{O}_{KGl_n}$ to itself.

Proof. It suffices to show that for each $(\alpha,\beta)\in S_n\times S_n$ the restriction of Φ_{univ} to the open set $X(\alpha,\beta)$ is a generalized isomorphism from $\oplus^n\mathcal{O}_{X(\alpha,\beta)}$ to itself. Let $\boldsymbol{z}(\alpha,\beta)$ ($\boldsymbol{y}(\alpha,\beta)$) be the upper (lower) triangular $n\times n$ matrix with 1 on the diagonal and entries $z_{ij}(\alpha,\beta)$ ($y_{ji}(\alpha,\beta)$) over (under) the diagonal ($1\le i<j\le n$). For $0\le i\le n$ we define

$$\begin{aligned} E_i(\alpha,\beta) &:= \boldsymbol{z}(\alpha,\beta)\cdot n_\beta^{-1}\cdot E_i|_{X(\alpha,\beta)} \quad ,\\ F_i(\alpha,\beta) &:= \boldsymbol{y}(\alpha,\beta)^{-1}\cdot n_\alpha^{-1}\cdot F_i|_{X(\alpha,\beta)} \quad . \end{aligned}$$

Here we interprete the matrices $\boldsymbol{z}(\alpha,\beta)\cdot n_\beta^{-1}$ and $\boldsymbol{y}(\alpha,\beta)^{-1}\cdot n_\alpha^{-1}$ as automorphisms of $\oplus^n\mathcal{K}_{X(\alpha,\beta)}$. Accordingly we view the sheaves $E_i(\alpha,\beta)$ and $F_i(\alpha,\beta)$ as subsheaves of $\oplus^n\mathcal{K}_{X(\alpha,\beta)}$. We have to show that the tupel

$$\begin{aligned} \Phi(\alpha,\beta) \;:=\; & (\mathcal{O}(Y_i(\alpha,\beta))\ ,\ \mathbf{1}_{\mathcal{O}(Y_i(\alpha,\beta))}\ ,\ \mathcal{O}(Z_i(\alpha,\beta))\ ,\ \mathbf{1}_{\mathcal{O}(Z_i(\alpha,\beta))}\ ,\\ & E_i(\alpha,\beta)\hookrightarrow\mathcal{O}(Z_i(\alpha,\beta))\cdot E_{i+1}(\alpha,\beta)\ ,\ E_i(\alpha,\beta)\hookleftarrow E_{i+1}(\alpha,\beta),\\ & F_{i+1}(\alpha,\beta)\hookrightarrow F_i(\alpha,\beta)\ ,\quad \mathcal{O}(Y_i(\alpha,\beta))\cdot F_{i+1}(\alpha,\beta)\hookleftarrow F_i(\alpha,\beta)\\ & (0\le i\le n-1)\ ,\\ & \boldsymbol{y}(\alpha,\beta)^{-1}n_\alpha^{-1}\boldsymbol{x}n_\beta\boldsymbol{z}(\alpha,\beta)^{-1}: E_n(\alpha,\beta)\xrightarrow{\sim}F_n(\alpha,\beta)) \end{aligned}$$

is a generalized isomorphism from $\oplus^n\mathcal{O}_{X(\alpha,\beta)}$ to itself.
We have for $0\le i\le n$ the following equality of subsheaves of $\oplus^n\mathcal{K}_{X(\alpha,\beta)}$:

$$\begin{aligned} E_i(\alpha,\beta) &= \bigoplus_{j=1}^{n-i}\mathcal{O}\left(-\sum_{\nu=0}^{i-1}Z_\nu(\alpha,\beta)\right)\oplus\bigoplus_{j=n-i+1}^{n}\mathcal{O}\left(-\sum_{\nu=0}^{n-j}Z_\nu(\alpha,\beta)\right) \quad ,\\ F_i(\alpha,\beta) &= \bigoplus_{j=1}^{i}\mathcal{O}\left(-\sum_{\nu=0}^{j-1}Y_\nu(\alpha,\beta)\right)\oplus\bigoplus_{j=i+1}^{n}\mathcal{O}\left(-\sum_{\nu=0}^{i-1}Y_\nu(\alpha,\beta)\right) \quad . \end{aligned}$$

This is easily checked by restricting both sides of the equations to the open subsets $X(\alpha,\beta,l)$, $(0\le l\le n)$ of $X(\alpha,\beta)$ and using 4.1. Observe that the morphisms

$$E_i(\alpha,\beta)\hookrightarrow\mathcal{O}(Z_i(\alpha,\beta))\cdot E_{i+1}(\alpha,\beta)\ ,\quad E_i(\alpha,\beta)\hookleftarrow E_{i+1}(\alpha,\beta)$$

are described by the matrices

$$\begin{bmatrix}\mathbb{I}_{n-i} & 0\\ 0 & \mu_i\mathbb{I}_i\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}\mu_i\mathbb{I}_{n-i} & 0\\ 0 & \mathbb{I}_i\end{bmatrix}\quad ,$$

and the morphisms

$$F_{i+1}(\alpha,\beta)\hookrightarrow F_i(\alpha,\beta)\ ,\quad \mathcal{O}(Y_i(\alpha,\beta))\cdot F_{i+1}(\alpha,\beta)\hookleftarrow F_i(\alpha,\beta)$$

by the matrices

$$\begin{bmatrix}\mathbb{I}_i & 0\\ 0 & \lambda_i\mathbb{I}_{n-i}\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}\lambda_i\mathbb{I}_i & 0\\ 0 & \mathbb{I}_{n-i}\end{bmatrix}$$

respectively, where we have abbreviated $\mathbf{1}_{\mathcal{O}(Y_i(\alpha,\beta))}$ by λ_i, and $\mathbf{1}_{\mathcal{O}(Z_i(\alpha,\beta))}$ by μ_i. Furthermore the matrix-decomposition on page 560 (for $k=n$)

shows that $\boldsymbol{y}(\alpha,\beta)^{-1}n_\alpha^{-1}\boldsymbol{x}n_\beta\boldsymbol{z}(\alpha,\beta)^{-1}$ is the diagonal matrix with entries $(t_1(\alpha,\beta)/t_0,\ldots,t_n(\alpha,\beta)/t_0)$. With this information at hand, it is easy to see that $\Phi(\alpha,\beta)$ is indeed a generalized isomorphism from $\oplus^n\mathcal{O}_{X(\alpha,\beta)}$ to itself. $\square$

Theorem 5.5. *Let S be a scheme and Φ a generalized isomorphism from $\oplus^n\mathcal{O}_S$ to itself. Then there is a unique morphism $f: S\to KGl_n$ such that $f^*\Phi_{univ}$ is equivalent to Φ. In other words, the scheme KGl_n together with Φ_{univ} represents the functor, which to each scheme S associates the set of equivalence classes of generalized isomorphisms from $\oplus^n\mathcal{O}_S$ to itself.*

The proof of the theorem will be given in section 7.

Corollary 5.6. *There is a (left) action of $Gl_n\times Gl_n$ on KGl_n, which extends the action $((\varphi,\psi),\Phi)\mapsto\psi\Phi\varphi^{-1}$ of $Gl_n\times Gl_n$ on Gl_n. The divisors Z_i and Y_i are invariant under this action.*

Proof. The the morphism $(\mathrm{Gl}_n\times\mathrm{Gl}_n)\times\mathrm{KGl}_n\to\mathrm{KGl}_n$ defining the action is given on S-valued points by

$$((\varphi,\psi),\Phi)\mapsto\Phi'\quad,$$

where Φ is a generalized isomorphism as in definition 5.2 from $\mathcal{E}_0=\oplus^n\mathcal{O}_S$ to $\mathcal{F}_0=\oplus^n\mathcal{O}_S$ and Φ' is the generalized isomorphism where for $2\le i\le n$ the bf-morphisms from $\mathcal{E}_i$ to $\mathcal{E}_{i-1}$, the ones from $\mathcal{F}_i$ to $\mathcal{F}_{i-1}$ and the isomorphism $\mathcal{E}_n\xrightarrow{\sim}\mathcal{F}_n$ are the same as in the tupel Φ, and where the bf-morphisms

$$\begin{array}{rl} & (\mathcal{M}_0,\mu_0,\ \mathcal{E}_1\to\mathcal{E}_0,\ \mathcal{M}_0\otimes\mathcal{E}_0\leftarrow\mathcal{E}_1,\ 0)\\ \text{and} & (\mathcal{L}_0,\lambda_0,\ \mathcal{F}_1\to\mathcal{F}_0,\ \mathcal{L}_0\otimes\mathcal{F}_0\leftarrow\mathcal{F}_1,\ 0)\end{array}$$

in the tupel Φ are replaced by the bf-morphisms

$$\begin{array}{rl} & (\mathcal{M}_0,\mu_0,\ \mathcal{E}_1\to\mathcal{E}_0\xrightarrow{\varphi}\mathcal{E}_0,\ \mathcal{M}_0\otimes\mathcal{E}_1\leftarrow\mathcal{E}_0\xleftarrow{\varphi^{-1}}\mathcal{E}_0,\ 0)\\ \text{and} & (\mathcal{L}_0,\lambda_0,\ \mathcal{F}_1\to\mathcal{F}_0\xrightarrow{\psi}\mathcal{F}_0,\ \mathcal{L}_0\otimes\mathcal{F}_1\leftarrow\mathcal{F}_0\xleftarrow{\psi^{-1}}\mathcal{F}_0,\ 0)\end{array}$$

respectively. The invariance of the divisors Z_i and Y_i is clear, since they are defined by the vanishing of μ_i and λ_i respectively. $\square$

6. Exterior powers

Lemma 6.1. *Let S be a scheme and $\mathcal{E}$, $\mathcal{F}$ two locally free $\mathcal{O}_S$-modules of rank n. Let*

$$g=(\mathcal{M},\ \mu,\quad\mathcal{E}\to\mathcal{F},\quad\mathcal{M}\otimes\mathcal{E}\leftarrow\mathcal{F},\quad r)$$

be a bf-morphism of rank r from $\mathcal{E}$ to $\mathcal{F}$. Then each point $x\in S$ has an open neighbourhood U such that over U there exist local frames $(e_1,\ldots,e_n)$ and $(f_1,\ldots,f_n)$ for $\mathcal{E}$ and $\mathcal{F}$ respectively with the property that the matrices for the morphisms

$$\mathcal{E}\longrightarrow\mathcal{F}\quad\text{and}\quad\mathcal{M}\otimes\mathcal{E}\longleftarrow\mathcal{F}$$

with respect to these frames are

$$\begin{bmatrix}\mathbb{I}_r & 0\\ 0 & \mu/m\mathbb{I}_{n-r}\end{bmatrix}\quad\text{and}\quad\begin{bmatrix}\mu\mathbb{I}_r & 0\\ 0 & m\mathbb{I}_{n-r}\end{bmatrix}$$

respectively, where m is a nowhere vanishing section of $\mathcal{M}$ over U.

Proof. Restricting to a neighbourhood of x, we may assume that the sheaves $\mathcal{M}, \mathcal{E}, \mathcal{F}$ are free. Let $(\tilde{e}_1, \ldots, \tilde{e}_n)$ and $(\tilde{f}_1, \ldots, \tilde{f}_n)$ be global frames for $\mathcal{E}$ and $\mathcal{F}$ respectively. After permutation of their elements, and restricting to a possibly smaller neighbourhood of x, we may further assume that the morphisms

$$\begin{array}{rcl} \langle \tilde{e}_1, \ldots, \tilde{e}_r \rangle & \longrightarrow & \mathcal{F}/\langle \tilde{f}_{r+1}, \ldots, \tilde{f}_n \rangle \\ \langle \tilde{f}_{r+1}, \ldots, \tilde{f}_n \rangle & \longrightarrow & \mathcal{M} \otimes \mathcal{E}/\langle \tilde{e}_1, \ldots, \tilde{e}_r \rangle \end{array}$$

induced by $g^\sharp$ and $g^\flat$ respectively, are isomorphisms. Let

$$\begin{array}{ll} \tilde{\mathcal{E}} := \langle \tilde{e}_1, \ldots, \tilde{e}_r \rangle \quad , & \tilde{\mathcal{E}}' := \ker(\mathcal{E} \to \mathcal{M} \otimes \mathcal{F}/\langle \tilde{f}_{r+1}, \ldots, \tilde{f}_n \rangle) \\ \tilde{\mathcal{F}} := \ker(\mathcal{F} \to \mathcal{E}/\langle \tilde{e}_1, \ldots, \tilde{e}_r \rangle) \quad , & \tilde{\mathcal{F}}' := \langle \tilde{f}_{r+1}, \ldots, \tilde{f}_n \rangle \ . \end{array}$$

Then we have direct-sum decompositions $\mathcal{E} = \tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}}'$, $\mathcal{F} = \tilde{\mathcal{F}} \oplus \tilde{\mathcal{F}}'$, which are respected by $g^\sharp$ and $g^\flat$. Let m be a nowhere vanishing section of $\mathcal{M}$. The frames $(e_1, \ldots, e_n)$, $(f_1, \ldots, f_n)$ of $\mathcal{E}$, $\mathcal{F}$, defined by setting $e_i := \tilde{e}_i$, $f_i := g^\sharp(\tilde{e}_i)$ for $1 \leq i \leq r$ and $e_i := (1/m)g^\flat(\tilde{f}_i)$, $f_i := \tilde{f}_i$ for $r+1 \leq i \leq n$, have the desired property. □

Proposition 6.2. *Let S be a scheme and $\mathcal{E}$, $\mathcal{F}$ two locally free $\mathcal{O}_S$-modules of rank n. Let*

$$g = (\mathcal{M},\, \mu, \quad \mathcal{E} \to \mathcal{F}, \quad \mathcal{M} \otimes \mathcal{E} \leftarrow \mathcal{F}, \quad i)$$

be a bf-morphism of rank i from $\mathcal{E}$ to $\mathcal{F}$ and let $1 \leq r \leq n$.

1. *There exists a unique morphism*

$$\wedge^r g : \wedge^r \mathcal{E} \longrightarrow (\mathcal{M}^\vee)^{\otimes \max(0, r-i)} \otimes \wedge^r \mathcal{F}$$

with the following property: If $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ are local frames for $\mathcal{E}$ and $\mathcal{F}$ respectively over an open set $U \subseteq S$, such that the matrices for the morphisms

$$\mathcal{E} \longrightarrow \mathcal{F} \quad \textit{and} \quad \mathcal{M} \otimes \mathcal{E} \longleftarrow \mathcal{F}$$

with respect to these frames are

$$\begin{bmatrix} \mathbb{I}_i & 0 \\ 0 & \mu/m\mathbb{I}_{n-i} \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \mu\mathbb{I}_i & 0 \\ 0 & m\mathbb{I}_{n-i} \end{bmatrix}$$

respectively (m being a nowhere vanishing section of $\mathcal{M}$ over U), then $(\wedge^r g)|_U$ takes the form

$$e_I \wedge e_J \mapsto m^{p-r} \otimes \mu^{\min(i,r)-p} \otimes f_I \wedge f_J \quad ,$$

where $I \subseteq \{1, \ldots, i\}$, $J \subseteq \{i+1, \ldots, n\}$ with $\sharp I = p$, $\sharp J = r-p$ and where $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$, if $I = \{i_1, \ldots, i_p\}$ and $i_1 < \cdots < i_p$. The e_J, f_I, f_J are defined analoguosly.

2. *Similarly, there exists a unique morphism*

$$\wedge^{-r} g : \wedge^r \mathcal{F} \longrightarrow \mathcal{M}^{\otimes \min(r,n-i)} \otimes \wedge^r \mathcal{E}$$

with the following property: If $(e_1, \ldots, e_n)$ *and* $(f_1, \ldots, f_n)$ *are local frames for* $\mathcal{E}$ *and* $\mathcal{F}$ *respectively over an open set* $U \subseteq S$*, such that the matrices for the morphisms*

$$\mathcal{F} \longrightarrow \mathcal{M} \otimes \mathcal{E} \quad \text{and} \quad \mathcal{E} \longleftarrow \mathcal{F}$$

with respect to these frames are

$$\begin{bmatrix} m\mathbb{I}_{n-i} & 0 \\ 0 & \mu\mathbb{I}_i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mu/m\mathbb{I}_{n-i} & 0 \\ 0 & \mathbb{I}_i \end{bmatrix}$$

respectively (m being a nowhere vanishing section of $\mathcal{M}$ *over* U*), then* $(\wedge^{-r} g)|_U$ *takes the form*

$$f_I \wedge f_J \mapsto m^p \otimes \mu^{\min(r,n-i)-p} \otimes e_I \wedge e_J \quad ,$$

where $I \subseteq \{1, \ldots, n-i\}$, $J \subseteq \{n-i+1, \ldots, n\}$ *with* $\sharp I = p$, $\sharp J = r - p$.

