Heegner Points and L-Series of Automorphic Cusp Forms of Drinfeld Type

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Abstract. In their famous article [Gr-Za], Gross and Zagier proved a formula relating heights of Heegner points on modular curves and derivatives of L-series of cusp forms.
We prove the function field analogue of this formula. The classical modular curves parametrizing isogenies of elliptic curves are now replaced by Drinfeld modular curves dealing with isogenies of Drinfeld modules. Cusp forms on the classical upper half plane are replaced by harmonic functions on the edges of a Bruhat-Tits tree.
As a corollary we prove the conjecture of Birch and Swinnerton-Dyer for certain elliptic curves over functions fields whose analytic rank is equal to 1.

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1 Introduction

Let $K = \mathbb{F}_q(T)$ be the rational function field over a finite field $\mathbb{F}_q$ of odd characteristic. In K we distinguish the polynomial ring $\mathbb{F}_q[T]$ and the place ∞. We consider harmonic functions f on $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$, the edges of the Bruhat-Tits tree of GL_2, which are invariant under $\Gamma_0(N)$ for $N \in \mathbb{F}_q[T]$. These are called automorphic cusp forms of Drinfeld type of level N (cf. section 2.1).
Let $L = K(\sqrt{D})$, with $\gcd(N, D) = 1$, be an imaginary quadratic extension of K (we assume that D is irreducible to make calculations technically easier). We attach to an automorphic cusp form f of Drinfeld type of level N, which is a newform, and to an element $\mathcal{A}$ in the class group of $O_L = \mathbb{F}_q[T][\sqrt{D}]$ an L-series $L(f, \mathcal{A}, s)$ (section 2.1).

We represent this L-series (normalized by a suitable factor $L^{(N,D)}(2s+1)$) as a Petersson product of f and a function Φ_s on $\Gamma_0(N) \setminus GL_2(K_\infty)/\Gamma_\infty K_\infty^*$ (sections 2.2 and 2.3). From this representation we get a functional equation for $L(f, \mathcal{A}, s)$ (Theorem 2.7.3 and Theorem 2.7.6), which shows in particular that $L(f, \mathcal{A}, s)$ has a zero at $s = 0$, if $\left[\dfrac{D}{N}\right] = 1$.

In this case, under the additional assumptions that N is square free and that each of its prime divisors is split in L, we evaluate the derivative of $L(f, \mathcal{A}, s)$ at $s = 0$. Since the function Φ_s is not harmonic in general, we apply a holomorphic projection formula (cf. section 2.4) to get

$$\frac{\partial}{\partial s}(L^{(N,D)}(2s+1)L(f, \mathcal{A}, s)) \mid_{s=0} = \int f \cdot \overline{\Psi_{\mathcal{A}}} \;\; (\text{if } \deg D \text{ is odd}),$$

resp.

$$\frac{\partial}{\partial s}(\frac{1}{1+q^{-s-1}} L^{(N,D)}(2s+1)L(f, \mathcal{A}, s)) \mid_{s=0} = \int f \cdot \overline{\Psi_{\mathcal{A}}} \;\; (\text{if } \deg D \text{ is even}),$$

where $\Psi_{\mathcal{A}}$ is an automorphic cusp form of Drinfeld type of level N. The Fourier coefficients $\Psi_{\mathcal{A}}^*(\pi_\infty^{\deg \lambda + 2}, \lambda)$ of $\Psi_{\mathcal{A}}$ are evaluated in sections 2.5, 2.6 and 2.8. The results are summarized in Theorem 2.8.2 and Theorem 2.8.3.

On the other hand let x be a Heegner point on the Drinfeld modular curve $X_0(N)$ with complex multiplication by $O_L = \mathbb{F}_q[T][\sqrt{D}]$. There exists a cusp form $g_{\mathcal{A}}$ of Drinfeld type of level N whose Fourier coefficients are given by (cf. Proposition 3.1.1):

$$g_{\mathcal{A}}^*(\pi_\infty^{\deg \lambda + 2}, \lambda) = q^{-\deg \lambda} \langle (x) - (\infty), T_\lambda((x)^{\sigma_{\mathcal{A}}} - (0)) \rangle,$$

where the automorphism $\sigma_{\mathcal{A}}$ belongs to the class $\mathcal{A}$ via class field theory, where T_λ is the Hecke operator attached to λ and where $\langle \; , \; \rangle$ denotes the global Néron-Tate height pairing of divisors on $X_0(N)$ over the Hilbert class field of L.

We want to compare the cusp forms $\Psi_{\mathcal{A}}$ and $g_{\mathcal{A}}$. Therefore we have to evaluate the height of Heegner points, which is the content of chapter 3. We evaluate the heights locally at each place of K. At the places belonging to the polynomial ring $\mathbb{F}_q[T]$ we use the modular interpretation of Heegner points by Drinfeld modules. Counting homomorphisms between different Drinfeld modules (similar to calculations in [Gr-Za]) yields the formula for these local heights (Corollary 3.4.10 and Proposition 3.4.13). At the place ∞ we construct a Green's function on the analytic upper half plane, which gives the local height in this case (Propositions 3.6.3, 3.6.5). Finally we evaluate the Fourier coefficients of $g_{\mathcal{A}}$ in Theorems 3.6.4 and 3.6.6.

In chapter 4 we compare the results on the derivatives of the L-series, i.e. the Fourier coefficients $\Psi_{\mathcal{A}}^*(\pi_\infty^{\deg \lambda + 2}, \lambda)$, and the result on the heights of Heegner points, i.e. the coefficients $g_{\mathcal{A}}^*(\pi_\infty^{\deg \lambda + 2}, \lambda)$, and get our main result (cf. Theorem 4.1.1 and Theorem 4.1.2): If $\gcd(\lambda, N) = 1$, then

$$\Psi_{\mathcal{A}}^*(\pi_\infty^{\deg \lambda + 2}, \lambda) = \frac{q-1}{2} \, q^{-(\deg D + 1)/2} \, g_{\mathcal{A}}^*(\pi_\infty^{\deg \lambda + 2}, \lambda) \;\; (\text{if } \deg D \text{ is odd}),$$

resp.

$$\Psi^*_{\mathcal{A}}(\pi_\infty^{\deg\lambda+2},\lambda) = \frac{q-1}{4}\, q^{-\deg D/2}\, g^*_{\mathcal{A}}(\pi_\infty^{\deg\lambda+2},\lambda)\ \ (\text{if } \deg D \text{ is even}).$$

We apply this result to elliptic curves. Let E be an elliptic curve over K with conductor $N\cdot\infty$, which has split multiplicative reduction at ∞, then E is modular, i.e. it belongs to an automorphic cusp form f of Drinfeld type of level N. In particular the L-series of E/K and of f satisfy $L(E,s+1) = L(f,s)$.
The L-series of E over the field $L = K(\sqrt{D})$ equals $L(E,s)L(E_D,s)$ and can be computed by

$$L(E,s+1)L(E_D,s+1) = \sum_{\mathcal{A}\in Cl(O_L)} L^{(N,D)}(2s+1)L(f,\mathcal{A},s),$$

if $\deg D$ is odd, or in the even case by

$$L(E,s+1)L(E_D,s+1) = \sum_{\mathcal{A}\in Cl(O_L)} \frac{1}{1+q^{-s-1}} L^{(N,D)}(2s+1)L(f,\mathcal{A},s).$$

This motivates the consideration of the L-series $L(f,\mathcal{A},s)$.
The functional equations for all $L(f,\mathcal{A},s)$ yield that $L(E,s)L(E_D,s)$ has a zero at $s=1$. In order to evaluate the first derivative, we consider a uniformization $\pi : X_0(N) \to E$ of the modular elliptic curve E and the Heegner point $P_L := \sum_{\mathcal{A}\in Cl(O_L)} \pi(x^{\sigma_{\mathcal{A}}})$. P_L is an L-rational point on E.
Our main result yields a formula relating the derivative of the L-series of E/L and the Néron-Tate height $\hat{h}_{E,L}(P_L)$ of the Heegner point on E over L (Theorem 4.2.1):

$$\frac{\partial}{\partial s}(L(E,s)L(E_D,s))\,|_{s=1} = \hat{h}_{E,L}(P_L)\; c(D)\; (\deg\pi)^{-1} \int\limits_{\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K^*_\infty} f\cdot\overline{f},$$

where the constant $c(D)$ equals $\frac{q-1}{2}\ q^{-(\deg D+1)/2}$ (if $\deg D$ is odd) or $\frac{q-1}{4}\ q^{-\deg D/2}$ (if $\deg D$ is even).
As a corollary (Corollary 4.2.2) we prove the conjecture of Birch and Swinnerton-Dyer for E/L, if its analytic rank is equal to 1.
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2 L-Series

2.1 Basic Definitions of L-Series

Let $\mathbb{F}_q$ be the finite field with $q = p^\alpha$ elements ($p \neq 2$), and let $K = \mathbb{F}_q(T)$ be the rational function field over $\mathbb{F}_q$. We distinguish the finite places given by the irreducible elements in the polynomial ring $\mathbb{F}_q[T]$ and the place ∞ of K. For ∞

we consider the completion K_∞ with normalized valuation v_∞ and valuation ring O_∞. We fix the prime $\pi_\infty = T^{-1}$, then $K_\infty = \mathbb{F}_q((\pi_\infty))$. In addition we define the following additive character ψ_∞ of K_∞: Take $\sigma : \mathbb{F}_q \to \mathbb{C}^*$ with $\sigma(a) = \exp(\frac{2\pi i}{p} Tr_{\mathbb{F}_q/\mathbb{F}_p}(a))$ and set $\psi_\infty(\sum a_i \pi_\infty^i) = \sigma(-a_1)$.
The oriented edges of the Bruhat-Tits tree of GL_2 over K_∞ are parametrized by the set $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$, where

$$\Gamma_\infty := \{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(O_\infty) \mid v_\infty(\gamma) > 0\}.$$

$GL_2(K_\infty)/\Gamma_\infty K_\infty^*$ can be represented by the two disjoint sets

$$\mathcal{T}_+ := \{\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix} \mid m \in \mathbb{Z}, u \in K_\infty/\pi_\infty^m O_\infty\} \tag{2.1.1}$$

and

$$\mathcal{T}_- := \{\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix} \mid m \in \mathbb{Z}, u \in K_\infty/\pi_\infty^m O_\infty\}. \tag{2.1.2}$$

Right multiplication by $\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}$ reverses the orientation of an edge.
We do not distinguish between matrices in $GL_2(K_\infty)$ and the corresponding classes in $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$.
We want to study functions on $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$. Special functions are defined in the following way: The groups $GL_2(\mathbb{F}_q[T])$ and $SL_2(\mathbb{F}_q[T])$ operate on $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$ by left multiplication. For $N \in \mathbb{F}_q[T]$ let

$$\Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_q[T]) \mid c \equiv 0 \bmod N\}$$

and $\Gamma_0^{(1)}(N) := \Gamma_0(N) \cap SL_2(\mathbb{F}_q[T])$.

Definition 2.1.1 *A function $f : GL_2(K_\infty)/\Gamma_\infty K_\infty^* \to \mathbb{C}$ is called an automorphic cusp form of Drinfeld type of level N if it satisfies the following conditions:*
i) f is harmonic, i.e.,

$$f(X\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}) = -f(X)$$

and

$$\sum_{\beta \in GL_2(O_\infty)/\Gamma_\infty} f(X\beta) = 0$$

for all $X \in GL_2(K_\infty)/\Gamma_\infty K_\infty^*$,
ii) f is invariant under $\Gamma_0(N)$, i.e.,

$$f(AX) = f(X)$$

for all $X \in GL_2(K_\infty)/\Gamma_\infty K_\infty^*$ and $A \in \Gamma_0(N)$,
iii) f has compact support modulo $\Gamma_0(N)$, i.e. there are only finitely many elements $\bar{X}$ in $\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*$ with $f(\bar{X}) \neq 0$.

Any function f on $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$ which is invariant under $\begin{pmatrix} 1 & \mathbb{F}_q[T] \\ 0 & 1 \end{pmatrix}$ has a Fourier expansion

$$f(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) = \sum_{\lambda \in \mathbb{F}_q[T]} f^*(\pi_\infty^m, \lambda)\psi_\infty(\lambda u) \tag{2.1.3}$$

with

$$f^*(\pi_\infty^m, \lambda) = \int\limits_{K_\infty/\mathbb{F}_q[T]} f(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\psi_\infty(-\lambda u)\, du,$$

where du is a Haar measure with $\int\limits_{K_\infty/\mathbb{F}_q[T]} du = 1$.

Since $\begin{pmatrix} 1 & \mathbb{F}_q[T] \\ 0 & 1 \end{pmatrix} \subset \Gamma_0(N)$ this applies to automorphic cusp forms. In this particular case the harmonicity conditions of Definition 2.1.1 imply

$$\begin{aligned} f^*(\pi_\infty^m, \lambda) &= 0, \text{ if } \lambda = 0 \text{ or if } \deg\lambda + 2 > m, \\ f^*(\pi_\infty^m, \lambda) &= q^{-m+\deg\lambda+2} f^*(\pi_\infty^{\deg\lambda+2}, \lambda), \text{ if } \lambda \neq 0 \text{ and } \deg\lambda + 2 \leq m. \end{aligned} \tag{2.1.4}$$

Hence we get the following:

REMARK 2.1.2 *All the Fourier coefficients of an automorphic cusp form f of Drinfeld type are uniquely determined by the coefficients $f^*(\pi_\infty^{\deg\lambda+2}, \lambda)$ for $\lambda \in \mathbb{F}_q[T]$.*

To an automorphic cusp form f one can attach an L-series $L(f, s)$ in the following way (cf. [We1], [We2]): Let $\mathfrak{m}$ be an effective divisor of K of degree n, then $\mathfrak{m} = (\lambda)_0 + (n - \deg\lambda)\infty$ with $\lambda \in \mathbb{F}_q[T]$, $\deg\lambda \leq n$. We define

$$f^*(\mathfrak{m}) = f^*(\pi_\infty^{n+2}, \lambda) \quad \text{and} \quad L(f, s) = \sum_{\mathfrak{m} \geq 0} f^*(\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-s}, \tag{2.1.5}$$

where $\mathrm{N}(\mathfrak{m})$ denotes the absolute norm of the divisor $\mathfrak{m}$.
The $\mathbb{C}$-vector space of automorphic cusp forms of Drinfeld type of level N is finite dimensional and it is equipped with a non-degenerate pairing, the Petersson product, given by

$$(f, g) \mapsto \int\limits_{\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*} f \cdot \overline{g}.$$

There is the notion of oldforms, i.e. linear combinations of forms $g(\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} X)$, where g is an automorphic cusp form of level M, $M|N$

and $M \neq N$, and d is a divisor of N/M. Automorphic cusp forms of Drinfeld type which are perpendicular under the Petersson product to all the oldforms are called newforms.

Important examples of newforms are the following: Let E be an elliptic curve over K with conductor $N \cdot \infty$, which has split multiplicative reduction at ∞, then E belongs to a newform f of level N such that the L-series of E satisfies ([De])

$$L(E, s+1) = L(f, s). \tag{2.1.6}$$

This newform is in addition an eigenform for all Hecke operators, but we do not assume this property in general.

From now on let f be an automorphic cusp form of level N which is a newform. Let L/K be an imaginary quadratic extension (i.e. a quadratic extension of K where ∞ is not split) in which each (finite) divisor of N is not ramified. Then there is a square free polynomial $D \in \mathbb{F}_q[T]$, prime to N with $L = K(\sqrt{D})$.

We assume in this paper that D is an irreducible polynomial. In principle all the arguments apply to the general case, but the details are technically more complicated. We distinguish two cases. In the first case the degree of D is odd, i.e. ∞ is ramified in L/K; in the second case the degree of D is even and its leading coefficient is not a square in $\mathbb{F}_q^*$, i.e. ∞ is inert in L/K.

The integral closure of $\mathbb{F}_q[T]$ in L is $O_L = \mathbb{F}_q[T][\sqrt{D}]$.

Let $\mathcal{A}$ be an element of the class group $Cl(O_L)$ of O_L. For an effective divisor $\mathfrak{m} = (\lambda)_0 + (n - \deg \lambda)\infty$ (as above) we define

$$r_{\mathcal{A}}(\mathfrak{m}) = \#\{\mathfrak{a} \in \mathcal{A} \mid \mathfrak{a} \text{ integral with } \mathrm{N}_{L/K}(\mathfrak{a}) = \lambda\, \mathbb{F}_q[T]\} \tag{2.1.7}$$

and hence we get the partial zeta function attached to $\mathcal{A}$ as

$$\zeta_{\mathcal{A}}(s) = \sum_{\mathfrak{m} \geq 0} r_{\mathcal{A}}(\mathfrak{m}) \mathrm{N}(\mathfrak{m})^{-s}. \tag{2.1.8}$$

For the calculations it is sometimes easier to define a function depending on elements of $\mathbb{F}_q[T]$ instead of divisors. We choose $\mathfrak{a}_0 \in \mathcal{A}^{-1}$ and $\lambda_0 \in K$ with $\mathrm{N}_{L/K}(\mathfrak{a}_0) = \lambda_0\, \mathbb{F}_q[T]$ and define

$$r_{\mathfrak{a}_0,\lambda_0}(\lambda) = \#\{\mu \in \mathfrak{a}_0 \mid \mathrm{N}_{L/K}(\mu) = \lambda_0 \lambda\}. \tag{2.1.9}$$

Then

$$r_{\mathcal{A}}(\mathfrak{m}) = \frac{1}{q-1} \sum_{\epsilon \in \mathbb{F}_q^*} r_{\mathfrak{a}_0,\lambda_0}(\epsilon\lambda).$$

The theta series is defined as

$$\Theta_{\mathfrak{a}_0,\lambda_0}\left(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}\right) = \sum_{\deg \lambda + 2 \leq m} r_{\mathfrak{a}_0,\lambda_0}(\lambda) \psi_\infty(\lambda u). \tag{2.1.10}$$

We will see later that the transformation rules of this theta series are the starting point of all our calculations.
Now we combine the L-series of a newform f (cf. (2.1.5)) and the partial zeta function of $\mathcal{A}$ (cf. (2.1.8)) to obtain the function

$$L(f,\mathcal{A},s)=\sum_{\mathfrak{m}\geq 0} f^*(\mathfrak{m})r_{\mathcal{A}}(\mathfrak{m})\mathrm{N}(\mathfrak{m})^{-s}. \tag{2.1.11}$$

For technical reasons we introduce

$$L^{(N,D)}(2s+1)=\frac{1}{q-1}\sum_{\substack{k\in\mathbb{F}_q[T]\\ \gcd(k,N)=1}}\left[\frac{D}{k}\right]q^{-(2s+1)\deg k}, \tag{2.1.12}$$

where $\left[\frac{D}{k}\right]$ denotes the Legendre resp. the Jacobi symbol for the polynomial ring $\mathbb{F}_q[T]$. For an irreducible $k\in\mathbb{F}_q[T]$ and a coprime $D\in\mathbb{F}_q[T]$ the Legendre symbol $\left[\frac{D}{k}\right]$ is by definition equal to 1 or -1 if D is or is not a square in $(\mathbb{F}_q[T]/k\,\mathbb{F}_q[T])^*$, respectively. If D is divisible by k, then $\left[\frac{D}{k}\right]$ equals 0. This definition is multiplicatively extended to the Jacobi symbol for arbitrary, not necessarily irreducible k, so e.g. $\left[\frac{D}{k}\right]=\left[\frac{D}{k_1}\right]\cdot\left[\frac{D}{k_2}\right]$ if $k=k_1\cdot k_2$.
In the first case, where $\deg D$ is odd, the function

$$L^{(N,D)}(2s+1)L(f,\mathcal{A},s)$$

is the focus of our interest; in the case of even degree it is the function

$$\frac{1}{1+q^{-s-1}}L^{(N,D)}(2s+1)L(f,\mathcal{A},s).$$

This is motivated by the following fact:

Proposition 2.1.3 *Let E be an elliptic curve with conductor $N\cdot\infty$ and corresponding newform f as above and let E_D be its twist by D. Then the following identities hold:*

$$L(E,s+1)L(E_D,s+1)=\sum_{\mathcal{A}\in Cl(O_L)}L^{(N,D)}(2s+1)L(f,\mathcal{A},s)$$

if $\deg D$ is odd, and

$$L(E,s+1)L(E_D,s+1)=\sum_{\mathcal{A}\in Cl(O_L)}\frac{1}{1+q^{-s-1}}L^{(N,D)}(2s+1)L(f,\mathcal{A},s)$$

if $\deg D$ is even.

It is not difficult to prove this fact using the definitions of the coefficients $f^*(\mathfrak{m})$ (cf. (2.1.5)) and $r_{\mathcal{A}}(\mathfrak{m})$ (cf. (2.1.7)) and the Euler products of the L-series of the elliptic curves.

2.2 RANKIN'S METHOD

The properties of the automorphic cusp form f yield

$$f^*(\pi_\infty^m,\lambda) = q^{-m+1} \sum_{u\in\pi_\infty/\pi_\infty^m} f(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\psi_\infty(-\lambda u). \tag{2.2.1}$$

We use this to calculate

$$L(f,\mathcal{A},s) = \frac{1}{q-1}\sum_{m=2}^{\infty}\Big(\sum_{\deg\lambda+2\le m} f^*(\pi_\infty^m,\lambda) r_{\mathfrak{a}_0,\lambda_0}(\lambda)\Big)\, q^{-(m-2)s}. \tag{2.2.2}$$

Now we distinguish the two cases.

2.2.1 $\deg D$ IS ODD

We continue with equations (2.2.1) and (2.2.2):

$$\begin{aligned} & L(f,\mathcal{A},s) \\ = \; & \frac{q}{q-1}\sum_{m=2}^{\infty}\sum_{u\in\pi_\infty/\pi_\infty^m} f(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\overline{\Theta_{\mathfrak{a}_0,\lambda_0}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})}\, q^{-m(s+1)+2s} \\ = \; & \frac{q}{q-1}\int\limits_{\mathbb{H}_\infty} f(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\overline{\Theta_{\mathfrak{a}_0,\lambda_0}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\, q^{-m(\bar{s}+1)+2\bar{s}}}, \end{aligned} \tag{2.2.3}$$

where

$$\mathbb{H}_\infty := \begin{pmatrix} 1 & \mathbb{F}_q[T] \\ 0 & 1 \end{pmatrix} \backslash \begin{pmatrix} K_\infty^* & K_\infty \\ 0 & 1 \end{pmatrix} / \begin{pmatrix} O_\infty^* & O_\infty \\ 0 & 1 \end{pmatrix}.$$

We consider the canonical mapping

$$\mathbb{H}_\infty \to \Gamma_0^{(1)}(ND)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^* =: G(ND),$$

which is surjective. We take the measure on $G(ND)$ which counts the size of the stabilizer of an element (cf. [Ge-Re], (4.8)). Then we get

$$\begin{aligned} L(f,\mathcal{A},s) \;=\; & \frac{q}{2(q-1)} \\ & \cdot \int\limits_{G(ND)} f(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\overline{\sum_M \Theta_{\mathfrak{a}_0,\lambda_0}(M\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\, q^{-m^*(\bar{s}+1)+2\bar{s}}} \end{aligned} \tag{2.2.4}$$

where the sum is taken over those $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} 1 & \mathbb{F}_q[T] \\ 0 & 1 \end{pmatrix}\backslash\Gamma_0^{(1)}(ND)$ with $M\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix} \in \mathcal{T}_+$, and where $m^* = m - 2v_\infty(cu+d)$.

Remark 2.2.1 *The definitions of* $\mathcal{T}_+$ *and* $\mathcal{T}_-$ *(cf. (2.1.1), (2.1.2)) yield:* $M\begin{pmatrix}\pi_\infty^m & u\\ 0 & 1\end{pmatrix} \in \mathcal{T}_+$ *if and only if* $v_\infty(c\pi_\infty^m) > v_\infty(cu+d)$.

In ([Rü1], Theorem 6.2) we showed that for those M satisfying $v_\infty(c\pi_\infty^m) > v_\infty(cu+d)$ one has the following transformation rule for the theta series (cf. (2.1.10)):

$$\Theta_{\mathfrak{a}_0,\lambda_0}(M\begin{pmatrix}\pi_\infty^m & u\\ 0 & 1\end{pmatrix}) = \Theta_{\mathfrak{a}_0,\lambda_0}(\begin{pmatrix}\pi_\infty^m & u\\ 0 & 1\end{pmatrix}) \left[\frac{d}{D}\right] \delta_{cu+d}\, q^{-v_\infty(cu+d)}, \tag{2.2.5}$$

where $\left[\frac{d}{D}\right]$ is the Legendre symbol (defined in section 2.1) and where δ_z denotes the local norm symbol at ∞, i.e., δ_z is equal to 1 if $z \in K_\infty^*$ is the norm of an element in the quadratic extension $K_\infty(\sqrt{D})/K_\infty$ and -1 otherwise.
Equations (2.2.4), (2.2.5) and the definition of $L^{(N,D)}$ (cf. (2.1.12)) yield:

$$L^{(N,D)}(2s+1)L(f,\mathcal{A},s) = \frac{q}{2(q-1)} \int\limits_{G(ND)} f\cdot \overline{\Theta_{\mathfrak{a}_0,\lambda_0} H_{1,\bar{s}}}$$

with

$$H_{1,s}(\begin{pmatrix}\pi_\infty^m & u\\ 0 & 1\end{pmatrix}) := q^{-m(s+1)+2s} \sum_{\substack{c,d\in\mathbb{F}_q[T]\\ c\equiv 0 \bmod ND\\ \gcd(d,N)=1\\ v_\infty(c\pi_\infty^m)>v_\infty(cu+d)}} \left[\frac{d}{D}\right] \delta_{cu+d}\, q^{v_\infty(cu+d)(2s+1)}.$$

We see that $\Theta_{\mathfrak{a}_0,\lambda_0}H_{1,s}$ is a function on $G(ND)$.
Let $\mu:\mathbb{F}_q[T]\to\{0,1,-1\}$ be the Moebius function with

$$\sum_{\substack{e\in\mathbb{F}_q[T]\\ e|n}} \mu(e) = 0 \text{ if } n\,\mathbb{F}_q[T] \neq \mathbb{F}_q[T],$$

and

$$\frac{1}{q-1}\sum_{e\in\mathbb{F}_q^*} \mu(e) = 1,$$

then $H_{1,s}(\begin{pmatrix}\pi_\infty^m & u\\ 0 & 1\end{pmatrix})$

$$= \frac{q^{-m(s+1)+2s}}{q-1} \sum_{e|N} \mu(e) \left[\frac{e}{D}\right] \delta_e\, q^{-(2s+1)\deg e} E_s^{(1)}(\begin{pmatrix}\frac{N\pi_\infty^m}{e} & \frac{Nu}{e}\\ 0 & 1\end{pmatrix})$$

with the Eisenstein series

$$E_s^{(1)}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) := \sum_{\substack{c,d\in\mathbb{F}_q[T] \\ c\equiv 0 \bmod D \\ v_\infty(c\pi_\infty^m)>v_\infty(cu+d)}} \left[\frac{d}{D}\right] \delta_{cu+d}\, q^{v_\infty(cu+d)(2s+1)}. \qquad (2.2.6)$$

For a divisor e of N the function

$$\Theta_{\mathfrak{a}_0,\lambda_0}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\, q^{-m(s+1)+2s} E_s^{(1)}(\begin{pmatrix} \frac{N\pi_\infty^m}{e} & \frac{Nu}{e} \\ 0 & 1 \end{pmatrix})$$

on $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$ is invariant under $\Gamma_0^{(1)}(\frac{ND}{e})$.
Since we assume that f is a newform of level N, it is orthogonal (with respect to the Petersson product) to functions of lower level. Therefore we get

PROPOSITION 2.2.2 *Let* $\deg D$ *be odd, then*

$$L^{(N,D)}(2s+1)L(f,\mathcal{A},s) = \frac{q}{2(q-1)} \int\limits_{G(ND)} f\cdot \overline{\Theta_{\mathfrak{a}_0,\lambda_0} H_{2,\bar{s}}}$$

with $H_{2,s}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})$

$$\begin{aligned} &:= q^{-m(s+1)+2s} E_s^{(1)}(\begin{pmatrix} N\pi_\infty^m & Nu \\ 0 & 1 \end{pmatrix}) \qquad (2.2.7) \\ &= q^{-m(s+1)+2s} \sum_{\substack{c,d\in\mathbb{F}_q[T] \\ c\equiv 0 \bmod D \\ v_\infty(cN\pi_\infty^m)>v_\infty(cNu+d)}} \left[\frac{d}{D}\right] \delta_{cNu+d}\, q^{v_\infty(cNu+d)(2s+1)}. \end{aligned}$$

.

2.2.2 $\deg D$ IS EVEN

We use equation (2.2.2) and the geometric series expansion of $1/(1+q^{-s-1})$ to evaluate

$$\frac{1}{1+q^{-s-1}} L(f,\mathcal{A},s) = \frac{1}{q-1} \cdot \sum_{m=2}^{\infty} q^{-(m-2)s} \sum_{l=2}^{m} (\sum_{\deg\lambda+2\leq l} f^*(\pi_\infty^l,\lambda) r_{\mathfrak{a}_0,\lambda_0}(\lambda))\, (-q^{-1})^{m-l}.$$

Since f is an automorphic cusp form and hence $f^*(\pi_\infty^l,\lambda) = q^{m-l} f^*(\pi_\infty^m,\lambda)$ (cf. (2.1.4)), we get

$$\frac{1}{1+q^{-s-1}} L(f,\mathcal{A},s) = \frac{1}{q-1} \cdot \sum_{m=2}^{\infty} (\sum_{\deg\lambda+2\leq m} f^*(\pi_\infty^m,\lambda) r_{\mathfrak{a}_0,\lambda_0}(\lambda))\, q^{-(m-2)s}\, \frac{(-1)^{m-\deg\lambda}+1}{2}.$$

If $r_{\mathfrak{a}_0,\lambda_0}(\lambda) \neq 0$, then $\deg\lambda \equiv \deg\lambda_0 \bmod 2$, because $\deg D$ is even. Now equation (2.2.1) yields

$$\frac{1}{1+q^{-s-1}}L(f,\mathcal{A},s) = \frac{q}{q-1}$$
$$\cdot \int_{\mathbb{H}_\infty} f(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\overline{\Theta_{\mathfrak{a}_0,\lambda_0}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\, q^{-m(\bar{s}+1)+2\bar{s}}}\ \frac{(-1)^{m-\deg\lambda_0}+1}{2}.$$

Thus the right side of this equation differs from (2.2.3) only by the factor $((-1)^{m-\deg\lambda_0}+1)/2$. But this factor is invariant under $GL_2(\mathbb{F}_q[T])$ and hence causes no problems here or in the next steps. Proceeding exactly as in the case where $\deg D$ is odd gives the following result:

Proposition 2.2.3 *Let $\deg D$ be even, then*

$$\frac{1}{1+q^{-s-1}}L^{(N,D)}(2s+1)L(f,\mathcal{A},s) = \frac{q}{2(q-1)}\cdot$$
$$\int_{G(ND)} f\cdot\overline{\Theta_{\mathfrak{a}_0,\lambda_0}H_{2,\bar{s}}}\ \frac{(-1)^{m-\deg\lambda_0}+1}{2}$$

with $H_{2,s}$ given by equation (2.2.7).

2.3 Computation of the Trace

The function $\Theta_{\mathfrak{a}_0,\lambda_0}H_{2,s}$ on $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$ is only invariant under $\Gamma_0^{(1)}(ND)$. To make it invariant under $\Gamma_0^{(1)}(N)$ we compute the trace with respect to the extension $\Gamma_0^{(1)}(ND)\backslash\Gamma_0^{(1)}(N)$. The trace from $\Gamma_0^{(1)}(N)$ to $\Gamma_0(N)$ is easy, this will be done at the very end of the calculations.

Since N and D are relatively prime, there are $\mu_1,\mu_2 \in \mathbb{F}_q[T]$ with $1=\mu_1 N + \mu_2 D$. The set

$$R = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -\mu_2 D & \mu_1 N \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix}\ (\lambda \bmod D)\} \qquad (2.3.1)$$

is therefore a set of representatives of $\Gamma_0^{(1)}(ND)\backslash\Gamma_0^{(1)}(N)$. Here we used the assumption that D is irreducible. In order to evaluate $\sum_{M\in R}\Theta_{\mathfrak{a}_0,\lambda_0}H_{2,s}(M\cdot)$, we treat $\Theta_{\mathfrak{a}_0,\lambda_0}$ and $H_{2,s}$ separately.

