Compact Moduli Spaces of Stable Sheaves over Non-Algebraic Surfaces

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Abstract. We show that under certain conditions on the topological invariants, the moduli spaces of stable bundles over polarized non-algebraic surfaces may be compactified by allowing at the border isomorphy classes of stable non-necessarily locally-free sheaves. As a consequence, when the base surface is a primary Kodaira surface, we obtain examples of moduli spaces of stable sheaves which are compact holomorphically symplectic manifolds.

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1 Introduction

Moduli spaces of stable vector bundles over polarized projective complex surfaces have been intensively studied. They admit projective compactifications which arise naturally as moduli spaces of semi-stable sheaves and a lot is known on their geometry. Apart from their intrinsic interest, these moduli spaces also provided a series of applications, the most spectacular of which being to Donaldson theory.

When one looks at non-algebraic complex surfaces, one still has a notion of stability for holomorphic vector bundles with respect to Gauduchon metrics on the surface and one gets the corresponding moduli spaces as open parts in the moduli spaces of simple sheaves. In order to compactify such a moduli space one may use the Kobayashi-Hitchin correspondence and the Uhlenbeck compactification of the moduli space of Hermite-Einstein connections. But the spaces one obtains in this way have a priori only a real-analytic structure. A different compactification method using isomorphy classes of vector bundles on blown-up surfaces is proposed by Buchdahl in [5] in the case of rank two vector

bundles or for topological invariants such that no properly semi-stable vector bundles exist.
In this paper we prove that under this last condition one may compactify the moduli space of stable vector bundles by considering the set of isomorphy classes of stable sheaves inside the moduli space of simple sheaves. See Theorem 4.3 for the precise formulation. In this way one gets a complex-analytic structure on the compactification. The idea of the proof is to show that the natural map from this set to the Uhlenbeck compactification of the moduli space of anti-self-dual connections is proper. We have restricted ourselves to the situation of anti-self-dual connections, rather than considering the more general Hermite-Einstein connections, since our main objective was to construct compactifications for moduli spaces of stable vector bundles over non-Kählerian surfaces. (In this case one can always reduce oneself to this situation by a suitable twist). In particular, when X is a primary Kodaira surface our compactness theorem combined with the existence results of [23] and [1] gives rise to moduli spaces which are holomorphically symplectic compact manifolds. Two ingredients are needed in the proof: a smoothness criterion for the moduli space of simple sheaves and a non-disconnecting property of the border of the Uhlenbeck compactification which follows from the gluing techniques of Taubes.
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2 PRELIMINARIES

Let X be a compact (non-singular) complex surface. By a result of Gauduchon any hermitian metric on X is conformally equivalent to a metric g with $\partial\bar{\partial}$-closed Kähler form ω. We call such a metric a GAUDUCHON METRIC and fix one on X. We shall call the couple (X, g) or (X, ω) a POLARIZED SURFACE and ω the POLARIZATION. One has then a notion of stability for torsion-free coherent sheaves.

DEFINITION 2.1 A torsion-free coherent sheaf $\mathcal{F}$ on X is called REDUCIBLE if it admits a coherent subsheaf $\mathcal{F}'$ with $0 < \operatorname{rank} \mathcal{F}' < \operatorname{rank} \mathcal{F}$, (and IRREDUCIBLE otherwise). A torsion-free sheaf $\mathcal{F}$ on X is called STABLY IRREDUCIBLE if every torsion-free sheaf $\mathcal{F}'$ with

$$\operatorname{rank}(\mathcal{F}') = \operatorname{rank}(\mathcal{F}),\ c_1(\mathcal{F}') = c_1(\mathcal{F}),\ c_2(\mathcal{F}') \leq c_2(\mathcal{F})$$

is irreducible.

Remark that if X is algebraic (and thus projective), every torsion-free coherent sheaf $\mathcal{F}$ on X is reducible. But by [2] and [22] there exist irreducible rank-two holomorphic vector bundles on any non-algebraic surface. Moreover stably irreducible bundles have been constructed on 2-dimensional tori and on primary Kodaira surfaces in [23], [24] and [1].

We recall that on a non-algebraic surface the DISCRIMINANT of a rank r torsion-free coherent sheaf which is defined by

$$\Delta(\mathcal{F}) = \frac{1}{r}\Big(c_2(\mathcal{F}) - \frac{(r-1)}{2r}c_1(\mathcal{F})^2\Big)$$

is non-negative [2].
Let $\mathcal{M}^{st}(E, L)$ denote the moduli space of stable holomorphic structures in a vector bundle E of rank $r > 1$, determinant $L \in \mathrm{Pic}(X)$ and second Chern class $c \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$. We consider the following condition on $(r, c_1(L), c)$:

(*) *every semi-stable vector bundle* $\mathcal{E}$ *with* $\mathrm{rank}(\mathcal{E}) = r$, $c_1(\mathcal{E}) = c_1(L)$ *and* $c_2(\mathcal{E}) \le c$ *is stable.*

Under this condition Buchdahl constructed a compactification of $\mathcal{M}^{st}(E, L)$ in [5]. We shall show that under this same condition one can compactify $\mathcal{M}^{st}(E, L)$ allowing simple coherent sheaves in the border. For simplicity we shall restrict ourselves to the case $\deg_\omega L = 0$. When $b_1(X)$ is odd we can always reduce ourselves to this case by a suitable twist with a topologically trivial line bundle; (see the following Remark).
The condition (*) takes a different aspect according to the parity of the first Betti number of X or equivalently, according to the existence or non-existence of a Kähler metric on X.

REMARK 2.2 (a) When $b_1(X)$ is odd (*) is equivalent to: "every torsion free sheaf $\mathcal{F}$ on X with $\mathrm{rank}(\mathcal{F}) = r$, $c_1(\mathcal{F}) = c_1(L)$ and $c_2(\mathcal{F}) \le c$ is irreducible", i.e. $(r, c_1(L), c)$ describes the topological invariants of a stably irreducible vector bundle.

(b) When $b_1(X)$ is even and $c_1(L)$ is not a torsion class in $H^2(X, \mathbb{Z}_r)$ one can find a Kähler metric g such that $(r, c_1(L), c)$ satisfies (*) for all c.

(c) When $b_1(X)$ is odd or when $\deg L = 0$, (*) implies $c < 0$.

(d) If $b_2(X) = 0$ then there is no torsion-free coherent sheaf on X whose invariants satisfy (*).

PROOF It is clear that the stable irreducibility condition is stronger than (*). Now if a sheaf $\mathcal{F}$ is not irreducible it admits some subsheaf $\mathcal{F}'$ with $0 < \mathrm{rank}\,\mathcal{F}' < \mathrm{rank}\,\mathcal{F}$. When $b_1(X)$ is odd the degree function $\deg_\omega : \mathrm{Pic}^0(X) \longrightarrow \mathbb{R}$ is surjective, so twisting by suitable invertible sheaves $L_1, L_2 \in \mathrm{Pic}^0(X)$ gives a semi-stable but not stable sheaf $(L_1 \otimes \mathcal{F}') \oplus (L_2 \otimes (\mathcal{F}/\mathcal{F}'))$ with the same Chern classes as $\mathcal{F}$. Since by taking double-duals the second Chern class decreases, we get a locally free sheaf

$$(L_1 \otimes (\mathcal{F}')^{\vee\vee}) \oplus (L_2 \otimes (\mathcal{F}/\mathcal{F}')^{\vee\vee})$$

which contradicts (*) for $(\mathrm{rank}(\mathcal{F}), c_1(\mathcal{F}), c_2(\mathcal{F}))$. This proves (a).