Proof. 1. An easy calculation shows that the morphism given by the prescription

$$e_I \wedge e_J \mapsto m^{p-r} \otimes \mu^{\min(i,r)-p} \otimes f_I \wedge f_J \quad ,$$

does not depend on the chosen local frames $(e_1, \ldots, e_n)$, $(f_1, \ldots, f_n)$. Therefore using 6.1, we may define $\wedge^r g$ by this local prescription.
2. This can be proven along the same lines as 1. Alternatively, it follows by applying 1. to the bf-morphism $(\mathcal{M},\ \mu,\ \mathcal{E}' \to \mathcal{F}',\ \mathcal{M} \otimes \mathcal{E}' \leftarrow \mathcal{F}',\ n-r)$ obtained from g by setting $\mathcal{E}' := \mathcal{F}$ and $\mathcal{F}' := \mathcal{M} \otimes \mathcal{E}$. □

In the situation of the above proposition 6.2, assume that $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ and $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, where $\operatorname{rk} \mathcal{E}_i = \operatorname{rk} \mathcal{F}_i =: n_i$ for $i = 1, 2$. Assume furthermore that the morphisms $\mathcal{E} \to \mathcal{F}$ and $\mathcal{F} \to \mathcal{M} \otimes \mathcal{E}$ both respect these direct-sum decompositions and that there are $i_1, i_2 \geq 0$ with $i_1 + i_2 = i$, such that the tupels

$$\begin{aligned} g_1 &:= (\mathcal{M},\ \mu,\ \mathcal{E}_1 \to \mathcal{F}_1,\ \mathcal{M} \otimes \mathcal{E}_1 \leftarrow \mathcal{F}_1,\ i_1) \\ \text{and} \quad g_2 &:= (\mathcal{M},\ \mu,\ \mathcal{E}_2 \to \mathcal{F}_2,\ \mathcal{M} \otimes \mathcal{E}_2 \leftarrow \mathcal{F}_2,\ i_2) \quad , \end{aligned}$$

induced by g, are also bf-morphisms. We write $g = g_1 \oplus g_2$. The following lemma says that exterior powers of bf-morphisms are compatible with direct sums whenever this makes sense.

LEMMA 6.3. *Let* $1 \leq r \leq n$ *and* $r = r_1 + r_2$ *for some* $r_1, r_2 \geq 0$.

1. *If* $\max(0, r-i) = \max(0, r_1 - i_1) + \max(0, r_2 - i_2)$*, then we have for every* $\epsilon_1 \in \Gamma(S, \wedge^{r_1} \mathcal{E}_1)$, $\epsilon_2 \in \Gamma(S, \wedge^{r_2} \mathcal{E}_2)$ *the following equality:*
$$(\wedge^r g)(\epsilon_1 \wedge \epsilon_2) = (\wedge^{r_1} g_1)(\epsilon_1) \wedge (\wedge^{r_2} g_2)(\epsilon_2) \quad .$$
2. *If* $\min(i, r) = \min(i_1, r_1) + \min(i_2, r_2)$*, then we have for every* $\omega_1 \in \Gamma(S, \wedge^{r_1} \mathcal{F}_1)$, $\omega_2 \in \Gamma(S, \wedge^{r_2} \mathcal{F}_2)$ *the following equality:*
$$(\wedge^{-r} g)(\omega_1 \wedge \omega_2) = (\wedge^{-r_1} g_1)(\omega_1) \wedge (\wedge^{-r_2} g_2)(\omega_2) \quad .$$

Proof. This follows immediately from the local description of $\wedge^r g$ and $\wedge^{-r} g$ respectively. □

DEFINITION 6.4. Let S be a scheme, $\mathcal{E}$ and $\mathcal{F}$ two localy free $\mathcal{O}_S$-modules of rank n and

$$\begin{aligned}\Phi \quad = \quad & (\mathcal{L}_i,\, \lambda_i,\, \mathcal{M}_i,\, \mu_i, \quad \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \quad \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\ & \mathcal{F}_{i+1} \to \mathcal{F}_i, \quad \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \le i \le n-1), \quad \mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n)\ ,\end{aligned}$$

a generalized isomorphism from $\mathcal{E} = \mathcal{E}_0$ to $\mathcal{F} = \mathcal{F}_0$. For $1 \le r \le n$ we define the r-th exterior power

$$\wedge^r \Phi : \quad \bigwedge^r \mathcal{E} \quad \longrightarrow \quad \bigotimes_{\nu=1}^{r} \left(\bigotimes_{i=0}^{\nu-1} \mathcal{L}_i^\vee \otimes \bigotimes_{i=0}^{n-\nu} \mathcal{M}_i \right) \otimes \bigwedge^r \mathcal{F}$$

of Φ as the composition

$$\wedge^r \Phi := (\wedge^{-r} g_0) \circ (\wedge^{-r} g_1) \circ \ldots \circ (\wedge^{-r} g_{n-1}) \circ (\wedge^r h_n) \circ (\wedge^r h_{n-1}) \circ \ldots \circ (\wedge^r h_0) \quad ,$$

where g_i and h_i are the bf-morphisms

$$\begin{aligned} & (\mathcal{M}_i,\, \mu_i, \quad \mathcal{E}_{i+1} \to \mathcal{E}_i, \quad \mathcal{M}_i \otimes \mathcal{E}_{i+1} \leftarrow \mathcal{E}_i, \quad \mathrm{i}) \\ \text{and} \qquad & (\mathcal{L}_i,\, \lambda_i, \quad \mathcal{F}_{i+1} \to \mathcal{F}_i, \quad \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i, \quad \mathrm{i}) \end{aligned}$$

respectively for $0 \le i \le n-1$, and where h_n is the isomorphism $\mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n$.

In the situation of the above definition consider especially the case, where $\mathcal{E} = \oplus^n \mathcal{O}_S$ and $\mathcal{F} = \oplus^n \mathcal{L}$ for some invertible $\mathcal{O}_S$-module $\mathcal{L}$. Then we have natural direct sum decompositions

$$\bigwedge^r \mathcal{E} = \bigoplus_B \mathcal{O}_S \quad \text{and} \quad \bigwedge^r \mathcal{F} = \bigoplus_A \mathcal{L}^r \quad ,$$

where A and B run through all subsets of cardinality r of $\{1, \ldots, n\}$. For two such subsets A and B, we denote by π_A (respectively by ι_B) the projection $\wedge^r \mathcal{F} \to \mathcal{L}^r$ onto the A-th component (respectively the inclusion $\mathcal{O}_S \hookrightarrow \wedge^r \mathcal{E}$ of the B-th component). Now we define

$$\det{}_{A,B} \Phi := \pi_A \circ (\wedge^r \Phi) \circ \iota_B : \quad \mathcal{O}_S \quad \longrightarrow \quad \bigotimes_{\nu=1}^{r} \left(\bigotimes_{i=0}^{\nu-1} \mathcal{L}_i^\vee \otimes \bigotimes_{i=0}^{n-\nu} \mathcal{M}_i \right) \otimes \mathcal{L}^r \quad .$$

LEMMA 6.5. *Let $(\alpha, \beta) \in S_n \times S_n$ and let $X(\alpha, \beta)$ be the open set of KGl_n defined in section 4. Let Φ_{univ} be the generalized isomorphism defined in 5.4. Then the sections $det_{\alpha[1,r],\beta[1,r]} \Phi_{univ}$ are nowhere vanishing on $X(\alpha, \beta)$ for $1 \le r \le n$.*

Proof. From the proof of 5.4 it follows readily that the restriction of $\det_{\alpha[1,r],\beta[1,r]} \Phi_{\mathrm{univ}}$ to $X(\alpha, \beta)$ is $\prod_{\nu=1}^r (t_\nu(\alpha, \beta)/t_0)$ as an element of

$$\Gamma\left(X(\alpha,\beta),\ \mathcal{O}\left(\sum_{\nu=1}^{r} \left(\sum_{i=0}^{n-\nu} Z_i - \sum_{i=0}^{\nu-1} Y_i \right) \right) \right) \subset \Gamma(X(\alpha,\beta),\ \mathcal{K}_{\mathrm{KGl}_n}) \quad .$$

On the other hand, 4.1 tells us that $\prod_{\nu=1}^r (t_\nu(\alpha,\beta)/t_0)$ is a generator of $\mathcal{O}(\sum_{\nu=1}^r(\sum_{i=0}^{n-\nu} Z_i - \sum_{i=0}^{\nu-1} Y_i))$ over $X(\alpha,\beta)$. □

7. Proof of theorem 5.5

Let S be a scheme, $\mathcal{L}$ an invertible $\mathcal{O}_S$-module. For $0 \le i \le n-1$ let $(\mathcal{L}_i, \lambda_i)$, $(\mathcal{M}_i, \mu_i)$ be invertible $\mathcal{O}_S$-modules together with global sections, such that the zero sets of λ_i and μ_j do not intersect for $i+j<n$. Given these data, we associate to every tupel $(\varphi_1,\ldots,\varphi_n)$ of isomorphisms

$$\varphi_r : \quad \bigotimes_{i=0}^{n-r} \mathcal{M}_i^\vee \xrightarrow{\sim} \bigotimes_{i=0}^{r-1} \mathcal{L}_i^\vee \otimes \mathcal{L} \qquad (1 \le r \le n)$$

the following generalized isomorphism from $\oplus^n \mathcal{O}_S$ to $\oplus^n \mathcal{L}$:

$$\begin{aligned} \Phi(\varphi_1,\ldots,\varphi_n) \quad := \quad & (\mathcal{L}_i,\, \lambda_i,\, \mathcal{M}_i,\, \mu_i,\, \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1},\, \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\ & \mathcal{F}_{i+1} \to \mathcal{F}_i,\, \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i\ (0 \le i \le n-1),\, \mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n)\ , \end{aligned}$$

where the locally free modules $\mathcal{E}_i$ and $\mathcal{F}_i$ are defined as

$$\begin{aligned} \mathcal{E}_i \quad &:= \quad \bigoplus_{j=1}^{n-i} \left(\bigotimes_{\nu=0}^{i-1} \mathcal{M}_\nu^\vee \right) \oplus \bigoplus_{j=n-i+1}^{n} \left(\bigotimes_{\nu=0}^{n-j} \mathcal{M}_\nu^\vee \right) \quad , \\ \mathcal{F}_i \quad &:= \quad \left(\bigoplus_{j=1}^{i} \left(\bigotimes_{\nu=0}^{j-1} \mathcal{L}_\nu^\vee \right) \oplus \bigoplus_{j=i+1}^{n} \left(\bigotimes_{\nu=0}^{i-1} \mathcal{L}_\nu^\vee \right) \right) \otimes \mathcal{L} \quad , \end{aligned}$$

the morphisms

$$\mathcal{E}_i \longrightarrow \mathcal{M}_i \otimes \mathcal{E}_{i+1} \quad \text{and} \quad \mathcal{E}_i \longleftarrow \mathcal{E}_{i+1}$$

are described by the matrices

$$\begin{bmatrix} \mathbb{I}_{n-i} & 0 \\ 0 & \mu_i \mathbb{I}_i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mu_i \mathbb{I}_{n-i} & 0 \\ 0 & \mathbb{I}_i \end{bmatrix} \quad ,$$

the morphisms

$$\mathcal{F}_{i+1} \longrightarrow \mathcal{F}_i \quad \text{and} \quad \mathcal{L}_i \otimes \mathcal{F}_{i+1} \longleftarrow \mathcal{F}_i$$

by the matrices

$$\begin{bmatrix} \mathbb{I}_i & 0 \\ 0 & \lambda_i \mathbb{I}_{n-i} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda_i \mathbb{I}_i & 0 \\ 0 & \mathbb{I}_{n-i} \end{bmatrix}$$

respectively, and the isomorphism $\mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n$ is given by the diagonal matrix with entries $(\varphi_1,\ldots,\varphi_n)$.

Definition 7.1. Let S be a scheme, $\mathcal{L}$ an invertible $\mathcal{O}_S$-module and

$$\begin{aligned} \Phi \quad := \quad & (\mathcal{L}_i,\, \lambda_i,\, \mathcal{M}_i,\, \mu_i, \quad \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \quad \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\ & \mathcal{F}_{i+1} \to \mathcal{F}_i, \quad \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \le i \le n-1), \quad \mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n) \end{aligned}$$

an arbitrary generalized isomorphism from $\mathcal{E}_0 = \oplus^n \mathcal{O}_S$ to $\mathcal{F}_0 = \oplus^n \mathcal{L}$. A *diagonalization of* Φ with respect to a pair $(\alpha, \beta) \in S_n \times S_n$ of permutations is a tupel $(u_i, v_i\ (0 \le i \le n), (\varphi_1, \ldots, \varphi_n))$ of isomorphisms

$$\begin{array}{rcll} u_i & : & \mathcal{E}_i \xrightarrow{\sim} \bigoplus_{j=1}^{n-i} \left(\bigotimes_{\nu=0}^{i-1} \mathcal{M}_\nu^\vee \right) \oplus \bigoplus_{j=n-i+1}^{n} \left(\bigotimes_{\nu=0}^{n-j} \mathcal{M}_\nu^\vee \right) & (0 \le i \le n) \\ v_i & : & \mathcal{F}_i \xrightarrow{\sim} \left(\bigoplus_{j=1}^{i} \left(\bigotimes_{\nu=0}^{j-1} \mathcal{L}_\nu^\vee \right) \oplus \bigoplus_{j=i+1}^{n} \left(\bigotimes_{\nu=0}^{i-1} \mathcal{L}_\nu^\vee \right) \right) \otimes \mathcal{L} & (0 \le i \le n) \\ \varphi_r & : & \bigotimes_{i=0}^{n-r} \mathcal{M}_i^\vee \xrightarrow{\sim} \bigotimes_{i=0}^{r-1} \mathcal{L}_i^\vee \otimes \mathcal{L} \qquad (1 \le r \le n) & \end{array}$$

such that $(u_i,\ v_i\ (0 \le i \le n))$ establishes a quasi-equivalence between Φ and $\Phi(\varphi_1, \ldots, \varphi_n)$ and such that

$$u_n \cdot n_\beta : \ \mathcal{E}_0 = \oplus^n \mathcal{O}_S \xrightarrow{\sim} \oplus^n \mathcal{O}_S \quad \text{and} \quad v_n \cdot n_\alpha : \ \mathcal{F}_0 = \oplus^n \mathcal{L} \xrightarrow{\sim} \oplus^n \mathcal{L}$$

are described by upper and lower triangular matrices respectively, with unit diagonal entries.

Definition 7.2. As in the above definition let Φ be a generalized isomorphism from $\oplus^n \mathcal{O}_S$ to $\oplus^n \mathcal{L}$. A pair $(\alpha, \beta) \in S_n \times S_n$ of permutations is called *admissible*, if for all $1 \le r \le n$ the global sections $\det_{\alpha[1,r], \beta[1,r]} \Phi$ of

$$\bigotimes_{\nu=1}^{r} \left(\bigotimes_{i=0}^{n-\nu} \mathcal{M}_i \otimes \bigotimes_{i=0}^{\nu-1} \mathcal{L}_i^\vee \right) \otimes \mathcal{L}^r$$

are nowhere vanishing on S.

Proposition 7.3. *Let S be a scheme, $\mathcal{L}$ an invertible $\mathcal{O}_S$-module and Φ a generalized isomorphism from $\oplus^n \mathcal{O}_S$ to $\oplus^n \mathcal{L}$. Then:*

1. *For $(\alpha, \beta) \in S_n \times S_n$ the following are equivalent:*
   (a) *there exists a diagonalization of Φ with respect to (α, β)*
   (b) *(α, β) is admissible for Φ*
2. *Every point of S has an open neighbourhood U, such that there is a diagonalization of $\Phi|_U$ with respect to some pair $(\alpha, \beta) \in S_n \times S_n$.*
3. *For a given pair $(\alpha, \beta) \in S_n \times S_n$ there is at most one diagonalization of Φ with respect to (α, β).*

Proof. Let

$$\begin{array}{rcl} \Phi & := & (\mathcal{L}_i, \lambda_i, \mathcal{M}_i, \mu_i, \quad \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \quad \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\ & & \mathcal{F}_{i+1} \to \mathcal{F}_i, \quad \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \le i \le n-1), \quad \mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n) \ , \end{array}$$

be a generalized isomorphism from $\mathcal{E}_0 = \oplus^n \mathcal{O}_S$ to $\mathcal{F}_0 = \oplus^n \mathcal{L}$. For $0 \le i \le n-1$ denote bf-morphisms as follows:

$$\begin{array}{rcl} g_i & := & (\mathcal{M}_i,\ \mu_i,\ \mathcal{E}_{i+1} \to \mathcal{E}_i,\ \mathcal{M}_i \otimes \mathcal{E}_{i+1} \leftarrow \mathcal{E}_i,\ i) \quad , \\ h_i & := & (\mathcal{L}_i,\ \lambda_i,\ \mathcal{F}_{i+1} \to \mathcal{F}_i,\ \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i,\ i) \quad , \end{array}$$

and let h_n the isomorphism $\mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n$.
1. We may assume that $\alpha = \beta = \mathrm{id}$. We show by induction on n that admissibility of $(\mathrm{id}, \mathrm{id})$ for Φ implies the diagonalizability of Φ with respect to $(\mathrm{id}, \mathrm{id})$. The case $n = 1$ is trivial, so assume $n \geq 2$. By assumption, the morphism

$$\det{}_{\{1\},\{1\}}\Phi : \ \mathcal{O}_S \longrightarrow \mathcal{L}_0^\vee \otimes \bigotimes_{i=0}^{n-1} \mathcal{M}_i \otimes \mathcal{L}$$

is an isomorphism. Let

$$\begin{array}{rcll} \tilde{\mathcal{E}}_i' & := & \ker(\pi_1 \circ (\wedge^1 h_0) \circ \ldots \circ (\wedge^{-1} g_i) : \ \mathcal{E}_i \longrightarrow \mathcal{L}_0^\vee \otimes \bigotimes_{j=i}^{n-1} \mathcal{M}_j \otimes \mathcal{L}) & \\ & & & (i \in [0,n]) \\ \tilde{\mathcal{F}}_i' & := & \ker(\pi_1 \circ (\wedge^1 h_0) \circ \ldots \circ (\wedge^1 h_{i-1}) : \ \mathcal{F}_i \longrightarrow \mathcal{L}_0^\vee \otimes \mathcal{L}) & (i \in [1,n]) \\ \tilde{\mathcal{F}}_0' & := & \ker(\pi_1 : \oplus^n \mathcal{L} \longrightarrow \mathcal{L}) = \oplus^{n-1}\mathcal{L} & \end{array}$$

and

$$\begin{array}{rcll} \tilde{\mathcal{E}}_i & := & \mathrm{im}((\wedge^{-1} g_{i-1}) \circ \ldots \circ (\wedge^{-1} g_0) \circ \iota_1 : \ \bigotimes_{j=0}^{i-1} \mathcal{M}_j^\vee \longrightarrow \mathcal{E}_i) & (i \in [0,n]) \\ \tilde{\mathcal{F}}_i & := & \mathrm{im}((\wedge^1 h_i) \circ \ldots \circ (\wedge^{-1} g_0) \circ \iota_1 : \ \bigotimes_{j=0}^{n-1} \mathcal{M}_j^\vee \longrightarrow \mathcal{F}_i) & (i \in [1,n]) \\ \tilde{\mathcal{F}}_0 & := & \mathrm{im}((\wedge^1 h_0) \circ \ldots \circ (\wedge^{-1} g_0) \circ \iota_1 : \ \mathcal{L}_0^\vee \otimes \bigotimes_{j=0}^{n-1} \mathcal{M}_j^\vee \longrightarrow \mathcal{F}_0) & \end{array}$$