From ([Rü1], Prop. 4.4) we get, if $m > v_\infty(u)$:

$$\Theta_{\mathfrak{a}_0,\lambda_0}(\begin{pmatrix} \frac{\pi_\infty^m}{u^2} & \frac{1}{u} \\ 0 & 1 \end{pmatrix}) = \Theta_{\mathfrak{a}_0,\lambda_0}(\begin{pmatrix} \frac{\pi_\infty^m}{D} & \frac{u}{D} \\ 0 & 1 \end{pmatrix})\delta_u\, q^{-v_\infty(u)}\delta_{-\lambda_0}q^{-\frac{1}{2}\deg D}\epsilon_0^{-1}$$

where $\epsilon_0 = 1$ if $\deg D$ is even and $\epsilon_0 = \delta_{-t}(-1)^{\alpha+1}\gamma(p)^\alpha$ ($q=p^\alpha$; $\gamma(p)=1$ if $p\equiv 1 \bmod 4$ or i otherwise) if $\deg D$ is odd. Then one evaluates

$$\Theta_{\mathfrak{a}_0,\lambda_0}(\begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix}\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) = \Theta_{\mathfrak{a}_0,\lambda_0}(\begin{pmatrix} \frac{\pi_\infty^m}{D} & \frac{-(u+\lambda)}{D} \\ 0 & 1 \end{pmatrix})\cdot \qquad (2.3.2)$$
$$\cdot\delta_{u+\lambda}\, q^{-v_\infty(u+\lambda)}\delta_{\lambda_0}\, q^{-\frac{1}{2}\deg D}\epsilon_0^{-1}.$$

Now (2.2.5) and (2.3.2) yield the operation of the matrices $M \in R$ (cf. (2.3.1)) on $\Theta_{\mathfrak{a}_0,\lambda_0}$.
The situation for $H_{2,s}$ and hence for the Eisenstein series $E_s^{(1)}$ (cf. (2.2.6)) is easier. Straightforward calculations (mainly transformations of the summation indices) yield:
If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_q[T])$ with $\gcd(c,D)=1$ and if $v_\infty(c\pi_\infty^m) > v_\infty(cu+d)$ then

$$E_s^{(1)}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) = E_s^{(D)}(\begin{pmatrix} \frac{\pi_\infty^m}{D} & \frac{u+c^*d}{D}) \\ 0 & 1 \end{pmatrix}) \left[\frac{c}{D}\right] \cdot \\ \cdot \delta_D\, q^{-(2s+1)\deg D} \delta_{cu+d}\, q^{-v_\infty(cu+d)(2s+1)} \tag{2.3.3}$$

with $c^* \equiv c^{-1} \bmod D$. Here $E_s^{(D)}$ is the Eisenstein series

$$E_s^{(D)}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) := \sum_{\substack{c,d\in\mathbb{F}_q[T] \\ v_\infty(c\pi_\infty^m)>v_\infty(cu+d)}} \left[\frac{c}{D}\right] \delta_{cu+d}\, q^{v_\infty(cu+d)(2s+1)}. \tag{2.3.4}$$

2.3.1 deg D is odd

We apply the results of this section ((2.3.2) and (2.3.3)) to Proposition 2.2.2. Let $G(N)$ be the set $\Gamma_0^{(1)}(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*$.

Proposition 2.3.1 *Let* $\deg D$ *be odd, then*

$$L^{(N,D)}(2s+1)L(f,\mathcal{A},s) = \frac{q}{2(q-1)} \int_{G(N)} f \cdot \overline{\Phi_{\bar{s}}^{(o)}}$$

with

$$\begin{aligned} \Phi_s^{(o)}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) &:= \sum_{M\in R} \Theta_{\mathfrak{a}_0,\lambda_0} H_{2,s}(M \begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) \qquad (2.3.5) \\ &= q^{-\deg D} \sum_{\lambda \bmod D} \Theta_{\mathfrak{a}_0,\lambda_0}(\begin{pmatrix} \frac{\pi_\infty^m}{D} & \frac{-(u+\lambda)}{D} \\ 0 & 1 \end{pmatrix}) \mathcal{E}_s(\begin{pmatrix} \frac{\pi_\infty^m}{D} & \frac{u+\lambda}{D} \\ 0 & 1 \end{pmatrix}) \end{aligned}$$

where $\mathcal{E}_s(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})$

$$\begin{aligned} := \ & q^{(s+1)\deg D+2s} q^{-m(s+1)} \Bigg[E_s^{(1)}(\begin{pmatrix} ND\pi_\infty^m & NDu \\ 0 & 1 \end{pmatrix}) \qquad (2.3.6) \\ & + E_s^{(D)}(\begin{pmatrix} N\pi_\infty^m & Nu \\ 0 & 1 \end{pmatrix}) \delta_{\lambda_0 DN}\, \epsilon_0^{-1} \left[\frac{D}{N}\right] q^{(-\frac{1}{2}-2s)\deg D} \Bigg]. \end{aligned}$$

2.3.2 $\deg D$ is even

We already mentioned that the factor $((-1)^{m-\deg\lambda_0}+1)/2$ is invariant under the whole group $GL_2(\mathbb{F}_q[T])$. Therefore it is not affected by the trace. Proposition 2.2.3 yields the following.

Proposition 2.3.2 *Let $\deg D$ be even, then*

$$\frac{1}{1+q^{-s-1}} L^{(N,D)}(2s+1)L(f,\mathcal{A},s) = \frac{q}{2(q-1)} \int\limits_{G(N)} f \cdot \overline{\Phi_{\bar{s}}^{(e)}}$$

with

$$\Phi_s^{(e)}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) := \Phi_s^{(o)}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) \frac{(-1)^{m-\deg\lambda_0}+1}{2}. \tag{2.3.7}$$

2.4 Holomorphic Projection

We want to evaluate an integral $\int_{G(N)} f\cdot\overline{\Phi}$, where f is our automorphic cusp form of Drinfeld type of level N (cf. section 2.1) and Φ is any function on $G(N) = \Gamma_0^{(1)}(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*$. Since the Petersson product is non-degenerate on cusp forms, we find an automorphic cusp form Ψ of Drinfeld type for $\Gamma_0^{(1)}(N)$ (one has to modify the definition of cusp forms to $\Gamma_0^{(1)}(N)$ in an obvious way) such that

$$\int\limits_{G(N)} g\cdot\overline{\Psi} = \int\limits_{G(N)} g\cdot\overline{\Phi}$$

for all cusp forms g.
If we set $g=f$ we obtain our result. In this section we want to show how one can compute the Fourier coefficients of Ψ from those of Φ. We already noticed that only the coefficients $\Psi^*(\pi_\infty^{\deg\lambda+2},\lambda)$ are important (cf. Remark 2.1.2).
For this we take $g=P_\lambda$, where P_λ ($\lambda\in\mathbb{F}_q[T],\lambda\neq 0$) are the Poincaré series introduced in [Rü2], and evaluate (cf. [Rü2], Prop. 14)

$$\int\limits_{G(N)} P_\lambda\cdot\overline{\Psi} = \frac{4}{q-1}\,\overline{\Psi^*(\pi_\infty^{\deg\lambda+2},\lambda)}. \tag{2.4.1}$$

On the other hand we calculate (with transformations as in the proof of [Rü2], Prop. 14)

$$\int\limits_{G(N)} P_\lambda\cdot\overline{\Phi} = 2\lim_{\sigma\to 1}\int\limits_{\mathbb{H}_\infty} g_{\lambda,\sigma}\cdot\overline{(\Phi-\widetilde{\Phi})} \tag{2.4.2}$$

where

$$g_{\lambda,\sigma}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) := \begin{cases} 0 & \text{if } \deg\lambda+2>m \\ q^{-m\sigma}\psi_\infty(\lambda u) & \text{if } \deg\lambda+2\le m \end{cases}$$

and where

$$\widetilde{\Phi}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) := \Phi(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}). \tag{2.4.3}$$

For these calculations we used again the canonical mapping (cf. section 2.2)

$$\mathbb{H}_\infty \to G(N).$$

Since $\mathbb{H}_\infty$ represents only the part $\mathcal{T}_+$ of $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$ (cf. section 2.1) and since Φ is not necessarily harmonic, we also have to consider the function $\widetilde{\Phi}$. Using the Fourier expansions

$$\begin{aligned} \Phi(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) &= \sum_\mu \Phi^*(\pi_\infty^m, \mu)\psi_\infty(\mu u) \\ \widetilde{\Phi}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) &= \sum_\mu \widetilde{\Phi}^*(\pi_\infty^m, \mu)\psi_\infty(\mu u) \end{aligned}$$

and the character relations for ψ_∞, (2.4.2) yields

$$\int_{G(N)} P_\lambda \cdot \overline{\Phi} = \frac{2}{q} \lim_{\sigma\to 0} \sum_{m=\deg\lambda+2}^{\infty} q^{-m\sigma} \overline{(\Phi^*(\pi_\infty^m, \lambda) - \widetilde{\Phi}^*(\pi_\infty^m, \lambda))}. \tag{2.4.4}$$

Finally, (2.4.1) and (2.4.4) prove:

PROPOSITION 2.4.1 *Let $\Phi : G(N) = \Gamma_0^{(1)}(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^* \to \mathbb{C}$ be any function, then there is an automorphic cusp form Ψ of Drinfeld type for $\Gamma_0^{(1)}(N)$ such that*

$$\int_{G(N)} f \cdot \overline{\Psi} = \int_{G(N)} f \cdot \overline{\Phi}.$$

The Fourier coefficients of Ψ can be evaluated by the formula

$$\Psi^*(\pi_\infty^{\deg\lambda+2}, \lambda) = \frac{q-1}{2q} \lim_{\sigma\to 0} \sum_{m=\deg\lambda+2}^{\infty} q^{-m\sigma}(\Phi^*(\pi_\infty^m, \lambda) - \widetilde{\Phi}^*(\pi_\infty^m, \lambda)),$$

where $\widetilde{\Phi}$ is defined in (2.4.3).

Problems could arise since the limit may not exist. We will see this in the following sections, where we apply this holomorphic projection formula to $\Phi_s^{(o)}$, $\Phi_s^{(e)}$ (cf. (2.3.5) and (2.3.7)) or their derivatives.

2.5 FOURIER EXPANSIONS OF $\Phi_s^{(o)}$ AND $\Phi_s^{(e)}$

In this section we evaluate the Fourier coefficients $\Phi_s^{(o)*}(\pi_\infty^m,\lambda)$ and $\Phi_s^{(e)*}(\pi_\infty^m,\lambda)$ (cf. (2.3.5) and (2.3.7)). The function $\Theta_{\mathfrak{a}_0,\lambda_0}$ is already defined by its coefficients $r_{\mathfrak{a}_0,\lambda_0}$. It remains to evaluate the coefficients of $\mathcal{E}_s$ (cf. (2.3.6)) and therefore of the Eisenstein series $E_s^{(1)}$ (cf. (2.2.6)) and $E_s^{(D)}$ (cf. (2.3.4)).

We introduce a "basic function" on $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$:

$$F_s(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) = \sum_\lambda F_s^*(\pi_\infty^m,\lambda)\psi_\infty(\lambda u) := \sum_{\substack{d\in\mathbb{F}_q[T] \\ m>v_\infty(u+d)}} \delta_{u+d}\, q^{v_\infty(u+d)(2s+1)}. \tag{2.5.1}$$

We recall that δ_z is the local norm symbol of z at ∞. At first we express the Eisenstein series in terms of F_s. Elementary transformations give

$$E_s^{(1)}(\begin{pmatrix} ND\pi_\infty^m & NDu \\ 0 & 1 \end{pmatrix}) = \sum_{\substack{d\in\mathbb{F}_q[T] \\ d\neq 0}} \left[\frac{D}{d}\right] q^{-(2s+1)\deg d} + \delta_D\, q^{-(2s+1)\deg D}$$

$$\cdot \sum_{\substack{\mu\in\mathbb{F}_q[T] \\ \mu\neq 0}} \left[\sum_{\substack{c|\mu \\ c\equiv 0 \bmod D}} F_s^*(cN\pi_\infty^m, \frac{\mu}{c}) \sum_{d \bmod D} \left[\frac{d}{D}\right] \psi_\infty(\frac{\mu}{c}\frac{d}{D}) \right] \psi_\infty(\mu N u).$$

The Gauss sum can be evaluated

$$\sum_{d \bmod D} \left[\frac{d}{D}\right] \psi_\infty(\lambda \frac{d}{D}) = \left[\frac{\lambda}{D}\right] \epsilon_0^{-1} q^{\frac{1}{2}\deg D},$$

where ϵ_0 is as in (2.3.2). Therefore

$$\begin{aligned} & E_s^{(1)}(\begin{pmatrix} ND\pi_\infty^m & NDu \\ 0 & 1 \end{pmatrix}) \\ = & \sum_{\substack{d\in\mathbb{F}_q[T] \\ d\neq 0}} \left[\frac{D}{d}\right] q^{-(2s+1)\deg d} + \epsilon_0^{-1}\delta_D\, q^{(-2s-\frac{1}{2})\deg D} \,. \\ & \cdot \sum_{\mu\neq 0} \sum_{\substack{c|\mu \\ c\equiv 0 \bmod D}} \left[\frac{\mu/c}{D}\right] F_s^*(cN\pi_\infty^m, \frac{\mu}{c})\psi_\infty(\mu N u). \end{aligned} \tag{2.5.2}$$

The same transformations as above yield

$$\begin{aligned} E_s^{(D)}(\begin{pmatrix} N\pi_\infty^m & Nu \\ 0 & 1 \end{pmatrix}) = & \sum_{\substack{d\in\mathbb{F}_q[T] \\ d\neq 0}} \left[\frac{d}{D}\right] F_s^*(dN\pi_\infty^m, 0) + \\ & + \sum_{\mu\neq 0} \sum_{c|\mu} \left[\frac{c}{D}\right] F_s^*(cN\pi_\infty^m, \frac{\mu}{c})\psi_\infty(\mu N u). \end{aligned} \tag{2.5.3}$$

Now we have to evaluate the Fourier coefficients of the "basic function" F_s (cf. (2.5.1)). This is not very difficult, though perhaps a little tedious to write down in detail. One starts with the definition of the coefficients

$$F_s^*(\pi_\infty^m, \lambda) = q^{-m+1} \sum_{u \in \pi_\infty/\pi_\infty^m} F_s(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\psi_\infty(-\lambda u)$$

and uses the character relations for ψ_∞. We do not carry it out in detail. As the local norm symbol δ_z behaves differently we have to distinguish again the two cases.

2.5.1 $\deg D$ is odd

L_∞/K_∞ is ramified and the local norm symbol for $z = e_z\pi_\infty^n + \dots$ is given by $\delta_z = \chi_2(e_z)\,\delta_T^n$ (χ_2 is the quadratic character on $\mathbb{F}_q^*$; we recall that $\pi_\infty = T^{-1}$). We get:

Lemma 2.5.1 *Let* $\deg D$ *be odd, then*

$$F_s^*(\pi_\infty^m, \mu) = \begin{cases} 0 & , \text{ if either } \mu = 0 \text{ or } \deg \mu + 2 > m \\ \epsilon_0^{-1} q^{\frac{1}{2}} \delta_\mu\, q^{2s(\deg \mu + 1)} & , \text{ if } \mu \neq 0 \text{ and } \deg\mu + 2 \leq m. \end{cases}$$

Now (2.5.2), (2.5.3), Lemma 2.5.1 and the definition of $\mathcal{E}_s$ in (2.3.6) give:

Proposition 2.5.2 *Let* $\deg D$ *be odd, then*

$$\mathcal{E}_s(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) = \sum_{\substack{\mu \in \mathbb{F}_q[T] \\ \deg(\mu N)+2 \leq m}} e_s(\pi_\infty^m, \mu)\psi_\infty(\mu N u)$$

with

$$e_s(\pi_\infty^m, 0) = q^{(s+1)\deg D + 2s - m(s+1)} \sum_{\substack{d \in \mathbb{F}_q[T] \\ d \neq 0}} \left[\frac{D}{d}\right] q^{-(2s+1)\deg d} \tag{2.5.4}$$

and ($\mu \neq 0$)

$$\begin{aligned} e_s(\pi_\infty^m, \mu) \quad = \quad & q^{(-s+\frac{1}{2})\deg D + 4s + \frac{1}{2} - m(s+1) + 2s \deg \mu} \,. \\ & \cdot \Big(\sum_{\substack{c \mid \mu \\ c \equiv 0 \bmod D}} \left[\frac{D}{\mu/c}\right] q^{-2s\deg c} + \delta_{\lambda_0 N \mu} \left[\frac{D}{N}\right] \sum_{c \mid \mu} \left[\frac{D}{c}\right] q^{-2s \deg c}\Big). \end{aligned} \tag{2.5.5}$$

2.5.2 $\deg D$ is even

L_∞/K_∞ is inert and the local norm symbol for $z = e_z\pi_\infty^n + \dots$ is given by $\delta_z = (-1)^n$.
We get:

Lemma 2.5.3 *Let* $\deg D$ *be even, then*

$$F_s^*(\pi_\infty^m, 0) = \frac{1-q}{q^{2s}+1}(-q^{2s})^m,$$

and ($\mu \neq 0$ *with* $\deg \mu + 2 \leq m$)

$$F_s^*(\pi_\infty^m, \mu) = \frac{(-q^{2s})^{\deg \mu + 1}}{q^{2s}+1}((1-q)(-q^{2s})^{m-\deg \mu - 1} - 1 - q^{2s+1}).$$

Again (2.5.2), (2.5.3), Lemma 2.5.3 and the definition of $\mathcal{E}_s$ in (2.3.6) give:

Proposition 2.5.4 *Let* $\deg D$ *be even, then*

$$\mathcal{E}_s(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) = \sum_{\substack{\mu \in \mathbb{F}_q[T] \\ \deg(\mu N) + 2 \leq m}} e_s(\pi_\infty^m, \mu)\psi_\infty(\mu N u)$$

with

$$\begin{aligned} e_s(\pi_\infty^m, 0) = q^{\deg D(s+1) - m(s+1) + 2s} (& \sum_{\substack{d \in \mathbb{F}_q[T] \\ d \neq 0}} \left[\frac{D}{d}\right] q^{-(2s+1)\deg d} \\ & + \frac{1-q}{q^{2s}+1} q^{\deg D(-\frac{1}{2} - 2s) + 2sm - 2s \deg N} \\ & (-1)^{\deg \lambda_0 + m} \left[\frac{D}{N}\right] \sum_{\substack{d \in \mathbb{F}_q[T] \\ d \neq 0}} \left[\frac{D}{d}\right] q^{-2s \deg d}) \end{aligned} \tag{2.5.6}$$

and ($\mu \neq 0$)

$$e_s(\pi_\infty^m, \mu) = q^{m(-s-1) + 2s + \deg D(-s + \frac{1}{2})} \cdot \tag{2.5.7}$$

$$\begin{aligned} \Big((\sum_{\substack{c \mid \mu \\ c \equiv 0 \bmod D}} \left[\frac{D}{\mu/c}\right] q^{-2s \deg c}) (& \frac{1-q}{q^{2s}+1} (-1)^{m - \deg N - \deg \mu} \; q^{2s(m - \deg N)} \\ & + \frac{q^{2s+1}+1}{q^{2s}+1} \; q^{2s(\deg \mu + 1)}) \\ + \left[\frac{D}{N}\right] (\sum_{c \mid \mu} \left[\frac{D}{c}\right] q^{-2s \deg c}) (& \frac{1-q}{q^{2s}+1} (-1)^{\deg \lambda_0 + m} \; q^{2s(m - \deg N)} \\ & + \frac{q^{2s+1}+1}{q^{2s}+1} (-1)^{\deg \lambda_0 + \deg N + \deg \mu} \; q^{2s(\deg \mu + 1)}) \Big). \end{aligned}$$

2.6 Fourier Expansions of $\widetilde{\Phi_s^{(o)}}$ and $\widetilde{\Phi_s^{(e)}}$

In accordance with (2.4.3) let $\widetilde{\Phi_s^{(o)}}$ (resp. $\widetilde{\Phi_s^{(e)}}$) on $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$ be defined as $\widetilde{\Phi_s^{(o)}}(X) = \Phi_s^{(o)}(X \begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix})$ (resp. $\widetilde{\Phi_s^{(e)}}(X) = \Phi_s^{(e)}(X \begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix})$).

The situation is more complicated than in the last section. To extend functions canonically from $\mathcal{T}_+$ (cf. (2.1.1)) to the whole of $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$ we need the following proposition.

PROPOSITION 2.6.1 *Let $\chi_D : (\mathbb{F}_q[T]/D\,\mathbb{F}_q[T])^* \to \mathbb{C}^*$ be a character modulo D and let $\chi_\infty : K_\infty^* \to \mathbb{C}^*$ be a character which vanishes on the subgroup of 1-units $O_\infty^{(1)} = \{x \in K_\infty^* \mid v_\infty(x-1) > 0\}$.*
Let $F : \mathcal{T}_+ \to \mathbb{C}$ be a function which satisfies

$$F(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) = F(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\,\chi_D(d)\,\chi_\infty(cu+d),$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(1)}(D)$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix} \in \mathcal{T}_+$.
Then F can be defined on $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$ with

$$F(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) = F(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\,\chi_D(d)\,\cdot$$
$$\cdot\begin{cases} \chi_\infty(cu+d) & ,\ \text{if } v_\infty(c\pi_\infty^m) > v_\infty(cu+d) \\ \chi_\infty(c^{-1}) & ,\ \text{if } v_\infty(c\pi_\infty^m) \le v_\infty(cu+d). \end{cases} \tag{2.6.1}$$

PROOF. We already know that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix} \in \mathcal{T}_+$ is equivalent to $v_\infty(c\pi_\infty^m) > v_\infty(cu+d)$ (cf. Remark 2.2.1). For each $X \in GL_2(K_\infty)/\Gamma_\infty K_\infty^*$ there is $A \in \Gamma_0^{(1)}(D)$ and $\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix} \in \mathcal{T}_+$ such that $X = A\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}$ in $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$. Then we define $F(X)$ by equation (2.6.1). The assumption on F guarantees that this definition is independent of the choice of A and $\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}$. □

We apply this proposition to $\Theta_{\mathfrak{a}_0,\lambda_0}$ (cf. (2.1.10)) and to the Eisenstein series. The Eisenstein series $E_s^{(i)}$ $(i = 1, D)$ (cf. (2.2.6), (2.3.4)) satisfy

$$E_s^{(i)}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) = E_s^{(i)}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})\,\cdot$$
$$\cdot\left[\frac{d}{D}\right]\delta_{cu+d}\,q^{-v_\infty(cu+d)(2s+1)}$$

if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(1)}(D)$ and $v_\infty(c\pi_\infty^m) > v_\infty(cu+d)$. We can apply Proposition 2.6.1 with $\chi_D(d) = \left[\frac{d}{D}\right]$ and $\chi_\infty(z) = \delta_z\,q^{-v_\infty(z)(2s+1)}$.
Hence

$$E_s^{(1)}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}) = \sum_{\substack{c,d\in\mathbb{F}_q[T] \\ c\equiv 0 \bmod D \\ v_\infty(c\pi_\infty^m)\le v_\infty(cu+d)}} \left[\frac{d}{D}\right]\delta_{-c}\,q^{v_\infty(c)(2s+1)}$$

and

$$E_s^{(D)}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}) = \sum_{\substack{c,d\in\mathbb{F}_q[T] \\ v_\infty(c\pi_\infty^m)\le v_\infty(cu+d)}} \left[\frac{c}{D}\right]\delta_{-c}\, q^{v_\infty(c)(2s+1)}.$$

We denote these functions by $\widetilde{E_s^{(1)}}$ and $\widetilde{E_s^{(D)}}$ as above. Starting with the definition of the Fourier coefficients we calculate

$$\begin{aligned}
&\widetilde{E_s^{(1)}}(\begin{pmatrix} ND\pi_\infty^m & NDu \\ 0 & 1 \end{pmatrix}) \\
&= \sum_{\substack{\mu\neq 0 \\ \deg(\mu N)+2\le m}} \Bigg[\epsilon_0\, q^{\deg D(-2s-\frac{1}{2})+\deg N(-2s)+1-m}\,\cdot \\
&\quad \cdot\delta_{\mu ND}\sum_{\substack{c|\mu \\ c\equiv 0 \bmod D}}\left[\frac{D}{\mu/c}\right]q^{-2s\deg c}\Bigg]\psi_\infty(\mu Nu) \qquad (2.6.2)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{E_s^{(D)}}(\begin{pmatrix} N\pi_\infty^m & Nu \\ 0 & 1 \end{pmatrix}) &= q^{\deg N(-2s)+1-m}\delta_{-N}\sum_{c\neq 0}\left[\frac{D}{c}\right]q^{-2s\deg c} + \\
&+ \sum_{\substack{\mu\neq 0 \\ \deg(\mu N)+2\le m}}\Bigg[q^{\deg N(-2s)+1-m}\delta_{-N}\sum_{c|\mu}\left[\frac{D}{c}\right]q^{-2s\deg c}\Bigg]\psi_\infty(\mu Nu). \quad (2.6.3)
\end{aligned}$$

In addition we have $\widetilde{q^{-m(s+1)}} = q^{-(1-m)(s+1)}$, therefore (2.6.2) and (2.6.3) give:

Proposition 2.6.2 *Let $\deg D$ be odd or even, then*

$$\widetilde{\mathcal{E}_s}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) := \mathcal{E}_s(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix}) = \sum_{\substack{\mu\in\mathbb{F}_q[T] \\ \deg(\mu N)+2\le m}}\widetilde{e_s}(\pi_\infty^m,\mu)\psi_\infty(\mu Nu)$$

with

$$\widetilde{e_s}(\pi_\infty^m, 0) = q^{\deg D(-s+\frac{1}{2})+\deg N(-2s)+ms+s}\epsilon_0^{-1}\,\delta_{\lambda_0}\left[\frac{D}{N}\right]\sum_{d\neq 0}\left[\frac{D}{c}\right]q^{-2s\deg d} \qquad (2.6.4)$$

and ($\mu\neq 0$)

$$\begin{aligned}
\widetilde{e_s}(\pi_\infty^m,\mu) &= q^{\deg D(-s+\frac{1}{2})+\deg N(-2s)+ms+s}\epsilon_0^{-1}\,\cdot \qquad (2.6.5) \\
&\Bigg(\delta_{\mu N}\sum_{\substack{c|\mu \\ c\equiv 0 \bmod D}}\left[\frac{D}{\mu/c}\right]q^{-2s\deg c} + \delta_{\lambda_0}\left[\frac{D}{N}\right]\sum_{c|\mu}\left[\frac{D}{c}\right]q^{-2s\deg c}\Bigg).
\end{aligned}$$

For $\Theta_{\mathfrak{a}_0,\lambda_0}$ it is not just straightforward calculation. In the following we make use of the fact that the Fourier coefficients $r_{\mathfrak{a}_0,\lambda_0}(\lambda)$ of $\Theta_{\mathfrak{a}_0,\lambda_0}$ are independent of π_∞^m if $\deg\lambda + 2 \le m$.

$\Theta_{\mathfrak{a}_0,\lambda_0}$ satisfies Proposition 2.6.1 with $\chi_D(d) = \left[\dfrac{d}{D}\right]$ and $\chi_\infty(z) = \delta_z\, q^{-v_\infty(z)}$ (cf. (2.2.5)). Again we denote $\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}(\cdot) = \Theta_{\mathfrak{a}_0,\lambda_0}(\cdot \begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix})$.

Let $\pi_\infty^m \in K_\infty^*$ and $u \in K_\infty$. Choose $c, d \in \mathbb{F}_q[T]$ with $c \equiv 0 \bmod D, \gcd(c,d) = 1$ and $v_\infty(u + \frac{d}{c}) \ge m+1$ and find $a, b \in \mathbb{F}_q[T]$ with $ad - bc = 1$. Then for all $k \in \mathbb{Z}$ with $k \le m+1$ there is the following identity in $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$:

$$\begin{pmatrix} \pi_\infty^k & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \frac{\pi_\infty^{1-k}}{c^2} & \frac{a}{c} \\ 0 & 1 \end{pmatrix}$$

We use this identity for $k = m$ and $k = m+1$. Then Proposition 2.6.1 gives

$$\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) - \widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}(\begin{pmatrix} \pi_\infty^{m+1} & u \\ 0 & 1 \end{pmatrix}) = \left[\frac{d}{D}\right]\delta_{-c}\, q^{v_\infty(c)} \cdot$$
$$\cdot \sum_{\deg\mu+2=1-m+2\deg c} r_{\mathfrak{a}_0,\lambda_0}(\mu)\psi_\infty(\mu\frac{a}{c}). \quad (2.6.6)$$

On the other hand we set $u_\epsilon = -\frac{d}{c} + \epsilon\pi_\infty^m$ for $\epsilon \in \mathbb{F}_q^*$, we compare $\frac{a}{c}$ with $\frac{au_\epsilon+b}{cu_\epsilon+d}$ and sum over all ϵ:

$$(q-1)\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) - \left[\frac{d}{D}\right]\delta_{-c}\, q^{v_\infty(c)}\sum_{\epsilon\in\mathbb{F}_q^*}\Theta_{\mathfrak{a}_0,\lambda_0}(\begin{pmatrix} \frac{\pi_\infty^{1-m}}{c^2} & \frac{au_\epsilon+b}{cu_\epsilon+d} \\ 0 & 1 \end{pmatrix})$$
$$= q\left[\frac{d}{D}\right]\delta_{-c}\, q^{v_\infty(c)}\sum_{\deg\mu+2=1-m+2\deg c} r_{\mathfrak{a}_0,\lambda_0}(\mu)\psi_\infty(\mu\frac{a}{c}). \quad (2.6.7)$$

Now

$$\begin{pmatrix} \frac{\pi_\infty^{1-m}}{c^2} & \frac{au_\epsilon+b}{cu_\epsilon+d} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \pi_\infty^{m+1} & u_\epsilon \\ 0 & 1 \end{pmatrix}$$

in $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$. We use this to evaluate the corresponding value of $\Theta_{\mathfrak{a}_0,\lambda_0}$. A combination of (2.6.6) and (2.6.7) therefore gives

$$q\,\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}(\begin{pmatrix} \pi_\infty^{m+1} & u \\ 0 & 1 \end{pmatrix}) - \widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix})$$
$$= \delta_{-\pi_\infty^m}\, q^{-m}\sum_{\epsilon\in\mathbb{F}_q^*}\delta_\epsilon\Theta_{\mathfrak{a}_0,\lambda_0}(\begin{pmatrix} \pi_\infty^{m+1} & u+\epsilon\pi_\infty^m \\ 0 & 1 \end{pmatrix}). \quad (2.6.8)$$

If we evaluate in (2.6.8) the Fourier coefficients at λ with $\deg\lambda + 2 \le m$, we get the recursion formula

$$q\,\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}^*(\pi_\infty^{m+1},\lambda) - \widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}^*(\pi_\infty^m,\lambda) = \delta_{\pi_\infty^m}\, q^{-m}\sum_{\epsilon\in\mathbb{F}_q^*}\delta_\epsilon r_{\mathfrak{a}_0,\lambda_0}(\lambda). \quad (2.6.9)$$

The Fourier coefficient in (2.6.8) at λ with $\deg\lambda + 2 = m + 1$ yields

$$q\,\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}^*(\pi_\infty^{\deg\lambda+2},\lambda) = \delta_{\pi_\infty^{\deg\lambda+1}} q^{-\deg\lambda-1}\sum_{\epsilon\in\mathbb{F}_q^*}\delta_\epsilon\psi_\infty(-\lambda\epsilon\pi_\infty^{\deg\lambda+1})r_{\mathfrak{a}_0,\lambda_0}(\lambda). \tag{2.6.10}$$

For $\lambda = 0$ we calculate

$$\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}^*(\pi_\infty,0) = q^{-\frac{1}{2}\deg D}\delta_{\lambda_0}\epsilon_0^{-1}\sum_{\deg\mu+2\leq\deg D} r_{\mathfrak{a}_0,\lambda_0}(\mu). \tag{2.6.11}$$

It is now obvious how one evaluates $\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}^*(\pi_\infty^m,\lambda)$ with the recursion formula (2.6.9) and the starting values (2.6.10) and (2.6.11). Here again we have to consider the two cases separately.

Proposition 2.6.3 *Let $\deg D$ be odd, then*

$$\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}(\begin{pmatrix}\pi_\infty^m & u\\ 0 & 1\end{pmatrix}) = \sum_{\deg\lambda+2\leq m}\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}^*(\pi_\infty^m,\lambda)\psi_\infty(\lambda u)$$

with

$$\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}^*(\pi_\infty^m,\lambda) = q^{\frac{1}{2}}q^{-m}\epsilon_0^{-1}\delta_{\lambda_0}r_{\mathfrak{a}_0,\lambda_0}(\lambda). \tag{2.6.12}$$

Proposition 2.6.4 *Let $\deg D$ be even, then*

$$\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}(\begin{pmatrix}\pi_\infty^m & u\\ 0 & 1\end{pmatrix}) = \sum_{\deg\lambda+2\leq m}\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}^*(\pi_\infty^m,\lambda)\psi_\infty(\lambda u)$$

with

$$\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}^*(\pi_\infty^m,\lambda) = q^{-m}(-1)^{\deg\lambda_0}\big(\frac{q+1}{2} + \frac{q-1}{2}(-1)^{m+\deg\lambda_0-1}\big)\, r_{\mathfrak{a}_0,\lambda_0}(\lambda). \tag{2.6.13}$$

2.7 Functional Equations

In this section we modify the representations of the L-series of Proposition 2.3.1 and Proposition 2.3.2. With these new formulas we can prove functional equations for the L-series. Later we will use them to get our final results.