For (b) it is enough to take a Kähler class ω such that

$$\omega(r' \cdot c_1(L) - r \cdot \alpha) \neq 0 \text{ for all } \alpha \in NS(X)/\mathrm{Tors}(NS(X))$$

and integers r' with $0 < r' < r$. This is possible since the Kähler cone is open in $H^{1,1}(X)$.
For (c) just consider $(L \otimes L_1) \oplus \mathcal{O}_X^{\otimes(r-1)}$ for a suitable $L_1 \in \mathrm{Pic}^0(X)$ in case $b_1(X)$ odd. Finally, suppose $b_2(X) = 0$. Then X admits no Kähler structure hence $b_1(X)$ is odd. If $\mathcal{F}$ were a coherent sheaf on X whose invariants satisfy (*) we should have

$$\Delta(\mathcal{F}) = \frac{1}{r}\Big(c_2 - \frac{(r-1)}{2r}c_1(L)^2\Big) = \frac{1}{r}c_2 < 0$$

contradicting the non-negativity of the discriminant. □

3 The moduli space of simple sheaves

The existence of a coarse moduli space Spl_X for simple (torsion-free) sheaves over a compact complex space has been proved in [12] ; see also [19]. The resulting complex space is in general non-Hausdorff but points representing stable sheaves with respect to some polarization on X are always separated.
In order to give a better description of the base of the versal deformation of a coherent sheaf $\mathcal{F}$ we need to compare it to the deformation of its determinant line bundle $\det \mathcal{F}$. We first establish

Proposition 3.1 *Let X be a nonsingular compact complex surface, $(S, 0)$ a complex space germ, $\mathcal{F}$ a coherent sheaf on $X \times S$ flat over S and $q : X \times S \to X$ the projection. If the central fiber $\mathcal{F}_0 := \mathcal{F}|_{X \times \{0\}}$ is torsion-free then there exists a locally free resolution of $\mathcal{F}$ over $X \times S$ of the form*

$$0 \longrightarrow q^*G \longrightarrow E \longrightarrow \mathcal{F} \longrightarrow 0$$

where G is a locally free sheaf on X.

Proof In [20] it is proven that a resolution of $\mathcal{F}_0$ of the form

$$0 \longrightarrow G \longrightarrow E_0 \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

exists on X with G and E_0 locally free on X as soon as the rank of G is large enough and

$$H^2(X, \mathcal{H}om(\mathcal{F}_0, G)) = 0.$$

We only have to notice that when $\mathcal{F}_0$ and G vary in some flat families over S then one can extend the above exact sequence over $X \times S$. We choose S to be Stein and denote by $p : X \times S \to S$ the projection.

From the spectral sequence relating the relative and global Ext-s we deduce the surjectivity of the natural map

$$\mathrm{Ext}^1(X \times S; \mathcal{F}, q^*G) \longrightarrow H^0(S, \mathcal{E}xt^1(p; \mathcal{F}, q^*G)).$$

We can apply the base change theorem for the relative Ext^1 sheaf if we know that $\mathrm{Ext}^2(X; \mathcal{F}_0, G) = 0$ (cf. [3] Korollar 1). But in the spectral sequence

$$H^p(X, \mathcal{E}xt^q(\mathcal{F}_0, G)) \Longrightarrow \mathrm{Ext}^{p+q}(X; \mathcal{F}_0, G)$$

relating the local Ext $-s$ to the global ones, all degree two terms vanish since $H^2(X; \mathcal{H}om(\mathcal{F}_0, G)) = 0$ by assumption. Thus by base change

$$\mathrm{Ext}^1(X; \mathcal{F}_0, G) \cong \mathcal{E}xt^1(p; \mathcal{F}, q^*G)_0 \Big/ \mathfrak{m}_{S,0} \cdot \mathcal{E}xt^1(p; \mathcal{F}, q^*G)$$

and the natural map

$$\mathrm{Ext}^1(X \times S; \mathcal{F}, q^*G) \longrightarrow \mathrm{Ext}^1(X; \mathcal{F}_0, G)$$

given by restriction is surjective. □

Let X, S and $\mathcal{F}$ be as above. One can use Proposition 3.1 to define a morphism

$$\det : (S, 0) \longrightarrow (\mathrm{Pic}(X), \det \mathcal{F}_0)$$

by associating to $\mathcal{F}$ its determinant line bundle $\det \mathcal{F}$.

The tangent space at the isomorphy class $[\mathcal{F}] \in Spl_X$ of a simple sheaf $\mathcal{F}$ is $\mathrm{Ext}^1(X; \mathcal{F}, \mathcal{F})$ since Spl_X is locally around $[\mathcal{F}]$ isomorphic to the base of the versal deformation of $\mathcal{F}$. The space of obstructions to the extension of a deformation of $\mathcal{F}$ is $\mathrm{Ext}^2(X; \mathcal{F}, \mathcal{F})$.

In order to state the next theorem which compares the deformations of $\mathcal{F}$ and $\det \mathcal{F}$, we have to recall the definition of the trace maps

$$tr^q : \mathrm{Ext}^q(X; \mathcal{F}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{O}_X).$$

When $\mathcal{F}$ is locally free one defines $tr_{\mathcal{F}} : \mathcal{E}nd(\mathcal{F}) \longrightarrow \mathcal{O}_X$ in the usual way by taking local trivializations of $\mathcal{F}$. Suppose now that $\mathcal{F}$ has a locally free resolution $F^\bullet$. (See [21] and [10] for more general situations.) Then one defines

$$tr_{F^\bullet} : \mathcal{H}om^\bullet(F^\bullet, F^\bullet) \longrightarrow \mathcal{O}_X$$

by

$$tr_{F^\bullet} |_{\mathcal{H}om(F^i, F^j)} = \begin{cases} (-1)^i tr_{F^i} , & \text{for } i = j \\ 0 , & \text{for } i \neq j. \end{cases}$$

Here we denoted by $\mathcal{H}om^\bullet(F^\bullet, F^\bullet)$ the complex having $\mathcal{H}om^n(F^\bullet, F^\bullet) = \bigoplus_i \mathcal{H}om(F^i, F^{i+n})$ and differential

$$d(\varphi) = d_{F^\bullet} \circ \varphi - (-1)^{\deg \varphi} \cdot \varphi \circ d_{F^\bullet}$$

for local sections $\varphi \in \mathcal{H}om^n(F^\bullet, F^\bullet)$. $tr_{F^\bullet}$ becomes a morphism of complexes if we see $\mathcal{O}_X$ as a complex concentrated in degree zero. Thus $tr_{F^\bullet}$ induces morphisms at hypercohomology level. Since the hypercohomology groups of $\mathrm{Hom}^\bullet(F^\bullet, F^\bullet)$ and of $\mathcal{O}_X$ are $\mathrm{Ext}^q(X; \mathcal{F}, \mathcal{F})$ and $H^q(X, \mathcal{O}_X)$ respectively, we get our desired maps

$$tr^q : \mathrm{Ext}^q(X; \mathcal{F}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{O}_X).$$

Using tr^0 over open sets of X we get a sheaf homomorphism $tr : \mathcal{E}nd(\mathcal{F}) \longrightarrow \mathcal{O}_X$. Let $\mathcal{E}nd_0(\mathcal{F})$ be its kernel. If one denotes the kernel of $tr^q : \mathrm{Ext}^q(X; \mathcal{F}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{O}_X)$ by $\mathrm{Ext}^q(X, \mathcal{F}, \mathcal{F})_0$ one gets natural maps $H^q(X, \mathcal{E}nd_0(\mathcal{F})) \longrightarrow \mathrm{Ext}^q(X, \mathcal{F}, \mathcal{F})_0$, which are isomorphisms for $\mathcal{F}$ locally free.