Then we have natural direct sum decompositions

$$\begin{array}{rcll} \mathcal{E}_i & = & \tilde{\mathcal{E}}_i \oplus \tilde{\mathcal{E}}_i' & (0 \leq i \leq n) \quad , \\ \mathcal{F}_i & = & \tilde{\mathcal{F}}_i \oplus \tilde{\mathcal{F}}_i' & (0 \leq i \leq n) \quad . \end{array}$$

Since the bf-morphisms g_i and h_i respect these decompositions, we can write $g_i = \tilde{g}_i \oplus \tilde{g}_i'$ and $h_i = \tilde{h}_i \oplus \tilde{h}_i'$ where $\tilde{g}_i$ (respectively $\tilde{g}_i'$, $\tilde{h}_i$, $\tilde{h}_i'$) is a bf-morphism from $\tilde{\mathcal{E}}_{i+1}$ to $\tilde{\mathcal{E}}_i$ (respectively from $\tilde{\mathcal{E}}_{i+1}'$ to $\tilde{\mathcal{E}}_i'$, from $\tilde{\mathcal{F}}_{i+1}$ to $\tilde{\mathcal{F}}_i$, from $\tilde{\mathcal{F}}_{i+1}'$ to $\tilde{\mathcal{F}}_i'$) for $0 \leq i \leq n-1$. By the same reason, we can write $h_n = \tilde{h}_n \oplus \tilde{h}_n'$, where $\tilde{h}_n : \tilde{\mathcal{E}}_n \xrightarrow{\sim} \tilde{\mathcal{F}}$ and $\tilde{h}_n' : \tilde{\mathcal{E}}_n' \xrightarrow{\sim} \tilde{\mathcal{F}}'$. Observe that $\mathrm{rk}\, \tilde{g}_i = 0$ and $\mathrm{rk}\, \tilde{g}_i' = i$ for $0 \leq i \leq n-1$ and that

$$\mathrm{rk}\, \tilde{h}_i = \begin{cases} 0\,, & \text{if} \quad i = 0 \\ 1\,, & \text{if} \quad i > 0 \end{cases} \quad , \quad \mathrm{rk}\, \tilde{h}_i' = \begin{cases} 0\,, & \text{if} \quad i = 0 \\ i-1\,, & \text{if} \quad i > 0 \end{cases} \quad .$$

Now we define

$$\begin{array}{rcl} \mathcal{L}' & := & \mathcal{L}\otimes\mathcal{L}_0^\vee \\ \mathcal{L}'_i & := & \mathcal{L}_{i+1} \quad , \quad \lambda'_i := \lambda_{i+1} \qquad (0\leq i\leq n-2) \\ \mathcal{M}'_i & := & \mathcal{M}_i \quad , \quad \mu'_i := \mu_i \qquad (0\leq i\leq n-2) \\ \mathcal{E}'_i & := & \tilde{\mathcal{E}}'_i \qquad (0\leq i\leq n-1) \\ \mathcal{F}'_i & := & \tilde{\mathcal{F}}'_{i+1} \qquad (0\leq i\leq n-1) \end{array}$$

where we identify $\mathcal{E}'_0$ with $\oplus^{n-1}\mathcal{O}_S$ via the isomorphism

$$\mathcal{E}'_0=\tilde{\mathcal{E}}'_0 \xrightarrow{\text{inclusion}} \mathcal{E}_0=\oplus^n\mathcal{O}_S \xrightarrow{\pi_{[2,n]}} \oplus^{n-1}\mathcal{O}_S \quad ,$$

and $\mathcal{F}'_0$ with $\oplus^{n-1}\mathcal{L}'$ via the isomorphism

$$\mathcal{F}'_0=\tilde{\mathcal{F}}'_1 \xrightarrow{\wedge^1\tilde{h}'_0} \mathcal{L}_0^\vee\otimes\oplus^{n-1}\mathcal{L}=\oplus^{n-1}\mathcal{L}' \quad .$$

Let

$$\begin{array}{rcl} \Phi' & := & (\mathcal{L}'_i,\,\lambda'_i,\,\mathcal{M}'_i,\,\mu'_i,\quad \mathcal{E}'_i\to\mathcal{M}'_i\otimes\mathcal{E}'_{i+1},\quad \mathcal{E}'_i\leftarrow\mathcal{E}'_{i+1}, \\ & & \mathcal{F}'_{i+1}\to\mathcal{F}'_i,\quad \mathcal{L}'_i\otimes\mathcal{F}'_{i+1}\leftarrow\mathcal{F}'_i \quad (0\leq i\leq n-2),\quad \mathcal{E}'_{n-1}\xrightarrow{\sim}\mathcal{F}'_{n-1})\ , \end{array}$$

where $\mathcal{E}'_{n-1}\xrightarrow{\sim}\mathcal{F}'_{n-1}$ is the composition

$$\mathcal{E}'_{n-1}=\tilde{\mathcal{E}}'_{n-1} \xrightarrow{\wedge^{-1}\tilde{g}'_{n-1}} \tilde{\mathcal{E}}'_n \xrightarrow{\tilde{h}'_n} \tilde{\mathcal{F}}'_n=\mathcal{F}'_{n-1} \quad ,$$

and where the other morphisms are the ones from the $\tilde{g}'_i$ and the $\tilde{h}'_i$. It is easy to see that Φ' is a generalized isomorphism from $\oplus^{n-1}\mathcal{O}_S$ to $\oplus^{n-1}\mathcal{L}'$. Furthermore, it follows from 6.3 that

$$(\wedge^r\Phi)(e_1\wedge\cdots\wedge e_r)=(\wedge^1\Phi)(e_1)\wedge(\wedge^{r-1}\Phi')(e'_1\wedge\cdots\wedge e'_{r-1}) \qquad (2\leq r\leq n)\ ,$$

where $(e_1,\ldots,e_n)\subset\Gamma(S,\mathcal{E}_0)$ and $(e'_1,\ldots,e'_{n-1})\subset\Gamma(S,\mathcal{E}'_0)$ are the canonical global frames of $\oplus^n\mathcal{O}_S$ and $\oplus^{n-1}\mathcal{O}_S$ respectively. Therefore we have

$$\det{}_{[1.r][1,r]}\Phi=\det{}_{\{1\}\{1\}}\Phi\otimes\det{}_{[1,r-1][1,r-1]}\Phi' \qquad (2\leq r\leq n)\ .$$

Since, by assumption, the sections $\det_{[1.r][1,r]}\Phi$ are nowhere vanishing, the above equation implies that the same is true also for the sections $\det_{[1,r-1][1,r-1]}\Phi'$ $(2\leq r\leq n)$. In other words, $(\mathrm{id},\mathrm{id})$ is admissible for Φ'. By induction-hypothesis, we conclude that there exists a diagonalization $(u'_i,\ v'_i,\ (0\leq i\leq n-1),\ (\varphi'_1,\ldots,\varphi'_{n-1}))$ of Φ' with respect to $(\mathrm{id},\mathrm{id})$.
Let

$$\begin{array}{rcl} \tilde{u}'_i & := & u'_i \qquad (0\leq i\leq n-1)\ , \\ \tilde{v}'_i & := & v'_{i-1} \qquad (1\leq i\leq n)\ , \end{array}$$

and

$$\begin{array}{rcl} \tilde{u}'_n & : & \tilde{\mathcal{E}}'_n \xrightarrow[\sim]{\tilde{g}^\sharp_{n-1}} \tilde{\mathcal{E}}'_{n-1}=\mathcal{E}'_{n-1} \xrightarrow[\sim]{u'_{n-1}} \bigoplus_{j=2}^n\left(\bigotimes_{\nu=0}^{n-j}\mathcal{M}^\vee\right) \quad , \\ \tilde{v}'_0 & : & \tilde{\mathcal{F}}'_0=\bigoplus^{n-1}\mathcal{L}=\mathcal{L}_0\otimes\mathcal{F}'_0 \xrightarrow[\sim]{v'_0} \bigoplus^{n-1}\mathcal{L} \quad . \end{array}$$

Observe that there are natural isomorphisms

$$\begin{aligned} \tilde{u}_i &: \tilde{\mathcal{E}}_i \xrightarrow{\sim} \bigotimes_{j=0}^{i-1} \mathcal{M}_j^\vee \qquad (0 \le i \le n) \\ \tilde{v}_i &: \tilde{\mathcal{F}}_i \xrightarrow{\sim} \bigotimes_{j=0}^{n-1} \mathcal{M}_j^\vee \xrightarrow{\det_{\{1\}\{1\}}\Phi} \mathcal{L} \otimes \mathcal{L}_0^\vee \qquad (1 \le i \le n) \\ \tilde{v}_0 &: \tilde{\mathcal{F}}_0 \xrightarrow{\sim} \mathcal{L}_0 \otimes \bigotimes_{j=0}^{n-1} \mathcal{M}_j^\vee \xrightarrow{\det_{\{1\}\{1\}}\Phi} \mathcal{L} \end{aligned}$$

We set $u_i := \tilde{u}_i \oplus \tilde{u}'_i$, $v_i := \tilde{v}_i \oplus \tilde{v}'_i$ for $0 \le i \le n$, and $\varphi_r := \varphi'_{r-1}$ for $2 \le r \le n$. Finally, we let $\varphi_1 : \bigotimes_{i=0}^{n-1} \mathcal{M}_i^\vee \xrightarrow{\sim} \mathcal{L} \otimes \mathcal{L}_0^\vee$ be the isomorphism induced by $\det_{\{1\}\{1\}}\Phi$. It is now easy to see that the tupel $(u_i,\ v_i\ (0 \le i \le n),\ (\varphi_1, \dots, \varphi_n))$ is a diagonalization of Φ with respect to $(\mathrm{id}, \mathrm{id})$.
Conversely, assume that there exists a diagonalization $(u_i,\ v_i\ (0 \le i \le n),\ (\varphi_1, \dots, \varphi_n))$ of Φ with respect to $(\mathrm{id}, \mathrm{id})$. Observe that the diagram

$$\begin{tikzcd} & \wedge^r \mathcal{E}_0 \arrow[r, "\wedge^r\Phi"] \arrow[d, "\wedge^r u_0"] & \mathcal{N}_r \otimes (\wedge^r \mathcal{F}_0) \arrow[d, "\wedge^r v_0"] \arrow[dr, "\pi_{[1,r]}"] & \\ \mathcal{O}_S \arrow[ur, "\iota_{[1,r]}"] \arrow[dr, "\iota_{[1,r]}"'] & & & \mathcal{N}_r \otimes \mathcal{L}^{\otimes r} \\ & \wedge^r(\oplus^n \mathcal{O}_S) \arrow[r, "\wedge^r\Phi(\varphi_1,\dots,\varphi_n)"] & \mathcal{N}_r \otimes (\wedge^r(\oplus^n \mathcal{L})) \arrow[ur, "\pi_{[1,r]}"'] & \end{tikzcd}$$

where $\mathcal{N}_r := \bigotimes_{\nu=1}^r (\bigotimes_{i=0}^{n-\nu} \mathcal{M}_i \otimes \bigotimes_{i=0}^{\nu-1} \mathcal{L}_i^\vee)$, is commutative for $1 \le r \le n$. Therefore we may assume that $\Phi = \Phi(\varphi_1, \dots, \varphi_n)$. But then $\det_{[1,r][1,r]}\Phi$ is the section induced by the isomorphism

$$\varphi_1 \otimes \dots \otimes \varphi_r : \bigotimes_{\nu=1}^{r} \bigotimes_{i=0}^{n-\nu} \mathcal{M}_i^\vee \xrightarrow{\sim} \bigotimes_{\nu=1}^{r} \bigotimes_{i=0}^{\nu-1} \mathcal{L}_i^\vee$$

for $1 \le r \le n$. In particular the $\det_{[1,r][1,r]}\Phi$ are nowhere vanishing on S, which is precisely what is required for the admissibility of $(\mathrm{id}, \mathrm{id})$ for Φ.
2. By 1, it suffices to show that in the case $S = \operatorname{Spec} k$ (k a field) there exists a pair $(\alpha, \beta) \in S_n \times S_n$ which is admissible for Φ. We apply induction on n, the case $n = 1$ being trivial.
It is an easy exercise in linear algebra, to show that the morphism

$$\wedge^1\Phi = (h_0^\flat)^{-1} \circ h_1^\sharp \dots \circ h_{n-1}^\sharp \circ h_n \circ g_{n-1}^\flat \circ \dots \circ g_0^\flat$$

has at least rank one. Consequently there exist indices $i_1, j_1 \in \{1, \dots, n\}$, such that the composition $\det_{\{i_1\}\{j_1\}}\Phi = \pi_{i_1} \circ (\wedge^1\Phi) \circ \iota_{j_1}$ is an isomorphism. Let the sheaves $\tilde{\mathcal{E}}'_i$, $\tilde{\mathcal{F}}'_i$ $(0 \le i \le n)$ be defined as on page 575, with π_1 replaced by π_{i_1}, and using these sheaves, let Φ' be defined as on page 576. This is a generalized isomophism from k^{n-1} to $\oplus^{n-1}(\mathcal{L}_0^\vee \otimes \mathcal{L})$. By induction-hypothesis there exists a pair $(\alpha', \beta') \in S_{n-1} \times S_{n-1}$, which is admissible for Φ'. Let $\alpha \in S_n$ be defined by

$$\alpha(r) := \begin{cases} i_1 \,, & \text{if} \quad r = 1 \\ \alpha'(r-1) + 1 \,, & \text{if} \quad 2 \le r \le n \end{cases} ,$$

and let $\beta \in S_n$ be defined analogously. As on page 576 we have

$$\det_{\alpha[1.r],\beta[1,r]}\Phi = \det_{\{i_1\}\{j_1\}}\Phi \otimes \det_{\alpha'[1,r-1],\beta'[1,r-1]}\Phi' \qquad (2 \le r \le n)$$

for $2 \le r \le n$, i.e. the pair (α, β) is admissible for Φ.
3. This follows from the proof of 1, since it is clear that the construction of the diagonalization there is unique. □

PROPOSITION 7.4. *Let S be a scheme and Φ a generalized isomorphism from $\oplus^n \mathcal{O}_S$ to itself.*

1. *If $(\alpha, \beta) \in S_n \times S_n$ is admissible for Φ, then there exists a unique morphism $S \to X(\alpha, \beta)$, such that the pull-back of Φ_{univ} to S by this morphism is equivalent to Φ.*
2. *We have the following description of $X(\alpha, \beta)$ as an open subset of KGl_n:*
$$X(\alpha,\beta) = \left\{x \in KGl_n \mid (det_{\alpha[1,r],\beta[1,r]}\Phi_{univ})(x) \neq 0 \quad for\ 1 \le r \le n\right\} \quad .$$
3. *If $(\alpha', \beta') \in S_n \times S_n$ is a further admissible pair for Φ, then the above morphism $S \to X(\alpha, \beta)$ factorizes over the inclusion $X(\alpha, \beta) \cap X(\alpha', \beta') \hookrightarrow X(\alpha, \beta)$.*

Proof. 1. Let

$$\begin{aligned} \Phi \quad = \quad & (\mathcal{L}_i,\, \lambda_i,\, \mathcal{M}_i,\, \mu_i, \quad \mathcal{E}_i \to \mathcal{M}_i \otimes \mathcal{E}_{i+1}, \quad \mathcal{E}_i \leftarrow \mathcal{E}_{i+1}, \\ & \mathcal{F}_{i+1} \to \mathcal{F}_i, \quad \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i \quad (0 \le i \le n-1), \quad \mathcal{E}_n \stackrel{\sim}{\to} \mathcal{F}_n) \ . \end{aligned}$$