2.7.1 $\deg D$ is odd

Since f is an automorphic cusp form of Drinfeld type and therefore satisfies (cf. Definition 2.1.1)

$$f(X\begin{pmatrix}0 & 1\\ \pi_\infty & 0\end{pmatrix}) = -f(X) \text{ for all } X\in GL_2(K_\infty)/\Gamma_\infty K_\infty^*,$$

we can transform the integral in Proposition 2.3.1, and get:

Lemma 2.7.1 *Let* $\deg D$ *be odd, then*

$$L^{(N,D)}(2s+1)L(f,\mathcal{A},s) = \frac{q}{4(q-1)} \int\limits_{G(N)} f \cdot \overline{F_{\bar{s}}^{(o)}}$$

with

$$F_s^{(o)}(X) := \Phi_s^{(o)}(X) - \widetilde{\Phi_s^{(o)}}(X),$$

whose Fourier coefficients are

$$F_s^{(o)*}(\pi_\infty^m, \lambda) = -\widetilde{F_s^{(o)}}^*(\pi_\infty^m, \lambda) = \Phi_s^{(o)*}(\pi_\infty^m, \lambda) - \widetilde{\Phi_s^{(o)}}^*(\pi_\infty^m, \lambda).$$

Now we evaluate $F_s^{(o)*}(\pi_\infty^m, \lambda)$. We start with the definition (cf. Proposition 2.3.1)

$$\begin{aligned} \Phi_s^{(o)}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) &= q^{-\deg D} \cdot \\ &\cdot \sum_{\lambda \bmod D} \Theta_{\mathfrak{a}_0,\lambda_0}(\begin{pmatrix} \frac{\pi_\infty^m}{D} & \frac{-(u+\lambda)}{D} \\ 0 & 1 \end{pmatrix}) \mathcal{E}_s(\begin{pmatrix} \frac{\pi_\infty^m}{D} & \frac{u+\lambda}{D} \\ 0 & 1 \end{pmatrix}) \end{aligned}$$

and use the Fourier coefficients of $\Theta_{\mathfrak{a}_0,\lambda_0}$ (cf. (2.1.10)) and $\mathcal{E}_s$ (cf. (2.3.6) and Proposition 2.5.2) to evaluate

$$\Phi_s^{(o)*}(\pi_\infty^m, \lambda) = \sum_{\substack{\mu \in \mathbb{F}_q[T] \\ \deg(\mu N)+2 \le m+\deg D}} r_{\mathfrak{a}_0,\lambda_0}(\mu N - \lambda D)\, e_s(\pi_\infty^{m+\deg D}, \mu). \quad (2.7.1)$$

On the other hand, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_q[T])$ with $b, c \equiv 0 \bmod D$ and if u is such that $v_\infty(u + d/c) \ge m$, then we have (using the transformation rules of $\Theta_{\mathfrak{a}_0,\lambda_0}$ and $\mathcal{E}_s$):

$$\widetilde{\Phi_s^{(o)}}(\begin{pmatrix} \pi_\infty^m & u \\ 0 & 1 \end{pmatrix}) = \Phi_s^{(o)}(\begin{pmatrix} \frac{\pi_\infty^{1-m}}{c^2} & \frac{a}{c} \\ 0 & 1 \end{pmatrix}).$$

When we expand this equation with Fourier coefficients, we get

$$\widetilde{\Phi_s^{(o)}}^*(\pi_\infty^m, \lambda) = \sum_{\substack{\mu \in \mathbb{F}_q[T] \\ \deg(\mu N)+2 \le m+\deg D}} \widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}^*(\pi_\infty^{m+\deg D}, \mu N - \lambda D)\, \widetilde{e_s}(\pi_\infty^{m+\deg D}, \mu)\, \delta_D. \quad (2.7.2)$$

Now we replace e_s, $\widetilde{\Theta_{\mathfrak{a}_0,\lambda_0}}^*$ and $\widetilde{e_s}$ in (2.7.1) and (2.7.2) by (2.5.5), (2.6.12) and (2.6.5), and we get:

Proposition 2.7.2 *Let* $\deg D$ *be odd and* $\deg\lambda + 2 \le m$, *then*

$$\begin{aligned}\Phi_s^{(o)*}(\pi_\infty^m,\lambda) - \widetilde{\Phi_s^{(o)}}^*(\pi_\infty^m,\lambda) &= r_{\mathfrak{a}_0,\lambda_0}(-\lambda D)\, q^{-m-\frac{1}{2}\deg D+\frac{1}{2}} \cdot \\ &\Big(\big(\sum_{\substack{d\in\mathbb{F}_q[T]\\ d\neq 0}} \left[\frac{D}{d}\right] q^{-(2s+1)\deg d}\big)\, q^{-ms+2s+\frac{1}{2}\deg D-\frac{1}{2}} \\ &-\big(\sum_{\substack{d\in\mathbb{F}_q[T]\\ d\neq 0}} \left[\frac{D}{d}\right] q^{-2s\deg d}\big) \left[\frac{D}{N}\right] q^{ms-2s\deg N+s}\Big)\end{aligned}$$

$$\begin{aligned}&+ \sum_{\substack{\mu\in\mathbb{F}_q[T]\,,\,\mu\neq 0\\ \deg(\mu N)+2\le m+\deg D}} r_{\mathfrak{a}_0,\lambda_0}(\mu N-\lambda D)\, q^{-m-\frac{1}{2}\deg D+\frac{1}{2}} \cdot \\ &\Big(\big(\sum_{\substack{c\mid\mu\\ c\equiv 0 \bmod D}} \left[\frac{D}{\mu/c}\right] q^{-2s\deg c}\big)\big(q^{-2s\deg D-ms+4s+2s\deg\mu} - \delta_{\lambda_0 N\mu}\, q^{ms-2s\deg N+s}\big) \\ &+\big(\sum_{c\mid\mu} \left[\frac{D}{c}\right] q^{-2s\deg c}\big) \left[\frac{D}{N}\right] \big(\delta_{\lambda_0 N\mu}\, q^{-2s\deg D-ms+4s+2s\deg\mu} - q^{ms-2s\deg N+s}\big)\Big).\end{aligned}$$

With these formulas we prove the following result:

Theorem 2.7.3 *Let* $\deg D$ *be odd, then*

$$\begin{aligned} q^{(\deg N+\deg D-\frac{5}{2})s}\, (\Phi_s^{(o)*}(\pi_\infty^m,\lambda) - \widetilde{\Phi_s^{(o)}}^*(\pi_\infty^m,\lambda)) &= \\ -\left[\frac{D}{N}\right] q^{(\deg N+\deg D-\frac{5}{2})(-s)}\, (\Phi_{-s}^{(o)*}(\pi_\infty^m,\lambda) - \widetilde{\Phi_{-s}^{(o)}}^*(\pi_\infty^m,\lambda)),&\end{aligned}$$

and therefore Lemma 2.7.1 implies that

$$Z(s) := q^{(\deg N+\deg D-\frac{5}{2})s} L^{(N,D)}(2s+1) L(f,\mathcal{A},s)$$

satisfies the functional equation

$$Z(s) = -\left[\frac{D}{N}\right] Z(-s).$$

Proof. One can verify the functional equation for $\Phi_s^{(o)*}(\pi_\infty^m,\lambda) - \widetilde{\Phi_s^{(o)}}^*(\pi_\infty^m,\lambda)$ independently for each summand (summation over $\mu\in\mathbb{F}_q[T]$) in the formula of Proposition 2.7.2 if one applies the following remarks:
a) For the first summand we mention (cf. [Ar]) that

$$L_D(s) := \frac{1}{q-1} \sum_{\substack{d\in\mathbb{F}_q[T]\\ d\neq 0}} \left[\frac{D}{d}\right] q^{-s\deg d} \tag{2.7.3}$$

is the L-series of the extension $K(\sqrt{D})/K$ and satisfies

$$L_D(2s+1) = q^{s(-2\deg D+2)-\frac{1}{2}\deg D+\frac{1}{2}} L_D(-2s). \tag{2.7.4}$$

b) Let $\mu \in \mathbb{F}_q[T]$, $\mu \neq 0$ with $r_{\mathfrak{a}_0,\lambda_0}(\mu N - \lambda D) \neq 0$. Then there is $\kappa \in L$ with $N_{L/K}(\kappa) = \lambda_0(\mu N - \lambda D)$ (cf. (2.1.9)). Hence we get $\left[\frac{D}{\mu}\right] = \left[\frac{D}{N}\right] \delta_{\lambda_0 N\mu}$. This implies

$$\sum_{c|\mu} \left[\frac{D}{c}\right] q^{-2s\deg c} = q^{-2s\deg\mu} \left[\frac{D}{N}\right] \delta_{\lambda_0 N\mu} \sum_{c|\mu} \left[\frac{D}{c}\right] q^{2s\deg c} \tag{2.7.5}$$

if $\mu \not\equiv 0 \bmod D$.
c) For $\mu \in \mathbb{F}_q[T]$ with $\mu \equiv 0 \bmod D$, it is easy to see that

$$\sum_{c|\mu} \left[\frac{D}{c}\right] q^{-2s\deg c} = q^{-2s\deg\mu} \sum_{\substack{c|\mu \\ c\equiv 0 \bmod D}} \left[\frac{D}{\mu/c}\right] q^{2s\deg c}. \quad \square \tag{2.7.6}$$

2.7.2 $\deg D$ IS EVEN

The automorphic cusp form f of Drinfeld type satisfies (cf. Definition 2.1.1)

$$\sum_{\beta\in GL_2(O_\infty)/\Gamma_\infty} f(X\beta) = 0 \text{ for all } X \in GL_2(K_\infty)/\Gamma_\infty K_\infty^*.$$

With this identity a transformation of the integral in Proposition 2.3.2 yields immediately:

LEMMA 2.7.4 *Let* $\deg D$ *be even, then*

$$\frac{1}{1+q^{-s-1}} L^{(N,D)}(2s+1) L(f,\mathcal{A},s) = \frac{q}{2(q-1)} \int_{G(N)} f \cdot \overline{F_{\bar{s}}^{(e)}}$$

with

$$F_s^{(e)}(X) := \frac{q}{q+1}\,\Phi_s^{(e)}(X) - \frac{1}{q+1} \sum_{\substack{\beta\in GL_2(O_\infty)/\Gamma_\infty \\ \beta\neq 1}} \Phi_s^{(e)}(X\beta),$$

whose Fourier coefficients are

$$F_s^{(e)*}(\pi_\infty^m,\lambda) = \begin{cases} \frac{q}{q+1}(\Phi_s^{(e)*}(\pi_\infty^m,\lambda) - \widetilde{\Phi_s^{(e)}}^*(\pi_\infty^{m+1},\lambda)), & \textit{if } m \equiv \deg\lambda_0 \bmod 2 \\ 0 & , \textit{if } m \not\equiv \deg\lambda_0 \bmod 2 \end{cases}$$

and

$$\widetilde{F_s^{(e)}}^*(\pi_\infty^m,\lambda) = \begin{cases} 0 & , \textit{if } m \equiv \deg\lambda_0 \bmod 2 \\ \frac{1}{q+1}(\widetilde{\Phi_s^{(e)}}^*(\pi_\infty^m,\lambda) - \Phi_s^{(e)*}(\pi_\infty^{m-1},\lambda)), & \textit{if } m \not\equiv \deg\lambda_0 \bmod 2 \\ & \textit{and } \deg\lambda + 2 < m \\ \widetilde{\Phi_s^{(e)}}^*(\pi_\infty^m,\lambda) & , \textit{if } m \not\equiv \deg\lambda_0 \bmod 2 \\ & \textit{and } \deg\lambda + 2 = m\,. \end{cases}$$

The calculations of $F_s^{(e)*}(\pi_\infty^m,\lambda)$ and $\widetilde{F_s^{(e)}}^*(\pi_\infty^m,\lambda)$ are similar to those above and use Propositions 2.5.4, 2.6.2 and 2.6.4 developed in the previous sections. We only give the results.

Proposition 2.7.5 *Let* $\deg D$ *be even, then*

$$\Phi_s^{(e)*}(\pi_\infty^m,\lambda)-\widetilde{\Phi_s^{(e)}}^*(\pi_\infty^{m+1},\lambda)=r_{\mathfrak{a}_0,\lambda_0}(-\lambda D)\,q^{-m-\frac12\deg D}\,\cdot$$
$$\Big(\big(\sum_{\substack{d\in\mathbb{F}_q[T]\\ d\neq 0}}\left[\frac{D}{d}\right]q^{-(2s+1)\deg d}\big)\,q^{-ms+2s+\frac12\deg D}$$
$$+\big(\sum_{\substack{d\in\mathbb{F}_q[T]\\ d\neq 0}}\left[\frac{D}{d}\right]q^{-2s\deg d}\big)\,\left[\frac{D}{N}\right]\,q^{ms-2s\deg N+2s}\frac{-q^{1-s}-q^s}{q^s+q^{-s}}\Big)$$
$$+\sum_{\substack{\mu\in\mathbb{F}_q[T]\,,\,\mu\neq0\\ \deg(\mu N)+2\le m+\deg D}}r_{\mathfrak{a}_0,\lambda_0}(\mu N-\lambda D)\,q^{-m-\frac12\deg D}\,\cdot$$
$$\Big(\big(\sum_{\substack{c|\mu\\ c\equiv 0\bmod D}}\left[\frac{D}{\mu/c}\right]q^{-2s\deg c}\big)\big((-1)^{\deg(\lambda_0N\mu)}q^{ms-2s\deg N+2s}\frac{-q^{1-s}-q^s}{q^s+q^{-s}}$$
$$+q^{-ms-2s\deg D+2s\deg\mu+4s}\frac{q^{-s}+q^{1+s}}{q^s+q^{-s}}\big)$$
$$+\big(\sum_{c|\mu}\left[\frac{D}{c}\right]q^{-2s\deg c}\big)\left[\frac{D}{N}\right]\big(q^{ms-2s\deg N+2s}\frac{-q^{1-s}-q^s}{q^s+q^{-s}}$$
$$+(-1)^{\deg(\lambda_0N\mu)}q^{-ms-2s\deg D+2s\deg\mu+4s}\frac{q^{-s}+q^{1+s}}{q^s+q^{-s}}\big)\Big),$$

if $m\equiv\deg\lambda_0 \bmod 2$*, and*

$$\widetilde{\Phi_s^{(e)}}^*(\pi_\infty^{\deg\lambda+2},\lambda)=\sum_{\substack{\mu\in\mathbb{F}_q[T]\,,\,\mu\neq0\\ \deg(\mu N)=\deg(\lambda D)}}r_{\mathfrak{a}_0,\lambda_0}(\mu N-\lambda D)\,q^{-\deg\lambda-1-\frac12\deg D}\,\cdot$$
$$\Big(-\big(\sum_{\substack{c|\mu\\ c\equiv0\bmod D}}\left[\frac{D}{\mu/c}\right]q^{-2s\deg c}\big)+\left[\frac{D}{N}\right]\big(\sum_{c|\mu}\left[\frac{D}{c}\right]q^{-2s\deg c}\big)\Big)$$
$$\cdot\,q^{s\deg\lambda-2s\deg N+3s}$$

if $\deg\lambda\not\equiv\deg\lambda_0 \bmod 2$.

The proof of the following functional equation is completely analogous to the proof in the first case. Parts b) and c) in the proof of Theorem 2.7.3 are the same, part a) has to be replaced by the functional equation for $\deg D$ even (cf. [Ar])

$$L_D(-2s+1) = \frac{1+q^{1-2s}}{1+q^{2s}}\, q^{\deg D(2s-\frac{1}{2})} L_D(2s). \tag{2.7.7}$$

We get

Theorem 2.7.6 *Let* $\deg D$ *be even, then*

$$\begin{aligned} &q^{(\deg N+\deg D-3)s}\,(\Phi_s^{(e)*}(\pi_\infty^m,\lambda) - \widetilde{\Phi_s^{(e)}}^*(\pi_\infty^{m+1},\lambda)) = \\ -\left[\frac{D}{N}\right] &q^{(\deg N+\deg D-3)(-s)}\,(\Phi_{-s}^{(e)*}(\pi_\infty^m,\lambda) - \widetilde{\Phi_{-s}^{(e)}}^*(\pi_\infty^{m+1},\lambda)), \end{aligned}$$

if $m \equiv \deg \lambda_0 \bmod 2$*, and therefore Lemma 2.7.4 implies that*

$$Z(s) := q^{(\deg N+\deg D-3)s} \frac{1}{1+q^{-s-1}} L^{(N,D)}(2s+1) L(f,\mathcal{A},s)$$

satisfies the functional equation

$$Z(s) = -\left[\frac{D}{N}\right] Z(-s).$$

2.8 Derivatives of L-Series

The functional equations in Theorem 2.7.3 and Theorem 2.7.6 show that the L-series have a zero at $s=0$, if $\left[\frac{D}{N}\right]=1$. From now on we assume that $\left[\frac{D}{N}\right]=1$, and we want to compute the derivatives of the L-series at $s=0$.

2.8.1 $\deg D$ is odd

The first calculations are straightforward, we will only sketch this procedure. We start with the representation of $L^{(N,D)}(2s+1)L(f,\mathcal{A},s)$ in Lemma 2.7.1, then we evaluate the derivatives $\frac{\partial}{\partial s}F_s^{(o)}\mid_{s=0}$ and $\frac{\partial}{\partial s}\widetilde{F_s^{(o)}}\mid_{s=0}$ from Proposition 2.7.2 by ordinary calculus. To simplify the formulas we introduce

$$t(\mu,D) := \begin{cases} 1 & \text{, if } \mu \equiv 0 \bmod D \\ 0 & \text{, if } \mu \not\equiv 0 \bmod D \end{cases}, \tag{2.8.1}$$

and we consider the function $L_D(s)$ defined in equation (2.7.3). It is known that

$$h_L := \#Cl(O_L) = L_D(0).$$

In addition we use equations (2.7.4), (2.7.5) and (2.7.6).

Then we apply the holomorphic projection formula of Proposition 2.4.1 and evaluate

$$\lim_{\sigma\to 0}\sum_{m=\deg\lambda+2}^{\infty} q^{-m\sigma}\Big(\frac{\partial}{\partial s}F_s^{(o)*}(\pi_\infty^m,\lambda)\mid_{s=0} - \frac{\partial}{\partial s}\widetilde{F_s^{(o)}}^{*}(\pi_\infty^m,\lambda)\mid_{s=0}\Big).$$

In Proposition 2.7.2 there is a summation over $\mu\in\mathbb{F}_q[T]$ with $\deg(\mu N)+2\le m+\deg D$. We divide this summation into two parts. The first sum is over those μ with $\deg(\mu N)\le\deg(\lambda D)$ and the second sum is over those μ with $\deg(\mu N)>\deg(\lambda D)$. This is done in view of the following lemma.

Lemma 2.8.1 *Let $\mu\in\mathbb{F}_q[T]$, $\mu\neq 0$ with $r_{\mathfrak{a}_0,\lambda_0}(\mu N-\lambda D)\neq 0$, then*
a)

$$\frac{1-\delta_{\lambda_0 N\mu}}{2}(t(\mu,D)-1)\Big(\sum_{c\mid\mu}\left[\frac{D}{c}\right]\Big)=0$$

and
b)

$$\delta_{\lambda_0 N\mu}=1 \textit{ if } \deg(\mu N)>\deg(\lambda D).$$

Proof. The proof is an immediate consequence of $\left[\frac{D}{\mu}\right]=\left[\frac{D}{N}\right]\delta_{\lambda_0 N\mu}$, which was shown in the proof of Theorem 2.7.3, part b). □

Now at the end of our calculations we have to apply the trace corresponding to $\Gamma_0^{(1)}(N)\subset\Gamma_0(N)$ to get a cusp form of level N (and not just a cusp form for the subgroup $\Gamma_0^{(1)}(N)$). We recall that

$$\sum_{\epsilon\in\mathbb{F}_q^*} r_{\mathfrak{a}_0,\lambda_0}(\epsilon\mu)=(q-1)r_{\mathcal{A}}((\mu)).$$

A heuristic consideration, based on the holomorphic projection formula of Proposition 2.4.1 and on our calculations, would then give:

$$\frac{\partial}{\partial s}(L^{(N,D)}(2s+1)L(f,\mathcal{A},s))\mid_{s=0}=\int_{\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*} f\cdot\overline{\Psi_{\mathcal{A}}},$$

where $\Psi_{\mathcal{A}}$ is an automorphic cusp form of Drinfeld type of level N with

$$\Psi^*_{\mathcal{A}}(\pi_\infty^{\deg\lambda+2},\lambda) = \frac{\ln q}{2}\, q^{-(\deg D+1)/2}\, q^{-\deg\lambda}$$

$$\cdot\Bigg\{(q-1)\, r_{\mathcal{A}}((\lambda))\, h_L\left(\deg N - \deg(\lambda D) - \frac{q+1}{2(q-1)} - \frac{2}{\ln q}\,\frac{L'_D(0)}{L_D(0)}\right)$$

$$+ \sum_{\substack{\mu\neq 0\\ \deg(\mu N)\leq\deg(\lambda D)}} r_{\mathcal{A}}((\mu N-\lambda D))\Bigg((\sum_{c|\mu}\left[\frac{D}{c}\right])\,(t(\mu,D)+1)\,\frac{1+\delta_{(\mu N-\lambda D)\mu N}}{2}\,\cdot$$

$$\cdot\,(\deg(\frac{\mu N}{\lambda D}) - \frac{q+1}{2(q-1)}) - (1-\delta_{(\mu N-\lambda D)\mu N})\,(\sum_{c|\mu}\left[\frac{D}{c}\right])\,\deg\mu$$

$$+\,(1-\delta_{(\mu N-\lambda D)\mu N})\,(t(\mu,D)+1)\,(\sum_{c|\mu}\left[\frac{D}{c}\right]\deg c)\Bigg)$$

$$-\,\frac{q+1}{2(q-1)}\lim_{\sigma\to 0}\sum_{\deg(\mu N)>\deg(\lambda D)} r_{\mathcal{A}}((\mu N-\lambda D))\,(\sum_{c|\mu}\left[\frac{D}{c}\right])\,(t(\mu,D)+1)\,\cdot$$

$$\cdot\, q^{(-\sigma-1)\deg(\frac{\mu N}{\lambda D})}\,\frac{(q-1)^2(q+q^{-\sigma})}{(q+1)(q^{\sigma+1}-1)^2}\, q^{(-\sigma)\deg\lambda}\Bigg\}, \tag{2.8.2}$$

provided the limit exists. But unfortunately this is not the case.
In order to get the final result, we proceed as follows:
1) We evaluate the pole of the limit in (2.8.2).
2) We find a function h on $\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*$, whose holomorphic projection formula gives the same pole part as in (2.8.2) and which is perpendicular to f under the Petersson product.
3) We replace $\frac{\partial}{\partial s}F_s^{(o)}\mid_{s=0}$ by $\frac{\partial}{\partial s}F_s^{(o)}\mid_{s=0} -h$ in the derivative of the equation in Lemma 2.7.1 and in our calculations.
We start with 1): From section 3.5.1 we get the following result (independently of these calculations):
Let $C_1 := 2(q-1)^2/[GL_2(\mathbb{F}_q[T]):\Gamma_0(N)]$, then the limit

$$\lim_{\sigma\to 0}\Bigg(\sum_{\deg(\mu N)>\deg(\lambda D)} r_{\mathcal{A}}((\mu N-\lambda D))\,(\sum_{c|\mu}\left[\frac{D}{c}\right])\,(t(\mu,D)+1)$$

$$\cdot\, q^{(-\sigma-1)\deg(\frac{\mu N}{\lambda D})} - C_1\, h_L\,(\sum_{a|\lambda} q^{\deg a})\frac{1}{1-q^{-\sigma}}\Bigg)$$

exists. But for this we have to adjust our assumptions. From now on N has to be square free with $\left[\frac{D}{P}\right]=1$ for each prime divisor P of N and we only consider those λ with $\gcd(\lambda,N)=1$.

We use this to calculate

$$\lim_{\sigma\to 0}\Bigg(\sum_{\deg(\mu N)>\deg(\lambda D)} r_{\mathcal{A}}((\mu N-\lambda D))\,(\sum_{c|\mu}\left[\frac{D}{c}\right])\,(t(\mu,D)+1)\,q^{(-\sigma-1)\deg(\frac{\mu N}{\lambda D})}$$

$$\cdot\frac{(q-1)^2(q+q^{-\sigma})}{(q+1)(q^{\sigma+1}-1)^2}\,q^{(-\sigma)\deg\lambda}-\;C_1\,h_L\,(\sum_{a|\lambda}q^{\deg a})\frac{q^{(-\sigma)(\deg\lambda+2)}}{1-q^{-\sigma}}\Bigg)$$

$$=\lim_{\sigma\to 0}\Bigg(\sum_{\deg(\mu N)>\deg(\lambda D)} r_{\mathcal{A}}((\mu N-\lambda D))\,(\sum_{c|\mu}\left[\frac{D}{c}\right])\,(t(\mu,D)+1)\,q^{(-\sigma-1)\deg(\frac{\mu N}{\lambda D})}$$

$$-\,C_1\,h_L\,(\sum_{a|\lambda}q^{\deg a})\frac{1}{1-q^{-\sigma}}\Bigg)+(\sum_{a|\lambda}q^{\deg a})\cdot C_2, \tag{2.8.3}$$

where C_2 is a certain constant.
2) To find the function h we introduce for $s>1$:

$$g_{0,s}(\begin{pmatrix}\pi_\infty^m & u\\ 0 & 1\end{pmatrix}):=-g_{0,s}(\begin{pmatrix}\pi_\infty^m & u\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ \pi_\infty & 0\end{pmatrix}):=q^{-ms}$$

and the Eisenstein series

$$G_s(X):=\sum_M g_{0,s}(M\cdot X),$$

where the sum is taken over $M\in\begin{pmatrix}1 & \mathbb{F}_q[T]\\ 0 & 1\end{pmatrix}\backslash SL_2(\mathbb{F}_q[T])$. Then G_s is a function on $GL_2(K_\infty)/\Gamma_\infty K_\infty^*$, which is invariant under $SL_2(\mathbb{F}_q[T])$ and which satisfies $\widetilde{G_s}=-G_s$. In addition G_s is perpendicular to cusp forms. This can be shown analogously to calculations in the proof of [Rü2], Proposition 14 (in fact G_s can be seen as a Poincaré series for $\mu=0$).
We evaluate the Fourier coefficients of G_s in a straightforward way (cf. proof of [Rü2], Proposition 8) and get for $\deg\lambda+2\le m$, $\lambda\neq 0$:

$$G_s^*(\pi_\infty^m,\lambda)=(\sum_{a|\lambda}q^{-(2s-1)\deg a})$$

$$\cdot\Big((1-q^{2s})\,q^{s(2\deg\lambda-m)-\deg\lambda}+(q^s+1)(q^{1-s}-1)\,q^{s(m-2)+1-m}\Big).$$

The coefficients $G_s^*(\pi_\infty^m,0)$ are not important, because they play no role in the holomorphic projection formula.
Now we define the Eisenstein series G by its Fourier coefficients

$$G^*(\pi_\infty^m,\lambda):=\lim_{s\to 1}\;G_s^*(\pi_\infty^m,\lambda),$$

and H by

$$H^*(\pi_\infty^m,\lambda):=\lim_{s\to 1}\;\frac{\partial}{\partial s}G_s^*(\pi_\infty^m,\lambda).$$

In the next step we evaluate the holomorphic projection formulas for G and H and we get

$$\lim_{\sigma\to 0}\sum_{m=\deg\lambda+2}^{\infty} q^{-m\sigma}(G^*(\pi_\infty^m,\lambda)-\widetilde{G}^*(\pi_\infty^m,\lambda)) = -2(q+1)\, q^{-\deg\lambda-1}(\sum_{a|\lambda} q^{\deg a}), \tag{2.8.4}$$

and

$$\lim_{\sigma\to 0}\sum_{m=\deg\lambda+2}^{\infty} q^{-m\sigma}(H^*(\pi_\infty^m,\lambda)-\widetilde{H}^*(\pi_\infty^m,\lambda)) = -2(q+1)\, q^{-\deg\lambda-1}\ln q$$
$$\cdot\Big(-(\sum_{a|\lambda} q^{\deg a}(\deg\lambda-2\deg a))+(\sum_{a|\lambda} q^{\deg a})\lim_{\sigma\to 0}\frac{q^{(-\sigma)(\deg\lambda+2)}}{1-q^{-\sigma}}$$
$$-\frac{1}{q+1}(\sum_{a|\lambda} q^{\deg a})\Big). \tag{2.8.5}$$

This construction is motivated by the fact that the limit in the last formula already occurred in equation (2.8.3).

3) Comparing (2.8.4) and (2.8.5) with (2.8.2) and (2.8.3) shows how to choose $h = a\cdot G + b\cdot H$ with $a, b\in\mathbb{C}$ to get the final result:

THEOREM 2.8.2 *Let D be irreducible of odd degree, and let N be square free with $\left[\frac{D}{P}\right]=1$ for each prime divisor P of N. For a newform f of level N we get:*

$$\frac{\partial}{\partial s}(L^{(N,D)}(2s+1)L(f,\mathcal{A},s))\,|_{s=0} = \int\limits_{\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*} f\cdot\overline{\Psi_{\mathcal{A}}},$$

where $\Psi_{\mathcal{A}}$ is a cusp form of level N, whose Fourier coefficients for λ with $\gcd(\lambda, N)=1$ *are given by*

$$\Psi^*_{\mathcal{A}}(\pi_\infty^{\deg\lambda+2},\lambda) = \frac{\ln q}{2}\, q^{-(\deg D+1)/2}\, q^{-\deg\lambda}$$

$$\cdot\Bigg\{ (q-1)\, r_{\mathcal{A}}((\lambda))\, h_L \left(\deg N - \deg(\lambda D) - \frac{q+1}{2(q-1)} - \frac{2}{\ln q}\,\frac{L'_D(0)}{L_D(0)} \right)$$

$$+ \sum_{\substack{\mu\neq 0\\ \deg(\mu N)\le\deg(\lambda D)}} r_{\mathcal{A}}((\mu N-\lambda D)) \Bigg((\sum_{c|\mu} \left[\frac{D}{c}\right])\ (t(\mu,D)+1)\ \frac{1+\delta_{(\mu N-\lambda D)\mu N}}{2}$$

$$\cdot\,(\deg(\frac{\mu N}{\lambda D}) - \frac{q+1}{2(q-1)}) - (1-\delta_{(\mu N-\lambda D)\mu N})\ (\sum_{c|\mu}\left[\frac{D}{c}\right])\ \deg\mu$$

$$+ (1-\delta_{(\mu N-\lambda D)\mu N})\ (t(\mu,D)+1)\ (\sum_{c|\mu}\left[\frac{D}{c}\right]\deg c)\Bigg)$$

$$-\frac{q+1}{2(q-1)}\lim_{s\to 0}\Bigg(\sum_{\deg(\mu N)>\deg(\lambda D)} r_{\mathcal{A}}((\mu N-\lambda D))\ (\sum_{c|\mu}\left[\frac{D}{c}\right])\ (t(\mu,D)+1)$$

$$\cdot\, q^{(-s-1)\deg(\frac{\mu N}{\lambda D})} - \frac{C_1\, h_L\ (\sum_{a|\lambda} q^{\deg a})}{1-q^{-s}}\Bigg)$$

$$-\frac{q+1}{2(q-1)}\, C_1\, h_L\ (\sum_{a|\lambda} q^{\deg a}(\deg\lambda - 2\deg a)) + (\sum_{a|\lambda} q^{\deg a})\ C_2\Bigg\}.$$

The following notation is used: $h_L = \#Cl(O_L)$, $L_D(s)$ *is as in (2.7.3),* $t(\mu,D)$ *is as in (2.8.1),*

$$C_1 = 2(q-1)^2/[GL_2(\mathbb{F}_q[T]) : \Gamma_0(N)],$$

and C_2 *is any constant (in particular independent of* λ*).*

2.8.2 $\deg D$ is even

Of course the programme is the same as above. We start with Lemma 2.7.4, and we get the same pole as in (2.8.2) with different constants. Here we use the result (cf. section 3.5.2):
Let $C_1 := (q^2-1)^2/(2\,q\,[GL_2(\mathbb{F}_q[T]) : \Gamma_0(N)])$, then the limit

$$\lim_{\sigma\to 0}\Bigg(\sum_{\deg(\mu N)>\deg(\lambda D)} r_{\mathcal{A}}((\mu N-\lambda D))\ (\sum_{c|\mu}\left[\frac{D}{c}\right])\ (t(\mu,D)+1)$$

$$\cdot\, q^{(-\sigma-1)\deg(\frac{\mu N}{\lambda D})} - C_1\, h_L\ (\sum_{a|\lambda} q^{\deg a})\frac{1}{1-q^{-\sigma}}\Bigg)$$

converges.

Again we take the Eisenstein series G and H to get the final result:

THEOREM 2.8.3 *Let D be irreducible of even degree, and let N be square free with $\left[\frac{D}{P}\right] = 1$ for each prime divisor P of N. For a newform f of level N we get:*

$$\frac{\partial}{\partial s}\left(\frac{1}{1+q^{-s-1}} L^{(N,D)}(2s+1) L(f,\mathcal{A},s)\right)\Big|_{s=0} = \int_{\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*} f \cdot \overline{\Psi_{\mathcal{A}}},$$

where $\Psi_{\mathcal{A}}$ is a cusp form of level N, whose Fourier coefficients for λ with $\gcd(\lambda, N) = 1$ *are given by*

$$\begin{aligned}
\Psi_{\mathcal{A}}^*(\pi_\infty^{\deg\lambda+2},\lambda) &= \frac{\ln q}{4}\, q^{-\deg D/2}\, q^{-\deg\lambda} \\
\cdot \Bigg\{ & r_{\mathcal{A}}((\lambda))\, h_L\, (q-1)\left(\deg N - \deg(\lambda D) - \frac{2q}{q^2-1} - \frac{2}{\ln q}\,\frac{L_D'(0)}{L_D(0)}\right) \\
+ \sum_{\substack{\mu\neq 0\\ \deg(\mu N)\le\deg(\lambda D)}} & r_{\mathcal{A}}((\mu N-\lambda D))\Bigg(\Big(\sum_{c|\mu}\left[\frac{D}{c}\right]\Big)\,(t(\mu,D)+1)\,\frac{1+\delta_{(\mu N-\lambda D)\mu N}}{2} \\
& \cdot\Big(\deg\big(\frac{\mu N}{\lambda D}\big) - \frac{2q}{q^2-1}\Big) - (1-\delta_{(\mu N-\lambda D)\mu N})\,\Big(\sum_{c|\mu}\left[\frac{D}{c}\right]\Big)\,\deg\mu \\
& + (1-\delta_{(\mu N-\lambda D)\mu N})\,(t(\mu,D)+1)\,\Big(\sum_{c|\mu}\left[\frac{D}{c}\right]\deg c\Big)\Bigg) \\
- \frac{2q}{q^2-1}\lim_{s\to 0} & \Bigg(\sum_{\deg(\mu N)>\deg(\lambda D)} r_{\mathcal{A}}((\mu N-\lambda D))\,\Big(\sum_{c|\mu}\left[\frac{D}{c}\right]\Big)\,(t(\mu,D)+1) \\
& \cdot q^{(-s-1)\deg(\frac{\mu N}{\lambda D})} - C_1\, h_L\,\Big(\sum_{a|\lambda} q^{\deg a}\Big)\frac{1}{1-q^{-s}}\Bigg) \\
& - \frac{2q}{q^2-1}\, C_1\, h_L\,\Big(\sum_{a|\lambda} q^{\deg a}(\deg\lambda - 2\deg a)\Big) + \Big(\sum_{a|\lambda} q^{\deg a}\Big)\, C_2 \Bigg\}.
\end{aligned}$$

The following notation is used: $h_L = \#Cl(O_L)$, $L_D(s)$ is as in (2.7.3), $t(\mu, D)$ is as in (2.8.1),

$$C_1 = (q^2-1)^2/(2\,q\,[GL_2(\mathbb{F}_q[T]):\Gamma_0(N)]),$$

and C_2 is any constant (in particular independent of λ).