This construction generalizes immediately to give trace maps

$$tr^q : \mathrm{Ext}^q(X; \mathcal{F}, \mathcal{F} \otimes N) \longrightarrow H^q(X, N)$$

for locally free sheaves N on X or for sheaves N such that $\mathcal{T}or_i^{\mathcal{O}_X}(N, \mathcal{F})$ vanish for $i > 0$.

The following Lemma is easy.

Lemma 3.2 *If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X allowing finite locally free resolutions and $u \in \mathrm{Ext}^p(X; \mathcal{F}, \mathcal{G})$, $v \in \mathrm{Ext}^q(X; \mathcal{G}, \mathcal{F})$ then*

$$tr^{p+q}(u \cdot v) = (-1)^{p \cdot q} tr^{p+q}(v \cdot u).$$

Theorem 3.3 *Let X be a compact complex surface, $(S, 0)$ be a germ of a complex space and $\mathcal{F}$ a coherent sheaf on $X \times S$ flat over S such that $\mathcal{F}_0 := \mathcal{F}\Big|_{X \times \{0\}}$ is torsion-free. The following hold.*

(a) *The tangent map of $\det : S \to \mathrm{Pic}(X)$ in 0 factorizes as*

$$T_0 S \xrightarrow{KS} \mathrm{Ext}^1(X; \mathcal{F}, \mathcal{F}) \xrightarrow{tr^1} H^1(X, \mathcal{O}_X) = T_{[\det \mathcal{F}_0]}(\mathrm{Pic}(X)).$$

(b) *If T is a zero-dimensional complex space such that $\mathcal{O}_{S,0} = \mathcal{O}_{T,0}/I$ for an ideal I of $\mathcal{O}_{T,0}$ with $I \cdot \mathfrak{m}_{T,0} = 0$, then the obstruction $ob(\mathcal{F}, T)$ to the extension of $\mathcal{F}$ to $X \times T$ is mapped by*

$$\begin{aligned} tr^2 \otimes_{\mathbb{C}} id_I : \mathrm{Ext}^2(X; \mathcal{F}_0, \mathcal{F}_0 \otimes_{\mathbb{C}} I) &\cong \mathrm{Ext}^2(X; \mathcal{F}_0, \mathcal{F}_0) \otimes_{\mathbb{C}} I \longrightarrow \\ \longrightarrow H^2(X, \mathcal{O}_X) \otimes_{\mathbb{C}} I &\cong \mathrm{Ext}^2(X; \det \mathcal{F}_0, (\det \mathcal{F}_0) \otimes_{\mathbb{C}} I) \end{aligned}$$

to the obstruction to the extension of $\det \mathcal{F}$ to $X \times T$ which is zero.

Proof (a) We may suppose that S is the double point $(0, \mathbb{C}[\epsilon])$. We define the Kodaira-Spencer map by means of the Atiyah class (cf. [9]).
For a complex space Y let $p_1, p_2 : Y \times Y \to Y$ be the projections and $\Delta \subset Y \times Y$ the diagonal. Tensoring the exact sequence

$$0 \longrightarrow \mathcal{I}_\Delta/\mathcal{I}_\Delta^2 \longrightarrow \mathcal{O}_{Y\times Y}/\mathcal{I}_\Delta^2 \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

by $p_2^*\mathcal{F}$ for $\mathcal{F}$ locally free on Y and applying $p_{1,*}$ gives an exact sequence on Y

$$0 \longrightarrow \mathcal{F} \otimes \Omega_Y \longrightarrow p_{1,*}(p_2^*\mathcal{F} \otimes (\mathcal{O}_{Y\times Y}/\mathcal{I}_\Delta^2)) \longrightarrow \mathcal{F} \longrightarrow 0.$$

The class $A(\mathcal{F}) \in \mathrm{Ext}^1(Y; \mathcal{F}, \mathcal{F} \otimes \Omega_Y)$ of this extension is called the Atiyah class of $\mathcal{F}$. When $\mathcal{F}$ is not locally free but admits a finite locally free resolution $F^\bullet$ one gets again a class $A(\mathcal{F})$ in $\mathrm{Ext}^1(Y; \mathcal{F}, \mathcal{F} \otimes \Omega_Y)$ seen as first cohomology group of $\mathcal{H}om^\bullet(F^\bullet, F^\bullet \otimes \Omega_Y)$.
Consider now $Y = X \times S$ with X and S as before, $p : Y \to S$, $q : Y \to X$ the projections and $\mathcal{F}$ as in the statement of the theorem.
he decomposition $\Omega_{X\times S} = q^*\Omega_X \oplus p^*\Omega_S$ induces

$$\begin{aligned}&\mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F} \otimes \Omega_{S\times X}) \cong \\ &\mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F} \otimes q^*\Omega_X) \oplus \mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F} \otimes p^*\Omega_S).\end{aligned}$$

The component $A_S(\mathcal{F})$ of $A(\mathcal{F})$ lying in $\mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F} \otimes p^*\Omega_S)$ induces the "tangent vector" at 0 to the deformation $\mathcal{F}$ through the isomorphisms

$$\begin{aligned}&\mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F} \otimes p^*\Omega_S) \cong \mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F} \otimes p^*\mathfrak{m}_{S,0}) \cong \\ &\mathrm{Ext}^1(X \times S;\, \mathcal{F}, \mathcal{F}_0) \cong \mathrm{Ext}^1(X; \mathcal{F}_0, \mathcal{F}_0).\end{aligned}$$

Applying now $tr^1 : \mathrm{Ext}^1(Y; \mathcal{F}, \mathcal{F} \otimes \Omega_Y) \to H^1(Y; \Omega_Y)$ to the Atiyah class $A(\mathcal{F})$ gives the first Chern class of $\mathcal{F}$, $c_1(\mathcal{F}) := tr^1(A(\mathcal{F}))$, (cf. [10], [21]).
It is known that

$$c_1(\mathcal{F}) = c_1(\det \mathcal{F}), \text{ i.e. } tr^1(A(\mathcal{F})) = tr^1(A(\det \mathcal{F})).$$