By proposition 7.3 there exists a diagonalization $(u_i,\ v_i\ (0 \le i \le n),\ (\varphi_1, \ldots, \varphi_n))$ of Φ with respect to (α, β). Let $a_{ji} \in \Gamma(S, \mathcal{O}_S)$ (respectively $b_{ij} \in \Gamma(S, \mathcal{O}_S)$) $(1 \le i < j \le n)$ be the nontrivial entries of the lower (respectively upper) triangular matrix $(v_0 \cdot n_\alpha)^{-1}$ (respectively $u_0 \cdot n_\beta$). Let $a : S \to U^-$ and $b : S \to U^+$ be the morphisms defined by $a^*(y_{ji}) = a_{ji}$ and $b^*(z_{ij}) = b_{ij}$ respectively. Furthermore, let $\varphi : S \to \widetilde{T}$ be the morphism induced by the tupel

$$(\mathcal{L}_i,\ \lambda_i,\ \mathcal{M}_i,\ \mu_i\ (0 \le i \le n-1), \varphi_r\ (1 \le r \le n)) \quad ,$$

(cf lemma 4.3). Thus we have a morphism

$$f : S \stackrel{(a,\varphi,b)}{\longrightarrow} U^- \times \widetilde{T} \times U^+ \stackrel{\sim}{\longrightarrow} X(\alpha, \beta) \quad ,$$

where the right isomorphism is the inverse of the one in lemma 4.4. It is clear that

$$\begin{aligned} f^*\mathcal{O}(Y_i(\alpha,\beta)) \cong \mathcal{L}_i \quad &, \quad f^*\mathbf{1}_{\mathcal{O}(Y_i(\alpha,\beta))} = \lambda_i \\ f^*\mathcal{O}(Z_i(\alpha,\beta)) \cong \mathcal{M}_i \quad &, \quad f^*\mathbf{1}_{\mathcal{O}(Z_i(\alpha,\beta))} = \mu_i \qquad (0 \le i \le n-1) \ . \end{aligned}$$

Denote by $(u_i',\ v_i'\ (0 \le i \le n),\ (\varphi_1', \ldots, \varphi_n'))$ the pull-back under f of the diagonalization of $\Phi_{\mathrm{univ}}|_{X(\alpha,\beta)}$, which exists by 6.5 and 7.3. By the uniquness of diagonalizations (cf 7.3), we have $u_0 = u_0'$, $v_0 = v_0'$ and $(\varphi_1, \ldots, \varphi_n) = (\varphi_1', \ldots, \varphi_n')$. Therefore the isomorphisms

$$\begin{aligned} (u_i')^{-1} \circ u_i \quad &: \quad \mathcal{E}_i \stackrel{\sim}{\longrightarrow} f^*E_i \\ (v_i')^{-1} \circ v_i \quad &: \quad \mathcal{F}_i \stackrel{\sim}{\longrightarrow} f^*F_i \end{aligned}$$

induce an equivalence between Φ and $f^*\Phi_{\mathrm{univ}}$. This proves the existence part of the proposition. For uniqueness, assume that $\tilde{f}$ is a further morphism from

S to $X(\alpha,\beta)$, such that Φ is equivalent to $\tilde{f}^*\Phi_{\text{univ}}$. Let $\overline{u}_i : \mathcal{E}_i \xrightarrow{\sim} \tilde{f}^*E_i$, $\overline{v}_i : \mathcal{F}_i \xrightarrow{\sim} \tilde{f}^*F_i$, $(0 \le i \le n)$ be an equivalence. Note that by definition $\overline{u}_0 = \mathrm{id}_{\oplus^n\mathcal{O}_S} = \overline{u}_0$. Let $(\tilde{u}_i,\ \tilde{v}_i\ (0 \le i \le n), (\tilde{\varphi}_1,\dots,\tilde{\varphi}_n))$ be the pull-back under $\tilde{f}$ of the diagonalization with respect to (α,β) of $\Phi_{\text{univ}}|_{X(\alpha,\beta)}$. Then $(\tilde{u}_i \circ \overline{u}_i,\ \tilde{v}_i \circ \overline{v}_i\ (0 \le i \le n), (\tilde{\varphi}_1,\dots,\tilde{\varphi}_n))$ is a diagonalization of Φ with respect to (α,β). By 7.3.3 we conclude that $\tilde{u}_0 = \tilde{u}_0 \circ \overline{u}_0 = u_0$, $\quad \tilde{v}_0 = \tilde{v}_0 \circ \overline{v}_0 = v_0$ and $(\tilde{\varphi}_1,\dots,\tilde{\varphi}_n) = (\varphi_1,\dots,\varphi_n)$. But this implies that the composite morphism

$$S \xrightarrow{\tilde{f}} X(\alpha,\beta) \xrightarrow{\sim} U^- \times \widetilde{T} \times U^+$$

equals (a,φ,b) and thus that $\tilde{f} = f$.

2. Denote for a moment by U the open subset of KGl_n, defined by the non-vanishing of $\det_{\alpha[1,r],\beta[1,r]}\Phi_{\text{univ}}$ for $1 \le r \le n$. We have already seen in 6.5 that $X(\alpha,\beta)$ is contained in U. Let $x \in U$. Since $X(\alpha,\beta)$ is dense in U, there exists a generalization $y \in X(\alpha,\beta)$ of x. Then there exists a morphism $f : S \to U$, where S is the Spec of a valuation ring, such that the special point of S is mapped to x and its generic point to y. By definition of U, the pair (α,β) is admissible for the generalized isomorphism $f^*\Phi_{\text{univ}}$. Therefore 1 tells us that there exists a morphism $f' : S \to X(\alpha,\beta)$, which coincides with f at the generic point of S. Since KGl_n is separabel, it follows that $f = f'$ and thus that $x \in X(\alpha,\beta)$.

3. This follows immediatelly from 2. □

Proof. (Of theorem 5.5). Let S be a scheme and Φ a generalized isomorphism from $\oplus^n\mathcal{O}_S$ to itself. By proposition 7.3, there is a covering of S by open sets U_i $(i \in I)$, and for every index $i \in I$ a pair $(\alpha_i,\beta_i) \in S_n \times S_n$, which is admissible for $\Phi|_{U_i}$. Proposition 7.4.1 now tells us that there exists for each $i \in I$ a unique morhpism $f_i : U_i \to X(\alpha_i,\beta_i)$ with the property that there is an equivalence, say u_i, from $\Phi|_{U_i}$ to $f_i^*\Phi_{\text{univ}}$. By proposition 7.4.2, the f_i glue together, to give a morphism $f : S \to \mathrm{KGl}_n$. It remains to show that also the u_i glue together, to give an overall equivalence from Φ to $f^*\Phi_{\text{univ}}$. For this, it suffices to show that for two generalized isomorphisms Φ and Φ' from $\oplus^n\mathcal{O}_S$ to itself there exists at most one equivalence from Φ to Φ'. The question being local, we may assume by proposition 7.3.2 that Φ' is diagonalizable with respect to some pair $(\alpha,\beta) \in S_n \times S_n$. Composing the diagonalization of Φ' with any equivalence from Φ to Φ' gives a diagonalization of Φ with respect to (α,β). Since different equivalences from Φ to Φ' would yield different diagonalizations of Φ, proposition 7.3.3 tells us that there exists at most one equivalence. □

8. Complete collineations

In this section we prove a modular property for the compactification $\overline{\mathrm{PGl}}_n$ of PGl_n and compare it with the results of other authors.

The scheme $\overline{\mathrm{PGl}}_n$ together with closed subschemes $\overline{\Delta}_r$ $(1 \le r \le n-1)$ is defined by successive blow ups as follows. Let $\overline{\Omega}^{(0)} := \mathrm{Proj}\,(\mathbb{Z}[x_{i,j}\ (1 \le i,j \le n)])$ and let $\overline{\Delta}_r^{(0)} := V^+((\det_{AB}(x_{ij}) \mid A,B \subseteq \{1,\dots,n\},\ \sharp A = \sharp B = r+1))$ $(1 \le$

$r \leq n-1$). Inductively, define $\overline{\Omega}^{(\nu)}$ as the blowing up of $\overline{\Omega}^{(\nu-1)}$ along $\overline{\Delta}_\nu^{(\nu-1)}$. The closed subscheme $\overline{\Delta}_r^{(\nu)} \subset \overline{\Omega}^{(\nu)}$ is by definition the strict (resp. total) transform of $\overline{\Delta}_r^{(\nu-1)}$ for $r \neq \nu$ (resp. $r = \nu$). By definition, $\overline{\mathrm{PGl}}_n := \overline{\Omega}^{(n-1)}$ and $\overline{\Delta}_r := \overline{\Delta}_r^{(n-1)}$ for $1 \leq r \leq n-1$.

The variety $\overline{\mathrm{PGl}}_n \times \mathrm{Spec}\,(\mathbb{C})$ is the so-called "wonderful compactification" of the homogenuos space $\mathrm{PGl}_{n,\mathbb{C}} = (\mathrm{PGl}_{n,\mathbb{C}} \times \mathrm{PGl}_{n,\mathbb{C}})/\mathrm{PGl}_{n,\mathbb{C}}$ (cf. [CP]). Vainsencher [V], Laksov [Lak2] and Thorup-Kleiman [TK] have given a modular description for (some of) the S-valued points of $\overline{\mathrm{PGl}}_n$. We will give a brief account of their results.

Let $R \subseteq [1, n-1]$ and let S be a scheme. Following the terminology of Vainsencher, an S-valued *complete collineation of type* R from a rank-n vector bundle E to a rank-n vector bundle F is a collection of morphisms

$$v_i : E_i \to \mathcal{N}_i \otimes F_i \quad (0 \leq i \leq k),$$

where $R = \{r_1, \ldots, r_k\}$, $0 =: r_0 < r_1 < \cdots < r_k < r_{k+1} := n$, the $\mathcal{N}_i$ are line bundles, the E_i, F_i are vector bundles on S and v_i has overall rank $r_{i+1} - r_i$; furthermore it is required that $E_0 = E$, $F_0 = F$, and $E_i = \ker(v_{i-1})$, $F_i = \mathcal{N}_{i-1}^\vee \otimes \mathrm{coker}(v_{i-1})$ for $1 \leq i \leq k$. Vainsencher proves that the locally closed subscheme $(\cap_{r\in R}\overline{\Delta}_r) \setminus \cup_{r\notin R}\overline{\Delta}_r$ of $\overline{\mathrm{PGl}}_n$ represents the functor which to each scheme S associates the set of isomorphism classes of S-valued complete collineations of type R from $\oplus^n\mathcal{O}_S$ to itself.

Laksov went further. He succeeded to give a modular description for those S-valued points of $\overline{\Delta}(R) := \cap_{r\in R}\overline{\Delta}_r$ for which the pull-back of the divisor $\sum_{r\notin R}\overline{\Delta}_r|_{\overline{\Delta}(R)}$ on $\overline{\Delta}(R)$ is a well-defined divisor on S. We refer the reader to [Lak2] for more details.

Finally, Thorup and Kleiman gave the following description for *all* S-valued points of $\overline{\mathrm{PGl}}_n$. A morphism u from $(\oplus^n\mathcal{O}_S) \otimes (\oplus^n\mathcal{O}_S)^\vee$ to a line bundle $\mathcal{L}$ is called a *divisorial form*, if for each $i \in [1, n]$ the image $\mathcal{M}_i(u)$ of the induced map $\wedge^i(\oplus^n\mathcal{O}_S) \otimes \wedge^i(\oplus^n\mathcal{O}_S)^\vee \to \mathcal{L}^{\otimes i}$ is an invertible sheaf. In this case denote by u^i the induced surjection $\wedge^i(\oplus^n\mathcal{O}_S) \otimes \wedge^i(\oplus^n\mathcal{O}_S)^\vee \to \mathcal{M}_i(u)$. Following Thorup and Kleiman, we define a *projectively complete bilinear form* as a tupel $\boldsymbol{u} = (u_1, \ldots, u_n)$, where $u_i : \wedge^i(\oplus^n\mathcal{O}_S) \otimes \wedge^i(\oplus^n\mathcal{O}_S)^\vee \to \mathcal{M}_i$ is a surjection onto an invertible sheaf $\mathcal{M}_i$ for $1 \leq i \leq n$, such that $\boldsymbol{u}$ is "locally the pull-back of a divisorial form". The last requirement means the following: For each point $x \in S$, there exists an open neighborhood U of x, a morphism from U to some scheme S' and a divisorial form $u : (\oplus^n\mathcal{O}_{S'}) \otimes (\oplus^n\mathcal{O}_{S'})^\vee \to \mathcal{L}'$ on S' such that the restriction of $\boldsymbol{u}$ to U is isomorphic to the pull-back of $(u^1, \ldots, u^n)$. Thorup and Kleiman show that $\overline{\mathrm{PGl}}_n$ represents the functor that to each scheme S associates the set of isomorphism classes of projectively complete bilinear forms on S.

None of these descriptions is completely satisfactory: Those of Vainsencher and Laksov deal only with special S-valued points and the description of Thorup-Kleiman is not explicit and is not truely modular, since the condition "to

be locally pull-back of a divisorial form" makes reference to the existence of morphisms between schemes.
The terminology in the following definition will be justified by the corollary 8.2 below.

Definition 8.1. Let S be a scheme and $\mathcal{E}$, $\mathcal{F}$ two locally free $\mathcal{O}_S$-modules of rank n. A *complete collineation* from $\mathcal{E}$ to $\mathcal{F}$ is a tupel

$$\Psi = (\mathcal{L}_i,\ \lambda_i,\ \mathcal{F}_{i+1} \to \mathcal{F}_i,\ \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i\ (0 \le i \le n-1)) \quad ,$$

where $\mathcal{E} = \mathcal{F}_n, \mathcal{F}_{n-1}, \ldots, \mathcal{F}_1, \mathcal{F}_0 = \mathcal{F}$ are locally free $\mathcal{O}_S$-modules of rank n, the tupels

$$(\mathcal{L}_i,\ \lambda_i,\ \mathcal{F}_{i+1} \to \mathcal{F}_i,\ \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i,\ i)$$

are bf-morphisms of rank i for $0 \le i \le n-1$ and $\lambda_0 = 0$, such that for each point $x \in S$ and index $i \in \{0, \ldots, n-1\}$ with the property that $\lambda_i(x) = 0$, the following holds:
If (f, g) is one of the following two pairs of morphisms:

$$\mathcal{F}_n[x] \xrightarrow{f} \mathcal{F}_{i+1}[x] \xrightarrow{g} \mathcal{F}_i[x] \ ,$$

$$((\otimes_{j=0}^{i} \mathcal{L}_j) \otimes \mathcal{F}_{i+1})[x] \xleftarrow{g} ((\otimes_{j=0}^{i-1} \mathcal{L}_j) \otimes \mathcal{F}_i)[x] \xleftarrow{f} \mathcal{F}_0[x] \ ,$$

then $\mathrm{im}(g \circ f) = \mathrm{im}(g)$.
Two complete collineations Ψ and Ψ' from $\mathcal{E}$ to $\mathcal{F}$ are called *equivalent*, if there are isomorphisms $\mathcal{L}_i \xrightarrow{\sim} \mathcal{L}'_i$, $\mathcal{F}_i \xrightarrow{\sim} \mathcal{F}'_i$, such that all the obvious diagrams commute and such that $\mathcal{F}_n \xrightarrow{\sim} \mathcal{F}'_n$ and $\mathcal{F}_0 \xrightarrow{\sim} \mathcal{F}'_0$ is the identity on $\mathcal{E}$ and $\mathcal{F}$.