3 Heights of Heegner Points

3.1 Heegner Points

Let $K = \mathbb{F}_q(T)$ be the rational function field over $\mathbb{F}_q$ as in the previous chapters. For every $N \in \mathbb{F}_q[T]$ there exists a coarse moduli scheme $Y_0(N)$ over $\mathbb{F}_q[T]$ parametrizing isomorphism classes of pairs (ϕ, ϕ') of Drinfeld modules of rank 2 together with a cyclic isogeny $u : \phi \to \phi'$ of degree N (cf. Lecture 2, [AB]). This means that $\ker u \simeq \mathbb{F}_q[T]/(N)$. $Y_0(N)$ can be compactified to a scheme $X_0(N)$ by adjoining a finite number of sections. The points on these sections can be interpreted as generalized Drinfeld modules (cf. Lecture 9 [AB]). The fibres of $X_0(N) \to \operatorname{Spec} \mathbb{F}_q[T]$ are regular outside the divisors of N. We will also need the structure of the fibres over such places, which are known only for N square free. So we will assume this condition on N for the whole chapter. By abuse of notation we often write $X_0(N)$ also for the generic fibre $X_0(N) \otimes K$.
For every $\lambda \in \mathbb{F}_q[T]$ there is a Hecke correspondence on $X_0(N)$. If $x \in X_0(N)$ is represented by two Drinfeld modules ϕ, ϕ' and a cyclic isogeny $u : \phi \to \phi'$, in which case we write $x = (\phi, \phi', u)$, then $T_\lambda(x) = \sum_C (x_C)$, where C runs over all cyclic $\mathbb{F}_q[T]$ submodules of ϕ isomorphic to $\mathbb{F}_q[T]/(\lambda)$ which intersect $\ker u$ trivially. x_C is the point corresponding to $(\phi/C \to \phi'/u(C))$. The Hecke algebra is the subalgebra of End $J_0(N)$, the endomorphisms of the Jacobian of $X_0(N)$, generated by the endomorphisms induced by the Hecke correspondences. For more details see for example [Ge3].
Now let $L = K(\sqrt{D})$ be an imaginary quadratic extension, where D is a polynomial in $\mathbb{F}_q[T]$. In the first part of this section we prove results for general D, later we specialize to D being irreducible. We choose $N \in \mathbb{F}_q[T]$ such that each of its prime divisors is split in L. Then in particular we have $\left[\dfrac{D}{N}\right] = 1$. Suppose that ϕ, ϕ' are two Drinfeld modules of rank 2 for the ring $\mathbb{F}_q[T]$ with complex multiplication by an order $O \subset O_L = \mathbb{F}_q[T][\sqrt{D}]$, i.e. End $\phi =$ End $\phi' = O$ and that $u : \phi \to \phi'$ is a cyclic isogeny of degree N. Then ϕ and ϕ' can be viewed as rank 1 Drinfeld modules over O. As explained in the paper ([Ha]) there is a natural action on rank 1 Drinfeld modules: If $\mathfrak{n} \subset O$ is an invertible ideal and ϕ is a rank 1 Drinfeld module then there is a Drinfeld module $\mathfrak{n} * \phi$ with an isogeny $\phi_{\mathfrak{n}} : \phi \to \mathfrak{n} * \phi$. As was remarked in ([Ha]) just before Proposition 8.3, every isogeny is of this form up to isomorphism. The explicit class field theory ([Ha]) shows that ϕ, ϕ' and the isogeny u can be defined (modulo isomorphisms) over the class field H_O of O, which is unramified outside the conductor $\mathfrak{f} := \{\alpha \in L : \alpha O_L \subset O\}$ of O and where ∞ is totally split. (For the maximal order O_L we will simply write H instead of H_{O_L}.) Therefore the triple (ϕ, ϕ', u) defines an H_O-rational point x on $X_0(N)$. This holds even though $X_0(N)$ is not a fine moduli space. These rational points x are called *Heegner points*. We will primarily consider Heegner points for the maximal order O_L but Heegner points corresponding to non-maximal orders will occur naturally, when we consider the operation of Hecke operators on the Heegner points.

Heegner points corresponding to a maximal order can also be described by the following data: Let K_∞ be the completion of K at ∞ and let C_∞ be the completion of the algebraic closure of K_∞. The category of Drinfeld modules of rank 2 over C_∞ is equivalent to the category of rank 2 lattices in C_∞. If ϕ, ϕ' correspond to lattices Λ, Λ', the isogenies are described by $\{c \in C_\infty^* : c\Lambda \subset \Lambda'\}$. If ϕ has complex multiplication by O_L, the corresponding lattice is isomorphic to an ideal $\mathfrak{a}$ in O_L. Now let $\mathfrak{n}|N$ be an ideal of O_L which contains exactly one prime divisor of every conjugated pair over the primes dividing N. If $\mathfrak{n}|\mathfrak{a}$, the ideal $\mathfrak{n}^{-1}\mathfrak{a}$ is integral and corresponds to another Drinfeld module ϕ' with complex multiplication. The inclusion $\mathfrak{a} \subset \mathfrak{n}^{-1}\mathfrak{a}$ defines a cyclic isogeny of degree N, because $\mathfrak{n}^{-1}\mathfrak{a}/\mathfrak{a} \simeq O_L/\mathfrak{n} \simeq \mathbb{F}_q[T]/(N)$.
The data describing the Heegner point x is the ideal class of $\mathfrak{a}$ and the ideal $\mathfrak{n}$. We get the following analytic realization of the Heegner point x.
Let $\Omega = C_\infty - K_\infty$ be the Drinfeld upper half plane. Then $X_0(N)$ is analytically given by the quotient $\Gamma_0(N) \setminus \Omega$ compactified by adjoining finitely many cusps. Let $z \in \Omega$ with

$$z = \frac{B + \sqrt{D}}{2A}, \ N|A, \ B^2 \equiv D \bmod A.$$

Then the lattice $\langle z, 1\rangle$ is isomorphic to the ideal $\mathfrak{a} = A\mathbb{F}_q[T] + (B+\sqrt{D})\mathbb{F}_q[T]$, which defines together with the ideal $\mathfrak{n} = N\mathbb{F}_q[T] + (\beta + \sqrt{D})\mathbb{F}_q[T]$ with $\beta \equiv B \bmod N$ a Heegner point.
Now we consider the global Néron-Tate height pairing on the H-rational points of the Jacobian $J_0(N)$ of $X_0(N)$. There is an embedding of $J_0(N)$ in the projective space $\mathbb{P}^{2^g-1}$ (Kummer embedding), where g is the genus of $X_0(N)$. The naive height on points in the projective space defines a height function h on $J_0(N)(H)$. The Néron-Tate height is the unique function $\hat{h}$, which differs from h by a bounded function and such that the map $\langle\,.\,\rangle : J_0(N) \times J_0(N) \to \mathbb{R}$ defined by $\langle D, E\rangle = (1/2)(\hat{h}(D+E) - \hat{h}(D) - \hat{h}(E))$ is bilinear. $\langle\,.\,\rangle$ is called the Néron-Tate height pairing (cf. [Gr1]). The pairing depends on H although we omit this in the notation. Whenever we consider height pairings over other fields than H, it will be explicitly mentioned.
The global pairing can be written as a sum $\sum_v \langle\,.\,\rangle_v$ running over all places v of H. For an irreducible polynomial $P \in \mathbb{F}_q[T]$ we write $\langle\,.\,\rangle_P$ for $\sum_{v|P}\langle\,.\,\rangle_v$. For the definition of the local pairing see [Gr1, 2.5]. We recall the relation of the local pairing at non-archimedian primes with the intersection product on a regular model (see [Gr1, 3]). Let v be some place of H and let H_v be the completion with valuation ring O_v. We write q_v for the cardinality of the residue field. Let X/H_v be a curve and $\mathcal{X}/O_v$ be a regular model of X. Suppose D, E are divisors of degree 0 on $X_0(N)$ with disjoint support. Let $\mathcal{F}_i$ be the fibre components of the special fibre of the regular model $\mathcal{X}$ and let $\tilde{D}, \tilde{E}$ be the horizontal divisors to D, E. (The horizontal divisor of a point in the generic fibre is just the Zariski closure of it in $\mathcal{X}$.) Let $(\,.\,)_v$ be the intersection product on $\mathcal{X}$, which is defined in the following way. Let $x \neq y$ be two distinct points on X and $\tilde{x}, \tilde{y}$ their closure in $\mathcal{X}$. For a point z in the special fibre we consider

the stalk $O_{\mathcal{X},z}$ of the structure sheaf in z. Let f_x, f_y be local equations for $\tilde{x}, \tilde{y}$ in z. Then $O_{\mathcal{X},z}/(f_x, f_y)$ is a module of finite length. The intersection number $(\tilde{x}\,.\,\tilde{y})_{v,z}$ is defined to be the length of the module $O_{\mathcal{X},z}/(f_x, f_y)$ and it is zero for almost all z. Let deg z be the degree of the residue field in z over the residue field of v. The intersection number is then $(\tilde{x}\,.\,\tilde{y})_v = \sum_z (\tilde{x}\,.\,\tilde{y})_{v,z} \cdot \deg z$.
Now return to the divisors D, E of degree 0. There exist $\alpha_i \in \mathbb{Q}$ such that

$$\langle D, E\rangle_v = -\ln q_v[(\tilde{D}\,.\,\tilde{E})_v + \sum_i \alpha_i(\mathcal{F}_i\,.\,\tilde{E})_v] \tag{3.1.1}$$

cf. [Gr1, 3]. The elements α_i are unique up to an additive constant, independent of i. In particular if $(\tilde{D}\,.\,\mathcal{F}_i)_v = 0$ for all i, the equation (3.1.1) is satisfied with $\alpha_i = 0$ for all i.
Let $x = (\phi, \phi', u)$ be a Heegner point on $X_0(N)$ for the maximal order O_L. We denote by $\sigma_{\mathcal{A}}$ the element in the Galois group of H/L which corresponds via class field theory to $\mathcal{A} \in Cl(O_L)$. Then $x^{\sigma_{\mathcal{A}}}$ is again a Heegner point for the maximal order. The cusps are given by the cosets $\Gamma_0(N) \setminus \mathbb{P}^1(K)$ and they are K-rational. If $\deg N > 0$ we have at least the two different cusps 0 and ∞. We get the divisors $(x) - (\infty)$ and $(x)^{\sigma_{\mathcal{A}}} - (0)$ of degree 0 on $X_0(N)$.
Let T_λ be a Hecke operator and let g be an automorphic cusp form of Drinfeld type of level N (cf. Definition (2.1.1). If we associate to (T_λ, g) the Fourier coefficient $(T_\lambda g)^*(\pi_\infty^2, 1)$, we get a bilinear map between the Hecke algebra and the space of cusp forms of level N. This map is a non-degenerate pairing ([Ge3, Thm. 3.17]). For $\gcd(\lambda, N) = 1$ we have

$$(T_\lambda g)^*(\pi_\infty^2, 1) = q^{\deg\lambda} g^*(\pi_\infty^{\deg\lambda+2}, \lambda).$$

This is the key to the proof of the following proposition as in [Gr-Za, V 1]

Proposition 3.1.1 *There is an automorphic cusp form $g_{\mathcal{A}}$ of Drinfeld type of level N such that*

$$\langle (x) - (\infty), T_\lambda((x)^{\sigma_{\mathcal{A}}} - (0))\rangle \;= q^{\deg\lambda} g_{\mathcal{A}}^*(\pi_\infty^{\deg\lambda+2}, \lambda)$$

for all $\lambda \in \mathbb{F}_q[T]$ with $\gcd(\lambda, N) = 1$.

We want to compare $g_{\mathcal{A}}$ with the cusp form $\Psi_{\mathcal{A}}$ of the previous section. Therefore we have to evaluate this global height pairing. As we compare the cusp forms only up to old forms, it suffices to calculate the height pairings above only for the Hecke operators with $\gcd(\lambda, N) = 1$. Thus we restrict to this case in the whole section.
The first objective of this section is to express the intersection number of the Heegner divisors on $X_0(N)$ at the finite places, i.e., those places corresponding to irreducible polynomials in $\mathbb{F}_q[T]$, by numbers of homomorphisms between the corresponding Drinfeld modules (Theorem 3.3.4).
For a place v of H we write H_v for the completion at v and O_v for the valuation ring. Let W be the completion of the maximal unramified extension of O_v and

π a uniformizing element of O_v (W, resp.). In order to calculate the local pairings

$$\langle (x)-(\infty)\,,\,T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle_v$$

we first describe the divisor $T_\lambda x^{\sigma_{\mathcal{A}}}$.

3.2 The divisor $T_\lambda x^{\sigma_{\mathcal{A}}}$

Definition 3.2.1 *If $x=(\phi,\phi',u)$ and $y=(\psi,\psi',v)$ are two points on $X_0(N)$, where ϕ,ϕ',ψ,ψ' are Drinfeld modules of rank 2 and $u:\phi\to\phi'$ and $v:\psi\to\psi'$ are cyclic isogenies of degree N, we define*

$$Hom_{\,R}(x,y):=\{(f,f')\in Hom_{\,R}(\phi,\psi)\times Hom_{\,R}(\phi',\psi')\,:\,vf=f'u\}$$

for any ring R where this is well defined, e.g. for R a local ring with algebraically closed residue field.

Consider a finite place v of H. Let H_v be the completion of H at v and let O_v be the valuation ring. Let W again be the completion of the maximal unramified extension of O_v.

Lemma 3.2.2 *Let $x=(\phi,\phi',\phi_{\mathfrak{n}})$ be a Heegner point for the maximal order and $\mathfrak{a}$ an integral ideal in the class $\mathcal{A}$, which corresponds to $\sigma_{\mathcal{A}}\in Gal\,(H/L)$ under the Artin homomorphism. Then*

$$Hom_{\,W}(x^{\sigma_{\mathcal{A}}},x)\simeq End_{\,W}(x)\cdot\mathfrak{a} \tag{3.2.1}$$

and

$$Hom_{\,W/\pi^n}(x^{\sigma_{\mathcal{A}}},x)\simeq End_{\,W/\pi^n}(x)\cdot\mathfrak{a} \tag{3.2.2}$$

for every $n\geq 1$ as left modules over the prevailing ring of endomorphisms.

Proof. It is enough to show the second assertion for all n, because for n sufficiently big the second and the first assertion coincide. We show the assertion for ϕ instead of x. To show it for x one only has to remark that the morphism defined below is compatible with the morphism $\phi_{\mathfrak{n}}$. We again assume that ϕ is defined over W and has leading coefficients in $\overline{\mathbb{F}_q}^*$. For brevity we write $R_n:=\mathrm{End}_{\,W/\pi^n}(\phi)$. Let Λ be the fraction field of W and $I_{\mathfrak{a}}$ be the left ideal in $\Lambda\{\tau\}$ generated by all ϕ_a with $a\in\mathfrak{a}$. This ideal is left principal and generated by some $\phi_{\mathfrak{a}}\in W\{\tau\}$ ([Ha], Prop. 7.5). So $I_{\mathfrak{a}}\cap R_n$ is a left ideal in R_n and we shall show that it is equal to the left ideal $R_n\mathfrak{a}$. The inclusion $R_n\mathfrak{a}\subset I_{\mathfrak{a}}\cap R_n$ is trivial. For the other inclusion we shall show $(I_{\mathfrak{a}}\cap R_n)\mathfrak{a}^{-1}\subset R_n$. Without loss of generality we shall assume that the image under the natural inclusion of $\mathfrak{a}$ is not divisible by π. Then for every $b\in\mathfrak{a}^{-1}$ there is a twisted power series ϕ_b in $W\{\{\tau\}\}$ such that for $a\in\mathfrak{a}$ we get $\phi_a\phi_b=\phi_{ab}$. Now let $f\in I_{\mathfrak{a}}\cap R_n$ and $b\in\mathfrak{a}^{-1}$, then $f=\sum f_i\phi_{a_i}$, for some $f_i\in W\{\tau\}$. So $f\phi_b=\sum f_i\phi_{a_ib}$, which is

a polynomial, because $a_i b \in O$. We also have $\phi_b\phi_a = \phi_a\phi_b$ for every $a \in \mathbb{F}_q[T]$, and therefore $f\phi_b\phi_a \equiv \phi_a f\phi_b \bmod \pi^n$. This implies $f \cdot b \in R_n$.
We know from (Thm. 8.5[Ha]) that there exists a $w \in W^*$ such that

$$\phi_{\mathfrak{a}}\phi_a = w^{-1}\phi_a^{\sigma_{\mathcal{A}}} w\phi_{\mathfrak{a}}$$

holds for every $a \in O$, i.e. $w\phi_{\mathfrak{a}} \in \mathrm{Hom}_{\,W}(\phi, \phi^\sigma)$. Now define an R_n-module homomorphism from $\mathrm{Hom}_{\,W/\pi^n}(\phi^{\sigma_{\mathcal{A}}}, \phi)$ to $R_n \cap I_{\mathfrak{a}}$ by the assignment $f \mapsto f \cdot w\phi_{\mathfrak{a}}$. On the other hand if $g \in W/\pi^n\{\tau\}$ such that $gw\phi_{\mathfrak{a}} =: u \in R_n$, then we have to show that $g \in \mathrm{Hom}_{\,W/\pi^n}(\phi^\sigma, \phi)$. We have

$$g\phi_a^\sigma w\phi_{\mathfrak{a}} = gw\phi_{\mathfrak{a}}\phi_a = u\phi_a = \phi_a u$$

for all $a \in \mathbb{F}_q[T]$, where the last equality holds, because $\mathbb{F}_q[T]$ is central in R_1 and therefore also in R_n for every $n \geq 1$. But $\phi_a u = \phi_a g w\phi_{\mathfrak{a}}$ and so

$$(g\phi_a^\sigma - \phi_a g)w\phi_{\mathfrak{a}} = 0.$$

$w\phi_{\mathfrak{a}}$ cannot be a zero divisor, because the leading coefficient of $\phi_{\mathfrak{a}}$ is 1. This finishes the proof of the lemma. $\square$
From this lemma we get the following result about the multiplicity of x in $T_\lambda x^{\sigma_{\mathcal{A}}}$. The proof is exactly the same as in the characteristic 0 case ([Gr-Za, (4.3)]).

Proposition 3.2.3 *Let $\sigma_{\mathcal{A}} \in Gal\ (H/L)$, let $\mathcal{A}$ be the ideal class corresponding to $\sigma_{\mathcal{A}}$ and let $\lambda \in \mathbb{F}_q[T]$. Then the multiplicity of x in the divisor $T_\lambda x^{\sigma_{\mathcal{A}}}$ is equal to the number $r_{\mathcal{A}}((\lambda))$ of integral ideals in the class of $\mathcal{A}$ with norm (λ).*

The points $y \in T_\lambda x^{\sigma_{\mathcal{A}}}$ are Heegner points for orders $O_y \subset O_L$. Let $\mathfrak{f} = \{\alpha \in L : \alpha O_L \subset O_y\}$ be the conductor of the order O_y. Let P be an irreducible, monic polynomial in $\mathbb{F}_q[T]$ and let $s = s(y, P)$ be the greatest integer with $P^s|\mathfrak{f}$. We call s the level of y at P. If $P \nmid \lambda$, we get $s = 0$, because $\mathfrak{f}|(\lambda)$. If $\lambda = P^t \cdot R$ with $P \nmid R$ and $t > 0$, then

$$T_\lambda x^{\sigma_{\mathcal{A}}} = \sum_{z \in T_R x^{\sigma_{\mathcal{A}}}} T_{P^t} z.$$

The following proposition tells us how often each level occurs in the divisor $T_{P^t} z$. For a proof see [Ti1].

Proposition 3.2.4 *Let $P \in \mathbb{F}_q[T]$ be irreducible and let z be a Heegner point of level 0 at P. Set $d = \deg P$ and $q_P = q^d$. Then the number of points of level s in the divisor $T_{P^t} z$ is equal to*

$$\left.\begin{array}{cl} (t-s+1)(q_P^s - q_P^{s-1}) & \textit{for } t \geq s \geq 1 \\ t+1 & \textit{for } s = 0 \end{array}\right\} \quad \textit{if } P \textit{ is split in } L/K;$$

$$\left.\begin{array}{cl} q_P^s + q_P^{s-1} & \textit{for } t \geq s \geq 1,\ s \equiv t \bmod 2 \\ 1 & \textit{for } s = 0,\ t \equiv 0 \bmod 2 \end{array}\right\} \quad \textit{if } P \textit{ is inert in } L/K;$$

$$q_P^s \quad \textit{for } t \geq s \geq 0 \quad \textit{if } P \textit{ is ramified in } L/K.$$

The next proposition shows where the points with level s are defined. The proof is given by D. Hayes ([Ha, Thm 8.10, Thm. 1.5])

Proposition 3.2.5 *Let P be any irreducible polynomial and let z be a Heegner point for an order O with conductor prime to N. Suppose that z has level 0 at P. Then*

1. *z is defined over H_O, the ring class field of O, which is unramified over H at P. The Galois group of H_O/H is isomorphic to the group of ideals in O_L modulo the principal ideals generated by some element $a \in O$ which is prime to the conductor of O.*

2. *Every $y \in T_{P^t}z$ of level s at P is defined over another class field H_s with $[H_s : H_O] = |(O_L/P^sO_L)^*|/|(\mathbb{F}_q[T]/P^s\mathbb{F}_q[T])^*|$, which is totally ramified at P over H_O.*

3.3 The finite places

For the calculations of height pairings at the finite places we want to make use of the modular interpretation of the points on the modular curve in every fibre including the fibres over the divisors of N. In contrast to the elliptic curve case, we do not know how these fibres look like if N is not square free. This is one reason why we confine ourselves to this case.
The first step is to describe the pairings at a finite place v by intersection products on a regular model of $X_0(N) \otimes K$. When $v \nmid N$ then $X_0(N) \otimes O_v$ is a regular model and when $v|N$ we take a regularization of $X_0(N) \otimes O_v$, which can be done by finitely many blow ups at the singular points. After that we use the modular interpretation to describe the intersection numbers by numbers of homomorphisms.
First we recall the structure of the fibres of $X_0(N)$ at the places over N (see [Ge2]).

Proposition 3.3.1 *For $N \in \mathbb{F}_q[T]$ square free, $N \notin \mathbb{F}_q$, the modular curve $X_0(N)$ over $\mathbb{F}_q[T]$ is regular outside N and outside the supersingular points in the fibres above prime divisors of N. Let P be any prime divisor of N of degree d. Then the special fibre over P consists of two copies of $X_0(N/P)$, which intersect transversally in the supersingular points. One of the components is the image of the map*

$$\begin{array}{rcl} X_0(N/P) \times \mathbb{F}_q[T]/P & \longrightarrow & X_0(N) \times \mathbb{F}_q[T]/P \\ (\phi, \phi', u) \bmod P & \longmapsto & (\phi, \phi'', \tau^d u) \bmod P, \end{array}$$

where τ^d is the Frobenius of $\mathbb{F}_q[T]/P$ regarded as isogeny of degree P. This component is the "local component", the other one is the "reduced component". The cusp 0 lies on the reduced component and ∞ lies on the local component.

Remarks. 1. We need not know what the resolutions of singular points are, because our horizontal divisors always intersect the fibres over N outside the supersingular points and the next proposition will show that no contribution from the fibre components of the regular model will occur.
2. Because $\gcd(D,N) = 1$, the regular model remains regular under base change to the Hilbert class field H/L as well as over the completion of the maximal unramified extension W of some completion O_v for a place v of H.

Proposition 3.3.2 *Let $x = (\phi, \phi', \phi_{\mathfrak{n}})$ be a Heegner point for the maximal order, $\sigma_{\mathcal{A}} \in Gal\ (H/L)$, $\mathcal{A}$ the corresponding ideal class, $\lambda \in \mathbb{F}_q[T]$, v a finite place of H of residue cardinality q_v. Suppose that $r_{\mathcal{A}}((\lambda)) = 0$, then*

$$\langle (x) - (\infty), T_\lambda((x)^{\sigma_{\mathcal{A}}} - (0))\rangle_v = -(x \,.\, T_\lambda x^{\sigma_{\mathcal{A}}})_v \ln q_v.$$

Proof. At first we check that the horizontal extension of one of the divisors in the pairing has zero intersection with both fibre components if $v|N$. It follows that the values α_i in 3.1.1 all vanish. Let $\mathfrak{n}$ be the ideal, such that $\phi_{\mathfrak{n}}$ is the cyclic isogeny defining x. If $v|\mathfrak{n}$, x intersects the fibre in the local component. If $v|\overline{\mathfrak{n}}$ then it intersects in the reduced component. Thus one of the divisors $(x) - (\infty), (x) - (0)$ has zero intersection with both fibre components. But the points in $T_\lambda x^{\sigma_{\mathcal{A}}}$ reduce to the same component as x. This shows that one of the divisors has zero intersection with all fibre components. This is trivially true for the places of good reduction ($v \nmid N$). The result now follows by linearity of the pairing and the fact that x can be represented by a Drinfeld module with good reduction, so it does not intersect with the cusps. $\square$

Proposition 3.3.3 *Let $x = (\phi \to \phi')$, $y = (\psi \to \psi')$ be two $W-$rational sections, i.e. horizontal divisors on $X_0(N)$ over W, which intersect properly and which reduce to regular points outside the cusps in the special fibre. Then*

$$(y \,.\, x)_v = \frac{1}{q-1} \sum_{n \geq 1} \#Isom_{\ W/\pi^n}(y, x).$$

Proof. Let $f : Y \longrightarrow X_0(N)$ be a fine moduli scheme (e.g. a supplementary full level $N'-$ structure with $\gcd(N', N) = 1$ and N' with at least two different prime factors.) Let y_0 be a pre-image of y and x_i the different pre-images of x, i.e. $f_*(y_0) = y$, $f^*(x) = \sum x_i$. Because f is proper, the projection formula [Sha, Lect.6,(7)] implies that

$$(y \,.\, x)_v = (f_* y_0 \,.\, x)_v = (y_0 \,.\, f^* x)_v = \sum_i (y_0 \,.\, x_i)_v.$$

If (ϕ, ϕ', u) is a representative of x, all the x_i are represented by (ϕ, ϕ', u, P, Q), where P, Q generates the N' torsion module. Every such point occurs with multiplicity $\#\mathrm{Aut}(x)/(q-1)$ in $f^*(x)$. The $q-1$ trivial automorphisms all give the same point in $f^*(x)$. Now let (ψ, ψ', v) be a representative of y and (ϕ, ϕ', u) a representative of x. Let y_0 be represented by (ψ, ψ', v, P, Q).

Then an isomorphism $f : (\psi, \psi', v) \to (\phi, \phi', u)$ defines an isomorphism $\hat{f} : (\psi, \psi', v, P, Q) \to (\phi, \phi', u, f(P), f(Q))$ and this is uniquely determined. If x_{i_0} is the class of
$(\phi, \phi', u, f(P), f(Q))$, we have

$$\#\mathrm{Isom}_{W/\pi^n}(y_0, x_i) = \begin{cases} 1 & \text{, if } x_i = x_{i_0} \\ 0 & \text{, otherwise.} \end{cases}$$

Now x_{i_0} occurs in $f^*(x)$ with multiplicity $\#\mathrm{Aut}(x)/(q-1)$ and therefore

$$\begin{aligned} \sum_i \#\mathrm{Isom}_{W/\pi^n}(y_0, x_i) &= \begin{cases} \dfrac{\#\mathrm{Aut}(x)}{q-1} & \text{, if } \#\mathrm{Isom}_{W/\pi^n}(y, x) \neq 0 \\ 0 & \text{, otherwise} \end{cases} \\ &= \frac{1}{q-1} \#\mathrm{Isom}_{W/\pi^n}(y, x). \end{aligned}$$

Therefore we only have to show that

$$(y_0 \,.\, x_i)_v = \sum_{n \geq 1} \#\mathrm{Isom}_{W/\pi^n}(y_0, x_i).$$

Let $Y \hookrightarrow \mathbb{P}^r_W$ be a projective embedding. Let $\mathbb{A}^r_W$ be an affine part, which contains the intersection point s of x_i and y_0. The coordinates with respect to this affine part are denoted by $y_0 = (\eta_1, \ldots, \eta_r)$ and $x_i = (\xi_{i1}, \ldots, \xi_{ir})$. In the local ring $\mathcal{O}_{Y,s}$ we have the local functions $z_j - \eta_j$, $z_j - \xi_{ij}$, respectively. The ideal generated by these functions contains all differences $(\eta_j - \xi_{ij})$ and therefore is the ideal $(\pi)^k$ with $k = \min_j v(\eta_j - \xi_{ij})$. From the definition of the intersection number we get

$$(y_0 \,.\, x_i)_v = \dim_{W/\pi}(\mathcal{O}_{Y,s}/(z_j - \eta_j, z_j - \xi_{ij})) = \dim_{W/\pi}(W/\pi^k W) = k.$$

On the other hand

$$\#\mathrm{Isom}_{W/\pi^n}(y_0, x_i) = \begin{cases} 0, & \eta_j \not\equiv \xi_{ij} \bmod \pi^n \text{ for some } j \\ 1, & \eta_j \equiv \xi_{ij} \bmod \pi^n \text{ for all } j, \end{cases}$$

because Y is a fine moduli scheme. It follows that

$$\sum_{n \geq 1} \#\mathrm{Isom}_{W/\pi^n}(y_0, x_i) = k.$$

□

The degree of an isogeny u between two Drinfeld modules ϕ, ϕ' is by definition the ideal IJ, if $\ker u \simeq \mathbb{F}_q[T]/I \oplus \mathbb{F}_q[T]/J$.

THEOREM 3.3.4 *Let $P \in \mathbb{F}_q[T]$ be irreducible , $v|P$ a place of H with local parameter π and W the completion of the maximal unramified extension of*

O_v. Let $x = (\phi, \phi', u)$ be a Heegner point for the maximal order O_L. Let $\sigma_{\mathcal{A}} \in$ Gal (H/L). For λ, N without common divisor and $r_{\mathcal{A}}((\lambda)) = 0$ we get

$$(x \,.\, T_\lambda x^{\sigma_{\mathcal{A}}})_v = \frac{1}{q-1} \sum_{n \geq 1} \# \mathit{Hom}_{\ W/\pi^n}(x^{\sigma_{\mathcal{A}}}, x)_{\deg \lambda}.$$

The subscript $\deg \lambda$ indicates that only homomorphisms of degree λ are counted. The sum is finite because $\# \mathit{Hom}_{\ W/\pi^n}(x^{\sigma_{\mathcal{A}}}, x)_{\deg \lambda} = 0$ for n sufficiently large, because $\mathit{Hom}_{\ W}(x^{\sigma_{\mathcal{A}}}, x) = \bigcap_{n \geq 1} \mathit{Hom}_{\ W/\pi^n}(x^{\sigma_{\mathcal{A}}}, x)$ and $\# \mathit{Hom}_{\ W}(x^{\sigma_{\mathcal{A}}}, x)_{\deg \lambda} = r_{\mathcal{A}}((\lambda)) = 0$ by assumption.