Now $\det \mathcal{F}$ is invertible so

$$tr^1 : \mathrm{Ext}^1(Y, \det \mathcal{F}, (\det \mathcal{F}) \otimes \Omega_Y)) \longrightarrow H^1(Y, \Omega_Y)$$

is just the canonical isomorphism. Since tr^1 is compatible with the decomposition $\Omega_{X\times S} = q^*\Omega_X \oplus p^*\Omega_S$ we get $tr^1(A_S(\mathcal{F})) = A_S(\det \mathcal{F})$ which proves (a).
(b) In order to simplify notation we drop the index 0 from $\mathcal{O}_{S,0}, \mathfrak{m}_{S,0}$, $\mathcal{O}_{T,0}$, $\mathfrak{m}_{T,0}$ and we use the same symbols $\mathcal{O}_S, \mathfrak{m}_S, \mathcal{O}_T, \mathfrak{m}_T$ for the respective pulled-back sheaves through the projections $X \times S \to S$, $X \times T \to T$.
There are two exact sequences of $\mathcal{O}_S$-modules:

$$\begin{aligned}&(1) \qquad 0 \longrightarrow \mathfrak{m}_S \longrightarrow \mathcal{O}_S \longrightarrow \mathbb{C} \longrightarrow 0,\\ &(2) \qquad 0 \longrightarrow I \longrightarrow \mathfrak{m}_T \longrightarrow \mathfrak{m}_S \longrightarrow 0.\end{aligned}$$

(Use $I \cdot \mathfrak{m}_T = 0$ in order to make $\mathfrak{m}_T$ an $\mathcal{O}_S$-module.)

Let $j : \mathbb{C} \to \mathcal{O}_S$ be the $\mathbb{C}$-vector space injection given by the $\mathbb{C}$-algebra structure of $\mathcal{O}_S$. j induces a splitting of (1). Since $\mathcal{F}$ is flat over S we get exact sequences over $X \times S$

$$0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow 0$$
$$0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} I \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_T \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S \longrightarrow 0$$

which remain exact as sequences over $\mathcal{O}_X$. Thus we get elements in $\mathrm{Ext}^1(X; \mathcal{F}_0, \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S)$ and $\mathrm{Ext}^1(X; \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S, \mathcal{F} \otimes_{\mathbb{C}} I)$ whose Yoneda composite $ob(\mathcal{F}, T)$ in $\mathrm{Ext}^2(X; \mathcal{F}_0, \mathcal{F} \otimes_{\mathbb{C}} I)$ is represented by the 2-fold exact sequence

$$0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} I \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_T \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

and is the obstruction to extending $\mathcal{F}$ from $X \times S$ to $X \times T$, as is well-known.

Consider now a resolution

$$0 \longrightarrow q^*G \longrightarrow E \longrightarrow \mathcal{F} \longrightarrow 0$$

of $\mathcal{F}$ as provided by Proposition 3.1, i.e. with G locally free on X and E locally free on $X \times S$. Our point is to compare $ob(\mathcal{F}, T)$ to $ob(E, T)$.

Since $\mathcal{F}$ is flat over S we get the following commutative diagrams with exact rows and columns by tensoring this resolution with the exact sequences (1) and (2):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & q^*G \otimes_{\mathbb{C}} \mathfrak{m}_S & \longrightarrow & q^*G & \longrightarrow & G_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E \otimes_{\mathcal{O}_S} \mathfrak{m}_S & \longrightarrow & E & \longrightarrow & E_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (1')$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & q^*G \otimes_{\mathbb{C}} I & \longrightarrow & q^*G \otimes_{\mathbb{C}} \mathfrak{m}_T & \longrightarrow & q^*G \otimes_{\mathbb{C}} \mathfrak{m}_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E \otimes_{\mathcal{O}_S} I & \longrightarrow & E \otimes_{\mathcal{O}_S} \mathfrak{m}_T & \longrightarrow & E \otimes_{\mathcal{O}_S} \mathfrak{m}_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_S} I & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_T & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (2')
$$

Using the section $j : \mathbb{C} \to \mathcal{O}_S$ we get an injective morphism of $\mathcal{O}_X$ sheaves

$$G_0 \xrightarrow{id_{q^*G} \otimes j} q^*G \otimes_{\mathbb{C}} \mathfrak{m}_T \longrightarrow E \otimes_{\mathcal{O}_S} \mathfrak{m}_T$$

which we call j_G.
From $(1')$ we get a short exact sequence over X in the obvious way

$$0 \longrightarrow (E \otimes_{\mathcal{O}_S} \mathfrak{m}_S) \oplus j_G(G_0) \longrightarrow E \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

Combining this with the middle row of $(2')$ we get a 2-fold extension

$$0 \longrightarrow (E \otimes_{\mathcal{O}_S} I) \oplus G_0 \longrightarrow (E \otimes_{\mathcal{O}_S} \mathfrak{m}_T) \oplus G_0 \longrightarrow E \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

whose class in $\mathrm{Ext}^2(X; \mathcal{F}_0, (E \otimes_{\mathcal{O}_S} I) \otimes G_0)$ we denote by u.
Let v be the surjection $E \to \mathcal{F}$ and

$$v' := \begin{pmatrix} v \otimes id_I \\ 0 \end{pmatrix} : (E \otimes_{\mathcal{O}_S} I) \oplus G_0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} I,$$
$$v'' = \begin{pmatrix} v_0 \\ 0 \end{pmatrix} : E_0 \oplus G_0 \longrightarrow \mathcal{F}_0,$$

the $\mathcal{O}_X$-morphisms induced by v.
The commutative diagrams

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & (E \otimes_{\mathcal{O}_S} I) \oplus G_0 & \longrightarrow & (E \otimes_{\mathcal{O}_S} \mathfrak{m}_T) \oplus G_0 & \longrightarrow & E & \longrightarrow & \mathcal{F}_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle \binom{v \otimes id_{\mathfrak{m}_T}}{0}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_S} I & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_T & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_0 & \longrightarrow & 0
\end{array}
$$

and

$$\begin{array}{ccccccccc}
0 \to & (E \otimes_{\mathcal{O}_S} I) \oplus G_0 & \to & (E \otimes_{\mathcal{O}_S} \mathfrak{m}_T) \oplus G_0 & \to & E \oplus G_0 & \to & E_0 \oplus G_0 & \to 0 \\
& \downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \binom{id}{j_G}} & & \downarrow{\scriptstyle v''} & \\
0 \to & (E \otimes_{\mathcal{O}_S} I) \oplus G_0 & \to & (E \otimes \mathfrak{m}_T) \oplus G_0 & \to & E & \to & \mathcal{F}_0 & \to 0
\end{array}$$

show that $ob(\mathcal{F}, T) = v' \cdot u$ and

$$(ob(E,T), 0) = u \cdot v'' \in \operatorname{Ext}^2(X; E_0 \oplus G_0, (E \otimes_{\mathcal{O}_S} I) \oplus G_0).$$