Corollary 8.2. *On $\overline{PGl}_n$ there exists a universal complete collineation Ψ_{univ} from $\oplus^n \mathcal{O}$ to itself, such that the pair $(\overline{PGl}_n, \Psi_{univ})$ represents the functor, which to every scheme S associates the set of equivalence classes of complete collineations from $\oplus^n \mathcal{O}_S$ to itself.*

Proof. Observe that $\overline{\mathrm{PGl}}_n$ is naturally isomorphic to the closed subscheme Y_0 of KGl_n. The restriction of Φ_{univ} to $\overline{\mathrm{PGl}}_n$ induces in an obvious way a complete collineation Ψ_{univ} of $\oplus^n \mathcal{O}$ to itself on $\overline{\mathrm{PGl}}_n$. The corollary now follows from theorem 5.5. □

We conclude this section by indicating how one can recover Vainsencher's and Thorup-Kleiman's description from corollary 8.2. Let S be a scheme and let

$$\Psi = (\mathcal{L}_i,\ \lambda_i,\ \mathcal{F}_{i+1} \to \mathcal{F}_i,\ \mathcal{L}_i \otimes \mathcal{F}_{i+1} \leftarrow \mathcal{F}_i\ (0 \le i \le n-1))$$

be a complete collineation from $\oplus^n \mathcal{O}_S$ to itself in the sense of definition 8.1.
First assume that there exists a subset R of $[1, n-1]$, such that the map $S \to \overline{\mathrm{PGl}}_n$ corresponding to Ψ factors through $(\cap_{r \in R} \overline{\Delta}_r) \setminus \cup_{r \notin R} \overline{\Delta}_r$. This means that λ_r is zero for $r \in R$ and is nowhere vanishing for $r \in [1, n-1] \setminus R$. As above, let $R = \{r_1, \ldots, r_k\}$, $0 =: r_0 < r_1 < \cdots < r_k < r_{k+1} := n$. For $0 \le i \le k$ let

$$\begin{aligned} E_i &:= \ker(\mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_{r_i}) \\ F_i &:= \mathcal{N}_i^\vee \otimes \ker(\mathcal{F}_{r_{i+1}} \to \mathcal{F}_{r_{i+1}-1} \to \cdots \to \mathcal{F}_{r_i}) \quad , \end{aligned}$$

where $\mathcal{N}_i := \otimes_{j=1}^i \mathcal{L}_{r_i}^\vee$. Observe that the data in Ψ provide natural maps

$$v_i : E_i \to \mathcal{N}_i \otimes F_i$$

of overall rank $r_{i+1} - r_i$ for $0 \le i \le k$. Furthermore we have natural isomorphisms $E_0 = \mathcal{F}_n = \oplus^n \mathcal{O}$, $F_0 \cong \mathcal{F}_0 = \oplus^n \mathcal{O}$, and for $1 \le i \le k$:

$$\begin{aligned} \ker(v_{i-1}) &= E_i \\ \operatorname{coker}(v_{i-1}) &\cong \operatorname{coker}(\mathcal{F}_n \to \mathcal{F}_{r_i}) = \operatorname{coker}(\mathcal{F}_{r_{i+1}} \to \mathcal{F}_{r_i}) \cong \\ &\cong \ker(\mathcal{L}_{r_i} \otimes \mathcal{F}_{r_{i+1}} \to \mathcal{L}_{r_i} \otimes \mathcal{F}_{r_i}) \cong \mathcal{N}_{i-1} \otimes F_i \quad . \end{aligned}$$

Thus, $(v_i)_{0 \le i \le k}$ is a complete homomorphism of type R in the sense of Vainsencher.

Now let Ψ be arbitrary. As in section 6, Ψ induces nowhere vanishing morphisms

$$\wedge^r \Psi : \wedge^r \mathcal{F}_n \to \left(\bigotimes_{\nu=1}^{r} \bigotimes_{i=0}^{\nu-1} \mathcal{L}_i^\vee \right) \otimes \wedge^r \mathcal{F}_0$$

and thus surjections

$$u_r : \wedge^r \mathcal{F}_n \otimes \wedge^r \mathcal{F}_0^\vee \to \bigotimes_{\nu=1}^{r} \bigotimes_{i=0}^{\nu-1} \mathcal{L}_i^\vee$$

for $1 \le r \le n$. The tupel $(u_r)_{1 \le r \le n}$ is a projectively complete bilinear form in the sense of Thorup-Kleiman. This follows from the fact that $\wedge^1 \Psi_{\mathrm{univ}}$ induces a divisorial form on $\overline{\mathrm{PGl}}_n$.

9. Geometry of the strata

In this section we need relative versions of the varieties KGl_n, $\overline{\mathrm{PGl}}_n$ and $\overline{O}_{I,J} := \cap_{i \in I} Z_i \cap \cap_{j \in J} Y_j$, where I and J are subsets of $[0, n-1]$. They are defined in the following theorem.

THEOREM 9.1. *Let T be a scheme and let $\mathcal{E}$ and $\mathcal{F}$ be two locally free $\mathcal{O}_T$-modules of rank n. For a T-scheme S we write $\mathcal{E}_S$ and $\mathcal{F}_S$ for the pull-back of E and F to S. Let I, J be two subsets of $[0, n-1]$ Consider the following contravariant functors from the category of T-schemes to the category of sets:*

$$\mathrm{KGL}(\mathcal{E}, \mathcal{F}) : \ S \ \mapsto \left\{ \begin{array}{l} \textit{equivalence classes of} \\ \textit{generalized isomorphisms} \\ \textit{from } \mathcal{E}_S \textit{ to } \mathcal{F}_S \end{array} \right\}$$

$$\overline{\mathrm{PGL}}(\mathcal{E}, \mathcal{F}) : \ S \ \mapsto \left\{ \begin{array}{l} \textit{equivalence classes of} \\ \textit{complete collineations} \\ \textit{from } \mathcal{E}_S \textit{ to } \mathcal{F}_S \end{array} \right\}$$

$$\overline{\mathbf{O}}_{I,J}(\mathcal{E}, \mathcal{F}) : \ S \ \mapsto \left\{ \begin{array}{l} \textit{equivalence classes of} \\ \textit{generalized isomorphisms } \Phi \\ \textit{as in definition 5.2 from} \\ \mathcal{E}_S \textit{ to } \mathcal{F}_S, \textit{ with } \mu_i = 0 \textit{ for } i \in I \\ \textit{and } \lambda_j = 0 \textit{ for } j \in J \end{array} \right\}$$

These functors are representable by smooth projective T-schemes, which we will call $KGl(\mathcal{E},\mathcal{F})$, $\overline{PGl}(\mathcal{E},\mathcal{F})$ and $\overline{O}_{I,J}(\mathcal{E},\mathcal{F})$ respectively.

Proof. In the case of $T = \operatorname{Spec}\mathbb{Z}$ and $\mathcal{E} = \mathcal{F} = \oplus^n \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}$, the theorem is a consequence of 5.5 and 8.2, where the representing objects are KGl_n, $\overline{\mathrm{PGl}}_n$ and $\overline{O}_{I,J}$ respectively. Let $T = \cup U_i$ an open covering such that there exist trivializations

$$\begin{array}{rcl} \xi_i & : & \mathcal{E}|_{U_i} \xrightarrow{\sim} \oplus^n \mathcal{O}_{U_i} \\ \zeta_i & : & F|_{U_i} \xrightarrow{\sim} \oplus^n \mathcal{O}_{U_i} \quad . \end{array}$$

Let $\mathrm{KGl}_{U_i} := \mathrm{KGl}_n \times_{\operatorname{Spec}\mathbb{Z}} U_i$ and $\pi_i : \mathrm{KGl}_{U_i} \to U_i$ the projection. By corollary 5.6, over the intersections $U_i \cap U_j$ the pairs $(\xi_i\xi_j^{-1}, \zeta_i\zeta_j^{-1})$ induce isomorphisms $\pi_i^{-1}(U_i\cap U_j) \xrightarrow{\sim} \pi_j^{-1}(U_i \cap U_j)$. These provide the data for the pieces KGl_{U_i} to glue together to define $\mathrm{KGl}(\mathcal{E},\mathcal{F})$. Using theorem 5.5 it is easy to check that $\mathrm{KGl}(\mathcal{E},\mathcal{F})$ has the required universal property. This proves the existence of $\mathrm{KGl}(\mathcal{E},\mathcal{F})$. The existence of $\overline{\mathrm{PGl}}(\mathcal{E},\mathcal{F})$ and of $\overline{O}_{I,J}(\mathcal{E},\mathcal{F})$ is proved analogously. $\square$

Definition 9.2. Let T be a scheme and $\mathcal{E}$ a locally free $\mathcal{O}_T$-module of rank n. Let $\mathbf{d} := (d_0,\dots,d_t)$, where $0 \le d_0 \le \dots \le d_t \le n$ Let $\mathrm{Fl}^{\mathbf{d}}(\mathcal{E})$ be the flag variety which represents the following contravariant functor from the category of T-schemes to the category of sets:

$$S \mapsto \left\{ \begin{array}{l} \text{All filtrations } F_0\mathcal{E} \subseteq \dots \subseteq F_t\mathcal{E}, \text{ where} \\ F_p\mathcal{E} \text{ is a subbundle of rank } d_p \text{ of } \mathcal{E}_S \\ \text{for } 0 \le p \le t \end{array} \right\}$$

Here as usual, a subbundle of $\mathcal{E}_S$ means a locally free subsheaf of $\mathcal{E}_S$, which is locally a direct summand.

After these preliminaries we can state the main result of this section, which descibes the structure of the schemes $\overline{O}_{I,J}$ defined above.

Theorem 9.3. *Let T be a scheme and let $\mathcal{E}$ and $\mathcal{F}$ be two locally free $\mathcal{O}_T$-modules of rank n. Let $I := \{i_1,\dots,i_r\}$ and $J := \{j_1,\dots,j_s\}$, where $i_1 + j_1 \ge n$ and $0 \le i_1 < \dots < i_{r+1} := n$, $0 \le j_1 < \dots < j_{s+1} := n$. Let $\mathbf{d} := (d_0,\dots,d_{r+s+1})$ and $\boldsymbol{\delta} := (\delta_0,\dots,\delta_{r+s+1})$, where*

$$d_p := \left\{ \begin{array}{lll} n - j_{s+1-p} & \textit{for} & 0 \le p \le s \\ i_{p-s} & \textit{for} & s+1 \le p \le r+s+1 \end{array} \right.$$

and $\delta_q := n - d_{r+s+1-q}$ for $0 \le q \le r+s+1$. Let

$$\begin{array}{rl} & 0 = U_0 \subseteq U_1 \subseteq \dots \subseteq U_{r+s+1} = \mathcal{E}_{Fl^{\mathbf{d}}(\mathcal{E})\times Fl^{\boldsymbol{\delta}}(\mathcal{F})} \\ \textit{and} & 0 = V_0 \subseteq V_1 \subseteq \dots \subseteq V_{r+s+1} = \mathcal{F}_{Fl^{\mathbf{d}}(\mathcal{E})\times Fl^{\boldsymbol{\delta}}(\mathcal{F})} \end{array}$$

be the pull back to $Fl^{\mathbf{d}}(\mathcal{E}) \times Fl^{\boldsymbol{\delta}}(\mathcal{F})$ of the universal flag on $Fl^{\mathbf{d}}(\mathcal{E})$ and $Fl^{\boldsymbol{\delta}}(\mathcal{F})$ respectively. Then there is a natural isomorphism

$$\overline{O}_{I,J}(\mathcal{E},\mathcal{F}) \xrightarrow{\sim} P_1 \underset{Fl}{\times} \dots \underset{Fl}{\times} P_r \underset{Fl}{\times} Q_s \underset{Fl}{\times} \dots \underset{Fl}{\times} Q_1 \underset{Fl}{\times} K' \quad ,$$

where $Fl := Fl^{\mathbf{d}}(\mathcal{E}) \times Fl^{\boldsymbol{\delta}}(\mathcal{F})$ *and where*

$$\begin{array}{lcll} P_p & := & \overline{PGl}(V_{r-p+1}/V_{r-p}, U_{s+p+1}/U_{s+p}) & (1 \leq p \leq r) \\ Q_q & := & \overline{PGl}(U_{s-q+1}/U_{s-q}, V_{r+q+1}/V_{r+q}) & (1 \leq q \leq s) \\ K' & := & KGl(U_{s+1}/U_s, V_{r+1}/V_r) & . \end{array}$$

Proof. The isomorphism

$$\overline{O}_{I,J} \cong P_1 \underset{\mathrm{Fl}}{\times} \ldots \underset{\mathrm{Fl}}{\times} P_r \underset{\mathrm{Fl}}{\times} Q_s \underset{\mathrm{Fl}}{\times} \ldots \underset{\mathrm{Fl}}{\times} Q_1 \underset{\mathrm{Fl}}{\times} K'$$

is given on S-valued points by the bijectiv correspondence

$$\Phi \longleftrightarrow ((F_\bullet\mathcal{E}, F_\bullet\mathcal{F}), \varphi_1, \ldots, \varphi_r, \psi_s, \ldots, \psi_1, \Phi') \quad ,$$

where

$$\Phi = \left(\begin{tikzcd} \mathcal{E}_S \arrow[r, bend left, "\otimes"] & \mathcal{E}_1 \arrow[l, "0"', "{(\mathcal{M}_0,\mu_0)}"] \arrow[r, bend left, "\otimes"] & \mathcal{E}_2 \arrow[l, "1"', "{(\mathcal{M}_1,\mu_1)}"] & \cdots & \mathcal{E}_{n-1} \arrow[r, bend left, "\otimes"] & \mathcal{E}_n \arrow[l, "n-1"', "{(\mathcal{M}_{n-1},\mu_{n-1})}"] \arrow[r, "\sim"] & \mathcal{F}_n \arrow[r, "n-1", "{(\mathcal{L}_{n-1},\lambda_{n-1})}"'] & \mathcal{F}_{n-1} \arrow[l, bend right, "\otimes"'] & \cdots & \mathcal{F}_2 \arrow[r, "1", "{(\mathcal{L}_1,\lambda_1)}"'] & \mathcal{F}_1 \arrow[l, bend right, "\otimes"'] \arrow[r, "0", "{(\mathcal{L}_0,\lambda_0)}"'] & \mathcal{F}_S \arrow[l, bend right, "\otimes"'] \end{tikzcd} \right)$$

is a generalized isomorphism from $\mathcal{E}_S$ to $\mathcal{F}_S$ with $\mu_i = \lambda_j = 0$ for $i \in I$ and $j \in J$,

$$\begin{array}{rl} & F_\bullet\mathcal{E} = (0 = F_0\mathcal{E} \subseteq \cdots \subseteq F_{r+s+1}\mathcal{E} = \mathcal{E}_S) \\ \text{and} & F_\bullet\mathcal{F} = (0 = F_0\mathcal{F} \subseteq \cdots \subseteq F_{r+s+1}\mathcal{F} = \mathcal{F}_S) \end{array}$$

are flags of type $\mathbf{d}$ and $\boldsymbol{\delta}$ in $\mathcal{E}_S$ and $\mathcal{F}_S$ respectively,

$$\varphi_p = \left(\begin{tikzcd} \mathcal{E}_0^{(p)} \arrow[r, bend left, "\otimes"] & \mathcal{E}_1^{(p)} \arrow[l, "0"', "{(\mathcal{M}_0^{(p)},\mu_0^{(p)})}"] & \cdots & \mathcal{E}_{m_p-1}^{(p)} \arrow[r, bend left, "\otimes"] & \mathcal{E}_{m_p}^{(p)} \arrow[l, "m_p"', "{(\mathcal{M}_{m_p}^{(p)},\mu_{m_p}^{(p)})}"] \end{tikzcd} \right)$$

is a complete collineation from $\mathcal{E}_{m_p}^{(p)} = F_{r-p+1}\mathcal{F}/F_{r-p}\mathcal{F}$ to $\mathcal{E}_0^{(p)} = F_{s+p+1}\mathcal{E}/F_{s+p}\mathcal{E}$ for $1 \leq p \leq r$,

$$\psi_q = \left(\begin{tikzcd} \mathcal{F}_{n_q}^{(q)} \arrow[r, "n_q", "{(\mathcal{L}_{n_q}^{(q)},\lambda_{n_q}^{(q)})}"'] & \mathcal{F}_{n_q-1}^{(q)} \arrow[l, bend right, "\otimes"'] & \cdots & \mathcal{F}_1^{(q)} \arrow[r, "0", "{(\mathcal{L}_0^{(q)},\lambda_0^{(q)})}"'] & \mathcal{F}_0^{(p)} \arrow[l, bend right, "\otimes"'] \end{tikzcd} \right)$$

is a complete collineation from $\mathcal{F}_{n_q}^{(q)} = F_{s-q+1}\mathcal{F}/F_{s-q}\mathcal{E}$ to $\mathcal{F}_0^{(q)} = F_{r+q+1}\mathcal{F}/F_{r+q}\mathcal{F}$ for $s \geq q \geq 1$ and $\Phi' =$

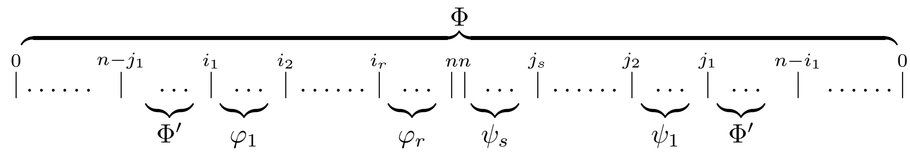
$$\left(\begin{tikzcd} \mathcal{E}'_0 \arrow[r, bend left, "\otimes"] & \mathcal{E}'_1 \arrow[l, "0"', "{(\mathcal{M}'_0,\mu'_0)}"] \arrow[r, bend left, "\otimes"] & \mathcal{E}'_2 \arrow[l, "1"', "{(\mathcal{M}'_1,\mu'_1)}"] & \cdots & \mathcal{E}'_{n'-1} \arrow[r, bend left, "\otimes"] & \mathcal{E}'_{n'} \arrow[l, "n'-1"', "{(\mathcal{M}'_{n'-1},\mu'_{n'-1})}"] \arrow[r, "\sim"] & \mathcal{F}'_{n'} \arrow[r, "n'-1", "{(\mathcal{L}'_{n'-1},\lambda'_{n'-1})}"'] & \mathcal{F}'_{n'-1} \arrow[l, bend right, "\otimes"'] & \cdots & \mathcal{F}'_2 \arrow[r, "1", "{(\mathcal{L}'_1,\lambda'_1)}"'] & \mathcal{F}'_1 \arrow[l, bend right, "\otimes"'] \arrow[r, "0", "{(\mathcal{L}'_0,\lambda'_0)}"'] & \mathcal{F}'_0 \arrow[l, bend right, "\otimes"'] \end{tikzcd} \right)$$

is a generalized isomorphism from $\mathcal{E}'_0 = F_{s+1}\mathcal{E}/F_s\mathcal{E}$ to $\mathcal{F}'_0 = F_{r+1}\mathcal{F}/F_r\mathcal{F}$.
The mapping

$$\Phi \mapsto ((F_\bullet\mathcal{E}, F_\bullet\mathcal{F}), \varphi_1, \ldots, \varphi_r, \psi_s, \ldots, \psi_1, \Phi')$$

is defined as follows: Let Φ as above be given. For convenience we set $\mathcal{E}_{n+1} := \mathcal{F}_n$, $\mathcal{F}_{n+1} := \mathcal{E}_n$ and we let $\mathcal{F}_{n+1} \to \mathcal{F}_n$ and $\mathcal{E}_n \leftarrow \mathcal{E}_{n+1}$ be the isomorphism $\mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n$ and its inverse respectively, whereas we let $\mathcal{E}_n \xrightarrow{\otimes} \mathcal{E}_{n+1}$

and $\mathcal{F}_{n+1} \twoheadleftarrow\!\!\otimes\!\!- \mathcal{F}_n$ both be the zero morphism. For what follows, the picture below may help to keep track of the indices:



Let $F_0\mathcal{E} = F_0\mathcal{F} := 0$, $F_{r+s+1}\mathcal{E} := \mathcal{E}$, $F_{r+s+1}\mathcal{F} := \mathcal{F}$ and

$$F_p\mathcal{E} \quad := \quad \begin{cases} \begin{pmatrix} \text{image of } \ker(\mathcal{E}_n \to \mathcal{F}_{j_{s-p+1}}) \text{ by} \\ \text{the morphism } \mathcal{E}_S \leftarrow \mathcal{E}_n \end{pmatrix}, & \text{if } 1 \le p \le s \\ \ker(\ \mathcal{E}_S -\!\!\otimes\!\!\twoheadrightarrow \mathcal{E}_{i_{p-s}+1}\ )\ , & \text{if } s+1 \le p \le r+s \end{cases}$$

$$F_q\mathcal{F} \quad := \quad \begin{cases} \begin{pmatrix} \text{image of } \ker(\mathcal{E}_{i_{r-q+1}} \leftarrow \mathcal{F}_n) \text{ by} \\ \text{the morphism } \mathcal{F}_n \to \mathcal{F}_S \end{pmatrix}, & \text{if } 1 \le q \le r \\ \ker(\ \mathcal{F}_{j_{q-r}+1} \twoheadleftarrow\!\!\otimes\!\!- \mathcal{F}_S\ )\ , & \text{if } r+1 \le q \le r+s \end{cases}$$

It is then clear from the definition of generalized isomorphisms that

$$\begin{aligned} & F_\bullet\mathcal{E} := (0 = F_0\mathcal{E} \subseteq \cdots \subseteq F_{r+s+1}\mathcal{E} = \mathcal{E}_S) \\ \text{and} \qquad & F_\bullet\mathcal{F} := (0 = F_0\mathcal{F} \subseteq \cdots \subseteq F_{r+s+1}\mathcal{F} = \mathcal{F}_S) \end{aligned}$$

are flags of type $\mathbf{d}$ and $\boldsymbol{\delta}$ in $\mathcal{E}_S$ and $\mathcal{F}_S$ respectively. Let $1 \le p \le r$. We set

$$\begin{aligned} \mathcal{E}_0^{(p)} &:= \ker(\ \mathcal{E} -\!\!\otimes\!\!\twoheadrightarrow \mathcal{E}_{i_{p+1}+1}\ )/\ker(\ \mathcal{E} -\!\!\otimes\!\!\twoheadrightarrow \mathcal{E}_{i_p+1}\ ) = F_{s+p+1}\mathcal{E}/F_{s+p}\mathcal{E} \\ \mathcal{M}_0^{(p)} &:= \bigotimes_{i=0}^{i_p} \mathcal{M}_i\ , \qquad \mu_0^{(p)} := 0 \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_k^{(p)} &:= \ker(\mathcal{E}_{i_p} \leftarrow \mathcal{E}_{i_p+k}) \cap \ker(\ \mathcal{E}_{i_p+k} -\!\!\otimes\!\!\twoheadrightarrow \mathcal{E}_{i_{p+1}+1}\ ) \quad (1 \le k \le m_p) \\ \mathcal{M}_k^{(p)} &:= \mathcal{M}_{i_p+k}\ , \qquad \mu_k^{(p)} := \mu_{i_p+k} \qquad\qquad\qquad (1 \le k \le m_p - 1) \end{aligned}$$

where $m_p = i_{p+1} - i_p$. Observe that the sheaves $\mathcal{E}_k^{(p)}$ thus defined are locally free of rank m_p. Indeed, this is clear for $k = 0$. For $k \ge 1$ it suffices to show that $\mathcal{E}_{i_p+k}$ is generated by the two subsheaves $\ker(\mathcal{E}_{i_p} \leftarrow \mathcal{E}_{i_p+k})$ and $\ker(\ \mathcal{E}_{i_p+k} -\!\!\otimes\!\!\twoheadrightarrow \mathcal{E}_{i_{p+1}+1}\ )$. For this in turn, it suffices to show that the image of $\ker(\ \mathcal{E}_{i_p+k} -\!\!\otimes\!\!\twoheadrightarrow \mathcal{E}_{i_{p+1}+1}\ )$ by the morphism $\mathcal{E}_{i_p} \leftarrow \mathcal{E}_{i_p+k}$ is $\operatorname{im}(\mathcal{E}_{i_p} \leftarrow \mathcal{E}_{i_p+k})$. But this is clear, since by the definition of generalized isomorphisms we have

$$\begin{aligned} & \ker(\ \mathcal{E}_{i_p+k} -\!\!\otimes\!\!\twoheadrightarrow \mathcal{E}_{i_{p+1}+1}\ ) \supseteq \operatorname{im}(\mathcal{E}_{i_p+k} \leftarrow \mathcal{E}_{i_{p+1}+1}) \\ \text{and} \qquad & \operatorname{im}(\mathcal{E}_{i_p} \leftarrow \mathcal{E}_{i_p+k}) = \operatorname{im}(\mathcal{E}_{i_p} \leftarrow \mathcal{E}_{i_{p+1}+1}) \quad . \end{aligned}$$

Since

$$\bigotimes_{i=0}^{i_p} \mathcal{M}_i^\vee \otimes (\mathcal{E}_S/F_{s+p}\mathcal{E}) = \operatorname{im}(\bigotimes_{i=0}^{i_p} \mathcal{M}_i^\vee \otimes \mathcal{E}_S \to \mathcal{E}_{i_p+1}) = \ker(\mathcal{E}_{i_p} \leftarrow \mathcal{E}_{i_p+1}) \quad ,$$

we have a natural isomorphism $\mathcal{E}_0^{(p)} = F_{s+p+1}\mathcal{E}/F_{s+p}\mathcal{E} \xrightarrow{\sim} \mathcal{M}_0^{(p)} \otimes \mathcal{E}_1^{(p)}$. We define $\mathcal{E}_0^{(p)} \leftarrow \mathcal{E}_1^{(p)}$ to be the zero morphism. Thus we have a bf-morphism $\mathcal{E}_0^{(p)} \xleftarrow[(\mathcal{M}_0^{(p)},\mu_0^{(p)}=0)]{0} \mathcal{E}_1^{(p)}$ of rank zero. For $1 \leq k \leq m_p - 1$ let $\mathcal{E}_k^{(p)} \xleftarrow[(\mathcal{M}_k^{(p)},\mu_k^{(p)})]{k} \mathcal{E}_{k+1}^{(p)}$ be the bf-morphism induced by the bf-morphism $\mathcal{E}_{i_p+k} \xleftarrow[(\mathcal{M}_{i_p+k},\mu_{i_p+k})]{i_p+k} \mathcal{E}_{i_p+k+1}$. Observe that $\ker(\mathcal{E}_{i_{p+1}} \mathrel{-\!\otimes\!\!\rightarrow} \mathcal{E}_{i_{p+1}+1}) = \mathrm{im}(\mathcal{E}_{i_{p+1}} \leftarrow \mathcal{E}_n) = \mathcal{E}_n/\ker(\mathcal{E}_{i_{p+1}} \leftarrow \mathcal{E}_n)$ and that the morphism $\mathcal{E}_n \to \mathcal{F}_S$ maps $\ker(\mathcal{E}_{i_p} \leftarrow \mathcal{E}_n)$ injectively into $\mathcal{F}_S$. Therefore we have a natural isomorphism $\mathcal{E}_{m_p}^{(p)} \cong F_{r-p+1}\mathcal{F}/F_{r-p}\mathcal{F}$ by which we identify these two sheaves. It is not difficult to see that

$$\varphi_p := \left(\mathcal{E}_0^{(p)} \xleftarrow[(\mathcal{M}_0^{(p)},\mu_0^{(p)})]{0} \mathcal{E}_1^{(p)} \quad \cdots \quad \mathcal{E}_{m_p-1}^{(p)} \xleftarrow[(\mathcal{M}_{m_p}^{(p)},\mu_{m_p}^{(p)})]{m_p} \mathcal{E}_{m_p}^{(p)} \right)$$

is a complete collineation in the sense of 8.1 from $\mathcal{E}_{m_p}^{(p)} = F_{r-p+1}\mathcal{F}/F_{r-p}\mathcal{F}$ to $\mathcal{E}_0^{(p)} = F_{s+p+1}\mathcal{E}/F_{s+p}\mathcal{E}$. In a completely symmetric way the generalized isomorphism Φ induces also complete collineations

$$\psi_q = \left(\mathcal{F}_{n_q}^{(q)} \xrightarrow[(\mathcal{L}_{n_q}^{(q)},\lambda_{n_q}^{(q)})]{n_q} \mathcal{F}_{n_q-1}^{(q)} \quad \cdots \quad \mathcal{F}_1^{(q)} \xrightarrow[(\mathcal{L}_0^{(q)},\lambda_0^{(q)})]{0} \mathcal{F}_0^{(p)} \right)$$

from $\mathcal{F}_{n_q}^{(q)} = F_{s-q+1}\mathcal{F}/F_{s-q}\mathcal{E}$ to $\mathcal{F}_0^{(q)} = F_{r+q+1}\mathcal{F}/F_{r+q}\mathcal{F}$ for $s \geq q \geq 1$. It remains to construct the generalized isomorphism Φ'. We set

$$\begin{aligned} \mathcal{E}'_k &:= \ker(\mathcal{E}_{n-j_1+k} \mathrel{-\!\otimes\!\!\rightarrow} \mathcal{E}_{i_1+1})/\mathrm{im}(\mathcal{E}_{n-j_1+k} \leftarrow \ker(\mathcal{E}_n \to \mathcal{F}_{j_1})) \\ \mathcal{F}'_k &:= \ker(\mathcal{F}_{j_1+1} \mathrel{\leftarrow\!\!\otimes\!-} \mathcal{F}_{n-i_1+k})/\mathrm{im}(\ker(\mathcal{E}_{i_1} \leftarrow \mathcal{F}_n) \to \mathcal{F}_{n-i_1+k})) \end{aligned}$$

for $0 \leq k \leq n' := i_1 + j_1 - n$ and

$$\begin{aligned} \mathcal{M}'_k &:= \mathcal{M}_{n-j_1+k} \ , \quad \mu'_k := \mu_{n-j_1+k} \\ \mathcal{L}'_k &:= \mathcal{L}_{n-i_1+k} \quad , \quad \lambda'_k := \lambda_{n-i_1+k} \end{aligned}$$

for $0 \leq k \leq n' - 1$. It is then clear that the $\mathcal{E}'_k$ and $\mathcal{F}'_k$ are locally free of rank $n' = i_1 + j_1 - n$. It follows from definition 5.2.2. that the μ_i and λ_j are nowhere vanishing for $0 \leq i \leq n - j_1 - 1$ and $n - i_1 - 1 \geq j \geq 0$. Therefore we may identify $\mathcal{E}_i$ with $\mathcal{E}_S$ and $\mathcal{F}_j$ with $\mathcal{F}_S$ for $0 \leq i \leq n - j_1 - 1$ and $n - i_1 - 1 \geq j \geq 0$ respectively. This implies in particular that we have $\mathcal{E}'_0 = F_{s+1}\mathcal{E}/F_s\mathcal{E}$ and $\mathcal{F}'_0 = F_{r+1}\mathcal{F}/F_r\mathcal{F}$. Let

$$\mathcal{E}'_k \xleftarrow[(\mathcal{M}'_k,\mu'_k)]{k} \mathcal{E}'_{k+1} \quad \text{and} \quad \mathcal{F}'_{k+1} \xrightarrow[(\mathcal{L}'_k,\lambda'_k)]{k} \mathcal{F}'_k$$

be the bf-morphisms induced by the bf-morphisms

$$\begin{tikzcd}[column sep=large] \mathcal{E}_{n-j_1+k} \arrow[r, bend left, "\otimes"] & \mathcal{E}_{n-j_1+k+1} \arrow[l, "n-j_1+k"', "(\mathcal{M}_{n-j_1+k},\mu_{n-j_1+k})"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=large] \mathcal{F}_{n-i_1+k+1} \arrow[r, "n-i_1+k", "(\mathcal{L}_{n-i_1+k},\lambda_{n-i_1+k})"'] & \mathcal{F}_{n-i_1+k} \arrow[l, bend right, "\otimes"'] \end{tikzcd}$$

respectively. We have

$$\ker(\mathcal{E}_{i_1} \overset{\otimes}{\longrightarrow} \mathcal{E}_{i_1+1}) = \operatorname{im}(\mathcal{E}_{i_1} \leftarrow \mathcal{E}_n) = \mathcal{E}_n / \ker(\mathcal{E}_{i_1} \leftarrow \mathcal{E}_n)$$

and therefore

$$\mathcal{E}'_{n'} = \mathcal{E}_n/(\ker(\mathcal{E}_{i_1} \leftarrow \mathcal{E}_n) + \ker(\mathcal{E}_n \to \mathcal{F}_{j_1})) \quad .$$

By the same argument:

$$\mathcal{F}'_{n'} = \mathcal{F}_n/(\ker(\mathcal{E}_{i_1} \leftarrow \mathcal{F}_n) + \ker(\mathcal{F}_n \to \mathcal{F}_{j_1})) \quad .$$

Thus the isomorphism $\mathcal{E}_n \overset{\sim}{\to} \mathcal{F}_n$ induces an isomorphism $\mathcal{E}'_{n'} \overset{\sim}{\to} \mathcal{F}'_{n'}$. Again it is not difficult to check that $\Phi' :=$

$$\left(\begin{tikzcd}[column sep=small] \mathcal{E}'_0 \arrow[r, bend left, "\otimes"] & \mathcal{E}'_1 \arrow[l, "0"', "(\mathcal{M}'_0,\mu'_0)"] \arrow[r, bend left, "\otimes"] & \mathcal{E}'_2 \arrow[l, "1"', "(\mathcal{M}'_1,\mu'_1)"] & \cdots & \mathcal{E}'_{n'-1} \arrow[r, bend left, "\otimes"] & \mathcal{E}'_{n'} \arrow[l, "n'-1"', "(\mathcal{M}'_{n'-1},\mu'_{n'-1})"] \arrow[r, "\sim"] & \mathcal{F}'_{n'} \arrow[r, "n'-1", "(\mathcal{L}'_{n'-1},\lambda'_{n'-1})"'] & \mathcal{F}'_{n'-1} \arrow[l, bend right, "\otimes"'] & \cdots & \mathcal{F}'_2 \arrow[r, "1", "(\mathcal{L}'_1,\lambda'_1)"'] & \mathcal{F}'_1 \arrow[l, bend right, "\otimes"'] \arrow[r, "0", "(\mathcal{L}'_0,\lambda'_0)"'] & \mathcal{F}'_0 \arrow[l, bend right, "\otimes"'] \end{tikzcd} \right)$$

is a generalized isomorphism from $\mathcal{E}'_0 = F_{s+1}\mathcal{E}/F_s\mathcal{E}$ to $\mathcal{F}'_0 = F_{r+1}\mathcal{F}/F_r\mathcal{F}$. This completes the construction of the mapping

$$\Phi \mapsto ((F_\bullet\mathcal{E}, F_\bullet\mathcal{F}), \varphi_1, \ldots, \varphi_r, \psi_s, \ldots, \psi_1, \Phi') \quad .$$