The rest of this section is used to prove this theorem. First of all we consider the easiest case, namely $P \nmid \lambda$. For the case $P | \lambda$ we need the Eichler-Shimura congruence and a congruence between points of level 0 and points of higher level. After that the formula of the theorem is proved at first for P split, then for P inert and finally for P ramified.
Suppose now that $P \nmid \lambda$. We have

$$\frac{d}{dt} \phi_\lambda(t) = \lambda \not\equiv 0 \bmod P,$$

and so the zeroes of $\phi_\lambda(t)$ are pairwise disjoint mod P. If $u : x^{\sigma_{\mathcal{A}}} \to x$ is an isogeny over W/π^n of degree λ, then u is uniquely determined up to automorphism of x by $\ker u(t) \subset \ker \phi_\lambda(t)$. For a fixed u we have a unique lifting to a submodule of $\ker \phi_\lambda(t)$ over W, i.e., there exists y and an isogeny $x^{\sigma_{\mathcal{A}}} \to y$ of degree λ, such that

$$\begin{tikzcd} x^{\sigma_{\mathcal{A}}} \arrow[r] \arrow[dr] & x \arrow[d, "\simeq"] \\ & y \end{tikzcd}$$

commutes. Therefore

$$\# \mathrm{Hom}_{\ W/\pi^n}(x^{\sigma_{\mathcal{A}}}, x)_{\deg \lambda} = \sum_{y \in T_\lambda x^{\sigma_{\mathcal{A}}}} \# \mathrm{Isom}_{\ W/\pi^n}(y, x).$$

By summation over n together with Proposition 3.3.3 the assertion of the theorem follows.
Now let $\lambda = P^t R$ with $t \geq 1$ and $P \nmid R$. The elements $y \in T_\lambda x^{\sigma_{\mathcal{A}}}$ are Heegner points of different levels and are defined over some extension H_s/H which is ramified at P. The analogue of the Eichler-Shimura congruence holds, i.e.

$$T_P \equiv F^* + F \bmod P,$$

where F is the Frobenius correspondence and F^* is its dual correspondence on $X_0(N)$. This can be shown in the following way. Let $u : \phi \to \phi'$ be a cyclic isogeny of degree P given as a $\tau-$polynomial. Then there exists a dual isogeny $v : \phi' \to \phi$, such that $\phi_P = v \cdot u$. Then either $v \equiv \tau^d \bmod P$ and consequently $\phi' \equiv \phi^F \bmod P$ or $u \equiv \tau^d \bmod P$ and consequently $\phi \equiv {\phi'}^F \bmod P$. This proves the Eichler-Shimura congruence.
By a simple induction we get

$$T_{P^t} \equiv F^{*t} + F^{*(t-1)}F + \cdots + F^* F^{t-1} + F^t \bmod P.$$

Lemma 3.3.5 *Let $y \in T_{P^t}z$. If s is the level of y, then y is defined over a class field H_s (cf. Proposition 3.2.5). Let $H_{s,v}$ be the completion at $v|P$ and W_s the valuation ring of the maximal unramified extension of $H_{s,v}$ with local parameter π_s. y is defined over W_s. If z has level 0 at P then it is defined over an unramified extension H_O/H (cf. Proposition 3.2.5), thus also over W. There exists a y_0 of level 0 defined over W, with $y \equiv y_0 \bmod \pi_s$.*

Proof. For P ramified or split in L/K let $\sigma_\mathfrak{p} \in \mathrm{Gal}\ (H_O/L)$ be the Frobenius of $\mathfrak{p}|P$ over L. For P inert let $\sigma_\mathfrak{p} = \sigma_P \in \mathrm{Gal}\ (H_O/K)$ be the Frobenius of P over K. Then $\sigma_\mathfrak{p}$ operates on ϕ. The definition of Frobenius yields $\phi^{\sigma_\mathfrak{p}} \equiv \phi^F \bmod \pi_s$ and $\phi^{\sigma_\mathfrak{p}^{-1}} \equiv \phi' \bmod \pi_s$ with ${\phi'}^F \equiv \phi \bmod \pi_s$.
Now let $y \in T_{P^t}z$. From the Eichler-Shimura congruence we get the existence of y' and i with $0 \leq i \leq t$, such that ${y'}^{F^i} \equiv z \bmod \pi_s$ and $y \equiv {y'}^{F^{t-i}} \bmod \pi_s$ therefore $y \equiv {y'}^{\sigma_\mathfrak{p}^{t-i}} \equiv z^{\sigma_\mathfrak{p}^{t-2i}} \bmod \pi_s$, so we can take $y_0 = z^{\sigma_\mathfrak{p}^{t-2i}}$. □
Remark. In the ramified and in the inert case we have $y_0 = z^{\sigma_\mathfrak{p}}$ for t odd or $y_0 = z$ for t even. This holds because , if P is inert in L/K it is a principal ideal generated by an element which does not divide the conductor of O. This implies, that $\sigma_\mathfrak{p}^2 = 1$ for $\sigma_\mathfrak{p} \in \mathrm{Gal}(H_O/K)$. If P is ramified in L/K we have that $\mathfrak{p}^2 = (P)$ is a principal ideal prime to the conductor. Therefore $\sigma_\mathfrak{p}^2 = 1$ also in this case.

Lemma 3.3.6 *Let $y \in T_{P^t}z$ with level $s \geq 1$ and y_0, π_s as in Lemma 3.3.5. Then*

$$y \not\equiv y_0 \bmod \pi_s^2.$$

Proof. The assertion is even true for the associated formal modules [Gr2, Prop. 5.3]. The formal module associated to a Drinfeld module is an extension of ϕ to a homomorphism $\phi^{(P)} : \mathbb{F}_q[T]_P \to W/\pi\{\{\tau\}\}$ where $\mathbb{F}_q[T]_P$ is the completion at P and $W/\pi\{\{\tau\}\}$ is the twisted power series ring. □
Now we can go on with the proof of Theorem 3.3.4. We treat the cases P split, P inert and P ramified separately.
Suppose at first that P is split. Then ϕ has ordinary reduction and therefore $\mathrm{Hom}\ _W(x^{\sigma_\mathcal{A}}, x) = \mathrm{Hom}\ _{W/\pi^n}(x^{\sigma_\mathcal{A}}, x)$ for all $n \geq 1$, because $\mathrm{Hom}\ _{W/\pi^n}(x, x) = \mathrm{Hom}\ _W(x, x) = O_L$ and $\mathrm{Hom}\ _{W/\pi^n}(x^{\sigma_\mathcal{A}}, x) = \mathfrak{a}$. By assumption we have $r_\mathcal{A}((\lambda)) = 0$, so $\mathrm{Hom}\ _{W/\pi^n}(x^{\sigma_\mathcal{A}}, x)_{\deg \lambda} = \emptyset$.

On the other hand let $y \in T_\lambda x^{\sigma_\mathcal{A}}$ with $x \equiv y \bmod \pi_s$. Then Lemma 3.3.5 gives $y \equiv y_0 \bmod \pi_s$ for a $y_0 \in T_\lambda x^{\sigma_\mathcal{A}}$ of level 0. Because $x \simeq y_0$ over W/π and therefore also over W, we get $x \simeq y_0$ over W, and so $x \in T_\lambda x^{\sigma_\mathcal{A}}$, which contradicts the assumption $r_\mathcal{A}((\lambda)) = 0$. Therefore we get

$$(x\;.\;T_\lambda x^{\sigma_\mathcal{A}})_v = 0.$$

If P is inert then let $y_0 = z^{\sigma_\mathfrak{p}}$ for t odd and $y_0 = z$ for t even, respectively, as in the remark following Lemma 3.3.5. Then Lemma 3.3.5 and Lemma 3.3.6 yield $y \equiv y_0 \bmod \pi_s$, $y \not\equiv y_0 \bmod \pi_s^2$.

Each y of level $s = s(y)$ is defined over W_s, which is ramified of degree $e_s = q_P^{(s-1)}(q_P + 1)$ (cf. Proposition 3.2.5(2)).

We distinguish between the intersection pairing over W and W_s. For the latter we write $(\,.\,)_{v,s}$. From the definition of the intersection multiplicity we get

$$(T_{P^t} z\;.\;x)_v = \frac{1}{e_s}(T_{P^t} z\;.\;x)_{v,s}$$

and further

$$\begin{aligned}
(T_{P^t} z\;.\;x)_v &= \frac{1}{e_s}\sum_{y \in T_{P^t} z} (y\;.\;x)_{v,s} = \qquad\qquad (3.3.1)\\
&= \sum_{\substack{s=0\\ s\equiv t \bmod 2}}^{t} \sum_{\substack{y\in T_{P^t} z\\ s(y)=s}} \frac{1}{(q-1)e_s} \sum_{n\geq 1} \#\mathrm{Isom}_{\,W_s/\pi_s^n}(y,x)\\
&= \begin{cases} \dfrac{1}{q-1}\displaystyle\sum_{n\geq 1}\#\mathrm{Isom}_{\,W/\pi^n}(z,x) \\ \quad + \displaystyle\sum_{\substack{s=1\\ s\equiv t \bmod 2}}^{t} \dfrac{\#\{y\in T_{P^t}z : s(y)=s\}}{(q-1)\cdot e_s}\#\mathrm{Isom}_{\,W/\pi}(y_0,x) & \text{, if } t \text{ is even}\\ \displaystyle\sum_{\substack{s=1\\ s\equiv t \bmod 2}}^{t} \dfrac{\#\{y\in T_{P^t}z : s(y)=s\}}{(q-1)\cdot e_s}\#\mathrm{Isom}_{\,W/\pi}(y_0,x) & \text{, if } t \text{ is odd}\end{cases}\\
&= \begin{cases} \dfrac{1}{q-1}\left(\displaystyle\sum_{n\geq 1}\#\mathrm{Isom}_{\,W/\pi^n}(z,x) + \cdot\dfrac{t}{2}\#\mathrm{Isom}_{\,W/\pi}(z,x)\right) & \text{, if } t \text{ is even}\\ \dfrac{1}{q-1}\cdot\dfrac{t+1}{2}\#\mathrm{Hom}_{\,W/\pi}(z,x)_{\deg P} & \text{, if } t \text{ is odd.}\end{cases}
\end{aligned}$$

Summing over all $z \in T_R x$ we obtain $(P \nmid R)$

$$(T_\lambda x^{\sigma_\mathcal{A}} \,.\, x)_v = \sum_{z \in T_R x^{\sigma_\mathcal{A}}} (T_{P^t} z \,.\, x)_v =$$

$$= \begin{cases} \dfrac{1}{q-1} \displaystyle\sum_{n \geq 1} \#\mathrm{Hom}\,_{W/\pi^n}(x^{\sigma_\mathcal{A}}, x)_{\deg R} & \\ + \dfrac{t}{2(q-1)} \#\mathrm{Hom}\,_{W/\pi}(x^{\sigma_\mathcal{A}}, x)_{\deg R} & \text{, if } t \text{ is even} \\ & \\ \dfrac{t+1}{2(q-1)} \#\mathrm{Hom}\,_{W/\pi}(x^{\sigma_\mathcal{A}}, x)_{\deg RP} & \text{, if } t \text{ is odd.} \end{cases}$$

LEMMA 3.3.7 *a) If t is even:*

$$\begin{aligned} \#Hom\,_{W/\pi^n}(x^{\sigma_\mathcal{A}}, x)_{\deg R} &= \#Hom\,_{W/\pi^{n+t/2}}(x^{\sigma_\mathcal{A}}, x)_{\deg \lambda} && \text{for } n \geq 1 \\ \#Hom\,_{W/\pi}(x^{\sigma_\mathcal{A}}, x)_{\deg R} &= \#Hom\,_{W/\pi^n}(x^{\sigma_\mathcal{A}}, x)_{\deg \lambda} && \text{for } n \leq t/2. \end{aligned}$$

b) If t is odd:

$$\begin{aligned} \#Hom\,_{W/\pi}(x^{\sigma_\mathcal{A}}, x)_{\deg RP} &= \#Hom\,_{W/\pi^{(t+1)/2}}(x^{\sigma_\mathcal{A}}, x)_{\deg \lambda} \\ \#Hom\,_{W/\pi}(x^{\sigma_\mathcal{A}}, x)_{\deg RP} &= \#Hom\,_{W/\pi^n}(x^{\sigma_\mathcal{A}}, x)_{\deg \lambda} \end{aligned}$$

for $n \leq (t+1)/2$.

PROOF. a) We have that $\phi_{P^{t/2}} \equiv \tau^{dt} \bmod \pi^{t/2}$ is an isogeny of degree P^t. If $u \in \mathrm{Hom}\,_{W/\pi^n}(x^{\sigma_\mathcal{A}}, x)_{\deg R}$, then $\phi_{P^{t/2}}(u\phi_x^{\sigma_\mathcal{A}} - \phi_x u) \equiv 0 \bmod \pi^{n+t/2}$ and therefore $\phi_{P^{t/2}} u \phi_x^{\sigma_\mathcal{A}} \equiv \phi_x \phi_{P^{t/2}} u \bmod \pi^{n+t/2}$, i.e.

$$\phi_{P^{t/2}} u \in \mathrm{Hom}\,_{W/\pi^{n+t/2}}(x^{\sigma_\mathcal{A}}, x)_{\deg \lambda}.$$

Now $P \nmid R$, thus $\pi \nmid u_0$ for $u \neq 0$, and $\phi_{P^{t/2}} u \equiv 0 \bmod \pi^{n+t/2}$ implies $u \equiv 0 \bmod \pi^n$, i.e. the map is injective.
Now let $u \in \mathrm{Hom}\,_{W/\pi^{n+t/2}}(x^{\sigma_\mathcal{A}}, x)_{\deg \lambda}$, then there is a splitting $u = u_1 \cdot u_2$ with an isogeny u_1 of degree P^t and an isogeny u_2 of degree R. We have $u_1 \equiv \tau^{dt} \bmod \pi^{t/2}$, therefore the map is also surjective. This also shows that

$$\mathrm{Hom}\,_{W/\pi}(x^{\sigma_\mathcal{A}}, x)_{\deg R} \longrightarrow \mathrm{Hom}\,_{W/\pi^n}(x^{\sigma_\mathcal{A}}, x)_{\deg \lambda}$$

is bijective for $n \leq t/2$, which implies a).

b) Analogous to a) with

$$\begin{array}{rcl} \mathrm{Hom}\,_{W/\pi}(x^{\sigma_\mathcal{A}}, x)_{\deg RP} & \longrightarrow & \mathrm{Hom}\,_{W/\pi^{(t+1)/2}}(x^{\sigma_\mathcal{A}}, x)_{\deg \lambda} \\ f & \longmapsto & \phi_{P^{(t-1)/2}} \cdot f. \end{array}$$

This completes the proof of Theorem 3.3.4, if P is inert.

Now let P be ramified. $\mathfrak{p}$ a prime in L over P. Then $y_0 = z^{\sigma_{\mathfrak{p}}}$ for t odd and $y_0 = z$ for t even, respectively, where $\sigma_{\mathfrak{p}}$ is now the Frobenius over L. Lemma 3.3.5 and Lemma 3.3.6 tell us again that $y \equiv y_0 \bmod \pi_s$, $y \not\equiv y_0 \bmod \pi_s^2$.

$$(T_\lambda x^{\sigma_{\mathcal{A}}} \,.\, x)_v = \tag{3.3.2}$$
$$= \begin{cases} \dfrac{1}{q-1} \displaystyle\sum_{n\geq 1} \#\mathrm{Isom}_{\,W/\pi^n}(z,x) + \dfrac{t}{q-1}\#\mathrm{Isom}_{\,W/\pi}(z,x) \\ \dfrac{1}{q-1} \displaystyle\sum_{n\geq 1} \#\mathrm{Isom}_{\,W/\pi^n}(z^{\sigma_{\mathfrak{p}}},x) + \dfrac{t}{q-1}\#\mathrm{Hom}_{\,W/\pi}(z^{\sigma_{\mathfrak{p}}},x)_{\deg P} \end{cases}$$

for t even or odd, resp. Summing over all $z \in T_R x^{\sigma_{\mathcal{A}}}$ yields

$$(T_\lambda x^{\sigma_{\mathcal{A}}} \,.\, x)_v = \sum_{z\in T_R x^{\sigma_{\mathcal{A}}}} (T_{P^t} z \,.\, x)_v =$$
$$= \begin{cases} \dfrac{1}{q-1} \displaystyle\sum_{n\geq 1} \#\mathrm{Hom}_{\,W/\pi^n}(x^{\sigma_{\mathcal{A}}},x)_{\deg R} \\ + \dfrac{t}{q-1}\#\mathrm{Hom}_{\,W/\pi}(x^{\sigma_{\mathcal{A}}},x)_{\deg R} & \text{, if } t \text{ is even} \\ \\ \dfrac{1}{q-1} \displaystyle\sum_{n\geq 1} \#\mathrm{Hom}_{\,W/\pi^n}(x^{\sigma_{\mathcal{A}}\sigma_{\mathfrak{p}}},x)_{\deg R} \\ + \dfrac{t}{q-1}\#\mathrm{Hom}_{\,W/\pi}(x^{\sigma_{\mathcal{A}}\sigma_{\mathfrak{p}}},x)_{\deg R} & \text{, if } t \text{ is odd.} \end{cases}$$

LEMMA 3.3.8 *a) For t even*

$$\begin{aligned} \#Hom_{\,W/\pi^n}(x^{\sigma_{\mathcal{A}}},x)_{\deg R} &= \#Hom_{\,W/\pi^{n+t}}(x^{\sigma_{\mathcal{A}}},x)_{\deg\lambda} && \textit{for } n\geq 1 \\ \#Hom_{\,W/\pi}(x^{\sigma_{\mathcal{A}}},x)_{\deg R} &= \#Hom_{\,W/\pi^n}(x^{\sigma_{\mathcal{A}}},x)_{\deg\lambda} && \textit{for } n\leq t. \end{aligned}$$

b) For t odd

$$\begin{aligned} \#Hom_{\,W/\pi^n}(x^{\sigma_{\mathcal{A}}\sigma_{\mathfrak{p}}},x)_{\deg R} &= \#Hom_{\,W/\pi^{(t+n)}}(x^{\sigma_{\mathcal{A}}},x)_{\deg\lambda} \\ \#Hom_{\,W/\pi}(x^{\sigma_{\mathcal{A}}\sigma_{\mathfrak{p}}},x)_{\deg R} &= \#Hom_{\,W/\pi^n}(x^{\sigma_{\mathcal{A}}},x)_{\deg\lambda} \end{aligned}$$

where the first equality holds for all $n \geq 1$ and the second one for all $n \leq t+1$.

PROOF. The proof is analogous to the proof of the previous lemma. Using the facts that $f \longmapsto \phi_{P^{t/2}} f$ for a) and $f \longmapsto \phi_{\mathfrak{p}^t} f$ for b) are bijections of the sets. □
Now Theorem 3.3.4 is also completely proved.

3.4 QUATERNIONS

Assume that P is a prime which is non split in L. Then ϕ has supersingular reduction and therefore End $_{W/\pi}(\phi)$ is a maximal order in a quaternion algebra

B over K with maximal subfield L, unramified outside P and ∞ and having invariants $\mathrm{inv}_P = 1/2$ and $\mathrm{inv}_\infty = -1/2$ (cf. [Ge4]). The reduced norm of B/K will be denoted by nr and the reduced trace by tr. The norm of L/K will be denoted by $\mathrm{N}_{L/K}(\cdot)$. Let $x = (\phi, \phi', \phi_{\mathfrak{n}})$ be a Heegner point. Then

$$R := \mathrm{End}_{\,W/\pi}(x) = \{f \in \mathrm{End}_{\,W/\pi}(\phi) : \phi_{\mathfrak{n}} f = g\phi_{\mathfrak{n}}, \text{ for some } g \in \mathrm{End}_{\,W/\pi}(\phi)\}$$

which is the same as $R = \mathrm{End}_{\,W/\pi}(\phi) \cap \mathrm{End}_{\,W/\pi}(\phi')$ in B. This is also an order in B, but is not a maximal one in general.
Let $\mathbb{F}_q[T]_P$ be the completion of $\mathbb{F}_q[T]$ at P. Then ϕ extends to a formal module

$$\phi^{(P)} : \mathbb{F}_q[T]_P \longrightarrow W/\pi\{\{\tau\}\}$$

where $W/\pi\{\{\tau\}\}$ is the twisted power series ring. Then we have the following analogue of the theorem of Serre-Tate [Dr] with $x^{(P)} = (\phi^{(P)}, {\phi'}^{(P)}, \phi_{\mathfrak{n}}^{(P)})$:

Lemma 3.4.1

$$End_{\,W/\pi^n}(x) = End_{\,W/\pi}(x) \cap End_{\,W/\pi^n}(x^{(P)}).$$

Proof. It suffices to show that

$$\mathrm{End}_{\,W/\pi^n}(\phi) = \mathrm{End}_{\,W/\pi}(\phi) \cap \mathrm{End}_{\,W/\pi^n}(\phi^{(P)}).$$

We use induction. Let $f \in \mathrm{End}_{\,W/\pi^n}(\phi) \cap \mathrm{End}_{\,W/\pi^{n+1}}(\phi^{(P)})$. We have to show that $f \in \mathrm{End}_{\,W/\pi^{n+1}}(\phi)$. Therefore let

$$f = f_0 + f_1\tau + \ldots + f_i\tau^i + \ldots \in W/\pi^{n+1}\{\{\tau\}\}$$

with $f\phi_a \equiv \phi_a f \bmod \pi^{n+1}$ for all $a \in \mathbb{F}_q[T]$ and assume that there is an $M \in \mathbb{N}$, such that $f_i \equiv 0 \bmod \pi^n$ for all $i \geq M$. Now ϕ has supersingular reduction at π, therefore

$$\phi_P = a_0 + a_1\tau + \ldots + a_{2d}\tau^{2d} \equiv a_{2d}\tau^{2d} \bmod \pi,$$

if $d = \deg P$. Now if $k > M + 2d$ then the $k-$th coefficient of $f\phi_P$ is

$$f_{k-2d}a_{2d}^{q^{k-2d}} + f_{k-2d+1}a_{2d-1}^{q^{k-2d+1}} + \cdots + f_k a_0^{q^k} \equiv f_{k-2d}a_{2d}^{q^{k-2d}} \bmod \pi^{n+1},$$

because $f_i \equiv 0 \bmod \pi^n$ and $a_i \equiv 0 \bmod \pi$ for $i < 2d$. On the other hand this coefficient is equal to the $k-$th coefficient of $\phi_P f$ which is

$$a_0 f_k + a_1 f_{k-1}^q + \cdots + a_{2d-1}f_{k-2d+1}^{q^{2d-1}} + a_{2d}f_{k-2d}^{q^{2d}} \equiv 0 \bmod \pi^{n+1}.$$

Here a_{2d} occurs only together with $f_{k-2d}^{q^{2d}}$ which vanishes modulo π^{n+1} because $d \geq 1$. Comparing both yields the assertion, namely $f_{k-2d} \equiv 0 \bmod \pi^{n+1}$ for all $k > M + 2d$. □

From Lemma 3.4.1 and the corresponding statement for formal groups ([Gr2]) we immediately get for the order R in the quaternion algebra B:

Proposition 3.4.2 *Let $\mathfrak{p}|P$ be a prime ideal in L and j in R with $B = L + Lj$. Then End $_{W/\pi^n}(x) =$*

$$\{b = b_1 + b_2 j \in R : D \cdot N_{L/K}(b_2) nr(j) \equiv 0 \bmod P(N_{L/K}(\mathfrak{p}))^{n-1}\}.$$

Together with 3.3.4 we get

Proposition 3.4.3 *Let x be a Heegner point for the maximal order O_L, $\sigma_{\mathcal{A}} \in$ Gal (H/L) and $\mathfrak{a}$ an ideal from the ideal class $\mathcal{A}$ corresponding to $\sigma_{\mathcal{A}}$. Let $R =$ End $_{W/\pi}(x)$ and suppose* $\gcd(\lambda, N) = 1$. *Then*

$$(x \,.\, T_\lambda x^{\sigma_{\mathcal{A}}})_v = \frac{1}{q-1} \sum_{\substack{b \in R\mathfrak{a}, b \notin L \\ nr(b) = \lambda N_{L/K}(\mathfrak{a})}} \left\{ \begin{array}{ll} \frac{1}{2}(1 + ord_P(nr(j) N_{L/K}(b_2))) & (P \text{ inert}) \\ ord_P(D\ nr(j) N_{L/K}(b_2)) & (P \text{ ramified}). \end{array} \right.$$

Proof. Theorem 3.3.4 yields

$$(x \,.\, T_\lambda x^{\sigma_{\mathcal{A}}})_v = \frac{1}{q-1} \sum_{n \geq 1} \#\text{Hom}_{\,W/\pi^n}(x^{\sigma_{\mathcal{A}}}, x)_{\deg \lambda}.$$

With Lemma 3.2.2 and Proposition 3.4.2 we obtain

$$\begin{aligned}
&(q-1)(x \,.\, T_\lambda x^{\sigma_{\mathcal{A}}})_v \\
&\quad = \sum_{n \geq 1} \#\{b = b_1 + b_2 j \in R\mathfrak{a} : \\
&\qquad (\mathrm{nr}(b)) = (\lambda \mathrm{N}_{L/K}(\mathfrak{a})), \;\; D \cdot \mathrm{N}_{L/K}(b_2)\mathrm{nr}(j) \equiv 0 \bmod P(\mathrm{N}_{L/K}(\mathfrak{p}))^{n-1}\} \\
&\quad = \sum_{\substack{b \in R\mathfrak{a} \\ (\mathrm{nr}(b)) = (\lambda \mathrm{N}_{L/K}(\mathfrak{a}))}} \left\{ \begin{array}{l} \#\{n : \mathrm{nr}(j)\mathrm{N}_{L/K}(b_2) \equiv 0 \bmod P^{2n-1}\}, \ (P \text{ inert}) \\ \#\{n : D\ \mathrm{nr}(j)\mathrm{N}_{L/K}(b_2) \equiv 0 \bmod P^n\}, \ (P \text{ ramified}) \end{array} \right. \\
&\quad = \sum_{\substack{b \in R\mathfrak{a} \\ (\mathrm{nr}(b)) = (\lambda \mathrm{N}_{L/K}(\mathfrak{a}))}} \left\{ \begin{array}{ll} \frac{1}{2}(1 + \mathrm{ord}_P(\mathrm{nr}(j)\mathrm{N}_{L/K}(b_2))) & (P \text{ inert}) \\ \mathrm{ord}_P(D\ \mathrm{nr}(j)\mathrm{N}_{L/K}(b_2)) & (P \text{ ramified}). \end{array} \right.
\end{aligned}$$

In the assertion of the proposition we sum only over $b \notin L$ or equivalently $b_2 \neq 0$. As we assume that $r_{\mathcal{A}}((\lambda)) = 0$ this makes no difference because the elements with $b_2 = 0$ correspond to homomorphisms defined over W. □

The next step towards our final formulae is to describe $R\mathfrak{a}$ explicitly. This can be done in almost the same way as in the paper of Gross and Zagier, therefore we omit the details.

First of all we want to describe the quaternion algebra by Hilbert symbols. This is obtained by class field theory.

Proposition 3.4.4 *Let P be monic and inert. Let ϵ_D be the leading coefficient of D. Then there exists a monic, irreducible polynomial $Q \neq P$ and $\epsilon \in \mathbb{F}_q^* - \mathbb{F}_q^2$, such that*

1. $\deg PQD$ *is odd,*

2. $\epsilon PQ \equiv 1 \bmod l$ *for all* $l|D$.

In terms of the Hilbert symbol this means

$$\left(\frac{D, \epsilon PQ}{l}\right) = \begin{cases} 1 & \text{for } l \nmid P\infty \\ -1 & \text{for } l = P \text{ or } l = \infty. \end{cases}$$

Furthermore D *is a quadratic residue modulo* Q*, i.e.* Q *is split in* L/K.

COROLLARY 3.4.5 B *is described by*

$$B \simeq (D, \epsilon PQ),\ B = L + Lj$$

with $j^2 = \epsilon PQ$.

We recall the following definition.

DEFINITION 3.4.6 *The level (or reduced discriminant)* rd *of an order* J *in a quaternion algebra* B *is defined by*

$$rd := n(\tilde{J})^{-1},$$

where $\tilde{J} = \{b \in B : tr(bJ) \subset \mathbb{F}_q[T]\}$ *is the complement of* J *and* $n(\tilde{J})$ *is the gcd of the norms of elements in* $\tilde{J}$.

Then we can show that R has level NP and O_L is optimally embedded in R, i.e. $R \cap L = O_L$.
The next step is to identify the order R in B.

PROPOSITION 3.4.7 *The set*

$$S = \{\alpha + \beta j : \alpha \in \mathfrak{d}^{-1},\ \beta \in \mathfrak{d}^{-1}\mathfrak{q}^{-1}\mathfrak{n},\ \alpha \equiv \beta \bmod O_{\mathfrak{f}}\ \forall \mathfrak{f}|\mathfrak{d}\}$$

is an order in $(D, \epsilon PQ)$ *of level* NP *and* O_L *is optimally embedded in* S*. Here* $\mathfrak{d} = (\sqrt{D})$ *is the different,* $\mathfrak{q}|Q$ *is a prime of* L *and* $O_{\mathfrak{f}}$ *is the localization of* O_L *at* $\mathfrak{f}$.

The proof is given by straightforward calculations (cf. Satz 3.18, [Ti1]).
Now R, S are both orders in which O_L is optimally embedded and sharing the same level. A Theorem of Eichler [Ei, Satz 7] states the existence of an ideal $\mathfrak{b}$ of O_L with $R\mathfrak{b} = \mathfrak{b}S$.
So if $\mathfrak{a}$ is an ideal in the class $\mathcal{A}$ corresponding to $\sigma_{\mathcal{A}} \in \mathrm{Gal}\ (H/L)$, and without loss of generality we assume that P is not a divisor of $\mathfrak{a}$, then

$$R\mathfrak{a} = \mathfrak{b}S\mathfrak{b}^{-1}\mathfrak{a} =$$

$$= \{\alpha + \beta j : \alpha \in \mathfrak{d}^{-1}\mathfrak{a}, \beta \in \mathfrak{d}^{-1}\mathfrak{q}^{-1}\mathfrak{n}\mathfrak{b}\overline{\mathfrak{b}}^{-1}\overline{\mathfrak{a}},\ \alpha \equiv (-1)^{\mathrm{ord}_{\mathfrak{f}}(\mathfrak{b})}\beta \bmod O_{\mathfrak{f}}\ \forall \mathfrak{f}|\mathfrak{d}\}.$$

The ideal class $\mathcal{B}$ of $\mathfrak{b}$ depends on the place $v|P$. But P is inert, therefore it is a principal prime ideal in L, and so P is totally split in H/L. The places over P are permuted transitively by the Galois group. If $\tau \in \mathrm{Gal}\,(H/L)$ and W_τ is the maximal unramified extension of O_{H,v^τ} and π_τ is a uniformizing parameter and $R_\tau = \mathrm{End}_{\,W_\tau/\pi_\tau}(x)$, then $R_\tau = \mathfrak{c}_\tau R \mathfrak{c}_\tau^{-1}$, where $\mathfrak{c}_\tau$ lies in the ideal class corresponding to τ. If $\mathfrak{b}_\tau$ is defined by $R_\tau \mathfrak{b}_\tau = \mathfrak{b}_\tau S$, it follows that $\mathfrak{b}_\tau = \mathfrak{b}\mathfrak{c}_\tau$. Now we can give a more explicit formula for the height pairing at inert primes. We define $d(\mu, D)$ to be the number of common prime factors of μ and D.

PROPOSITION 3.4.8 *For P inert we get the formula:*

$$\begin{aligned}
&\langle (x) - (\infty)\ ,\ T_\lambda((x)^{\sigma_{\mathcal{A}}} - (0))\rangle_P \\
&= \sum_{v|P} -(x\ .\ T_\lambda x^{\sigma_{\mathcal{A}}})_v \ln q_v \\
&= -u^2 \frac{1}{q-1} \ln q\ \deg P \sum_{\substack{\mu \in \mathbb{F}_q[T]-\{0\} \\ \deg \mu \leq \deg \lambda D - \deg NP}} ord_P(P^2\mu) \cdot r_{\mathcal{A}}((NP\mu - \lambda D)) \cdot \\
&\qquad 2^{d(\mu,D)} R_{\{\mathcal{A}[\mathfrak{q}\mathfrak{n}]\}}((\mu)) \frac{1 - \delta_{(NP\mu - \lambda D)NP\mu}}{2}
\end{aligned}$$

with

$$u = |O_L^*/\mathbb{F}_q^*| = \begin{cases} q+1, & \textit{if } \deg D = 0 \\ 1, & \textit{otherwise.} \end{cases}$$

Here δ is again the local norm symbol at ∞ (cf. the definition of δ in section 2.2 following equation (2.2.5)). $R_{\{\mathcal{A}[\mathfrak{q}\mathfrak{n}]\}}(\mu)$ denotes the number of integral ideals $\mathfrak{c}$, which lie in a class differing from the class $\mathcal{A}[\mathfrak{q}\mathfrak{n}]$ by a square in the class group and with norm (μ).