We may restrict ourselves to the situation when I is generated by one element. Then we have canonical isomorphisms of $\mathcal{O}_X$-modules $E_0 \cong E \otimes_{\mathcal{O}_S} I$ and $\mathcal{F}_0 \cong \mathcal{F} \otimes_{\mathcal{O}_S} I$. By these one may identify v' and v''. Now the Lemma 3.2 on the graded symmetry of the trace map with respect to the Yoneda pairing gives $tr^2(ob(\mathcal{F}, T)) = tr^2(ob(E, T))$.
But E is locally free and the assertion (b) of the theorem may be proved for it as in the projective case by a cocycle computation.
Thus $tr^2(ob(E, T)) = ob(\det E)$ and since $\det(E) = (\det \mathcal{F}) \otimes q^*(\det G)$ and $q^*(\det G)$ is trivially extendable, the assertion (b) is true for $\mathcal{F}$ as well. □
The theorem should be true in a more general context. In fact the proof of (a) is valid for any compact complex manifold X and flat sheaf $\mathcal{F}$ over $X \times S$. Our proof of (b) is in a way symmetric to the proof of Mukai in [17] who uses a resolution for $\mathcal{F}$ of a special form in the projective case.
NOTATION For a compact complex surface X and an element L in $\operatorname{Pic}(X)$ we denote by $Spl_X(L)$ the fiber of the morphism $\det : Spl_X \to \operatorname{Pic}(X)$ over L.

COROLLARY 3.4 *For a compact complex surface X and $L \in \operatorname{Pic}(X)$ the tangent space to $Spl_X(L)$ at an isomorphy class $[F]$ of a simple torsion-free sheaf F with $[\det F] = L$ is $\operatorname{Ext}^1(X; F, F)_0$. When $\operatorname{Ext}^2(X; F, F)_0 = 0$, $Spl_X(L)$ and Spl_X are smooth of dimensions*

$$\dim \operatorname{Ext}^1(X; F, F)_0 = 2\operatorname{rank}(F)^2 \Delta(F) - (\operatorname{rank}(F)^2 - 1)\chi(\mathcal{O}_X)$$

and

$$\dim \operatorname{Ext}^1(X; F, F) = \dim \operatorname{Ext}^1(X; F, F)_0 + h^1(\mathcal{O}_X)$$

respectively.

We end this paragraph by a remark on the symplectic structure of the moduli space Spl_X when X is symplectic.
Recall that a complex manifold M is called HOLOMORPHICALLY SYMPLECTIC if it admits a global nondegenerate closed holomorphic two-form ω. For a surface X, being holomorphically symplectic thus means that the canonical line bundle K_X is trivial. For such an X, Spl_X is smooth and holomorphically symplectic

as well. The smoothness follows immediately from the above Corollary and a two-form ω is defined at $[F]$ on Spl_X as the composition:

$$T_{[F]}Spl_X \times T_{[F]}Spl_X \cong \mathrm{Ext}^1(X;F,F) \times \mathrm{Ext}^1(X;F,F) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^2(X;F,F) \xrightarrow{tr^2} H^2(X,\mathcal{O}_X) \cong H^2(X,K_X) \cong \mathbb{C}.$$

It can be shown exactly as in the algebraic case that ω is closed and nondegenerate on Spl_X (cf. [17], [9]). Moreover, it is easy to see that the restriction of ω to the fibers $Spl_X(L)$ of $\det : Spl_X \to \mathrm{Pic}(X)$ remains nondegenerate, in other words that $Spl_X(L)$ are holomorphically symplectic subvarieties of Spl_X.

4 The moduli space of ASD connections and the comparison map

4.1 The moduli space of anti-self-dual connections

In this subsection we recall some results about the moduli spaces of anti-self-dual connections in the context we shall need. The reader is referred to [6], [8] and [14] for a thorough treatment of these questions.

We start with a compact complex surface X equipped with a Gauduchon metric g and a differential (complex) vector bundle E with a hermitian metric h in its fibers. The space of all C^∞ unitary connections on E is an affine space modeled on $\mathcal{A}^1(X, End(E,h))$ and the C^∞ unitary automorphism group $\mathcal{G}$, also called gauge-group, operates on it. Here $End(E,h)$ is the bundle of skew-hermitian endomorphisms of (E,h). The subset of anti-self-dual connections is invariant under the action of the gauge-group and we denote the corresponding quotient by

$$\mathcal{M}^{ASD} = \mathcal{M}^{ASD}(E).$$

A unitary connection A on E is called reducible if E admits a splitting in two parallel sub-bundles.

We use as in the previous section the determinant map

$$\det : \mathcal{M}^{ASD}(E) \longrightarrow \mathcal{M}^{ASD}(\det E)$$

which associates to A the connection $\det A$ in $\det E$. This is a fiber bundle over $\mathcal{M}^{ASD}(\det E)$ with fibers $\mathcal{M}^{ASD}(E,[a])$ where $[a]$ denotes the gauge equivalence class of the unitary connection a in $\det E$. We denote by $\mathcal{M}^{st}(E) = \mathcal{M}^{st}_g(E)$ the moduli space of stable holomorphic structures in E and by $\mathcal{M}^{st}(E,L)$ the fiber of the determinant map $\det : \mathcal{M}^{st}(E) \longrightarrow \mathrm{Pic}(X)$ over an element L of $\mathrm{Pic}(X)$. Then one has the following formulation of the Kobayashi-Hitchin correspondence.

Theorem 4.1 *Let X be a compact complex surface, g a Gauduchon metric on X, E a differentiable vector bundle over X, a an anti-self-dual connection on $\det E$ (with respect to g) and L the element in $\mathrm{Pic}(X)$ given by $\bar\partial_a$ on $\det E$. Then $\mathcal{M}^{st}(E,L)$ is an open part of $Spl_X(L)$ and the mapping $A \mapsto \bar\partial_A$ gives* *rise to a real-analytic isomorphism between the moduli space* $\mathcal{M}^{ASD,*}(E,[a])$ *of irreducible anti-self-dual connections which induce* $[a]$ *on* $\det E$ *and* $\mathcal{M}^{st}(E,L)$.

We may also look at $\mathcal{M}^{ASD}(E,[a])$ in the following way. We consider all anti-self-dual connections inducing a fixed connection a on $\det E$ and factor by those gauge transformations in $\mathcal{G}$ which preserve a. This is the same as taking gauge transformations of (E,h) which induce a constant multiple of the identity on $\det E$. Since constant multiples of the identity leave each connection invariant, whether on $\det E$ or on E, we may as well consider the action of the subgroup of $\mathcal{G}$ inducing the identity on $\det E$. We denote this group by $S\mathcal{G}$, the quotient space by $\mathcal{M}^{ASD}(E,a)$ and by $\mathcal{M}^{ASD,*}(E,a)$ the part consisting of irreducible connections. There is a natural injective map

$$\mathcal{M}^{ASD}(E,a) \longrightarrow \mathcal{M}^{ASD}(E,[a])$$

which associates to an $S\mathcal{G}$-equivalence class of a connection A its $\mathcal{G}$-equivalence class. The surjectivity of this map depends on the possibility to lift any unitary gauge transformation of $\det E$ to a gauge transformation of E. This possibility exists if E has a rank-one differential sub-bundle, in particular when $r := \operatorname{rank} E > 2$, since then E has a trivial sub-bundle of rank $r-2$. In this case one constructs a lifting by putting in this rank-one component the given automorphism of $\det E$ and the identity on the orthogonal complement. A lifting also exists for all gauge transformations of $(\det E, \det h)$ admitting an r-th root. More precisely, denoting the gauge group of $(\det E, \det h)$ by $\mathcal{U}(1)$, it is easy to see that the elements of the subgroup $\mathcal{U}(1)^r := \{u^r \mid u \in \mathcal{U}(1)\}$ can be lifted to elements of $\mathcal{G}$. Since the obstruction to taking r-th roots in $\mathcal{U}(1)$ lies in $H^1(X,\mathbb{Z}_r)$, as one deduces from the corresponding short exact sequence, we see that $\mathcal{U}(1)^r$ has finite index in $\mathcal{U}(1)$. From this it is not difficult to infer that $\mathcal{M}^{ASD}(E,[a])$ is isomorphic to a topologically disjoint union of finitely many parts of the form $\mathcal{M}^{ASD}(E,a_k)$ with $[a_k]=[a]$ for all k.