We proceed by constructing the inverse of this mapping. Let data $((F_\bullet\mathcal{E}, F_\bullet\mathcal{F}), \varphi_1, \ldots, \varphi_r, \psi_s, \ldots, \psi_1, \Phi')$ be given. Let $\mathcal{E}_i := \mathcal{E}_S$ and $\mathcal{F}_j := \mathcal{F}_S$ for $0 \le i \le n - j_1$ and $n - i_1 \ge j \ge 0$ respectively. Now let $n - j_1 + 1 \le i \le i_1$ and $j_1 \ge j \ge n - i_1 + 1$. Let $\tilde{\mathcal{E}}_i$ and $\tilde{\mathcal{F}}_j$ be defined by the cartesian diagrams

$$\begin{tikzcd} \tilde{\mathcal{E}}_i \arrow[r] \arrow[d] & \mathcal{E}'_{i+j_1-n} \arrow[d] \\ F_{s+1}\mathcal{E} \arrow[r, two heads] & F_{s+1}\mathcal{E}/F_s\mathcal{E} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \tilde{\mathcal{F}}_j \arrow[r] \arrow[d] & \mathcal{F}'_{j+i_1-n} \arrow[d] \\ F_{r+1}\mathcal{F} \arrow[r, two heads] & F_{r+1}\mathcal{F}/F_r\mathcal{F} \end{tikzcd}$$

respectively. For a moment let $\mathcal{M} := \bigotimes_{k=0}^{i+j_1-n-1} \mathcal{M}'_k$. We have a commutative diagram

$$\begin{tikzcd} \mathcal{M}^\vee \otimes F_{s+1}\mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}'_{i+j_1-n} \arrow[d] \\ F_{s+1}\mathcal{E} \arrow[r, two heads] & F_{s+1}\mathcal{E}/F_s\mathcal{E} \end{tikzcd} \tag{$*$}$$

where the left vertical arrow is induced by $\bigotimes_{k=0}^{i+j_1-n-1} \mu'_k : \mathcal{O}_S \to \mathcal{M}$ and the upper horizontal arrow is the composition

$$\mathcal{M}^\vee \otimes F_{s+1}\mathcal{E} \to \mathcal{M}^\vee \otimes F_{s+1}\mathcal{E}/F_s\mathcal{E} = \mathcal{M}^\vee \otimes \mathcal{E}'_0 \to \mathcal{E}'_{i+j_1-n}$$

The diagram $(*)$ induces a morphism $\mathcal{M}^\vee \otimes F_{s+1}\mathcal{E} \to \tilde{\mathcal{E}}_i$. Analogously, we have a morphism $\mathcal{L}^\vee \otimes F_{r+1}\mathcal{F} \to \tilde{\mathcal{F}}_j$, where we have employed the abbreviation $\mathcal{L} := \bigotimes_{k=0}^{j+i_1-n-1} \mathcal{L}'_k$. Let $\mathcal{E}_i$ and $\mathcal{F}_j$ be defined by the cocartesian diagrams

$$\begin{tikzcd} \mathcal{M}^\vee \otimes F_{s+1}\mathcal{E} \arrow[r, hook] \arrow[d] & \mathcal{M}^\vee \otimes \mathcal{E}_S \arrow[d] \\ \tilde{\mathcal{E}}_i \arrow[r] & \mathcal{E}_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{L}^\vee \otimes F_{r+1}\mathcal{F} \arrow[r, hook] \arrow[d] & \mathcal{L}^\vee \otimes \mathcal{F}_S \arrow[d] \\ \tilde{\mathcal{F}}_j \arrow[r] & \mathcal{F}_j \end{tikzcd}$$

respectively.
We define $\mathcal{E}_n = \mathcal{F}_n$ by the cartesian diagram

$$\begin{tikzcd} \mathcal{E}_n = \mathcal{F}_n \arrow[rr] \arrow[d] & & \tilde{\mathcal{F}}_{j_1} \arrow[d, two heads] \\ \tilde{\mathcal{E}}_{i_1} \arrow[r, two heads] & \mathcal{E}'_{i_1+j_1-n} \arrow[r, "\cong"] & \mathcal{F}'_{i_1+j_1-n} \end{tikzcd}$$

Observe that the composed morphism $\mathcal{E}_n \to \tilde{\mathcal{E}}_{i_1} \to F_{s+1}\mathcal{E}$ maps the submodule $\ker(\mathcal{E}_n \to \tilde{\mathcal{F}}_{j_1})$ of $\mathcal{E}_n$ isomorphically onto the submodule $F_s\mathcal{E}$ of $F_{s+1}\mathcal{E}$. Therefore we have canonical injections

$$F_p\mathcal{E} \hookrightarrow F_s\mathcal{E} \xrightarrow{\sim} \ker(\mathcal{E}_n \to \tilde{\mathcal{F}}_{j_1}) \hookrightarrow \mathcal{E}_n$$

for $0 \leq p \leq s$. Analogously, we have canonical injections

$$F_q\mathcal{F} \hookrightarrow F_r\mathcal{F} \xrightarrow{\sim} \ker(\mathcal{F}_n \to \tilde{\mathcal{E}}_{i_1}) \hookrightarrow \mathcal{F}_n$$

for $0 \leq q \leq r$.
Now let $1 \leq p \leq r$, $i_p + 1 \leq i \leq i_{p+1}$ and $s \geq q \geq 1$, $j_{q+1} \geq j \geq j_q + 1$. We want to define $\mathcal{E}_i$ and $\mathcal{F}_j$ in this case. Let first $\tilde{\mathcal{E}}_i$ and $\tilde{\mathcal{F}}_j$ be defined by the cocartesian diagrams

$$\begin{tikzcd} \mathcal{M}^\vee \otimes F_{s+p+1}\mathcal{E}/F_{s+p}\mathcal{E} \arrow[r, two heads] \arrow[d, hook] & \mathcal{E}^{(p)}_{i-i_p} \arrow[d] \\ \mathcal{M}^\vee \otimes \mathcal{E}_S/F_{s+p}\mathcal{E} \arrow[r] & \tilde{\mathcal{E}}_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{L}^\vee \otimes F_{r+q+1}\mathcal{F}/F_{r+q}\mathcal{E} \arrow[r, two heads] \arrow[d, hook] & \mathcal{F}^{(q)}_{j-j_q} \arrow[d] \\ \mathcal{L}^\vee \otimes \mathcal{F}_S/F_{r+q}\mathcal{F} \arrow[r] & \tilde{\mathcal{F}}_j \end{tikzcd}$$

where we have set $\mathcal{M} := \bigotimes_{k=0}^{i-i_p-1} \mathcal{M}^{(p)}_k$ and $\mathcal{L} := \bigotimes_{k=0}^{j-j_q-1} \mathcal{L}^{(q)}_k$. Let furthermore $\hat{\mathcal{E}}_i$ and $\hat{\mathcal{F}}_j$ be defined by the cocartesian diagrams

$$\begin{tikzcd} F_{r-p+1}\mathcal{F}/F_{r-p}\mathcal{F} \arrow[r] \arrow[d, hook] & \mathcal{E}^{(p)}_{i-i_p} \arrow[d] \\ \mathcal{F}_n/F_{r-p}\mathcal{F} \arrow[r] & \hat{\mathcal{E}}_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F_{s-q+1}\mathcal{E}/F_{s-q}\mathcal{E} \arrow[r] \arrow[d, hook] & \mathcal{F}^{(q)}_{j-j_q} \arrow[d] \\ \mathcal{E}_n/F_{s-q}\mathcal{E} \arrow[r] & \hat{\mathcal{F}}_j \end{tikzcd}$$

Now we define $\mathcal{E}_i$ and $\mathcal{F}_j$ by the cocartesian diagrams

$$\begin{tikzcd} \mathcal{E}^{(p)}_{i-i_p} \arrow[r] \arrow[d] & \tilde{\mathcal{E}}_i \arrow[d] \\ \hat{\mathcal{E}}_i \arrow[r] & \mathcal{E}_i \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{F}^{(q)}_{j-j_q} \arrow[r] \arrow[d] & \tilde{\mathcal{F}}_j \arrow[d] \\ \hat{\mathcal{F}}_j \arrow[r] & \mathcal{F}_j \end{tikzcd}$$

respectively. For $p = r$ and $i = i_{r+1} = n$ this gives formally a new definition of $\mathcal{E}_n$, but it is clear that we have a canonical isomorphism between the two $\mathcal{E}_n$'s. A similar remark applies to $\mathcal{F}_n$.
We define the invertible sheaves $\mathcal{M}_i$ together with their respective sections μ_i as follows:

$$\begin{array}{lcl} \mathcal{M}_i & := & \mathcal{O}_S \ , \quad \mu_i := 1 \qquad (0 \le i \le n - j_1 - 1) \\ \mathcal{M}_i & := & \mathcal{M}'_{i+j_1-n} \ , \quad \mu_i := \mu'_{i+j_1-n} \qquad (n - j_1 \le i \le i_1 - 1) \\ \mathcal{M}_i & := & \mathcal{M}^{(p)}_{i-i_p} \ , \quad \mu_i := \mu^{(p)}_{i-i_p} \qquad (1 \le p \le r \ , \ \ i_p < i < i_{p+1}) \end{array}$$

$$\begin{array}{lcl} \mathcal{M}_{i_1} & := & \mathcal{M}^{(1)}_0 \otimes \bigotimes_{k=0}^{i_1+j_1-n-1} (\mathcal{M}'_k)^\vee \ , \quad \mu_{i_1} := 0 \\ \mathcal{M}_{i_p} & := & \mathcal{M}^{(p)}_0 \otimes \bigotimes_{k=0}^{i_p-i_{p-1}-1} (\mathcal{M}^{(p-1)}_k)^\vee \ , \quad \mu_{i_p} := 0 \qquad (2 \le p \le r) \end{array}$$

Let the $\mathcal{L}_j$ and λ_j be defined symmetrically (i.e. by replacing in the above definition the letter $\mathcal{M}$ with $\mathcal{L}$, μ with λ, i with j, j with i and r with s).
It remains to define the bf-morphisms

$$\begin{tikzcd} \mathcal{E}_i \arrow[r, bend left, "\otimes"] & \mathcal{E}_{i+1} \arrow[l, "i"', "(\mathcal{M}_i,\mu_i)"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{F}_{j+1} \arrow[r, "j", "(\mathcal{L}_j,\lambda_j)"'] & \mathcal{F}_j \arrow[l, bend right, "\otimes"'] \end{tikzcd}$$

for $n - j_1 \le i \le n - 1$ and $n - 1 \ge j \ge n - i_1$. Again we restrict ourselves to the left hand side, since the right hand side is obtained by the symmetric construction. For $n - j_1 \le i \le i_1 - 1$ (respectively for $1 \le p \le r \ , \ i_p \le i \le i_{p+1} - 1$) the bf-morphism $\begin{tikzcd} \mathcal{E}_i \arrow[r, bend left, "\otimes"] & \mathcal{E}_{i+1} \arrow[l] \end{tikzcd}$ is induced in an obvious way by the bf-morphism $\begin{tikzcd} \mathcal{E}'_{i+j_1-n} \arrow[r, bend left, "\otimes"] & \mathcal{E}'_{i+j_1-n+1} \arrow[l] \end{tikzcd}$ (respectively by the bf-morphism $\begin{tikzcd} \mathcal{E}^{(p)}_{i-i_p} \arrow[r, bend left, "\otimes"] & \mathcal{E}^{(p)}_{i-i_p+1} \arrow[l] \end{tikzcd}$). For the definition of the bf-morphism $\begin{tikzcd} \mathcal{E}_{i_1} \arrow[r, bend left, "\otimes"] & \mathcal{E}_{i_1+1} \arrow[l] \end{tikzcd}$ consider the two canonical exact sequences

$$0 \longrightarrow \tilde{\mathcal{E}}_{i_1} \longrightarrow \mathcal{E}_{i_1} \longrightarrow \mathcal{M}^\vee \otimes \mathcal{E}_S / F_{s+1}\mathcal{E} \longrightarrow 0$$

$$0 \longrightarrow \tilde{\mathcal{E}}_{i_1+1} \longrightarrow \mathcal{E}_{i_1+1} \longrightarrow \hat{\mathcal{E}}_{i_1+1}/\mathcal{E}^{(1)}_1 \longrightarrow 0$$

where $\mathcal{M} := \bigotimes_{k=0}^{i_1-1} \mathcal{M}_k$. Observe that we have canonical isomorphisms

$$\tilde{\mathcal{E}}_{i_1} \xrightarrow[\cong]{a} \mathcal{F}_n/F_r\mathcal{F} \xrightarrow[\cong]{b} \hat{\mathcal{E}}_{i_1+1}/\mathcal{E}_1^{(1)}$$

$$\tilde{\mathcal{E}}_{i_1+1} \xrightarrow[\cong]{c} \mathcal{M}_{i_1}^\vee \otimes \mathcal{M}^\vee \otimes \mathcal{E}_S/F_{s+1}\mathcal{E}$$

The isomorphism a follows from the observation we made after the definition of $\mathcal{E}_n = \mathcal{F}_n$, namely that the composed morphism $\mathcal{F}_n \to \tilde{\mathcal{F}}_{j_1} \to F_{r+1}\mathcal{F}$ maps $\ker(\mathcal{E}_n \to \tilde{\mathcal{E}}_{i_1})$ isomorphically to $F_r\mathcal{F}$. The isomorphism b comes from the fact that for $i = i_1+1$ the left vertical arrow in the defining diagram for $\hat{\mathcal{E}}_i$ vanishes, and the isomorphism c follows since for $i = i_1 + 1$ the left vertical arrow in the defining diagram for $\tilde{\mathcal{E}}_i$ is an isomorphism. Thus we have morphisms

$$\mathcal{E}_{i_1} \twoheadrightarrow \mathcal{M}^\vee \otimes \mathcal{E}_S/F_{s+1} \xrightarrow{c^{-1}} \mathcal{M}_{i_1} \otimes \tilde{\mathcal{E}}_{i_1+1} \hookrightarrow \mathcal{M}_{i_1} \otimes \mathcal{E}_{i_1+1}$$

$$\mathcal{E}_{i_1} \hookleftarrow \tilde{\mathcal{E}}_{i_1} \xleftarrow{a^{-1}b^{-1}} \hat{\mathcal{E}}_{i_1+1}/\mathcal{E}_1^{(1)} \twoheadleftarrow \mathcal{E}_{i_1+1}$$

which make up the bf-morphism $\begin{tikzcd} \mathcal{E}_{i_1} \arrow[r, bend left, "\otimes"] & \mathcal{E}_{i_1+1} \arrow[l] \end{tikzcd}$. For $2 \leq p \leq r$ the bf-morphism $\begin{tikzcd} \mathcal{E}_{i_p} \arrow[r, bend left, "\otimes"] & \mathcal{E}_{i_p+1} \arrow[l] \end{tikzcd}$ is constructed similarly from the exact sequences

$$0 \longrightarrow \hat{\mathcal{E}}_{i_p} \longrightarrow \mathcal{E}_{i_p} \longrightarrow \tilde{\mathcal{E}}_{i_p}/\mathcal{E}_{i_p-i_{p-1}}^{(p-1)} \longrightarrow 0$$

$$0 \longrightarrow \tilde{\mathcal{E}}_{i_p+1} \longrightarrow \mathcal{E}_{i_p+1} \longrightarrow \hat{\mathcal{E}}_{i_p+1}/\mathcal{E}_1^{(p)} \longrightarrow 0$$

and the canonical isomorphisms

$$\hat{\mathcal{E}}_{i_p} \xrightarrow{\cong} \mathcal{F}_n/F_{r-p+1}\mathcal{F} \xrightarrow{\cong} \hat{\mathcal{E}}_{i_p+1}/\mathcal{E}_1^{(p)}$$

$$\tilde{\mathcal{E}}_{i_p+1} \xrightarrow{\cong} \mathcal{M}_{i_p}^\vee \otimes \mathcal{M}^\vee \otimes \mathcal{E}/F_{s+p}\mathcal{E} \xrightarrow{\cong} \mathcal{M}_{i_p}^\vee \otimes \tilde{\mathcal{E}}_{i_p}/\mathcal{E}_{i_p-i_{p-1}}^{(p-1)}$$

where $\mathcal{M} := \bigotimes_{k=0}^{i_p-1} \mathcal{M}_k$.
This completes the construction of

$$\Phi = \left(\begin{tikzcd}[column sep=small] \mathcal{E}_S \arrow[r, bend left, "\otimes"] & \mathcal{E}_1 \arrow[l, "0"', "(\mathcal{M}_0,\mu_0)"] \arrow[r, bend left, "\otimes"] & \mathcal{E}_2 \arrow[l, "1"', "(\mathcal{M}_1,\mu_1)"] & \cdots & \mathcal{E}_{n-1} \arrow[r, bend left, "\otimes"] & \mathcal{E}_n \arrow[l, "n-1"', "(\mathcal{M}_{n-1},\mu_{n-1})"] \arrow[r, "\sim"] & \mathcal{F}_n \arrow[r, "n-1", "(\mathcal{L}_{n-1},\lambda_{n-1})"'] & \mathcal{F}_{n-1} \arrow[l, bend right, "\otimes"'] & \cdots & \mathcal{F}_2 \arrow[r, "1", "(\mathcal{L}_1,\lambda_1)"'] & \mathcal{F}_1 \arrow[l, bend right, "\otimes"'] \arrow[r, "0", "(\mathcal{L}_0,\lambda_0)"'] & \mathcal{F}_S \arrow[l, bend right, "\otimes"'] \end{tikzcd} \right)$$

It is not difficult to see that Φ is a generalized isomorphism from $\mathcal{E}_S$ to $\mathcal{F}_S$ and that the mapping $((F_\bullet\mathcal{E}, F_\bullet\mathcal{F}), \varphi_1, \ldots, \varphi_r, \psi_s, \ldots, \psi_1, \Phi') \mapsto \Phi$ is inverse to the mapping $\Phi \mapsto ((F_\bullet\mathcal{E}, F_\bullet\mathcal{F}), \varphi_1, \ldots, \varphi_r, \psi_s, \ldots, \psi_1, \Phi')$ constructed before. We leave the details to the reader. $\square$

In the situation of theorem 9.3 we denote by $\mathrm{Gl}(\mathcal{E})$ the group scheme over T, whose S-valued points are the automorphisms of $\mathcal{E}_S$. There is a natural left operation of $\mathrm{Gl}(\mathcal{E}) \times_T \mathrm{Gl}(\mathcal{F})$ on $\mathrm{KGl}(\mathcal{E},\mathcal{F})$, which is given on S-valued points by

$$(f,g)\left(\begin{tikzcd}\mathcal{E}_S \arrow[r, bend left, "u^\flat \otimes"] & \mathcal{E}_1 \arrow[l, "u^\sharp"] \end{tikzcd} \quad \cdots \quad \begin{tikzcd}\mathcal{F}_1 \arrow[r, "v^\sharp"'] & \mathcal{F}_S \arrow[l, bend right, "v^\flat \otimes"'] \end{tikzcd}\right) := \left(\begin{tikzcd}\mathcal{E}_S \arrow[r, bend left, "u^\flat\circ f^{-1} \otimes"] & \mathcal{E}_1 \arrow[l, "f\circ u^\sharp"] \end{tikzcd} \quad \cdots \quad \begin{tikzcd}\mathcal{F}_1 \arrow[r, "g\circ v^\sharp"'] & \mathcal{F}_S \arrow[l, bend right, "v^\flat\circ g^{-1} \otimes"'] \end{tikzcd}\right)$$

Corollary 9.4. *The orbits of the $Gl(\mathcal{E})\times_T Gl(\mathcal{F})$-operation on $KGl(\mathcal{E},\mathcal{F})$ are the locally closed subvarieties*

$$O_{I,J}(\mathcal{E},\mathcal{F}) := \overline{O}_{I,J}(\mathcal{E},\mathcal{F}) \setminus \left(\bigcup_{i\notin\mathcal{I}} Z_i(\mathcal{E},\mathcal{F}) \cup \bigcup_{j\notin\mathcal{J}} Y_j(\mathcal{E},\mathcal{F})\right) \quad ,$$

where $I, J \subseteq [0, n-1]$ with $\min I + \min J \geq n$, and where $Z_i(\mathcal{E},\mathcal{F}) := \overline{O}_{\{i\},\emptyset}(\mathcal{E},\mathcal{F})$ and $Y_j(\mathcal{E},\mathcal{F}) := \overline{O}_{\emptyset,\{j\}}(\mathcal{E},\mathcal{F})$.