PROOF. Let $\mathfrak{a}$ be a fixed ideal in $\mathcal{A}$ and let λ_0 be a fixed generator of $N_{L/K}(\mathfrak{a})$. We calculate the height pairing using Proposition 3.4.3 together with the explicit description of $R\mathfrak{a}$.
If $b = \alpha + \beta j \in R_\tau \mathfrak{a}$, i.e. $\alpha \in \mathfrak{d}^{-1}\mathfrak{a}, \beta \in \mathfrak{d}^{-1}\mathfrak{q}^{-1}\mathfrak{n}\mathfrak{b}_\tau \overline{\mathfrak{b}}_\tau^{-1} \overline{\mathfrak{a}}$, $\alpha \equiv (-1)^{\mathrm{ord}_{\mathfrak{f}}(\mathfrak{b})}\beta \bmod O_{\mathfrak{f}}$, we define

$$\mathfrak{c} \quad := \quad (\beta)\mathfrak{d}\mathfrak{q}\mathfrak{n}^{-1}\mathfrak{b}_\tau^{-1}\overline{\mathfrak{b}_\tau}\overline{\mathfrak{a}}^{-1} \in [\mathfrak{q}\mathfrak{n}^{-1}]\mathcal{B}^2\mathcal{A}$$

and

$$\nu := -\mathrm{N}_{L/K}(\alpha) D \lambda_0^{-1} \in \mathbb{F}_q[T]$$

$$\mu := -\epsilon \mathrm{N}_{L/K}(\beta) DQN^{-1}\lambda_0^{-1} \in \mathbb{F}_q[T].$$

Then $\mathfrak{c}$ is integral and

$$\mathrm{nr}(\alpha + \beta j) = \mathrm{N}_{L/K}(\alpha) - \epsilon PQ \mathrm{N}_{L/K}(\beta) = (-\nu + NP\mu)D^{-1}\lambda_0$$

thus

$$(\mathrm{nr}(\alpha+\beta j)) = (\lambda\lambda_0) \iff (-\nu + NP\mu) = (D\lambda) \iff \nu = NP\mu - \tilde{\epsilon}D\lambda$$

for a uniquely determined $\tilde{\epsilon} \in \mathbb{F}_q^*$.
Now if $\mu \in \mathbb{F}_q[T]$ and $\epsilon \in \mathbb{F}_q^*$ are given, then the number of $\alpha \in \mathfrak{d}^{-1}\mathfrak{a}$ with $\mathrm{N}_{L/K}(\alpha) = -\nu D^{-1}\lambda_0$ is $r_{\mathfrak{a},\lambda_0}(NP\mu - \tilde{\epsilon}D\lambda)$.
β is determined by the integral ideal $\mathfrak{c}$ up to multiplication with elements of ${O_L}^*$.
If $\deg D = 0$ there are no further restrictions on α, β. Now suppose $\deg D > 0$. We have that $\tilde{\epsilon}\lambda\lambda_0 = \mathrm{N}_{L/K}(\alpha) - \epsilon PQ\mathrm{N}_{L/K}(\beta)$ is integral and that $\epsilon PQ \equiv 1 \bmod f$ for all $f|D$. Therefore $\alpha \equiv \pm\beta \bmod O_{\mathfrak{f}}$.
Let $(\sqrt{D}) = \mathfrak{p}_1 \cdots \mathfrak{p}_t$, we can modify $\mathfrak{b}_\tau$ modulo squares of classes to find $\mathfrak{b}$ with

$$\mathfrak{b} = \mathfrak{b}_\tau \cdot \mathfrak{p}_1^{\epsilon_1} \cdots \mathfrak{p}_t^{\epsilon_t}$$

with $\epsilon_i \in \{0, 1\}$ such that

$$\alpha \equiv (-1)^{\mathrm{ord}_{\mathfrak{p}_i}(\mathfrak{b})}\beta \bmod O_{\mathfrak{p}_i}.$$

The ϵ_i are uniquely determined if $\beta \notin O_{\mathfrak{f}}$, which is the case exactly when $\mathfrak{p}_i \nmid \mu$. If $\beta \in O_{\mathfrak{f}}$ then both choices of ϵ_i give the correct congruence. Thus there are $2^{d(\mu,D)}$ ideal classes which differ from the class of $\mathfrak{b}_\tau$ only by classes of order 2 and which have the given congruence for α and β. The only exception to this is when $D|\mu$. In this case all ϵ_i can be chosen arbitrarily, so for each d-tuple $(\epsilon_1, \ldots, \epsilon_d)$ also $(-\epsilon_1, \ldots, -\epsilon_d)$ is possible but both give the same class. The number of classes is therefore divided by two. On the other hand the congruences fix β, except when all congruences are trivial. So the number of pairs (α, β) doubles in the latter case.
The existence of β is equivalent to $\epsilon^{-1}\mu Q^{-1}N\lambda_0$ being a norm of an element in L^*. As we already know that it is the norm of an ideal, we get the following local condition:

$$\epsilon^{-1}\mu Q^{-1}N\lambda_0 \in N_{L/K}(L^*) \iff \delta_{\epsilon^{-1}\mu Q^{-1}N\lambda_0} = 1.$$

By definition of Q we have $\left(\dfrac{D, \epsilon PQ}{\infty}\right) = -1$. Therefore the condition is equivalent to $\delta_{\mu PN\lambda_0} = -1$.
For a given α the number of β in some class $\mathfrak{b}_\tau$ is then

$$2^{d(\mu,D)}\, R_{\{\mathcal{A}[\mathfrak{qn}]\}}(\mu)\frac{1-\delta_{\mu PN\lambda_0}}{2}.$$

This shows that

$$\begin{aligned}&\langle (x)-(\infty)\ ,\ T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle_P\\ =\ & -u\frac{1}{q-1}\ln q_v \sum_{\substack{\mu\in\mathbb{F}_q[T]-\{0\}\\ \deg\mu\le\deg\lambda D-\deg NP}}\ \sum_{\tilde\epsilon\in\mathbb{F}_q^*}\frac{1}{2}(1+\mathrm{ord}_P(P\mu))\cdot\\ & r_{\mathfrak{a},\lambda_0}(NP\mu-\tilde\epsilon\lambda D)\cdot 2^{d(\mu,D)}R_{\{\mathcal{A}[\mathfrak{qn}]\}}((\mu))\frac{1-\delta_{NP\mu\lambda_0}}{2}.\end{aligned}$$

If $r_{\mathfrak{a},\lambda_0}((NP\mu-\tilde\epsilon\lambda D))\neq 0$ then $\delta_{\lambda_0}=\delta_{NP\mu-\tilde\epsilon\lambda D}$, $\ln q_v=2\deg P\ln q$. If we substitute this and change $\mu\mapsto\tilde\epsilon\mu$ we get

$$\begin{aligned}&\langle (x)-(\infty)\ ,\ T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle_P\\ =\ & -u\deg P\ \ln q \sum_{\substack{\mu\in\mathbb{F}_q[T]-\{0\}\\ \deg\mu\le\deg\lambda D-\deg NP}}\mathrm{ord}_P(P^2\mu)\sum_{\tilde\epsilon\in\mathbb{F}_q^*}r_{\mathfrak{a},\lambda_0}(\tilde\epsilon(NP\mu-\lambda D))\cdot\\ & 2^{d(\mu,D)}R_{\{\mathcal{A}[\mathfrak{qn}]\}}((\mu))\frac{1-\delta_{NP\mu(NP\mu-\lambda D)}}{2}.\end{aligned}$$

Using the identity

$$\frac{1}{q-1}\sum_{\tilde\epsilon\in\mathbb{F}_q^*}r_{\mathfrak{a},\lambda_0}(\tilde\epsilon(NP\mu-\lambda D))=ur_{\mathcal{A}}((NP\mu-\lambda D))$$

we get the formula of the proposition. □

We specialize this result to the case where D is irreducible. Then $u=1$ because $\deg D>0$. $d(\mu,D)=t(\mu,D)=0$ or 1 for $t(\mu,D)$ defined in (2.8.1) and therefore $2^{d(\mu,D)}=t(\mu,D)+1$.

Lemma 3.4.9 *If D is irreducible, then*

$$R_{\{\mathcal{A}[\mathfrak{qn}]\}}((\mu))\frac{1-\delta_{-NQ\mu\lambda_0}}{2}=\frac{1}{q-1}\sum_{c|\mu}\left[\frac{D}{c}\right]\frac{1-\delta_{-NQ\mu\lambda_0}}{2}.$$

Proof. One has to show that

$$R_{\{\mathcal{A}[\mathfrak{qn}]\}}((\mu))\frac{1-\delta_{-NQ\mu\lambda_0}}{2}=\#\{\mathfrak{c}\text{ integral ideal}:\mathrm{N}_{L/K}(\mathfrak{c})=(\mu)\}.$$

Then the assertion follows by comparing the coefficients of both sides of $\zeta_L(s)=\zeta_K(s)L_D(s)$. If $\deg D$ is odd then every class is a square in the class group and we are done. If $\deg D$ is even and if $-NQ\mu\lambda_0$ is a norm, then $\deg\mathrm{N}_{L/K}(\mathfrak{a}_0\mathfrak{qn})\equiv\deg\mu \bmod 2$. $\#\{\mathfrak{c}\text{ integral ideal}:\mathrm{N}_{L/K}(\mathfrak{c})=(\mu)\}$ is the sum of $r_{\tilde{\mathcal{A}}}((\mu))$ over all classes $\tilde{\mathcal{A}}$, which is equal to the sum over all square classes if $\mu\equiv 0 \bmod 2$ and equal to the sum over all non-square classes if $\mu\not\equiv 0 \bmod 2$. In any case this is $R_{\mathcal{A}[\mathfrak{qn}]}(\mu)$. □

From this lemma the following corollary follows.

COROLLARY 3.4.10 *Let P be inert and D be irreducible. Then*

$$\begin{aligned}
&\langle (x)-(\infty)\ ,\ T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle_P \\
&= \sum_{v|P} -(x\ .\ T_\lambda x^{\sigma_{\mathcal{A}}})_v \ln q_v \\
&= -\frac{\ln q}{q-1} \sum_{\substack{\mu\in\mathbb{F}_q[T]-\{0\}\\ \deg\mu\le\deg\lambda D-\deg NP}} \deg P\, ord_P(P^2\mu)\cdot r_{\mathcal{A}}((NP\mu-\lambda D))\cdot \\
&\qquad (t(\mu,D)+1)\left(\sum_{c|\mu}\left[\frac{D}{c}\right]\right)\frac{1-\delta_{(NP\mu-\lambda D)NP\mu}}{2}.
\end{aligned}$$

If P is ramified we can get a similar formula arguing in the same way as for the inert case. Let $\mathfrak{p}|P$ be a prime over P and let f $(=1$ or $2)$ be the order of the place in the class group. Then $\mathfrak{p}$ splits in H into h/f factors all of which have residue degree f over the residue field of $\mathfrak{p}$.

PROPOSITION 3.4.11 *There exists $\epsilon\in\mathbb{F}_q^*-\mathbb{F}_q^2$ and a monic polynomial $Q\in\mathbb{F}_q[T]$ with $\deg QD$ odd, such that $\epsilon Q\equiv 1 \bmod l$ for all $l|D$, $l\neq P$ and*

$$\left(\frac{\epsilon Q}{P}\right) = -1.$$

Also Q is split in L/K, $B\simeq(D,\epsilon Q)$ and $B=L+Lj$ with $j^2=\epsilon Q$.

PROPOSITION 3.4.12 *The order*

$$S=\{\alpha+\beta j : \alpha\in\mathfrak{p}\mathfrak{d}^{-1},\ \beta\in\mathfrak{p}\mathfrak{d}^{-1}\mathfrak{q}^{-1}\mathfrak{n},\ \alpha\equiv\beta \bmod O_{\mathfrak{f}}\ \forall\mathfrak{f}|\mathfrak{d}\}$$

in $(D,\epsilon Q)$ has level $N\cdot P$ and O_L is optimally embedded in S.

From this and the Theorem of Eichler we get

$$R_\tau\mathfrak{a}=\mathfrak{b}_\tau S\mathfrak{b}_\tau^{-1}\mathfrak{a}=$$

$$=\{\alpha+\beta j:\alpha\in\mathfrak{p}\mathfrak{d}^{-1}\mathfrak{a},\beta\in\mathfrak{p}\mathfrak{d}^{-1}\mathfrak{q}^{-1}\mathfrak{n}\mathfrak{b}_\tau\overline{\mathfrak{b}}_\tau^{-1}\overline{\mathfrak{a}},\ \alpha\equiv(-1)^{\mathrm{ord}_{\mathfrak{f}}(\mathfrak{b}_\tau)}\beta \bmod O_{\mathfrak{f}}\}.$$

In the same way as for the inert primes we can show:

PROPOSITION 3.4.13 *Assume again that $r_{\mathcal{A}}((\lambda))=0$. Let P be ramified. Then $\deg D>0$ and $u=1$. We have:*

$$\begin{aligned}
&\langle (x)-(\infty)\ ,\ T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle_P \\
&= \sum_{v|P} -(x\ .\ T_\lambda x^{\sigma_{\mathcal{A}}})_v \ln q_v \\
&= -\frac{\ln q}{q-1}\deg P \sum_{\substack{\mu\in\mathbb{F}_q[T]-\{0\}\\ \deg\mu\le\deg\lambda D-\deg NP}} ord_P(P\mu)\cdot r_{\mathcal{A}}((NP\mu-\lambda D))\cdot \\
&\qquad 2^{d(\mu,D)}R_{\{\mathcal{A}[\mathfrak{q}\mathfrak{n}]\}}(P\mu)\frac{1-\delta_{(NP\mu-\lambda D)NP\mu}}{2}.
\end{aligned}$$

If D is irreducible we get the formula:

$$\begin{aligned}
&\langle (x)-(\infty)\ ,\ T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle_P \\
&= -\frac{\ln q}{q-1}\deg P \sum_{\substack{\mu\in\mathbb{F}_q[T]-\{0\}\\ \deg\mu\le\deg\lambda D-\deg NP}} ord_P(P\mu)\cdot r_{\mathcal{A}}((NP\mu-\lambda D))\cdot \\
&\qquad 2\left(\sum_{\mu|c}\left[\frac{D}{c}\right]\right)\frac{1-\delta_{(NP\mu-\lambda D)NP\mu}}{2}.
\end{aligned}$$

Proof. The proof is just as in the inert case. The only thing we want to mention is that all classes for $\mathfrak{b}$ are counted. But the sum runs only over the $v|P$, so only over the h/f classes mod $\mathfrak{p}$. On the other hand there is a factor f from $\ln q_v = f\cdot\deg P\ln q$ which compensates for this. $\square$

Now we sum up the formulae for all finite P in the case that D is irreducible.

Theorem 3.4.14 *Let $N\in\mathbb{F}_q[T]$ square free, $D\in\mathbb{F}_q[T]$ irreducible and $D\equiv b^2 \bmod N$ for some $b\in\mathbb{F}_q[T]$. Let $L=K(\sqrt{D})$ and let H denote the Hilbert class field of L. Let $\sigma_{\mathcal{A}}\in Gal(H/L)$ and suppose that $\mathcal{A}$ is the corresponding ideal class.*

Let $\lambda\in\mathbb{F}_q[T]$ be such that $gcd(\lambda,N)=1$ and $r_{\mathcal{A}}((\lambda))=0$. Then

$$\begin{aligned}
&\sum_{P\neq\infty}\langle (x)-(\infty)\ ,\ T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle_P \\
&= -\frac{\ln q}{q-1}\sum_{\substack{\mu\in\mathbb{F}_q[T]-\{0\}\\ \deg\mu N\le\deg\lambda D}} r_{\mathcal{A}}((\mu N-\lambda D))\cdot(1-\delta_{(\mu N-\lambda D)\mu N}) \\
&\qquad\Big[-(t(\mu,D)+1)\left(\sum_{c|\mu}\left[\frac{D}{c}\right]\deg c\right)+\left(\sum_{c|\mu}\left[\frac{D}{c}\right]\deg\mu\right)\Big].
\end{aligned}$$

Proof. The sum over all $P\neq\infty$ of the formulae in Proposition 3.4.8 and in Proposition 3.4.13 gives

$$\begin{aligned}
&\sum_{P\neq\infty}\langle (x)-(\infty)\ ,\ T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle_P \\
&= -\frac{\ln q}{q-1}\sum_{\substack{\mu\in\mathbb{F}_q[T]-\{0\}\\ \deg\mu N\le\deg\lambda D}} r_{\mathcal{A}}((\mu N-\lambda D))\cdot(1-\delta_{(\mu N-\lambda D)\mu N}) \\
&\qquad\left[(t(\mu,D)+1)\sum_{\substack{P|\mu\\ P\text{ inert}}}\deg P\,\mathrm{ord}_P(P\mu)\left(\sum_{c|\frac{\mu}{p}}\left[\frac{D}{c}\right]\right)\right. \\
&\qquad\left.+2\,\mathrm{ord}_D(\mu)\,\deg D\left(\sum_{c|\frac{\mu}{D}}\left[\frac{D}{c}\right]\right)\right].
\end{aligned}$$

Some calculations with the Dirichlet character show that

$$\sum_{\substack{P|\mu \\ P \text{ inert}}} \deg P \operatorname{ord}_P(P\mu) \left(\sum_{c|\frac{\mu}{p}} \left[\frac{D}{c}\right] \right)$$

$$= \left(\sum_{c|\mu} \left[\frac{D}{c}\right] \right) \deg\mu - \operatorname{ord}_D(\mu) \deg D \left(\sum_{c|\mu} \left[\frac{D}{c}\right] \right)$$

$$-2 \left(\sum_{c|\mu} \left[\frac{D}{c}\right] \deg c \right).$$

Substituting this yields

$$\sum_{P\neq\infty} \langle (x) - (\infty) \, , \, T_\lambda((x)^{\sigma_{\mathcal{A}}} - (0)) \rangle_P$$

$$= -\frac{\ln q}{q-1} \sum_{\substack{\mu\in\mathbb{F}_q[T]-\{0\} \\ \deg\mu N \leq \deg\lambda D}} r_{\mathcal{A}}((\mu N - \lambda D)) \cdot (1 - \delta_{(\mu N - \lambda D)\mu N})$$

$$\cdot \Big[(t(\mu, D)+1) \left(\sum_{c|\mu} \left[\frac{D}{c}\right] \right) \deg\mu$$

$$- \deg D \; (t(\mu,D)+1) \operatorname{ord}_D(\mu) \left(\sum_{c|\mu} \left[\frac{D}{c}\right] \right)$$

$$-2(t(\mu,D)+1) \left(\sum_{c|\mu} \left[\frac{D}{c}\right] \deg c \right) + 2 \operatorname{ord}_D(\mu) \deg D \left(\sum_{c|\mu} \left[\frac{D}{c}\right] \right) \Big].$$

For $r_{\mathcal{A}}((\mu N - \lambda D)) \neq 0$ we observe that (cf. Lemma 2.8.1):

$$(1 - \delta_{(\mu N - \lambda D)\mu N})(t(\mu,D) - 1) \left(\sum_{c|\frac{\mu}{D}} \left[\frac{D}{c}\right] \right) = 0$$

from which the theorem follows. □

3.5 The local height pairing at ∞

At first we assume that $r_{\mathcal{A}}((\lambda)) = 0$. The local height pairing at places over ∞ can be calculated by Green's functions as described in [Ti3]. This approach is based on the general formula (3.1.1). This means that there are contributions coming from the intersection of horizontal divisors and from the intersection with the fibre components. In contrast to [Ti3] here we always consider Γ as a subgroup of $GL_2(\mathbb{F}_q[T])$ instead of $PGL_2(\mathbb{F}_q[T])$, so the formulae differ by a factor $q-1$.

The cases $\deg D$ odd and $\deg D$ even will be treated separately starting with the former case. In the whole section we again assume that D is irreducible. We write $|z|_i = \min\{|z-y| : y \in K_\infty\}$ and

$$d(z,z') = \log_q \frac{|z-z'|^2}{|z|_i|z'|_i}.$$

3.5.1 deg D odd

If z, z' are two elements in Ω with $\log_q |z|_i$, $\log_q |z'|_i \notin \mathbb{Z}$ and which represent $L_\infty-$ rational points on the algebraic curve $X_0(N)$ then by definition of the Green's function G ([Ti3, Def 2]) and Theorem 2 together with Proposition 8 of [Ti3] we have

$$\begin{aligned}
&\langle (z)-(\infty),(z')-(0)\rangle_{L_\infty} \\
&= (-\ln q)(G(z,z') - G(z',\infty) - G(0,z) + G(0,\infty)) \\
&= \frac{-\ln q}{q-1}\Bigg[\sum_{\substack{\gamma\in\Gamma \\ d(\gamma z,z')\le 0}} \Big(\frac{q+1}{2(q-1)} - d(\gamma z,z')\Big) \\
&\quad + \lim_{s\to 1}\Big[\frac{q+1}{2(q-1)} \sum_{\substack{\gamma\in\Gamma \\ d(\gamma z,z')>0}} q^{-d(\gamma z,z')s} - \frac{2\kappa(q-1)}{1-q^{1-s}}\Big] \\
&\quad - \lim_{s\to 1}\Big[q^{1/2}(q^2-1)(Ei_s^{(N)}(z') + Ei_s^{(N)}(\frac{1}{Nz})) - \frac{4\kappa(q-1)}{1-q^{1-s}}\Big]\Bigg]
\end{aligned}$$

with $\kappa := \dfrac{q^2-1}{2\,[GL_2(\mathbb{F}_q[T]):\Gamma_0(N)]}$. Here $Ei_s^{(N)}(z)$ is the Eisenstein series

$$Ei_s^{(N)}(z) = |z|_i^s \sum_{\substack{(c,d),\, N|c \\ \gcd(c,d)=1}} |cz+d|^{-2s}.$$

We define

$$R_N := \{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{F}_q[T]) : N|c,\, \det\begin{pmatrix} a & b \\ c & d\end{pmatrix} \neq 0\}.$$

If $\lambda \in \mathbb{F}_q[T]$, $\lambda \neq 0$ we get

$$\begin{aligned}
&\langle (z)-(\infty), T_\lambda((z')-(0))\rangle_{L_\infty} \\
&= \frac{-\ln q}{q-1}\Bigg[\sum_{\substack{\gamma\in R_N,\, (\det\gamma)=(\lambda)\\ d(z,\gamma z')\le 0}} \Big(\frac{q+1}{2(q-1)} - d(z,\gamma z')\Big) \\
&\quad + \lim_{s\to 1}\Big[\frac{q+1}{2(q-1)} \sum_{\substack{\gamma\in R_N,\, (\det\gamma)=(\lambda)\\ d(z,\gamma z')> 0}} q^{-d(z,\gamma z')s} - \frac{2\kappa\sigma_1(\lambda)}{1-q^{1-s}}\Big] \\
&\quad - \lim_{s\to 1}\Big[q^{1/2}(q+1)\Big(q^{\deg\lambda s}\sigma_{1-2s}(\lambda) Ei_s^{(N)}(z') + \sigma_1(\lambda) Ei_s^{(N)}(\frac{1}{Nz})\Big) \\
&\quad - \frac{4\kappa\sigma_1(\lambda)}{1-q^{1-s}}\Big]\Bigg]
\end{aligned}$$

with $\sigma_s(\lambda) := \sum_{a|\lambda} q^{\deg a\, s}$ for any s. Now we specialize z to be a Heegner point and z' to be a conjugate under the Galois group.

Let $\lambda \in \mathbb{F}_q[T]\setminus\{0\}$ with $r_{\mathcal{A}}((\lambda)) = 0$. Let $\mathfrak{n}$ be an ideal with $\mathfrak{n}\bar{\mathfrak{n}} = (N)$. For $j = 1, 2$ let $\mathfrak{a}_j = A_j\mathbb{F}_q[T] + (B_j + \sqrt{D})\mathbb{F}_q[T]$ be two ideals contained in $\mathfrak{n}$ with $\mathrm{N}_{L/K}(\mathfrak{a}_j) = (A_j)$ and let $\mathcal{A}_j$ be the corresponding ideal classes. Then this data defines two Heegner points which are represented in the upper half plane by $\tau_j = \dfrac{-B_j+\sqrt{D}}{2A_j}$. We have that $\log_q|\tau_j|_i = \log_q|\sqrt{D}/A_j| \notin \mathbb{Z}$.

If $\mathcal{A}$ is an ideal class and $\sigma_{\mathcal{A}} \in \mathrm{Gal}(H/L)$ the corresponding automorphism we get

$$\begin{aligned}
&\langle (\tau)-(\infty), T_\lambda((\tau)^{\sigma_{\mathcal{A}}}-(0))\rangle_\infty \\
&= \sum_{v|\infty} \langle (\tau)-(\infty), T_\lambda((\tau)^{\sigma_{\mathcal{A}}}-(0))\rangle_v \\
&= \sum_{\substack{\mathcal{A}_1,\mathcal{A}_2\in Cl(O_L)\\ \mathcal{A}_1\mathcal{A}_2^{-1}=\mathcal{A}}} \langle (\tau_{\mathcal{A}_1})-(\infty), T_\lambda((\tau_{\mathcal{A}_2})-(0))\rangle_{L_\infty} \\
&= \frac{-\ln q}{q-1}\Bigg[\lim_{s\to 1}\Big[F_1(\mathcal{A},s) - \frac{2\kappa h_L\sigma_1(\lambda)}{1-q^{1-s}}\Big] \\
&\quad - \lim_{s\to 1}\Big[F_2(\mathcal{A},s) - \frac{4\kappa h_L\sigma_1(\lambda)}{1-q^{1-s}}\Big]\Bigg] \qquad (3.5.1)
\end{aligned}$$

where τ is one of the $\tau_{\mathcal{A}_j}$, $\kappa = \dfrac{(q^2-1)}{2[\mathrm{GL}_2(\mathbb{F}_q[T]) : \Gamma_0(N)]}$ and

$$\begin{aligned} F_1(\mathcal{A}, s) \quad := \quad & \sum_{\substack{\mathcal{A}_1, \mathcal{A}_2 \in Cl(O_L) \\ \mathcal{A}_1 \mathcal{A}_2^{-1} = \mathcal{A}}} \Big[\sum_{\substack{\gamma \in R_N,\, (\det \gamma) = (\lambda) \\ d(\gamma \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2}) \leq 0}} \Big(\frac{q+1}{2(q-1)} - d(\gamma \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2}) \Big) \\ & + \frac{q+1}{2(q-1)} \sum_{\substack{\gamma \in R_N,\, (\det \gamma) = (\lambda) \\ d(\gamma \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2}) > 0}} q^{-d(\gamma \tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2}) s} \Big] \end{aligned}$$

and

$$\begin{aligned} F_2(\mathcal{A}, s) := & \sum_{\substack{\mathcal{A}_1, \mathcal{A}_2 \in Cl(O_L) \\ \mathcal{A}_1 \mathcal{A}_2^{-1} = \mathcal{A}}} q^{1/2}(q+1) \cdot \\ & \cdot \Big[q^{\deg \lambda s} \sigma_{1-2s}(\lambda) Ei_s^{(N)}(\tau_{\mathcal{A}_2}) + \sigma_1(\lambda) Ei_s^{(N)}\big(\frac{1}{N \tau_{\mathcal{A}_1}}\big) \Big]. \end{aligned} \tag{3.5.2}$$

At first we calculate the function $F_1(\mathcal{A}, s)$. The following proposition combined with the convergence of the limits in (3.5.1) implies the existence of the limits in section 2.8 (cf. the corresponding remark there).

Proposition 3.5.1 *The following equation for F_1 holds:*

$$\begin{aligned} F_1(\mathcal{A}, s) \quad = \quad & \sum_{\substack{\mu \neq 0 \\ \deg(\mu N) \leq \deg(\lambda D)}} r_{\mathcal{A}}((\mu N - \lambda D)) \left(\sum_{c | \mu} \left[\frac{D}{c} \right] \right) (t(\mu, D) + 1) \\ & \cdot \frac{1 + \delta_{(\mu N - \lambda D) \mu N}}{2} \left(-\deg \mu N + \deg \lambda D + \frac{q+1}{2(q-1)} \right) \\ & + \frac{q+1}{2(q-1)} \sum_{\substack{\mu \in \mathbb{F}_q[T] - \{0\} \\ \deg \mu N > \deg \lambda D}} r_{\mathcal{A}}((\mu N - \lambda D)) \left(\sum_{c | \mu} \left[\frac{D}{c} \right] \right) \\ & \cdot (t(\mu, D) + 1) q^{-(\deg \mu N - \deg \lambda D) s}. \end{aligned}$$

Proof. We define

$$M(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{n}) = \{(\alpha, \beta) \in \mathfrak{a}_1^{-1} \overline{\mathfrak{a}}_2^{-1} \times \mathfrak{a}_1^{-1} \mathfrak{a}_2^{-1} \mathfrak{n} |\, A_1 A_2 (\alpha - \beta) \in \sqrt{D} \mathbb{F}_q[T][\sqrt{D}]\}$$

By calculations analogous to [Gr-Za], II (3.6)-(3.10) the map

$$\begin{aligned} R_N \quad & \longrightarrow \quad M(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{n}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad & \longmapsto \quad (\alpha = c \tau_1 \overline{\tau}_2 + d \overline{\tau}_2 - a \tau_1 - b,\, \beta = c \tau_1 \tau_2 + d \tau_2 - a \tau_1 - b) \end{aligned}$$

is a bijection and $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -A_1A_2(\mathrm{N}_{L/K}(\alpha) - \mathrm{N}_{L/K}(\beta))D^{-1}$.
For $\lambda \in \mathbb{F}_q[T]$, $\lambda \neq 0$ we get

$$\begin{aligned} &\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in R_N, (ad-bc) = (\lambda)\} \\ &\simeq \quad \{(\alpha,\beta) \in M(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{n}) | (\frac{-A_1A_2(\mathrm{N}_{L/K}(\alpha) - \mathrm{N}_{L/K}(\beta))}{D}) = (\lambda)\}. \end{aligned}$$

We set $\mu = \mathrm{N}_{L/K}(\beta)/A_1^{-1}A_2^{-1}N \in \mathbb{F}_q[T]$, then $d(\gamma\tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2}) = \deg\mu N - \deg\lambda D$. Then it follows that

$$\begin{aligned} &\sum_{\substack{\gamma\in R_N \\ (\det\gamma)=(\lambda)}} \Big(\frac{q+1}{2(q-1)} - d(\gamma\tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2})\Big) \\ &= \sum_{\mu\in\mathbb{F}_q[T]-\{0\}} \Big(\frac{q+1}{2(q-1)} - \deg\mu N + \deg\lambda D\Big) \cdot \\ &\qquad \#\{(\alpha,\beta) \in M(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{n})| \\ &\qquad (\frac{-A_1A_2(\mathrm{N}_{L/K}(\alpha) - \mathrm{N}_{L/K}(\beta))}{D}) = (\lambda),\ \frac{\mathrm{N}_{L/K}(\beta)}{A_1^{-1}A_2^{-1}N} = \mu\}. \end{aligned}$$

We see that

$$\begin{aligned} &\#\{(\alpha,\beta) \in M(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{n})| \\ &\qquad (\frac{-A_1A_2(\mathrm{N}_{L/K}(\alpha) - \mathrm{N}_{L/K}(\beta))}{D}) = (\lambda),\ \frac{\mathrm{N}_{L/K}(\beta)}{A_1^{-1}A_2^{-1}N} = \mu\} \\ &= \quad \#\{\beta \in \mathfrak{a}_1^{-1}\mathfrak{a}_2^{-1}\mathfrak{n} | \frac{\mathrm{N}_{L/K}(\beta)}{A_1^{-1}A_2^{-1}N} = \mu\} \cdot \\ &\qquad \#\{\alpha \in \mathfrak{a}_1^{-1}\overline{\mathfrak{a}}_2^{-1} | (-A_1A_2(\mathrm{N}_{L/K}(\alpha) - \mathrm{N}_{L/K}(\beta))) = (\lambda D)\} \cdot \\ &\qquad \cdot\frac{1}{2}(t(\mu,D)+1) \\ &= \quad r_{\mathfrak{a}_1^{-1}\mathfrak{a}_2^{-1}\mathfrak{n}, A_1^{-1}A_2^{-1}N}(\mu) \cdot \sum_{\epsilon\in\mathbb{F}_q^*} r_{\mathfrak{a}_1^{-1}\overline{\mathfrak{a}}_2^{-1}, A_1^{-1}A_2^{-1}}(\mu N - \epsilon\lambda D) \cdot \frac{1}{2}(t(\mu,D)+1). \end{aligned}$$

Now we set $\mathfrak{a}_2 = \overline{\mathfrak{a}}_1^{-1}\overline{\mathfrak{a}}_0^{-1}$ and $A_2 = A_1^{-1}\lambda_0^{-1}$, summing over all classes we get for the first part of the formula in the proposition:

$$\begin{aligned} &\sum_{\substack{\mathcal{A}_1,\mathcal{A}_2\in Cl(O_L) \\ \mathcal{A}_1\mathcal{A}_2^{-1}=\mathcal{A}}} \sum_{\substack{\gamma\in R_N, \\ (\det\gamma)=(\lambda)}} \Big(\frac{q+1}{2(q-1)} - d(\gamma\tau_{\mathcal{A}_1}, \tau_{\mathcal{A}_2})\Big) \\ &= \sum_{\mu\in\mathbb{F}_q[T]-\{0\}} \Big(\frac{q+1}{2(q-1)} - \deg\mu N + \deg\lambda D\Big) \cdot \\ &\qquad \Big(\sum_{\mathcal{A}_1\in Cl(O_L)} r_{\mathfrak{a}_1^{-1}\overline{\mathfrak{a}}_1\overline{\mathfrak{a}}_0\mathfrak{n},\lambda_0 N}(\mu)\Big) \cdot \Big(\sum_{\epsilon\in\mathbb{F}_q^*} r_{\mathfrak{a}_0,\lambda_0}(\mu N - \epsilon\lambda D)\Big)\frac{1}{2}(t(\mu,D)+1). \end{aligned}$$

Since $\deg D$ is odd we see that

$$r_{\mathfrak{a}_1^{-1}\overline{\mathfrak{a}}_1\overline{\mathfrak{a}}_0\mathfrak{n},\lambda_0 N}(\mu) = r_{\mathcal{A}_1^2[\mathfrak{a}_0\overline{\mathfrak{n}}]}((\mu))(\delta_{\lambda_0 N\mu}+1).$$

The class number is odd, and therefore every class is a square. Hence

$$\sum_{\mathcal{A}_1\in Cl(O_L)} r_{\mathfrak{a}_1^{-1}\overline{\mathfrak{a}}_1\overline{\mathfrak{a}}_0\mathfrak{n},\lambda_0 N}(\mu) = \sum_{\mathcal{B}\in Cl(O_L)} r_{\mathcal{B}}((\mu))(\delta_{\lambda_0 N\mu}+1) = \frac{\delta_{\lambda_0 N\mu}+1}{q-1}\sum_{c|\mu}\left[\frac{D}{c}\right].$$

We use this equation, we change the order of the summation, and we continue with our formula

$$\begin{aligned}
&\sum_{\substack{\mathcal{A}_1,\mathcal{A}_2\in Cl(O_L)\\ \mathcal{A}_1\mathcal{A}_2^{-1}=\mathcal{A}}}\ \sum_{\substack{\gamma\in R_N,\\ (\det\gamma)=(\lambda)}} \Big(\frac{q+1}{2(q-1)} - d(\gamma\tau_{\mathcal{A}_1},\tau_{\mathcal{A}_2})\Big)\\
&= \sum_{\epsilon\in\mathbb{F}_q^*}\ \sum_{\mu\in\mathbb{F}_q[T]-\{0\}} \Big(\frac{q+1}{2(q-1)} - \deg\mu N + \deg\lambda D\Big)\cdot\\
&\quad \frac{1}{q-1}\Big(\sum_{c|\mu}\left[\frac{D}{c}\right]\Big)(\delta_{\lambda_0 N\epsilon\mu}+1)\, r_{\mathfrak{a}_0,\lambda_0}\left(\epsilon(\mu N-\lambda D)\right)\frac{1}{2}(t(\mu,D)+1).
\end{aligned}$$

If $r_{\mathfrak{a}_0,\lambda_0}\left(\epsilon(\mu N-\lambda D)\right)\neq 0$, then $\lambda_0\epsilon(\mu N-\lambda D)$ is a norm and $\delta_{\lambda_0 N\epsilon\mu} = \delta_{\mu N(\mu N-\lambda D)}$. In addition we use the relation

$$\sum_{\epsilon\in\mathbb{F}_q^*} r_{\mathfrak{a}_0,\lambda_0}\left(\epsilon(\mu N-\lambda D)\right) = (q-1)r_{\mathcal{A}}((\mu N-\lambda D)).$$

This proves the first part of the formula in the proposition. The same calculations hold with $q^{-d(\gamma\tau_{\mathcal{A}_1},\tau_{\mathcal{A}_2})s}$ instead of $d(\gamma\tau_{\mathcal{A}_1},\tau_{\mathcal{A}_2})$. If $\deg\mu N>\deg\lambda D$ we have $\delta_{\mu N(\mu N-\lambda D)}=1$. Therefore the second part of the formula is also true. $\square$

Now we continue with the calculation of the function $F_2(\mathcal{A},s)$ defined in equation (3.5.2).