4.2 The Uhlenbeck compactification

We continue by stating some results we need on the Uhlenbeck compactification of the moduli space of anti-self-dual connections. References for this material are [6] and [8].

Let (X,g) and (E,h) be as in 4.1. For each non-negative integer k we consider hermitian bundles (E_{-k},h_{-k}) on X with $\operatorname{rank} E_{-k} = \operatorname{rank} E =: r, (\det E_{-k}, \det h_{-k}) \cong (\det E, \det h)$, $c_2(E_{-k}) = c_2(E) - k$. Set

$$\bar{\mathcal{M}}^U(E) := \bigcup_{k\in\mathbb{N}} (\mathcal{M}^{ASD}(E_{-k}) \times S^k X)$$

$$\bar{\mathcal{M}}^U(E,[a]) := \bigcup_{k\in\mathbb{N}} (\mathcal{M}^{ASD}(E_{-k},[a]) \times S^k X)$$

$$\bar{\mathcal{M}}^U(E,a) := \bigcup_{k\in\mathbb{N}} (\mathcal{M}^{ASD}(E_{-k},a) \times S^k X)$$

where S^kX is the k-th symmetric power of X. The elements of these spaces are called IDEAL CONNECTIONS. The unions are finite since the second Chern class of a hermitian vector bundle admitting an anti-self-dual connection is bounded below (by $\frac{1}{2}c_1^2$).
To an element $([A],Z) \in \bar{\mathcal{M}}^U(E)$ one associates a Borel measure

$$\mu([A],Z) := |F_A|^2 + 8\pi^2\delta_Z$$

where δ_Z is the Dirac measure whose mass at a point x of X equals the multiplicity $m_x(Z)$ of x in Z. We denote by $m(Z)$ the total multiplicity of Z. A topology for $\bar{\mathcal{M}}^U(E)$ is determined by the following neighborhood basis for $([A],Z)$:

$$\begin{aligned} V_{U,N,\epsilon}([A],Z) = \{([A'],Z') \in \bar{\mathcal{M}}^U(E) \mid \mu([A'],Z') \in U \text{ and there is an} \\ L_3^2\text{-isomorphism } \psi : E_{-m(Z)} \mid_{X\setminus N} \longrightarrow E_{-m(Z')} \mid_{X\setminus N} \\ \text{such that } \|A - \psi^*(A')\|_{L_2^2(X\setminus N)} < \epsilon\} \end{aligned}$$

where $\epsilon > 0$ and U and N are neighborhoods of $\mu([A],Z)$ and $\operatorname{supp}(\delta_Z)$ respectively. This topology is first-countable and Hausdorff and induces the usual topology on each $\mathcal{M}^{ASD}(E_{-k}) \times S^kX$. Most importantly, by the weak compactness theorem of Uhlenbeck $\bar{\mathcal{M}}^U(E)$ is compact when endowed with this topology, $\mathcal{M}^{ASD}(E)$ is an open part of $\bar{\mathcal{M}}^U(E)$ and its closure $\bar{\mathcal{M}}^{ASD}(E)$ inside $\bar{\mathcal{M}}^U(E)$ is called the UHLENBECK COMPACTIFICATION of $\mathcal{M}^{ASD}(E)$. Analogous statements are valid for $\mathcal{M}^{ASD}(E,[a])$ and $\mathcal{M}^{ASD}(E,a)$.
Using a technique due to Taubes, one can obtain a neighborhood of an irreducible ideal connection $([A],Z)$ in the border of $\mathcal{M}^{ASD}(E,a)$ by gluing to A "concentrated" $SU(r)$ anti-self-dual connections over S^4. One obtains "cone bundle neighborhoods" for each such ideal connection $([A],Z)$ when $H^2(X,\mathcal{E}nd_0(E_{\bar\partial_A})) = 0$. For the precise statements and the proofs we refer the reader to [6] chapters 7 and 8 and to [8] 3.4. As a consequence of this description and of the connectivity of the moduli spaces of $SU(r)$ anti-self-dual connections over S^4 (see [15]) we have the following weaker property which will suffice to our needs.

PROPOSITION 4.2 *Around an irreducible ideal connection $([A],Z)$ with $H^2(X,\mathcal{E}nd_0(E_{\bar\partial_A})) = 0$ the border of the Uhlenbeck compactification $\bar{\mathcal{M}}^{ASD}(E,a)$ is* LOCALLY NON-DISCONNECTING *in $\bar{\mathcal{M}}^{ASD}(E,a)$, i.e. there exist arbitrarily small neighborhoods V of $([A],Z)$ in $\bar{\mathcal{M}}^{ASD}(E,a)$ with $V \bigcap \mathcal{M}^{ASD}(E,a)$ connected.*

Note that for $SU(2)$ connections a lot more has been proved, [7], [18]. In this case the Uhlenbeck compactification is the completion of the space of anti-self-dual connections with respect to a natural Riemannian metric.

4.3 The comparison map

We fix (X, g) a compact complex surface together with a Gauduchon metric on it, (E, h) a hermitian vector bundle over X, a an unitary anti-self-dual connection on $(\det E, \det h)$ and denote by L the (isomorphy class of the) holomorphic line bundle induced by $\bar{\partial}_a$ on $\det E$. Let $c_2 := c_2(E)$ and $r := \operatorname{rank} E$. We denote by $\mathcal{M}^{st}(r, L, c_2)$ the subset of Spl_X consisting of isomorphy classes of non-necessarily locally free sheaves F (with respect to g) with $\operatorname{rank} F = r, \det F = L, c_2(F) = c_2$.
In 4.1 we have mentioned the existence of a real-analytic isomorphism between $\mathcal{M}^{st}(E, L)$ and $\mathcal{M}^{ASD,*}(E, [a])$. When X is algebraic, $\operatorname{rank} E = 2$ and a is the trivial connection this isomorphism has been extended to a continuous map from the Gieseker compactification of $\mathcal{M}^{st}(E, \mathcal{O})$ to the Uhlenbeck compactification of $\mathcal{M}^{ASD}(E, 0)$ in [16] and [13]. The proof given in [16] adapts without difficulty to our case to show the continuity of the natural extension

$$\Phi : \mathcal{M}^{st}(r, L, c_2) \longrightarrow \bar{\mathcal{M}}^U(E, [a]).$$

Φ is defined by $\Phi([F]) = ([A], Z)$, where A is the unique unitary anti-self-dual connection inducing the holomorphic structure on $F^{\vee\vee}$ and Z describes the singularity set of F with multiplicities $m_x(Z) := \dim_{\mathbb{C}}(F_x^{\vee\vee}/F_x)$ for $x \in X$. The main result of this paragraph asserts that under certain conditions for X and E this map is proper as well.