Proof. The S-valued points of $O_{I,J}(\mathcal{E},\mathcal{F})$ are the generalized isomorphisms

$$\Phi = \left(\begin{tikzcd}[column sep=small]\mathcal{E}_S \arrow[r, bend left, "\otimes"] & \mathcal{E}_1 \arrow[l, "0"'] \arrow[r, bend left, "\otimes"] & \mathcal{E}_2 \arrow[l, "1"'] & \cdots & \mathcal{E}_{n-1} \arrow[r, bend left, "\otimes"] & \mathcal{E}_n \arrow[l, "n-1"'] \arrow[r, "\sim"] & \mathcal{F}_n \arrow[r, "n-1"] & \mathcal{F}_{n-1} \arrow[l, bend right, "\otimes"'] & \cdots & \mathcal{F}_2 \arrow[r, "1"] & \mathcal{F}_1 \arrow[l, bend right, "\otimes"'] \arrow[r, "0"] & \mathcal{F}_S \arrow[l, bend right, "\otimes"'] \end{tikzcd}\right)$$
$$(\mathcal{M}_0,\mu_0)(\mathcal{M}_1,\mu_1) \qquad (\mathcal{M}_{n-1},\mu_{n-1}) \quad (\mathcal{L}_{n-1},\lambda_{n-1}) \qquad (\mathcal{L}_1,\lambda_1)\ (\mathcal{L}_0,\lambda_0)$$

where $\mu_i = \lambda_j = 0$ for $i \in I$ and $j \in J$ and where μ_i, λ_j are nowhere vanishing for $i \notin I$, $j \notin J$. It is clear that $O_{I,J}(\mathcal{E},\mathcal{F})$ is invariant under the operation of $\mathrm{Gl}(\mathcal{E}) \times_T \mathrm{Gl}(\mathcal{F})$. From the proof of theorem 9.3 it follows that we have the following isomorphism

$$O_{I,J}(\mathcal{E},\mathcal{F}) \cong \overset{o}{P}_1 \underset{\mathrm{Fl}}{\times} \ldots \underset{\mathrm{Fl}}{\times} \overset{o}{P}_r \underset{\mathrm{Fl}}{\times} \overset{o}{Q}_s \underset{\mathrm{Fl}}{\times} \ldots \underset{\mathrm{Fl}}{\times} \overset{o}{Q}_1 \underset{\mathrm{Fl}}{\times} \overset{o}{K'} \quad ,$$

where

$$\begin{aligned} \overset{o}{P}_p &:= \mathrm{PGl}(V_{r-p+1}/V_{r-p}, U_{s+p+1}/U_{s+p}) \quad (1 \leq p \leq r) \\ \overset{o}{Q}_q &:= \mathrm{PGl}(U_{s-q+1}/U_{s-q}, V_{r+q+1}/V_{r+q}) \quad (1 \leq q \leq s) \\ \overset{o}{K'} &:= \mathrm{Isom}(U_{s+1}/U_s, V_{r+1}/V_r) \quad . \end{aligned}$$

There is a left $\mathrm{Gl}(\mathcal{E}) \times_T \mathrm{Gl}(\mathcal{F})$-operation on the right-hand side of this isomorphism, given on S-valued points by

$$\begin{aligned} &(f,g)((F_\bullet\mathcal{E}, F_\bullet\mathcal{F}), \varphi_1, \ldots, \varphi_r, \psi_s, \ldots, \psi_1, \Phi') := \\ &((f(F_\bullet\mathcal{E}), g(F_\bullet\mathcal{F})), f^{-1}\varphi_1 g, \ldots, f^{-1}\varphi_r g, g\psi_s f^{-1}, \ldots, g\psi_1 f^{-1}, g\Phi' f^{-1}), \end{aligned}$$

where φ_p is an isomorphism (up to multiplication by an invertible section of $\mathcal{O}_S$) from $F_{r-p+1}\mathcal{F}/F_{r-p}\mathcal{F}$ to $F_{s+p+1}\mathcal{E}/F_{s+p}\mathcal{E}$ for $1 \leq p \leq r$, ψ_q an isomorphism (up to multiplication by an invertible section of $\mathcal{O}_S$) from $F_{s-q+1}\mathcal{E}/F_{s-q}\mathcal{E}$ to $F_{r+q+1}\mathcal{F}/F_{r+q}\mathcal{F}$ for $s \geq q \geq 1$ and Φ' is an isomorphism from $F_{s+1}\mathcal{E}/F_s\mathcal{E}$ to $F_{r+1}\mathcal{F}/F_r\mathcal{F}$. It is easy to see that this operation is transitiv and that the isomorphism

$$O_{I,J}(\mathcal{E},\mathcal{F}) \cong \overset{o}{P}_1 \underset{\mathrm{Fl}}{\times} \ldots \underset{\mathrm{Fl}}{\times} \overset{o}{P}_r \underset{\mathrm{Fl}}{\times} \overset{o}{Q}_s \underset{\mathrm{Fl}}{\times} \ldots \underset{\mathrm{Fl}}{\times} \overset{o}{Q}_1 \underset{\mathrm{Fl}}{\times} \overset{o}{K'} \quad ,$$

is $\mathrm{Gl}(\mathcal{E}) \times_T \mathrm{Gl}(\mathcal{F})$-equivariant. □

10. A morphism of KGL_n onto the Grassmannian compactification of the general linear group

Let V be an n-dimensional vector space over some field. As mentioned in the introduction, there is another natural compactification of the general linear group $\mathrm{Gl}(V)$: The Grassmannian $\mathrm{Gr}_n(V \oplus V)$ of n-dimensional subspaces of a $V \oplus V$-dimensional vector space. The embedding $\mathrm{Gl}(V) \hookrightarrow \mathrm{Gr}_n(V \oplus V)$ is given by associating to an automorphism $V \xrightarrow{\sim} V$ its graph in $V \oplus V$. We will see in this section that there exists a natural morphism from $\mathrm{KGl}(V)$ to $\mathrm{Gr}_n(V \oplus V)$. Our motivation here is to obtain a better understanding of the relation between the Gieseker-type degeneration of moduli spaces of vector bundles and the torsion-free sheaves approach as developed in [NS] and [S2].
As in the previous section, we work over an arbitrary base scheme T. Let $\mathcal{E}$, $\mathcal{F}$ be two locally free $\mathcal{O}_T$-modules of rank n. Denote by $\mathrm{Gr}_n(\mathcal{E} \oplus \mathcal{F})$ the Grassmanian variety over T which parametrizes subbundles of rank n of $\mathcal{E} \oplus \mathcal{F}$. Let S be a T-scheme and let

$$\Phi = \left(\mathcal{E}_S \xleftarrow[(\mathcal{M}_0,\mu_0)]{0} \mathcal{E}_1 \xleftarrow[(\mathcal{M}_1,\mu_1)]{1} \mathcal{E}_2 \quad \cdots \quad \mathcal{E}_{n-1} \xleftarrow[(\mathcal{M}_{n-1},\mu_{n-1})]{n-1} \mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n \xrightarrow[(\mathcal{L}_{n-1},\lambda_{n-1})]{n-1} \mathcal{F}_{n-1} \quad \cdots \quad \mathcal{F}_2 \xrightarrow[(\mathcal{L}_1,\lambda_1)]{1} \mathcal{F}_1 \xrightarrow[(\mathcal{L}_0,\lambda_0)]{0} \mathcal{F}_S \right)$$

be a generalized isomorphism from $\mathcal{E}_S$ to $\mathcal{F}_S$. By 5.2.2, the morphism $\mathcal{E}_n \to \mathcal{E}_S \oplus \mathcal{F}_S$ induced by the two composed morphisms

$$\mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_S$$
$$\mathcal{E}_n \xrightarrow{\sim} \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_S$$

is a subbundle of $\mathcal{E}_S \oplus \mathcal{F}_S$. Let

$$\mathrm{KGl}(\mathcal{E},\mathcal{F}) \to \mathrm{Gr}_n(\mathcal{E} \oplus \mathcal{F})$$

be the morphism, which on S-valued points is given by $\Phi \mapsto (\mathcal{E}_n \to \mathcal{E}_S \oplus \mathcal{F}_S)$. Observe that the following diagram commutes

$$\begin{tikzcd} & \mathrm{Isom}(\mathcal{E},\mathcal{F}) \arrow[dl, hook'] \arrow[dr, hook] & \\ \mathrm{KGl}(\mathcal{E},\mathcal{F}) \arrow[rr] & & \mathrm{Gr}_n(\mathcal{E} \oplus \mathcal{F}) \end{tikzcd}$$

and that furthermore all the arrows in this diagram are equivariant with respect to the natural action of $\mathrm{Gl}(\mathcal{E}) \times_T \mathrm{Gl}(\mathcal{F})$ on the three schemes. In the next proposition we compute the fibres of the morphism $\mathrm{KGl}(\mathcal{E},\mathcal{F}) \to \mathrm{Gr}_n(\mathcal{E} \oplus \mathcal{F})$.

PROPOSITION 10.1. *Let S' be a T-scheme and let $\mathcal{H} \hookrightarrow \mathcal{E}_{S'} \oplus \mathcal{F}_{S'}$ be an S'-valued point of $Gr_n(\mathcal{E},\mathcal{F})$ such that $im(\mathcal{H} \to \mathcal{E}_{S'})$ and $im(\mathcal{H} \to \mathcal{F}_{S'})$ are subbundles of $\mathcal{E}_{S'}$ and $\mathcal{F}_{S'}$ respectively. Then the fibre product*

$$KGl(\mathcal{E},\mathcal{F}) \underset{Gr_n(\mathcal{E}\oplus\mathcal{F})}{\times} S'$$

is isomorphic to

$\overline{PGl}(ker(\mathcal{H} \to \mathcal{E}_{S'}), coker(\mathcal{H} \to \mathcal{E}_{S'})) \times_{S'} \overline{PGl}(ker(\mathcal{H} \to \mathcal{F}_{S'}), coker(\mathcal{H} \to \mathcal{F}_{S'}))$,

where by convention $\overline{PGl}(\mathcal{N}, \mathcal{N}) := S'$ *for the zero-sheaf* $\mathcal{N} = 0$ *on* S'.

Proof. Let S be an S'-scheme. An S-valued point of the fibre product $\mathrm{KGl}(\mathcal{E}, \mathcal{F}) \times_{\mathrm{Gr}_n(\mathcal{E}\oplus\mathcal{F})} S'$ is given by a generalized isomorphism

$$\Phi = \left(\begin{tikzcd} \mathcal{E}_S \arrow[r, bend left, "\otimes"] & \mathcal{E}_1 \arrow[l, "0"', "(\mathcal{M}_0,\mu_0)"] \arrow[r, bend left, "\otimes"] & \mathcal{E}_2 \arrow[l, "1"', "(\mathcal{M}_1,\mu_1)"] & \cdots & \mathcal{E}_{n-1} \arrow[r, bend left, "\otimes"] & \mathcal{E}_n \arrow[l, "n-1"', "(\mathcal{M}_{n-1},\mu_{n-1})"] \arrow[r, "\sim"] & \mathcal{F}_n \arrow[r, "n-1", "(\mathcal{L}_{n-1},\lambda_{n-1})"'] & \mathcal{F}_{n-1} \arrow[l, bend right, "\otimes"'] & \cdots & \mathcal{F}_2 \arrow[r, "1", "(\mathcal{L}_1,\lambda_1)"'] & \mathcal{F}_1 \arrow[l, bend right, "\otimes"'] \arrow[r, "0", "(\mathcal{L}_0,\lambda_0)"'] & \mathcal{F}_S \arrow[l, bend right, "\otimes"'] \end{tikzcd} \right)$$

from $\mathcal{E}_S$ to $\mathcal{F}_S$ such that the induced morphism $\mathcal{E}_n \hookrightarrow \mathcal{E}_S \oplus \mathcal{F}_S$ identifies $\mathcal{E}_n$ with the subbundle $\mathcal{H}_S$. Let i_1 and j_1 be the ranks of $\mathrm{im}(\mathcal{H} \to \mathcal{E}_{S'})$ and $\mathrm{im}(\mathcal{H} \to F_{S'})$ respectively. Observe that $i_1 + j_1 \geq n$. We restrict ourselves to the case, where i_1 and j_1 are both strictly smaller than n. (The cases where one or both of i_1, j_1 are equal to n are proved analogously). Then the sections $\mu_0, \ldots, \mu_{i_1-1}$ and $\lambda_{j_1-1}, \ldots, \lambda_0$ are invertible and $\mu_{i_1} = \lambda_{j_1} = 0$. From the proof of theorem 9.3 it follows that such a Φ may be given by a tupel $((F_\bullet\mathcal{E}, F_\bullet\mathcal{F}), \varphi, \psi, \Phi')$ where

$$\begin{aligned} F_\bullet\mathcal{E} &= (0 = F_0\mathcal{E} \subseteq F_1\mathcal{E} \subseteq F_2\mathcal{E} \subseteq F_3\mathcal{E} = \mathcal{E}_S) \\ F_\bullet\mathcal{F} &= (0 = F_0\mathcal{F} \subseteq F_1\mathcal{F} \subseteq F_2\mathcal{F} \subseteq F_3\mathcal{F} = \mathcal{F}_S) \end{aligned}$$

are the filtrations given by

$$\begin{aligned} F_1\mathcal{E} &:= \mathrm{im}(\ker(\mathcal{H}_S \to \mathcal{F}_S) \to \mathcal{E}_S) \\ F_2\mathcal{E} &:= \mathrm{im}(\mathcal{H}_S \to \mathcal{E}_S) \\ F_1\mathcal{F} &:= \mathrm{im}(\ker(\mathcal{H}_S \to \mathcal{E}_S) \to \mathcal{F}_S) \\ F_2\mathcal{F} &:= \mathrm{im}(\mathcal{H}_S \to \mathcal{F}_S) \quad , \end{aligned}$$

φ is a complete collineation from $F_1\mathcal{F}/F_0\mathcal{F} \cong \ker(\mathcal{H}_S \to \mathcal{E}_S)$ to $F_3\mathcal{E}/F_2\mathcal{E} \cong \mathrm{coker}(\mathcal{H}_S \to \mathcal{E}_S)$, ψ is a complete collineation from $F_1\mathcal{E}/F_0\mathcal{E} \cong \ker(\mathcal{H}_S \to \mathcal{F}_S)$ to $F_3\mathcal{F}/F_2\mathcal{F} \cong \mathrm{coker}(\mathcal{H}_S \to \mathcal{F}_S)$, and Φ' is the isomorphism

$$F_2\mathcal{E}/F_1\mathcal{E} \xrightarrow{\sim} \mathcal{H}_S/(\ker(\mathcal{H}_S \to \mathcal{E}_S) + \ker(\mathcal{H}_S \to \mathcal{F}_S)) \xrightarrow{\sim} F_2\mathcal{F}/F_1\mathcal{F} \quad .$$

We see in particular that the tupel $((F_\bullet\mathcal{E}, F_\bullet\mathcal{F}), \varphi, \psi, \Phi')$ is already determined by the subbundle $\mathcal{H}_S \hookrightarrow \mathcal{E}_S \oplus \mathcal{F}_S$ (i.e. by the morphism $S \to S'$) and the pair (φ, ψ). $\square$

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