Proposition 3.5.2 *For the function $F_2(\mathcal{A},s)$ the following formula holds*

$$\begin{aligned}
&\lim_{s\to 1}\left[F_2(\mathcal{A},s) - \frac{4\kappa h_L\sigma_1(\lambda)}{1-q^{1-s}}\right]\\
&\quad = C\Big(\sum_{a|\lambda} q^{\deg a}\Big) + 2\kappa h_L\sum_{a|\lambda}(\deg\lambda - 2\deg a)q^{\deg a}
\end{aligned}$$

with

$$\begin{aligned}
C &:= -4\kappa h_L\Bigg(\deg N - \sum_{\substack{(P)|(N)\\ (P)\ prime}}\deg P(1+q^{\deg P})^{-1}\\
&\qquad -\frac{\deg D}{2} - \frac{2}{q-1} - \frac{1}{\ln q}\frac{L_D'(1)}{L_D(1)}\Bigg).
\end{aligned}$$

Proof. $Ei_s^{(N)}(\tau)$ is invariant under the non trivial automorphism of L/K. From $\overline{\tau}_{\mathcal{A},\mathfrak{n}} = \tau_{\mathcal{A}^{-1},\overline{\mathfrak{n}}}$ and $1/(N\tau_{\mathcal{A}_1,\mathfrak{n}}) = \tau_{\mathcal{A}_1[\mathfrak{n}],\overline{\mathfrak{n}}}$ it follows that

$$\sum_{\mathcal{A}\in Cl(O_L)} Ei_s^{(N)}(\frac{1}{N\tau_{\mathcal{A}}}) = \sum_{\mathcal{A}\in Cl(O_L)} Ei_s^{(N)}(\tau_{\mathcal{A}}).$$

$Ei^{(N)}$ can be expressed through the Eisenstein series $Ei^{(1)}$ by (cf. Lemma 7, [Ti3])

$$Ei_s^{(N)}(\tau_{\mathcal{A}}) = |N|^{-s} \prod_{\substack{(P)|(N)\\ (P)\text{ prime}}} (1-|P|^{-2s})^{-1}\Big(\sum_{\substack{\delta|N\\ \delta \operatorname{mod}\mathbb{F}_q^*}} \mu(\delta)|\delta|^{-s}\Big) Ei_s^{(1)}(\frac{N}{\delta}\tau_{\mathcal{A}}).$$

$(N/\delta)\tau_{\mathcal{A}}$ are Heegner points for δ instead of N with the same discriminant. Immediately from the definitions we get

$$Ei_s^{(1)}(\tau) = (1-q^{1-2s})|\sqrt{D}|^s \zeta_L(\mathcal{A},s)$$

where $\zeta_L(\mathcal{A},s)$ is the partial $\zeta-$function to the class $\mathcal{A}$. This yields

$$\begin{aligned}&\sum_{\mathcal{A}\in Cl(O_L)} Ei_s^{(N)}(\tau_{\mathcal{A}})\\ &= |N|^{-s} \prod_{\substack{(P)|(N)\\ (P)\text{ prime}}} (1-|P|^{-2s})^{-1}\Big(\sum_{\substack{\delta|N\\ \delta \operatorname{mod}\mathbb{F}_q^*}} \mu(\delta)|\delta|^{-s}\Big)\\ &\quad (1-q^{1-2s})|\sqrt{D}|^s \sum_{\mathcal{A}} \zeta_L(\mathcal{A},s).\end{aligned}$$

We have $\sum_{\mathcal{A}} \zeta_L(\mathcal{A},s) = \zeta_L(s)(1-q^{-s}) = L_D(s)/(1-q^{1-s})$. This gives

$$\begin{aligned}F_2(\mathcal{A},s) &= |N|^{-s} \prod_{\substack{P|N\\ P \operatorname{mod}\mathbb{F}_q^*}} (1+|P|^{-s})^{-1} q^{1/2}(q+1)\frac{1-q^{1-2s}}{1-q^{1-s}}\\ &\quad \cdot|\sqrt{D}|^s L_D(s)(\sigma_1(\lambda)+|\lambda|^s\sigma_{1-2s}(\lambda)).\end{aligned}$$

Now a straightforward calculation gives the desired result. □

3.5.2 deg D even

For the case where the degree of D is even we proceed in almost the same way as for the case of odd degree, so we only need mention here the statements and the differences in the proofs.
We start again with the general formula for the local height pairing at infinity

for two points given by $z, z' \in \Omega$ of [Ti3] (Thm.1, Prop. 8,9):

$$\begin{aligned}
&\langle (z)-(\infty),(z')-(0)\rangle_{L_\infty} \\
&= \frac{-2\ln q}{q-1}\Bigg[\sum_{\substack{\gamma\in\Gamma\\ d(\gamma z,z')\le 0}}\Big(\frac{q}{q^2-1}-\frac{1}{2}d(\gamma z,z')\Big) \\
&\quad+\lim_{s\to 1}\Big[\frac{q}{q^2-1}\sum_{\substack{\gamma\in\Gamma\\ d(\gamma z,z')>0}} q^{-d(\gamma z,z')s}-\frac{\kappa(q-1)}{1-q^{1-s}}\Big] \\
&\quad-\lim_{s\to 1}\Big[q(q-1)(Ei_s^{(N)}(z')+Ei_s^{(N)}(\frac{1}{Nz}))-\frac{2\kappa(q-1)}{1-q^{1-s}}\Big]\Bigg].
\end{aligned}$$

Again we take $\tau_{\mathcal{A}_j}\in\Omega$ to be elements corresponding to the different ideal classes $\mathcal{A}_j$. If τ is one of these $\tau_{\mathcal{A}_j}$ we get

$$\begin{aligned}
&\langle (\tau)-(\infty),T_\lambda((\tau)^{\sigma_{\mathcal{A}}}-(0))\rangle_\infty \\
&= \frac{-\ln q}{q-1}\Bigg[\lim_{s\to 1}\Big[F_1(\mathcal{A},s)-\frac{2\kappa h_L\sigma_1(\lambda)}{1-q^{1-s}}\Big] \\
&\quad-\lim_{s\to 1}\Big[F_2(\mathcal{A},s)-\frac{4\kappa h_L\sigma_1(\lambda)}{1-q^{1-s}}\Big]\Bigg]
\end{aligned}$$

with $\kappa := \dfrac{q^2-1}{2\,[GL_2(\mathbb{F}_q[T]):\Gamma_0(N)]}$ as above and the modified functions F_1, F_2

$$\begin{aligned}
F_1(\mathcal{A},s) &:= \sum_{\substack{\mathcal{A}_1,\mathcal{A}_2\in Cl(O_L)\\ \mathcal{A}_1\mathcal{A}_2^{-1}=\mathcal{A}}}\Big[\sum_{\substack{\gamma\in R_N,\,(\det\gamma)=(\lambda)\\ d(\gamma\tau_{\mathcal{A}_1},\tau_{\mathcal{A}_2})\le 0}}\Big(\frac{2q}{q^2-1}-d(\gamma\tau_{\mathcal{A}_1},\tau_{\mathcal{A}_2})\Big) \\
&\quad+\frac{2q}{q^2-1}\sum_{\substack{\gamma\in R_N,\,(\det\gamma)=(\lambda)\\ d(\gamma\tau_{\mathcal{A}_1},\tau_{\mathcal{A}_2})>0}} q^{-d(\gamma\tau_{\mathcal{A}_1},\tau_{\mathcal{A}_2})s}\Big]
\end{aligned}$$

and

$$F_2(\mathcal{A},s) := \sum_{\substack{\mathcal{A}_1,\mathcal{A}_2\in Cl(O_L)\\ \mathcal{A}_1\mathcal{A}_2^{-1}=\mathcal{A}}} q\Big[q^{\deg\lambda s}\sigma_{1-2s}(\lambda)Ei_s^{(N)}(\tau_{\mathcal{A}_2})+\sigma_1(\lambda)Ei_s^{(N)}(\frac{1}{N\tau_{\mathcal{A}_1}})\Big].$$

With these definitions we get:

Proposition 3.5.3 *The following equation for F_1 holds*

$$\begin{aligned} F_1(\mathcal{A},s) &= \sum_{\substack{\mu\neq 0\\ \deg(\mu N)\leq\deg(\lambda D)}} r_{\mathcal{A}}((\mu N-\lambda D))\left(\sum_{c|\mu}\left[\frac{D}{c}\right]\right)(t(\mu,D)+1)\\ &\quad\cdot\frac{1+\delta_{(\mu N-\lambda D)N\mu}}{2}\left(-\deg\mu N+\deg\lambda D+\frac{2q}{q^2-1}\right)\\ &\quad+\frac{2q}{q^2-1}\sum_{\substack{\mu\in\mathbb{F}_q[T]-\{0\}\\ \deg\mu N>\deg\lambda D}} r_{\mathcal{A}}((\mu N-\lambda D))\left(\sum_{c|\mu}\left[\frac{D}{c}\right]\right)\\ &\quad\cdot(t(\mu,D)+1)q^{-(\deg\mu N-\deg\lambda D)s}. \end{aligned}$$

Proof. The proof of this proposition differs from the corresponding Proposition 3.5.1 only slightly. We start with

$$\sum_{\mu\neq 0}\Big(\frac{2q}{q^2-1}-\deg\mu N+\deg\lambda D\Big)\,r_{\mathcal{A}}((\mu N-\lambda D))\cdot$$
$$\left(\sum_{c|\mu}\left[\frac{D}{c}\right]\right)(\delta_{\mu N(\mu N-\lambda D)}+1)\,\frac{1}{2}(t(\mu,D)+1).$$

Since D is irreducible with even degree, the ideal class number is divisible by 2 exactly once. Hence the set $\{\mathcal{A}^2|\mathcal{A}\in Cl(O_L)\}$ is equal to the set

$$\{\mathcal{B}\in Cl(O_L)|\deg\mathrm{N}_{L/K}(\mathfrak{b})\text{ is even for all }\mathfrak{b}\in\mathcal{B}\}.$$

This yields:

$$\begin{aligned} \sum_{\mathcal{A}_1\in Cl(O_L)} r_{\mathcal{A}_1^2[\overline{\mathfrak{a}}_0\mathfrak{n}]}((\mu)) &= \sum_{\mathcal{B}\in Cl(O_L)} r_{\mathcal{B}}((\mu))(\delta_{\lambda_0 N\mu}+1)\\ &= \frac{1}{q-1}\Big(\sum_{c|\mu}\left[\frac{D}{c}\right]\Big)(\delta_{\lambda_0 N\mu}+1). \end{aligned}$$

Using similar arguments as in Proposition 3.5.1 we get for our first sum:

$$\sum_{\mu\neq 0}\Big(\frac{2q}{q^2-1}-\deg\mu N+\deg\lambda D\Big)\Big(\sum_{\mathcal{A}_1\in Cl(O_L)}\frac{1}{q-1}\sum_{\epsilon_1\in\mathbb{F}_q^*} r_{\mathfrak{a}_1^{-1}\overline{\mathfrak{a}}_1\overline{\mathfrak{a}}_0\mathfrak{n},\lambda_0 N}(\epsilon_1\mu)\Big)$$
$$\cdot\Big(\sum_{\epsilon\in\mathbb{F}_q^*} r_{\mathfrak{a}_0,\lambda_0}(\mu N-\epsilon\lambda D)\Big)\,\frac{1}{2}(t(\mu,D)+1).$$

Each $\epsilon_1\in\mathbb{F}_q^*$ is norm at the extension L/K, i.e., $\epsilon_1=\mathrm{N}_{L/K}(\kappa)$. The divisor of κ is of the form $(\kappa)=\mathfrak{b}^{-1}\overline{\mathfrak{b}}$. This proves

$$r_{\mathfrak{a}_1^{-1}\overline{\mathfrak{a}}_1\overline{\mathfrak{a}}_0\mathfrak{n},\lambda_0 N}(\mu)=r_{(\mathfrak{a}_1\mathfrak{b})^{-1}\overline{\mathfrak{a}_1\mathfrak{b}}\overline{\mathfrak{a}}_0\mathfrak{n},\lambda_0 N}(\epsilon_1\mu).$$

Therefore an appropriate choice of the ideals $\mathfrak{a}_1$ yields for our sum:

$$\sum_{\mu\neq 0}\Big(\frac{2q}{q^2-1}-\deg\mu N+\deg\lambda D\Big)\Big(\sum_{\mathcal{A}_1\in Cl(O_L)} r_{\mathfrak{a}_1^{-1}\overline{\mathfrak{a}}_1\overline{\mathfrak{a}}_0\mathfrak{n},\lambda_0 N}(\mu)\Big)\cdot$$
$$\Big(\sum_{\epsilon\in\mathbb{F}_q^*} r_{\mathfrak{a}_0,\lambda_0}(\mu N-\epsilon\lambda D)\Big)\frac{1}{2}(t(\mu,D)+1).$$

The rest follows in the same way as in the proof of Proposition 3.5.1. □
The formula for $F_2(\mathcal{A},s)$ can be calculated in exactly the same way, so we only write down the result.

Proposition 3.5.4 *For the function $F_2(\mathcal{A},s)$ the following formula holds*

$$\begin{aligned}&\lim_{s\to 1}\left[F_2(\mathcal{A},s)-\frac{4\kappa h_L\sigma_1(\lambda)}{1-q^{1-s}}\right]\\ =\;& C\Big(\sum_{a\mid\lambda}q^{\deg a}\Big)+2\kappa h_L\sum_{a\mid\lambda}(\deg\lambda-2\deg a)q^{\deg a}\end{aligned}$$

with

$$C:=-4\kappa h_L\left(\deg N-\sum_{\substack{(P)\mid(N)\\(P)\ prime}}\deg P(1+q^{\deg P})^{-1}\right.$$
$$\left.-\frac{\deg D}{2}-\frac{q+3}{q^2-1}-\frac{1}{\ln q}\frac{L_D'(1)}{L_D(1)}\right).$$

3.6 Modification if $r_{\mathcal{A}}((\lambda))\neq 0$

So far we have only evaluated heights and intersection numbers, when the divisors involved have a disjoint support. In order to get a final result we must also define and compute self-intersection numbers.
Let X be a complete, non-singular, irreducible curve defined over a global function field F over $\mathbb{F}_q$, and let x be a F-rational point on X. Let $\mathcal{X}$ be a regular model of X over $\mathbb{P}^1_F$. We call l_x a *local parameter at x* if l_x generates the prime ideal corresponding to x in the local ring at x in the generic fibre. Let $\tilde{x}$ be the Zariski closure of x in $\mathcal{X}$. If π is a local parameter of a fibre corresponding to a valuation v, we call l_x a *local parameter at x for the valuation v*, if l_x together with π generate the maximal ideal corresponding to the intersection point of $\tilde{x}$ with the fibre over v. Now fix a local parameter l_x at x. Then we define for each normalized valuation v of F the local self-intersection number of x as

$$(x\,.\,x)_v:=\lim_{y\to x}((x\,.\,y)_v-v(l_x(y)))=\lim_{y\to x}((x\,.\,y)_v+\frac{1}{\deg v}\log_q|l_x(y)|_v),\tag{3.6.1}$$

where $\deg v$ is defined as usual, and where the absolute value is given by $|\alpha|_v := q^{-\deg v \cdot v(\alpha)}$, according to the product formula of the function field F.
The definition (3.6.1) and the definition of the ordinary intersection number $(x \,.\, y)_v$ (cf. section 3.1) show immediately that $(x \,.\, x)_v = 0$, if l_x is a local parameter at x for the valuation v.
In the next step we have to choose the local parameter l_x in our situation. The curve $X_0(1)$ is the projective line parametrized by the j-invariant of a Drinfeld module of rank 2. We recall that a Drinfeld module of rank 2 over $\mathbb{F}_q[T]$ is given by an additive polynomial $\Phi_T(X) = TX + gX^q + \Delta X^{q^2}$ with discriminant Δ and j-invariant $j = g^{q+1}/\Delta$.
We let Y_1 be the projective line given by the parameter u with $u^{q+1} = j$. Then $Y_1/X_0(1)$ is an extension of degree $q+1$, where only the elliptic points and cusps (i.e. zeroes and poles of j) are ramified (These facts and the definition of elliptic points can be found in [Ge1] or in other textbooks on Drinfeld modules).
Let Y_N be the composite of Y_1 and $X_0(N)$, we get the following diagram:

$$\begin{tikzcd} & Y_N \arrow[dl] \arrow[d] \\ Y_1 \arrow[d] & X_0(N) \arrow[dl] \\ X_0(1) & \end{tikzcd}$$

On Y_N we choose for a point y the local parameter $l_y := u - u(y)$. The self-intersection numbers on $X_0(N)$ will then be evaluated with this local parameter on Y_N and with the projection formula for the extension $Y_N/X_0(N)$.
We distinguish different cases for the valuations v:

3.6.1 $v \nmid N \cdot \infty$

Let x be a Heegner point on $X_0(N)$, defined locally over W as in section 3.1, and let $y_1, \ldots, y_t$ be the points on Y_N lying over x. Then the projection formula yields

$$(x \,.\, x)_v = (y_1 \,.\, y_1)_v + (y_1 \,.\, y_2)_v + \cdots + (y_1 \,.\, y_t)_v.$$

Since the covering $Y_N/X_0(N)$ is unramified outside the elliptic points and cusps and outside the divisors of $N \cdot \infty$, we see that $u - u(y_1)$ is a local parameter of y_1 at v. Hence $(y_1 \,.\, y_1)_v = 0$ by the above remark. Since

$$(y_1 \,.\, y_j)_v = \frac{1}{q-1} \sum_{k=1}^{\infty} \#\mathrm{Isom}_{W/\pi^k}(y_1, y_j)$$

for $j \neq 1$ (Proposition 3.3.3), we therefore get

$$(x \,.\, x)_v = \frac{1}{q-1} \sum_{k=1}^{\infty} (\#\mathrm{Aut}_{W/\pi^k}(x) - \#\mathrm{Aut}_W(x)). \tag{3.6.2}$$

As mentioned at the end of the proof of Proposition 3.4.3 the automorphisms not defined over W correspond to the elements $b \in R\mathfrak{a}$, $b = b_1 + b_2 j$ with $b_2 \neq 0$ which corresponds to $\mu \neq 0$ in the formulae of Corollary 3.4.10 and Proposition 3.4.13. So these formulae already count only the "new" part, i.e. without counting homomorphisms over W. Thus if $\lambda \in \mathbb{F}_q[T]$ is prime to P the formulae for the local height pairings $\langle (x) - (\infty), T_\lambda(x)^{\sigma_\mathcal{A}} - (0) \rangle_P$ of Corollary 3.4.10 and Proposition 3.4.13 remain valid. This is not true however, if $v|\lambda$. We write as before $\lambda = P^t R$ with $P \nmid R$. For the points of level $s > 0$ it is not correct to take only the "new" part. So we have to add the contribution from homomorphisms over W for these points to the formulae of Corollary 3.4.10 and Proposition 3.4.13.

For P inert we look at the last line of (3.3.1) We get a contribution of

$$\frac{1}{q-1}\begin{cases} \dfrac{t}{2}\#\ \mathrm{Isom}_W(z,x)\,\#\{z \in T_R x^{\sigma_\mathcal{A}}\} & \text{if } t \text{ is even} \\ \dfrac{t+1}{2}\#\ \mathrm{Hom}_W(z,x)_{\deg P}\,\#\{z \in T_R x^{\sigma_\mathcal{A}}\} & \text{if } t \text{ is odd} \end{cases}$$

which is $(t/2) r_\mathbf{1}((P^t)) r_\mathcal{A}((R))$ if t is even and 0 if t is odd. In both cases this is equal to $(\mathrm{ord}_P(\lambda)/2) r_\mathcal{A}((\lambda))$.

If P is ramified we get in a similar way from (3.3.2) a contribution of $\mathrm{ord}_P(\lambda) r_\mathcal{A}((\lambda))$.

If P is split we have $t+1$ points of level 0 in $T_\lambda x^{\sigma_\mathcal{A}}$, where x is just one of them (cf. Proposition 3.2.4). From the $t-s+1$ divisors of points of level $s > 0$ there is at most one whose points are congruent to x. Summing over all levels shows that the correction term in this case is $\mathrm{ord}_P(\lambda) r_\mathcal{A}((\lambda)) k_\mathfrak{p}$, where $k_\mathfrak{p}$ is a number less or equal to t, and $k_\mathfrak{p} + k_{\bar{\mathfrak{p}}} = \mathrm{ord}_P(\lambda)$.

3.6.2 $v|N$

Let x be a Heegner point on $X_0(N)$ represented by the pair of ideals $(\mathfrak{a}, \mathfrak{a}\mathfrak{n}^{-1})$, where $\mathfrak{n}$ is a divisor of N in L (cf. section 3.1).

a) Suppose that $v|\mathfrak{n}$, in particular let v divide the prime divisor $\mathfrak{p}$ of norm $\mathrm{N}_{L/K}(\mathfrak{p})$. The Artin reciprocity law in explicit class field theory ([Ha, (8.7)]) uses the fundamental congruence

$$j(\mathfrak{a}\mathfrak{p}^{-1}) \equiv j(\mathfrak{a})^{\mathrm{N}_{L/K}(\mathfrak{p})} \bmod v.$$

From this we see that $u - u(y_1)$ is again a local parameter of y_1 at v for a point y_1 on Y_N lying over x. Hence the calculations of the previous section, in particular equation (3.6.2), remain true in this situation.

b) Suppose that $v|\bar{\mathfrak{n}}$. Then the calculations of a) show that $w_N(u - u(y_1))$ is a local parameter of y_1 at v, where w_N denotes the canonical involution on $X_0(N)$ and Y_N. Hence

$$(y_1 \,.\, y_1)_v = v\Big(\frac{\partial w_N(u)}{\partial u}(u(y_1))\Big). \tag{3.6.3}$$

Using the fact that $u^{q+1} = j$ we get

$$(\frac{\partial w_N(u)}{\partial u})^{q+1} = (\frac{\partial w_N(j)}{\partial j})^{q+1} (\frac{j}{w_N(j)})^q. \tag{3.6.4}$$

If the Heegner point x is represented by $\tau \in \Omega$, then $w_N(j)(\tau) = j(N\tau)$. And we can evaluate the right hand side of (3.6.4) with $\frac{\partial w_N(j)}{\partial z}(\tau)$ and $\frac{\partial j}{\partial z}(\tau)$. For $\frac{\partial j}{\partial z}$ we use the definition $j = g^{q+1}/\Delta$ and get

$$\frac{\partial j}{\partial z} = \frac{g^q}{\Delta^2} (\frac{\partial g}{\partial z}\Delta - g\frac{\partial \Delta}{\partial z}).$$

$\frac{\partial g}{\partial z}\Delta - g\frac{\partial \Delta}{\partial z}$ can be expressed in terms of Δ (cf. equation (3.6.11)). For $\frac{\partial w_N(j)}{\partial z}$ we perform similar calculations. Hence we get

$$(\frac{\partial w_N(u)}{\partial u}(u(y_1)))^{q^2-1} = N^{q^2-1} (\frac{\Delta(N\tau)}{\Delta(\tau)})^2. \tag{3.6.5}$$

$\Delta(\tau)/\Delta(N\tau)$ is algebraic over L and its divisor is equal to $\bar{\mathfrak{n}}^{q^2-1}$ (we get this by calculations analogous to those in [Deu], sect. 13). With this fact and with (3.6.5) we can evaluate the value in (3.6.3). Together we get

Lemma 3.6.1 *If $v|\mathfrak{n}$, then*

$$(x \,.\, x)_v = 0,$$

and if $v|\bar{\mathfrak{n}}$, then

$$(x \,.\, x)_v = -v(N) = -1.$$

We can now summarize the results of the first two cases. We want to evaluate the height of Heegner points as in section 3.4, but without any restriction on $r_{\mathcal{A}}((\lambda))$. We combine the calculations in section 3.4 with the contributions from subsection 3.6.1 and Lemma 3.6.1, and we get

Proposition 3.6.2

$$\begin{aligned}\sum_{P\neq\infty} \langle (x) - (\infty), T_\lambda((x)^{\sigma_{\mathcal{A}}} - (0))\rangle_P \;&=\; \frac{\ln q}{q-1} \\ &\cdot\Bigg\{ (q-1)\, r_{\mathcal{A}}((\lambda))\, h_L\, (\deg N - \deg\lambda) \\ + \sum_{\substack{\mu\neq 0\\ \deg(\mu N)\leq\deg(\lambda D)}} & r_{\mathcal{A}}((\mu N - \lambda D))\, (1 - \delta_{(\mu N - \lambda D)\mu N}) \\ \cdot\left((t(\mu, D) + 1)(\sum_{c|\mu} \left[\frac{D}{c}\right] \deg c) - (\sum_{c|\mu} \left[\frac{D}{c}\right]) \deg\mu \right)&\Bigg\}.\end{aligned}$$

3.6.3 $v|\infty$

Let x be a Heegner point on $X_0(N)$ represented by $\tau \in \Omega$, and let $y_1, \dots, y_t$ be the points on Y_N lying over x. As above the projection formula yields

$$(x\,.\,x)_v = (y_1\,.\,y_1)_v + (y_1\,.\,y_2)_v + \cdots + (y_1\,.\,y_t)_v.$$

The self-intersection number on Y_N is by definition given as

$$(y_1\,.\,y_1)_v := \lim_{\tilde{y}\to y_1}((y_1\,.\,\tilde{y})_v - v(u(\tilde{y}) - u(y_1))).$$

Therefore, if $\tilde{y}$ on Y_N is mapped to $\tilde{x}$ on $X_0(N)$, we get

$$(x\,.\,x)_v := \lim_{\tilde{x}\to x}((x\,.\,\tilde{x})_v - v(u(\tilde{y}) - u(y_1))). \tag{3.6.6}$$

The point x is represented by $\tau \in \Omega$, let in addition $\tilde{x}$ be represented by $\tilde{\tau} \in \Omega$. At first we treat the case where $\deg D$ is odd. The local height pairing of x and $\tilde{x}$ at v is given by the Green's function $G(\tilde{\tau}, \tau)$ ([Ti3], cf. also section 3.5):

$$\begin{aligned} G(\tilde{\tau},\tau) &= \frac{1}{q-1} \sum_{\substack{\gamma\in\Gamma_0(N)\\ |\tau-\gamma\tilde{\tau}|^2\le|\tau|_i|\gamma\tilde{\tau}|_i}} \Big(\frac{q+1}{2(q-1)} - \log_q \frac{|\tau-\gamma\tilde{\tau}|^2}{|\tau|_i|\gamma\tilde{\tau}|_i}\Big) \\ &+ \frac{q+1}{2(q-1)^2} \lim_{\sigma\to 1}\Big(\sum_{\substack{\gamma\in\Gamma_0(N)\\ |\tau-\gamma\tilde{\tau}|^2>|\tau|_i|\gamma\tilde{\tau}|_i}} \big(\frac{|\tau-\gamma\tilde{\tau}|^2}{|\tau|_i|\gamma\tilde{\tau}|_i}\big)^{-\sigma} - \frac{C_1}{1-q^{1-\sigma}}\Big), \end{aligned} \tag{3.6.7}$$

where we normalize the absolute value such that $|f| = q^{\deg f} = q^{-v(f)}$ for $f \in \mathbb{F}_q[T]$.

The Green's function $G(\tilde{\tau}, \tau)$ contains two parts, the intersection number $(\tilde{x}\,.\,x)_v$ and the contribution of the fibre components (cf. (3.1.1)). We must replace $(\tilde{x}\,.\,x)_v$ by the self intersection number $(x\,.\,x)_v$. The contribution of the fibre components remains unchanged.

We have $u^{q+1} = j$ and $j = j(z)$ for $z \in \Omega$, this yields

$$\lim_{\tilde{y}\to y_1}(v(u(\tilde{y}) - u(y_1))) = v(\frac{\partial u}{\partial \tau}) + \lim_{\tilde{\tau}\to\tau}(v(\tilde{\tau} - \tau)). \tag{3.6.8}$$

Here $\frac{\partial u}{\partial \tau}$ only represents the two derivatives $\frac{\partial u}{\partial j}$ and $\frac{\partial j}{\partial \tau}$, we do not assume that Y_N is a quotient of Ω.

Now (3.6.6), (3.6.7) and (3.6.8) show that we have to replace $G(\tilde{\tau}, \tau)$ by

$$\begin{aligned} G(\tau,\tau) &:= \frac{1}{q-1} \sum_{\substack{\gamma\in\Gamma_0(N),\gamma\tau\ne\tau\\ |\tau-\gamma\tau|^2\le|\tau|_i|\gamma\tau|_i}} \Big(\frac{q+1}{2(q-1)} - \log_q \frac{|\tau-\gamma\tau|^2}{|\tau|_i|\gamma\tau|_i}\Big) \\ &+ \frac{q+1}{2(q-1)^2} \lim_{\sigma\to 1}\Big(\sum_{\substack{\gamma\in\Gamma_0(N)\\ |\tau-\gamma\tau|^2>|\tau|_i|\gamma\tau|_i}} \big(\frac{|\tau-\gamma\tau|^2}{|\tau|_i|\gamma\tau|_i}\big)^{-\sigma} - \frac{C_1}{1-q^{1-\sigma}}\Big) \\ &+ \frac{q+1}{2(q-1)} + 2\log_q(|\tau|_i|\frac{\partial u}{\partial \tau}|), \end{aligned}$$

When we compare the results in section 3.5 with these formulas, we get

$$\begin{aligned}\langle (x)-(\infty), T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle_\infty = &\text{ right hand side of eq. (3.5.1)}\\ &-2\ \ln q\ r_{\mathcal{A}}((\lambda))\sum_\tau \big(\log_q(|\tau|_i|\frac{\partial u}{\partial \tau}|)+\frac{q+1}{4(q-1)}\big),\end{aligned} \tag{3.6.9}$$

where we sum over all τ corresponding to the classes in O_L. We denote the second sum in (3.6.9) by S, which we will evaluate now.
We use the definitions $u^{q+1}=j$ and $j=g^{q+1}/\Delta$ to evaluate

$$(\frac{\partial u}{\partial z})^{q^2-1}=\Delta^{2-q^2-q}(\frac{\partial g}{\partial z}\Delta-g\frac{\partial \Delta}{\partial z})^{q^2-1}. \tag{3.6.10}$$

$(\frac{\partial g}{\partial z}\Delta-g\frac{\partial \Delta}{\partial z})^{q-1}$ is a modular form of weight $q(q^2-1)$ for the group $GL_2(\mathbb{F}_q[T])$, and it is therefore a polynomial in g and Δ (cf. [Go]). The evaluation of the expansion around the cusp yields the identity

$$(\frac{\partial g}{\partial z}\Delta-g\frac{\partial \Delta}{\partial z})^{q-1}=-\bar{\pi}^{1-q}\Delta^q, \tag{3.6.11}$$

where $\bar{\pi}$ is a well-defined element (cf. [Ge3]) with $\log_q|\bar{\pi}|=q/(q-1)$.
Now (3.6.10) and (3.6.11) yield

$$\log_q(|\tau|_i|\frac{\partial u}{\partial \tau}|)=\log_q(|\Delta(\tau)|^{2/(q^2-1)}|\tau|_i)-\frac{q}{q-1}.$$

Since $|\Delta(\tau)|^{2/(q^2-1)}|\tau|_i$ is invariant under $GL_2(\mathbb{F}_q[T])$, we can assume that τ satisfies $|\tau|=|\tau|_i>1$. For these τ we use the product formula for Δ (for all the details concerning the product formula we refer to [Ge3]):

$$\Delta(\tau)=-\bar{\pi}^{q^2-1}t(\tau)^{q-1}\prod_{\substack{a\in\mathbb{F}_q[T]\\ \text{monic}}} f_a(t(\tau))^{(q^2-1)(q-1)},$$

where

$$t(\tau)=(\bar{\pi}\tau\prod_{\substack{l\in\mathbb{F}_q[T]\\ l\neq 0}}(1-\frac{\tau}{l}))^{-1},$$

and where f_a are well-defined polynomials. Using the definitions of f_a and $t(\tau)$ we can show that in our case (i.e. $\tau\in K_\infty(\sqrt{D})$, $\deg D$ odd, $|\tau|=|\tau|_i>1$)

$$\log_q|t(\tau)|=-|\tau|_i q^{1/2}\frac{q+1}{2(q-1)}$$

and

$$\log_q|f_a(t(\tau))|=0.$$

Therefore

$$\log_q|\Delta(\tau)|=q(q+1)-\frac{1}{2}q^{1/2}(q+1)|\tau|_i. \tag{3.6.12}$$

Hence we get

$$\log_q(|\tau|_i|\frac{\partial u}{\partial \tau}|) = \frac{q}{q-1} - \frac{q^{1/2}}{q-1}|\tau|_i + \log_q |\tau|_i. \tag{3.6.13}$$

Now the definition of S in (3.6.9) and equation (3.6.13) yield

$$S = -2 \ \ln q \ r_{\mathcal{A}}((\lambda))\Big(\frac{5q+1}{4(q-1)}h_L + \sum_\tau(-\frac{q^{1/2}|\tau|_i}{q-1} + \log_q |\tau|_i)\Big). \tag{3.6.14}$$

On the other hand we consider the Eisenstein series

$$Ei_s(\tau) := \sum_{\substack{c,d\in\mathbb{F}_q[T]\\(c,d)\neq(0,0)}} \frac{|\tau|_i^s}{|c\tau+d|^{2s}}. \tag{3.6.15}$$

We see that

$$\sum_\tau Ei_s(\tau) = \frac{(q-1)|\sqrt{D}|^s}{1-q^{1-s}}L_D(s), \tag{3.6.16}$$

where $L_D(s)$ again is the non-trivial L-series of the extension L/K. Straightforward calculations of the sum in (3.6.15) show that $Ei_s(\tau)$ can be expressed as a rational function:

$$Ei_s(\tau) = (q-1)\frac{|\tau|_i^s}{1-q^{1-2s}} + \frac{q^{1/2}|\tau|_i^{1-s}}{1-q^{2-2s}}\big((q-1)^2\frac{q^{-s}}{1-q^{1-2s}} + q - 1\big). \tag{3.6.17}$$

With (3.6.16) and (3.6.17) we can evaluate the following term

$$\frac{2}{\ln q}\frac{L'_D(0)}{L_D(0)} = -\deg D + \frac{2q}{q-1} - \frac{2}{h_L}\sum_\tau\big(\frac{q^{1/2}|\tau|_i}{q-1} - \log_q |\tau|_i\big). \tag{3.6.18}$$

We compare equation (3.6.14) coming from values of Δ and equation (3.6.18) dealing with Eisenstein series, to get

$$S = \ln q \ r_{\mathcal{A}}((\lambda)) \ h_L\big(-\deg D - \frac{2}{\ln q}\frac{L'_D(0)}{L_D(0)} - \frac{q+1}{2(q-1)}\big). \tag{3.6.19}$$

This result can be seen as the Kronecker limit formula for function fields.