Theorem 4.3 *Let X be a non-algebraic compact complex surface which has either Kodaira dimension $\operatorname{kod}(X) = -\infty$ or has trivial canonical bundle and let g be a Gauduchon metric on X. Let (E, h) be a hermitian vector bundle over X, $r := \operatorname{rank} E$, $c_2 := c_2(E)$, a an unitary anti-self-dual connection on $(\det E, \det h)$ and L the holomorphic line bundle induced by $\bar{\partial}_a$ on $\det E$. If $(r, c_1(L), c_2)$ satisfies condition (*) from section 2 then the following hold:*

(a) *the natural map $\Phi : \mathcal{M}^{st}(r, L, c_2) \longrightarrow \bar{\mathcal{M}}^U(E, [a])$ is continuous and proper,*

(b) *any unitary automorphism of $(\det E, \det h)$ lifts to an automorphism of (E, h) and*

(c) *$\mathcal{M}^{st}(r, L, c_2)$ is a compact complex (Hausdorff) manifold.*

Proof
Under the Theorem's assumptions we prove the following claims.
Claim 1. Spl_X is smooth and of the expected dimension at points $[F]$ of $\mathcal{M}^{st}(r, L, c_2)$.
By Corollary 3.4 for such a stable sheaf F we have to check that $Ext^2(X; F, F)_0 = 0$. When K_X is trivial this is equivalent to $\dim(Ext^2(X; F, F)) = 1$ and by Serre duality further to $\dim(Hom(X; F, F)) = 1$ which holds since stable sheaves are simple. So let now X be non-algebraic and

$\mathrm{kod}(X) = -\infty$. By surface classification $b_1(X)$ must be odd and Remark 2.2 shows that F is irreducible. In this case we shall show that $Ext^2(X; F, F) = 0$. By Serre duality we have $Ext^2(X; F, F) \cong \mathrm{Hom}(X; F, F \otimes K_X)^*$. By taking double duals $\mathrm{Hom}(X; F, F \otimes K_X)$ injects into $\mathrm{Hom}(X; F^{\vee\vee}, F^{\vee\vee} \otimes K_X)$. Suppose φ is a non-zero homomorphism $\varphi : F^{\vee\vee} \to F^{\vee\vee} \otimes K_X$. Then $\det \varphi : \det F^{\vee\vee} \to (\det F^{\vee\vee}) \otimes K_X^{\otimes r}$ cannot vanish identically since F is irreducible. Thus it induces a non-zero section of $K_X^{\otimes r}$ contradicting $\mathrm{kod}(X) = -\infty$.
Claim 2. $\mathcal{M}^{st}(r, L, c_2)$ *is open in* Spl_X.
This claim is known to be true over the open part of Spl_X parameterizing simple locally free sheaves and holds possibly in all generality. Here we give an ad-hoc proof.
If b_1 is odd or if the degree function $\deg_g : Pic(X) \longrightarrow \mathbb{R}$ vanishes identically the assertion follows from the condition (*). Suppose now that X is non-algebraic with b_1 even and trivial canonical bundle. Let F be a torsion-free sheaf on X with $\mathrm{rank}\, F = r, \det F = L$ and $c_2(F) = c_2$. If F is not stable then F sits in a short exact sequence

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0$$

with F_1, F_2 torsion-free coherent sheaves on X. Let $r_1 := \mathrm{rank}\, F_1, r_2 := \mathrm{rank}\, F_2$. We first show that the possible values for $\deg F_1$ lie in a discrete subset of $\mathbb{R}$. An easy computation gives

$$-\frac{c_1(F_1)^2}{r_1} - \frac{c_1(F_2)^2}{r_2} = -\frac{c_1(F)^2}{r} + 2r\Delta(F) - 2r_1\Delta(F_1) - 2r_2\Delta(F_2).$$

Since all discriminants are non-negative we get

$$-\frac{c_1(F_1)^2}{r_1} - \frac{c_1(F_2)^2}{r_2} \leq -\frac{c_1(F)^2}{r} + 2r\Delta(F).$$

In particular $c_1(F_1)^2$ is bounded by a constant depending only on $(r, c_1(L), c_2)$. Since X is non-algebraic the intersection form on $NS(X)$ is negative semi-definite. In fact, by [4] $NS(X)/Tors(NS(X))$ can be written as a direct sum $N \bigoplus I$ where the intersection form is negative definite on N, I is the isotropy subgroup for the intersection form and I is cyclic. We denote by c a generator of I. It follows the existence of a finite number of classes b in N for which one can have $c_1(F_1) = b + \alpha c$ modulo torsion, with $\alpha \in \mathbb{N}$. Thus $\deg F_1 = \deg b + \alpha \deg c$ lies in a discrete subset of $\mathbb{R}$.
Let now $b \in NS(X)$ be such that $0 < \deg b \leq |\deg F_1|$ for all possible subsheaves F_1 as above with $\deg F_1 \neq 0$. We consider the torsion-free stable central fiber $\mathcal{F}_0$ of a family of sheaves $\mathcal{F}$ on $X \times S$ flat over S. Suppose that $\mathrm{rank}(\mathcal{F}_0) = r, \det \mathcal{F}_0 = L, c_2(\mathcal{F}_0) = c_2$. We choose an irreducible vector bundle G on X with $c_1(G) = -b$. Then $H^2(X, \mathcal{H}om(\mathcal{F}_0, G)) = 0$, so if $\mathrm{rank}\, G$ is large enough we can apply Proposition 3.1 to get an extension

$$0 \longrightarrow q^*G \longrightarrow E \longrightarrow \mathcal{F} \longrightarrow 0$$

with E locally free on $X \times S$, for a possibly smaller S. (As in Proposition 3.1 we have denoted by q the projection $X \times S \longrightarrow S$.) It is easy to check that E_0 doesn't have any subsheaf of degree larger than $-\deg b$. Thus E_0 is stable. Hence small deformations of E_0 are stable as well. As a consequence we get that small deformations of $\mathcal{F}_0$ will be stable. Indeed, it is enough to consider for a destabilizing subsheaf F_1 of F_s, for $s \in S$, the induced extension

$$0 \longrightarrow G \longrightarrow E_1 \longrightarrow F_1 \longrightarrow 0.$$

Then E_1 is a subsheaf of E_s with $\deg E_1 = \deg G + \deg F_1 \geq 0$. This contradicts the stability of E_s.

Claim 3. Any neighborhood in Spl_X of a point $[F]$ of $\mathcal{M}^{st}(r, L, c_2)$ contains isomorphy classes of locally free sheaves.