We summarize Propositions 3.5.1 and 3.5.2 and the result (3.6.9), (3.6.19) about S in the following proposition.

PROPOSITION 3.6.3 *Let* $\deg D$ *be odd, then*

$$\begin{aligned}
\langle (x)-(\infty), T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle_\infty \;&=\; \frac{\ln q}{q-1} \\
\cdot\Bigg\{ (q-1)\, r_{\mathcal{A}}((\lambda))\; h_L \left(-\deg D - \frac{q+1}{2(q-1)} - \frac{2}{\ln q}\,\frac{L_D'(0)}{L_D(0)}\right) \\
+ \sum_{\substack{\mu\neq 0\\ \deg(\mu N)\leq \deg(\lambda D)}} r_{\mathcal{A}}((\mu N-\lambda D))(\sum_{c\mid\mu}\left[\frac{D}{c}\right])\;(t(\mu,D)+1)\;\frac{1+\delta_{(\mu N-\lambda D)\mu N}}{2} \\
\cdot\,(\deg(\frac{\mu N}{\lambda D}) - \frac{q+1}{2(q-1)}) \\
-\frac{q+1}{2(q-1)}\lim_{\sigma\to 0}\Bigg(\sum_{\deg(\mu N)>\deg(\lambda D)} r_{\mathcal{A}}((\mu N-\lambda D))\;(\sum_{c\mid\mu}\left[\frac{D}{c}\right])\;(t(\mu,D)+1) \\
\cdot\, q^{(-\sigma-1)\deg(\frac{\mu N}{\lambda D})} - \frac{C_1\; h_L\;(\sum_{a\mid\lambda} q^{\deg a})}{1-q^{-\sigma}}\Bigg) \\
-\frac{q+1}{2(q-1)}\;C_1\;h_L\;(\sum_{a\mid\lambda} q^{\deg a}(\deg\lambda - 2\deg a)) + (\sum_{a\mid\lambda} q^{\deg a})\;C_2\Bigg\}
\end{aligned}$$

with

$$C_1 := \frac{2(q-1)^2}{[GL_2(\mathbb{F}_q[T]) : \Gamma_0(N)]}$$

and

$$C_2 := -\frac{2(q^2-1)}{[GL_2(\mathbb{F}_q[T]):\Gamma_0(N)]}\;h_L\left(\deg N - \sum_{\substack{(P)\mid(N)\\ (P)\ prime}}\frac{\deg P}{q^{\deg P}+1} - \frac{\deg D}{2} - \frac{2}{q-1} - \frac{1}{\ln q}\frac{L_D'(1)}{L_D(1)}\right).$$

Combining this with the results for the finite primes finally gives:

THEOREM 3.6.4 *Let* $\deg D$ *be odd and let* $g_{\mathcal{A}}$ *be the Drinfeld automorphic cusp form of Proposition 3.1.1. Then* $g_{\mathcal{A}}$ *has the Fourier coefficients for all* $\lambda\in\mathbb{F}_q[T]$

with $\gcd(\lambda, N) = 1$*:*

$$
\begin{aligned}
&g^*_{\mathcal{A}}(\pi_\infty^{\deg\lambda+2}, \lambda) = \frac{\ln q}{q-1}\, q^{-\deg\lambda} \\
&\cdot\Bigg\{ (q-1)\, r_{\mathcal{A}}((\lambda))\, h_L \left(\deg N - \deg(\lambda D) - \frac{q+1}{2(q-1)} - \frac{2}{\ln q}\, \frac{L'_D(0)}{L_D(0)} \right) \\
&+ \sum_{\substack{\mu\neq 0 \\ \deg(\mu N)\leq \deg(\lambda D)}} r_{\mathcal{A}}((\mu N - \lambda D)) \Bigg((\sum_{c|\mu} \left[\frac{D}{c}\right])\, (t(\mu, D)+1)\, \frac{1+\delta_{(\mu N-\lambda D)\mu N}}{2} \\
&\quad \cdot (\deg(\frac{\mu N}{\lambda D}) - \frac{q+1}{2(q-1)}) - (1-\delta_{(\mu N-\lambda D)\mu N})\, (\sum_{c|\mu}\left[\frac{D}{c}\right])\, \deg\mu \\
&\quad + (1-\delta_{(\mu N-\lambda D)\mu N})\, (t(\mu, D)+1)\, (\sum_{c|\mu}\left[\frac{D}{c}\right] \deg c) \Bigg) \\
&- \frac{q+1}{2(q-1)} \lim_{\sigma\to 0} \Bigg(\sum_{\deg(\mu N)>\deg(\lambda D)} r_{\mathcal{A}}((\mu N-\lambda D))\, (\sum_{c|\mu}\left[\frac{D}{c}\right])\, (t(\mu, D)+1) \\
&\quad \cdot q^{(-\sigma-1)\deg(\frac{\mu N}{\lambda D})} - \frac{C_1\, h_L\, (\sum_{a|\lambda} q^{\deg a})}{1-q^{-\sigma}} \Bigg) \\
&- \frac{q+1}{2(q-1)}\, C_1\, h_L\, (\sum_{a|\lambda} q^{\deg a}(\deg\lambda - 2\deg a)) + (\sum_{a|\lambda} q^{\deg a})\, C_2 \Bigg\}
\end{aligned}
$$

with

$$C_1 := \frac{2(q-1)^2}{[GL_2(\mathbb{F}_q[T]) : \Gamma_0(N)]}$$

and

$$
\begin{aligned}
C_2 := -\frac{2(q^2-1)}{[GL_2(\mathbb{F}_q[T]) : \Gamma_0(N)]}\, h_L \Bigg(&\deg N - \sum_{\substack{(P)|(N) \\ (P)\ prime}} \frac{\deg P}{q^{\deg P}+1} - \frac{\deg D}{2} \\
&- \frac{2}{q-1} - \frac{1}{\ln q}\frac{L'_D(1)}{L_D(1)} \Bigg).
\end{aligned}
$$

If $\deg D$ is even, the calculations are the same. We will present only the differences in the formulas to the first case. The calculations with the corresponding Green's function (cf. (3.6.9)) give

$$S = -2\, \ln q\, r_{\mathcal{A}}((\lambda)) \sum_\tau \big(\log_q(|\tau|_i |\frac{\partial u}{\partial \tau}|) + \frac{q}{q^2-1} \big).$$

Equation (3.6.12) has to be replaced by

$$\log_q |\Delta(\tau)| = q(q+1) - q\, |\tau|_i.$$

Hence (3.6.14) has the form

$$S = -2\ \ln q\, r_{\mathcal{A}}((\lambda))\Big(\frac{q^2+2q}{q^2-1}h_L + \sum_\tau (-\frac{2q|\tau|_i}{q^2-1} + \log_q|\tau|_i)\Big).$$

The definition (3.6.15) and the relation (3.6.16) remain unchanged, but the rational expression (3.6.17) becomes

$$Ei_s(\tau) = (q-1)\frac{|\tau|_i^s}{1-q^{1-2s}} + \frac{q|\tau|_i^{1-s}}{1-q^{2-2s}}\big((q-1)^2\frac{q^{-2s}}{1-q^{1-2s}} + q - 1\big).$$

Equation (3.6.18) has to be replaced by

$$\frac{2}{\ln q}\frac{L_D'(0)}{L_D(0)} = -\deg D + \frac{2q^2+2q}{q^2-1} - \frac{2}{h_L}\sum_\tau \big(\frac{2q|\tau|_i}{q^2-1} - \log_q|\tau|_i\big).$$

And finally we get:

Proposition 3.6.5 *Let* $\deg D$ *be even, then*

$$\begin{aligned}
\langle (x)-(\infty), T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle_\infty \ &= \ \frac{\ln q}{q-1} \\
\cdot\Bigg\{ (q-1)\ r_{\mathcal{A}}((\lambda))\ h_L\ \Bigg(-\deg D - \frac{2q}{q^2-1} - \frac{2}{\ln q}\,\frac{L_D'(0)}{L_D(0)}\Bigg) \\
+ \sum_{\substack{\mu\neq 0 \\ \deg(\mu N)\leq\deg(\lambda D)}} r_{\mathcal{A}}((\mu N-\lambda D))(\sum_{c|\mu}\left[\frac{D}{c}\right])\ (t(\mu,D)+1)\ \frac{1+\delta_{(\mu N-\lambda D)\mu N}}{2} \\
\cdot\ (\deg(\frac{\mu N}{\lambda D}) - \frac{2q}{q^2-1}) \\
-\frac{2q}{q^2-1}\lim_{\sigma\to 0}\Bigg(\sum_{\deg(\mu N)>\deg(\lambda D)} r_{\mathcal{A}}((\mu N-\lambda D))\ (\sum_{c|\mu}\left[\frac{D}{c}\right])\ (t(\mu,D)+1) \\
\cdot\ q^{(-\sigma-1)\deg(\frac{\mu N}{\lambda D})} - \frac{C_1\ h_L\ (\sum_{a|\lambda} q^{\deg a})}{1-q^{-\sigma}}\Bigg) \\
-\frac{2q}{q^2-1}\ C_1\ h_L\ (\sum_{a|\lambda} q^{\deg a}(\deg\lambda - 2\deg a)) + (\sum_{a|\lambda} q^{\deg a})\ C_2\Bigg\}
\end{aligned}$$

with

$$C_1 := \frac{(q^2-1)^2}{2q[GL_2(\mathbb{F}_q[T]):\Gamma_0(N)]}$$

and

$$C_2 := -\frac{2(q^2-1)}{[GL_2(\mathbb{F}_q[T]) : \Gamma_0(N)]}\ h_L \left(\deg N - \sum_{\substack{(P)|(N)\\ (P)\ prime}} \frac{\deg P}{q^{\deg P}+1} - \frac{\deg D}{2} - \frac{q+3}{q^2-1} - \frac{1}{\ln q}\frac{L'_D(1)}{L_D(1)} \right).$$

Combining this with the results for the finite places yields:

Theorem 3.6.6 *Let* $\deg D$ *be even and let* $g_{\mathcal{A}}$ *be the Drinfeld automorphic cusp form of Proposition 3.1.1. Then* $g_{\mathcal{A}}$ *has the Fourier coefficients for all* $\lambda \in \mathbb{F}_q[T]$ *with* $\gcd(\lambda, N) = 1$*:*

$$\begin{aligned}
g^*_{\mathcal{A}}(\pi_\infty^{\deg\lambda+2}, \lambda) &= \frac{\ln q}{q-1}\, q^{-\deg\lambda} \\
&\cdot \Bigg\{ r_{\mathcal{A}}((\lambda))\ h_L\ (q-1) \left(\deg N - \deg(\lambda D) - \frac{2q}{q^2-1} - \frac{2}{\ln q}\ \frac{L'_D(0)}{L_D(0)} \right) \\
&+ \sum_{\substack{\mu\neq 0\\ \deg(\mu N)\leq\deg(\lambda D)}} r_{\mathcal{A}}((\mu N - \lambda D)) \Bigg((\sum_{c|\mu} \left[\frac{D}{c}\right])\ (t(\mu, D)+1)\ \frac{1+\delta_{(\mu N-\lambda D)\mu N}}{2} \\
&\cdot (\deg(\frac{\mu N}{\lambda D}) - \frac{2q}{q^2-1}) - (1-\delta_{(\mu N-\lambda D)\mu N})\ (\sum_{c|\mu}\left[\frac{D}{c}\right])\ \deg\mu \\
&+ (1-\delta_{(\mu N-\lambda D)\mu N})\ (t(\mu, D)+1)\ (\sum_{c|\mu}\left[\frac{D}{c}\right]\deg c) \Bigg) \\
&- \frac{2q}{q^2-1}\lim_{\sigma\to 0} \Bigg(\sum_{\deg(\mu N)>\deg(\lambda D)} r_{\mathcal{A}}((\mu N-\lambda D))\ (\sum_{c|\mu}\left[\frac{D}{c}\right])\ (t(\mu, D)+1) \\
&\cdot q^{(-\sigma-1)\deg(\frac{\mu N}{\lambda D})} - C_1\ h_L\ (\sum_{a|\lambda} q^{\deg a}) \frac{1}{1-q^{-\sigma}} \Bigg) \\
&- \frac{2q}{q^2-1}\ C_1\ h_L\ (\sum_{a|\lambda} q^{\deg a}(\deg\lambda - 2\deg a)) + (\sum_{a|\lambda} q^{\deg a})\ C_2 \Bigg\}
\end{aligned}$$

with

$$C_1 := \frac{(q^2-1)^2}{2q[GL_2(\mathbb{F}_q[T]) : \Gamma_0(N)]}$$

and

$$C_2 := -\frac{2(q^2-1)}{[GL_2(\mathbb{F}_q[T]):\Gamma_0(N)]}\, h_L \left(\deg N - \sum_{\substack{(P)|(N)\\ (P)\ prime}} \frac{\deg P}{q^{\deg P}+1} - \frac{\deg D}{2} - \frac{q+3}{q^2-1} - \frac{1}{\ln q}\frac{L_D'(1)}{L_D(1)} \right).$$

4 Conclusion

4.1 Main Results

Now we combine the previous chapters on L-series (chapter 2) and on Heegner points (chapter 3). We recall the assumptions: $D \in \mathbb{F}_q[T]$ is an irreducible polynomial and $N \in \mathbb{F}_q[T]$ is a square free polynomial, whose prime divisors are split in the imaginary quadratic extension $K(\sqrt{D})/K$.
If $\deg D$ is odd, we evaluated in Theorem 2.8.2 the Fourier coefficients $\Psi^*_{\mathcal{A}}(\pi_\infty^{\deg\lambda+2},\lambda)$ of an automorphic cusp form $\Psi_{\mathcal{A}}$ of Drinfeld type with

$$\frac{\partial}{\partial s}(L^{(N,D)}(2s+1)L(f,\mathcal{A},s))\mid_{s=0} = \int\limits_{\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*} f\cdot\overline{\Psi_{\mathcal{A}}}.$$

On the other hand, in Theorem 3.6.4 we obtained the Fourier coefficients of $g_{\mathcal{A}}$, which are defined as

$$g^*_{\mathcal{A}}(\pi_\infty^{\deg\lambda+2},\lambda) = q^{-\deg\lambda}\langle (x)-(\infty), T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle.$$

If we compare the two formulas, we see that

$$\Psi^*_{\mathcal{A}}(\pi_\infty^{\deg\lambda+2},\lambda) = \frac{q-1}{2}\, q^{-(\deg D+1)/2}\, g^*_{\mathcal{A}}(\pi_\infty^{\deg\lambda+2},\lambda)$$

for all $\lambda\in\mathbb{F}_q[T]$ with $\gcd(\lambda,N)=1$. Hence the two automorphic cusp forms $\Psi_{\mathcal{A}}$ and $(q-1)/2\cdot q^{-(\deg D+1)/2}\, g_{\mathcal{A}}$ differ only by an old form. Since f is a newform, the occurring old form does not affect the integral. And this can be summarized by the following main result:

Theorem 4.1.1 *Let $\deg D$ be odd. Let x be a Heegner point on $X_0(N)$ with complex multiplication by $O_L=\mathbb{F}_q[T][\sqrt{D}]$, let $\mathcal{A}\in Cl(O_L)$, and let $g_{\mathcal{A}}$ be the automorphic cusp form of Drinfeld type of level N, which is given by*

$$(T_\lambda g_{\mathcal{A}})^*(\pi_\infty^2,1) = \langle (x)-(\infty), T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle$$

for all $\lambda\in\mathbb{F}_q[T]$. Let f be a newform of level N, then

$$\frac{\partial}{\partial s}(L^{(N,D)}(2s+1)L(f,\mathcal{A},s))\mid_{s=0} = \frac{q-1}{2}\, q^{-(\deg D+1)/2} \int\limits_{\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*} f\cdot\overline{g_{\mathcal{A}}}.$$

If $\deg D$ is even, we have to compare Theorem 2.8.3 and Theorem 3.6.6. Let $\Psi_{\mathcal{A}}$ be defined by

$$\frac{\partial}{\partial s}\left(\frac{1}{1+q^{-s-1}} L^{(N,D)}(2s+1)L(f,\mathcal{A},s)\right)|_{s=0} = \int_{\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*} f\cdot\overline{\Psi_{\mathcal{A}}}.$$

The Fourier coefficients of $\Psi_{\mathcal{A}}$ and $g_{\mathcal{A}}$ satisfy

$$\Psi_{\mathcal{A}}^*(\pi_\infty^{\deg\lambda+2},\lambda) = \frac{q-1}{4}\, q^{-\deg D/2}\, g_{\mathcal{A}}^*(\pi_\infty^{\deg\lambda+2},\lambda)$$

for all $\lambda\in\mathbb{F}_q[T]$ with $\gcd(\lambda,N)=1$. Hence we have

THEOREM 4.1.2 *Let $\deg D$ be even. Let x be a Heegner point on $X_0(N)$ with complex multiplication by $O_L=\mathbb{F}_q[T][\sqrt{D}]$, let $\mathcal{A}\in Cl(O_L)$, and let $g_{\mathcal{A}}$ be the automorphic cusp form of Drinfeld type of level N, which is given by*

$$(T_\lambda g_{\mathcal{A}})^*(\pi_\infty^2,1) = \langle (x)-(\infty), T_\lambda((x)^{\sigma_{\mathcal{A}}}-(0))\rangle$$

for all $\lambda\in\mathbb{F}_q[T]$. Let f be a newform of level N, then

$$\frac{\partial}{\partial s}\left(\frac{1}{1+q^{-s-1}} L^{(N,D)}(2s+1)L(f,\mathcal{A},s)\right)|_{s=0} = \frac{q-1}{4}\, q^{-\deg D/2}\int_{\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*} f\cdot\overline{g_{\mathcal{A}}}.$$

4.2 APPLICATION TO ELLIPTIC CURVES

We want to apply our main results to elliptic curves. Therefore we assume in addition that the newform f is an eigenform for all Hecke operators. So far we haven't required this condition in our calculations.
Let χ be a character of the class group $Cl(O_L)$. If $\deg D$ is odd, we define

$$L(f,\chi,s) := \sum_{\mathcal{A}\in Cl(O_L)} \chi(\mathcal{A})\; L^{(N,D)}(2s+1)L(f,\mathcal{A},s).$$

Then Theorem 4.1.1 yields immediately

$$\frac{\partial}{\partial s}L(f,\chi,s)\,|_{s=0} = \frac{q-1}{2}\, q^{-(\deg D+1)/2}\int_{\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*} f\cdot\overline{\sum_{\mathcal{A}}\chi(\mathcal{A})g_{\mathcal{A}}}. \tag{4.2.1}$$

Note that $\sum_{\mathcal{A}}\chi(\mathcal{A})g_{\mathcal{A}}$ is an automorphic cusp form which satisfies (cf. the definition of $g_{\mathcal{A}}$ in Proposition 3.1.1)

$$(T\sum_{\mathcal{A}}\chi(\mathcal{A})g_{\mathcal{A}})^*(\pi_\infty^2,1) = \sum_{\mathcal{A}}\chi(\mathcal{A})\langle (x)-(\infty), T((x)^{\sigma_{\mathcal{A}}}-(0))\rangle \tag{4.2.2}$$

for each Hecke operator T. Exactly the same calculations as in [Gr-Za], p. 308 show that (4.2.2) can be computed as

$$(T\sum_{\mathcal{A}}\chi(\mathcal{A})g_{\mathcal{A}})^*(\pi_\infty^2,1)=h_L^{-1}\langle c_\chi, Tc_\chi\rangle,$$

where $c_\chi = \sum_{\mathcal{A}}\chi^{-1}(\mathcal{A})((x)-(\infty))^{\sigma_{\mathcal{A}}}$ is an element in the Jacobian $J_0(N)(H)\otimes\mathbb{C}$. Here we used the fact that $(0)-(\infty)$ is an element of finite order in $J_0(N)$ (cf. [Ge2, Satz 4.1]).

Let $\{f_i\}$ be a basis of the space of automorphic cusp forms of Drinfeld type of level N which consists of normalized newforms together with a basis of the space of oldforms. We assume that $f_1 = f$. And let $c_\chi = \sum_i c_\chi^{(i)}$ be the decomposition of c_χ in f_i-isotypical components (i.e. components, where the Hecke operators act by multiplication of the corresponding Hecke eigenvalues). Then again as in [Gr-Za], p. 308 we get

$$\sum_{\mathcal{A}}\chi(\mathcal{A})g_{\mathcal{A}} = h_L^{-1}\sum_{i,j}\langle c_\chi^{(i)}, c_\chi^{(j)}\rangle f_j. \tag{4.2.3}$$

Since f is a newform, we have $\langle c_\chi^{(i)}, c_\chi^{(1)}\rangle = 0$ for $i\neq 1$. Then equations (4.2.1) and (4.2.3) yield:

Corollary of Theorem 4.1.1 *If* $\deg D$ *is odd, then*

$$\frac{\partial}{\partial s}L(f,\chi,s)\mid_{s=0} \;=\; \frac{q-1}{2}\,q^{-(\deg D+1)/2}\;\cdot\; h_L^{-1}\,\langle c_\chi^{(1)}, c_\chi^{(1)}\rangle \int\limits_{\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*} f\cdot\overline{f}.$$

If $\deg D$ is even we define

$$L(f,\chi,s) := \sum_{\mathcal{A}\in Cl(O_L)}\chi(\mathcal{A})\,\frac{1}{1+q^{-s-1}}L^{(N,D)}(2s+1)L(f,\mathcal{A},s),$$

and we get analogously:

Corollary of Theorem 4.1.2 *If* $\deg D$ *is even, then*

$$\frac{\partial}{\partial s}L(f,\chi,s)\mid_{s=0} \;=\; \frac{q-1}{4}\,q^{-\deg D/2}\;\cdot\; h_L^{-1}\,\langle c_\chi^{(1)}, c_\chi^{(1)}\rangle \int\limits_{\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*} f\cdot\overline{f}.$$

Now let E be an elliptic curve with conductor $N\cdot\infty$, which has split multiplicative reduction at ∞, and let f be the corresponding newform as in section 2.1. We have already seen that the L-series of E over the imaginary quadratic field L

satisfies (with the notations of this section) $L(E,s{+}1)L(E_D,s{+}1) = L(f,\chi_0,s)$, where χ_0 is the trivial character of $Cl(O_L)$.
Let $\pi : X_0(N) \to E$ be a uniformization (cf. [Ge-Re], (8)) which maps the point ∞ on $X_0(N)$ to the zero on E. The two homomorphisms $\pi_* : J_0(N) \to E$ and $\pi^* : E \to J_0(N)$ are related by the formula $\pi_* \circ \pi^* = \deg \pi$. On $J_0(N)$ we consider the elliptic curve $E' = \pi^*(E)$. Then $\pi_{*|E'}$ and π^* are dual isogenies of E and E', in particular we get

$$\pi^* \circ \pi_{*|E'} = \deg \pi. \tag{4.2.4}$$

For a Heegner point x on $X_0(N)$ let $P_L := \sum_{\mathcal{A}\in Cl(O_L)} \pi(x^{\sigma_{\mathcal{A}}})$ be the corresponding Heegner point on E. The component $c^{(1)}_{\chi_0}$ lies on E' and we have

$$\pi_{*|E'}(c^{(1)}_{\chi_0}) = P_L. \tag{4.2.5}$$

The points P_L on E and $c^{(1)}_{\chi_0}$ on $J_0(N)$ are both defined over the field L. Let $\hat{h}_{E,L}$ be the canonical Néron-Tate height of E over L, analogously we consider $\hat{h}_{J_0(N),L}$. If we apply the projection formula and equations (4.2.4) and (4.2.5) we get

$$\begin{aligned} \deg\pi \cdot \hat{h}_{J_0(N),L}(c^{(1)}_{\chi_0}) &= \langle \deg\pi \cdot c^{(1)}_{\chi_0}, c^{(1)}_{\chi_0}\rangle_{J_0(N),L} \\ &= \langle \pi^* \circ \pi_{*|E'}(c^{(1)}_{\chi_0}), c^{(1)}_{\chi_0}\rangle_{J_0(N),L} \\ &= \langle \pi_{*|E'}(c^{(1)}_{\chi_0}), \pi_{*|E'}(c^{(1)}_{\chi_0})\rangle_{E,L} \\ &= \hat{h}_{E,L}(P_L). \end{aligned} \tag{4.2.6}$$

Since the height pairing $\langle\ ,\ \rangle$ is normalized for the Hilbert class field H (cf. section 3.1), we have

$$\langle c^{(1)}_{\chi_0}, c^{(1)}_{\chi_0}\rangle = h_L \cdot \hat{h}_{J_0(N),L}(c^{(1)}_{\chi_0}). \tag{4.2.7}$$

Now (4.2.6), (4.2.7) and the two corollaries yield in the case of elliptic curves:

Theorem 4.2.1 *Let E be an elliptic curve with conductor $N\cdot\infty$, which has split multiplicative reduction at ∞, with corresponding newform f as above, let $P_L \in E(L)$ be the Heegner point given by the parametrization $\pi : X_0(N) \to E$. Then the derivative of the L-series of E over L and the canonical height $\hat{h}_{E,L}(P_L)$ are related by the formula*

$$\frac{\partial}{\partial s}(L(E,s)L(E_D,s))\,|_{s=1} = \hat{h}_{E,L}(P_L)\; c(D)\; (\deg\pi)^{-1} \int\limits_{\Gamma_0(N)\backslash GL_2(K_\infty)/\Gamma_\infty K_\infty^*} f\cdot\overline{f},$$

where the constant

$$c(D) := \begin{cases} \frac{q-1}{2}\, q^{-(\deg D+1)/2} & \text{if } \deg D \text{ is odd,} \\ \frac{q-1}{4}\, q^{-\deg D/2} & \text{if } \deg D \text{ is even.} \end{cases}$$

Finally we mention just one consequence of Theorem 4.2.1. The L-series $L(E,s)\cdot L(E_D,s)$ of E over the field L has a zero at $s=1$ according to the functional equations of section 2.7. In the function field case it is known ([Ta], [Sh]) that the analytic rank of E/L is not smaller than the Mordell-Weil rank of $E(L)$. Therefore Theorem 4.2.1 implies

Corollary 4.2.2 *If*

$$\frac{\partial}{\partial s}(L(E,s)L(E_D,s))\mid_{s=1}\neq 0,$$

then the Birch and Swinnerton-Dyer conjecture is true for E, i.e. the analytic rank and the Mordell-Weil rank of E/L are both equal to 1.

Remarks.
1) In [Br] Brown proved the Birch and Swinnerton-Dyer conjecture in the case that the Heegner point has infinite order. And he conjectured that this assumption is true if and only if the first derivative of the L-series does not vanish at the point 1. Theorem 4.2.1 proves his conjecture.
2) Milne ([Mi]) showed that the equality of the analytic rank and the Mordell-Weil rank implies even the strong Birch and Swinnerton-Dyer conjecture. Therefore in our case the assumption of Corollary 4.2.2 implies

$$\frac{\partial}{\partial s}L^*(E/L,s)\mid_{s=1}=\frac{\#Ш\cdot\hat{h}_{E,L}(P_0)}{(\#E(L)_{\mathrm{tors}})^2},$$

where $L^*(E/L,s)$ is the modified L-series of the elliptic curve E over the field L (cf. [Ta], [Mi]), P_0 is a generator of the free part of the Mordell-Weil group $E(L)$ and $Ш$ is the Tate-Shafarevich group of E/L.

References

[AB] *Proceedings of the Workshop on Drinfeld Modules, Modular Schemes and Applications in Alden-Biesen*, Gekeler et al. (Eds.), World Scientific, 1996.

[Ar] E. Artin, *Quadratische Körper im Gebiete der höheren Kongruenzen II.*, Math. Zeitschrift 19 (1924), 207-246.

[Br] M. L. Brown, *On a conjecture of Tate for elliptic surfaces over finite fields*, Proc. London Math. Soc.(3), 69 (1994), no.3, 489-514

[De] P. Deligne, *Les constantes des équations fonctionelles des fonctions L*, in: Modular functions of one variable II, LNM 349, Springer (1973), 501-597.

[Deu] M. Deuring, *Die Klassenkörper der komplexen Multiplikation*, Enzyklopädie der Math. Wiss. Band I, 2. Teil, Heft 10, Teil II.

[Dr] V. G. Drinfeld, *Coverings of $p-$adic symmetric regions*, Funct. Anal. Appl. 10 (1976), 29-40.

[De-Hu] P. Deligne, D. Husemöller, *Survey of Drinfeld modules*, Contemp. Math. 67 (1987), 25-91.

[Ei] M. Eichler, *Zur Zahlentheorie der Quaternionen-Algebren*, Journal reine angew. Math. 195 (1956), 127-151.

[Ge1] E.-U. Gekeler, *Drinfeld modular curves*, LNM 1231, Springer, 1986.

[Ge2] E.- U. Gekeler, *Über Drinfeldsche Modulkurven vom Hecke-Typ*, Compositio Mathematica 57 (1986), 219-236.

[Ge3] E.- U. Gekeler, *Analytic Construction of Weil Curves over Function Fields*, J. Th. Nomb. Bordeaux 7 (1995), 27-49.

[Ge4] E.- U. Gekeler, *On Finite Drinfeld Modules*, J. of Algebra 141 (1991), 187-203.

[Ge-Re] E.-U. Gekeler, M. Reversat, *Jacobians of Drinfeld modular curves*, Journal reine angew. Math. 476 (1996), 27-93.

[Ger-vdP] L. Gerritzen, M. van der Put, *Schottky groups and Mumford curves*, LNM 817, Springer.

[Go] D. Goss, *Modular Forms for* $\mathbb{F}_r[T]$, Journal reine angew. Math. 317 (1980), 16-39.

[Gr1] B. Gross, *Local heights on curves* in "Arithmetic geometry", (ed. Cornell/Silverman) Springer 1986.

[Gr2] B. Gross, *On canonical and quasi-canonical liftings*, Invent Math. 84 (1986), 321-326.

[Gr-Za] B. Gross, D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. 84 (1986), 225-320.

[Ha] D. R. Hayes, *Explicit class field theory in global function fields*, Studies in Algebra and Number Theory, Advances in Mathematics supplementary studies vol.6, Academic Press (1979), 173-217.

[Ka-Ma] N. Katz, B. Mazur, *Arithmetic moduli of elliptic curves*, Princeton, Princeton University Press, 1985.

[Mi] J. S. Milne, *On a conjecture of Artin-Tate*, Ann. of Math. 102 (1975), 517-533

[Rü1] H.-G. Rück, *Theta Series of Imaginary Quadratic Function Fields*, manuscripta math. 88 (1995), 387-407.

[Rü2] H.-G. Rück, *Poincaré Series of Drinfeld Type*, Arch. Math. 68 (1997), 190-201.

[Rü3] H.-G. Rück, *L-series of automorphic cusp forms of Drinfeld type*, Proceedings of the Workshop on Drinfeld modules, modular schemes and applications, Alden-Biesen, World Scientific, 1996, 311-329.

[Rü-Ti] H.-G. Rück, U. Tipp, *A Gross-Zagier formula for function fields*, in "Algorithmic Algebra and Number Theory" (ed. Matzat et al.), Lectures given at a DFG-Conference at Heidelberg University in October 1997, Springer.

[Sha] I. R. Shafarevich, *Lectures on minimal models and birational transformations of two dimensional schemes*, Tata Institute, Bombay (1966).

[Sh] T. Shioda, *Some remarks on elliptic curves over function fields*, Astérisque 209 (1992), 99-114.

[Ta] J. Tate, *On the conjectures of Birch and Swinnerton-Dyer and a geometric analog*, Sém. Bourbaki 1965/66 in: Dix Exposés sur la Cohomologie des Schemas, Amsterdam, North Holland, 1968.

[Ti1] U. Tipp, *Höhenberechnungen von Heegnerpunkten auf Drinfeldschen Modulkurven*, Dissertation, Preprint No.10, IEM Essen, 1996.

[Ti2] U. Tipp, *Local height pairings of Heegner points on Drinfeld modular curves*, Proceedings of the Workshop on Drinfeld modules, modular schemes and applications, Alden-Biesen, World Scientific, 1996, 344-356.

[Ti3] U.Tipp, *Green's functions for Drinfeld modular curves*, J. of Number Theory 77 (1999), 171-199.

[We1] A. Weil, *Dirichlet Series and Automorphic Forms*, LNM 189, Springer 1971.

[We2] A. Weil, *On the Analogue of the Modular group in Characteristic p*, in "Functional Analysis, etc", Proc. Conf. in honor of M. Stone, Springer 1970, 211-223.

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