The proof goes as in the algebraic case by considering the "double-dual stratification" and making a dimension estimate. Here is a sketch of it.

If one takes a flat family $\mathcal{F}$ of torsion free sheaves on X over a reduced base S, one may consider for each fiber $\mathcal{F}_s$, $s \in S$, the injection into the double-dual $\mathcal{F}_s^{\vee\vee} := \mathcal{H}om(\mathcal{H}om(\mathcal{F}_s, \mathcal{O}_{X\times\{s\}}), \mathcal{O}_{X\times\{s\}})$. The double-duals form a flat family over some Zariski-open subset of S. To see this consider first $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O}_{X\times S})$. Since $\mathcal{F}$ is flat over S, one gets $(\mathcal{F}_s)^\vee = \mathcal{F}_s^\vee$. $\mathcal{F}^\vee$ is flat over the complement of a proper analytic subset of S and one repeats the procedure to obtain $\mathcal{F}^{\vee\vee}$ and $\mathcal{F}^{\vee\vee}/\mathcal{F}$ flat over some Zariski open subset S' of S. Over $X \times S'$, $\mathcal{F}^{\vee\vee}$ is locally free and $(\mathcal{F}^{\vee\vee}/\mathcal{F})_s = \mathcal{F}_s^{\vee\vee}/\mathcal{F}_s$ for $s \in S'$. Take now S a neighborhood of $[F]$ in $\mathcal{M}^{st}(r, L, c_2)$. Suppose that

$$\text{length}(\mathcal{F}_{s_0}^{\vee\vee}/\mathcal{F}_{s_0}) = k > 0$$

for some $s_0 \in S'$. Taking S' smaller around s_0 if necessary, we find a morphism ϕ from S' to a neighborhood T of $[\mathcal{F}_{s_0}^{\vee\vee}]$ in $\mathcal{M}^{st}(r, L, c_2 - k)$ such that there exists a locally free universal family $\mathcal{E}$ on $X \times T$ with $\mathcal{E}_{t_0} \cong \mathcal{F}_{s_0}^{\vee\vee}$ for some $t_0 \in T$ and $(\text{id}_X \times \phi)^*\mathcal{E} = \mathcal{F}^{\vee\vee}$. Let D be the relative Douady space of quotients of length k of the fibers of $\mathcal{E}$ and let $\pi : D \longrightarrow T$ be the projection. There exists an universal quotient $\mathcal{Q}$ of $(\text{id}_X \times \pi)^*\mathcal{E}$ on $X \times D$. Since $\mathcal{F}^{\vee\vee}/\mathcal{F}$ is flat over S', ϕ lifts to a morphism $\tilde{\phi} : S' \longrightarrow D$ with $(id_X \times \tilde{\phi})^*\mathcal{Q} = \mathcal{F}^{\vee\vee}/\mathcal{F}$. By the universality of S' there exists also a morphism (of germs) $\psi : D \longrightarrow S'$ with $(\text{id}_X \times \psi)^*\mathcal{F} = \text{Ker}((\text{id}_X \times \pi)^*\mathcal{E} \longrightarrow \mathcal{Q})$. One sees now that $\psi \circ \tilde{\phi}$ must be an isomorphism, in particular $\dim S' \leq \dim D$. Since S' and T have the expected dimensions, it is enough to compute now the relative dimension of D over T. This is $k(r+1)$. On the other side by Corollary 3.4 $\dim S' - \dim T = 2kr$. This forces $r = 1$ which is excluded by hypothesis.

After these preparations of a relatively general nature we get to the actual proof of the Theorem. We start with (b).

If $b_2^-(X)$ denotes the number of negative eigenvalues of the intersection form on $H^2(X, \mathbb{R})$, then for our surface X we have $b_2^-(X) > 0$. This is clear when K_X is trivial by classification and follows from the index theorem and Remark 2.2 (d) when $b_1(X)$ is odd. In particular, taking $p \in H^2(X, \mathbb{Z})$ with $p^2 < 0$ one

constructs topologically split rank two vector bundles F with given first Chern class l and arbitrarily large second Chern class: just consider $(L \otimes P^{\otimes n}) \oplus (P^*)^{\otimes n}$ where L and P are line bundles with $c_1(L) = l$, $c_1(P) = p$ and $n \in \mathbb{N}$. If E has rank two we take F with $\det F \cong \det E$ and $c_2(F) \geq c_2(E) = c_2$. (When $r > 2$ assertion (b) is trivial; cf. section 4.1). We consider an anti-self-dual connection A in E inducing a on $\det E$ and $Z \subset X$ consisting of $c_2(F) - c_2(E)$ distinct points. By the computations from the proof of Claim 1 we see that A is irreducible and $H^2_{A,0} = 0$. Using the gluing procedure mentioned in section 4.2 , one sees that a neighborhood of $([A], Z)$ in $\bar{\mathcal{M}}^U(F, [a])$ contains classes of irreducible anti-self-dual connections in F. We have seen in section 4.1 that any unitary automorphism of $\det F$ lifts to an unitary automorphism u of F. If we take a sequence of anti-self-dual connections (A_n) in F with $\det A_n = a$ and $([A_n])$ converging to $([A], Z)$, we get by applying u a limit connection B for subsequence of $(u(A_n))$. Since $\bar{\mathcal{M}}^U(F, [a])$ is Hausdorff, there exists an unitary automorphism $\tilde{u}$ of E with $\tilde{u}^*(B) = A$. It is clear that $\tilde{u}$ induces the original automorphism u on $\det F \cong \det E$.
We leave the proof of the following elementary topological lemma to the reader.

Lemma 4.4 *Let $\pi : Z \longrightarrow Y$ be a continuous surjective map between Hausdorff topological spaces. Suppose Z locally compact, Y locally connected and that there is a locally non-disconnecting closed subset Y_1 of Y with $Z_1 := \pi^{-1}(Y_1)$ compact and $\overset{\circ}{Z}_1 = \emptyset$. Suppose further that π restricts to a homeomorphism*

$$\pi\,|_{Z \setminus Z_1, Y \setminus Y_1} : Z \setminus Z_1 \longrightarrow Y \setminus Y_1.$$

Then for any neighborhood V of Z_1 in Z, $\pi(V)$ is a neighborhood of Y_1 in Y. If in addition Y is compact, then Z is compact as well.

We complete now the proof of the Theorem by induction on c_2. For fixed r and $c_1(E)$, $c_2(E)$ is bounded below if E is to admit an anti-self-dual connection. If we take c_2 minimal, then $\mathcal{M}^{st}(r, L, c_2) = \mathcal{M}^{st}(E, L)$ and $\mathcal{M}^{ASD,*}(E, [a]) = \mathcal{M}^{ASD}(E, [a])$ is compact. From Theorem 4.1 we obtain that Φ is a homeomorphism in this case.
Take now c_2 arbitrary but such that the hypotheses of the Theorem hold and assume that the assertions of the Theorem are true for any smaller c_2. We apply Lemma 4.4 to the following situation:

$$Z := \mathcal{M}^{st}(r, L, c_2),\ Y := \bar{\mathcal{M}}^U(E, [a]) = \bar{\mathcal{M}}^{ASD}(E, [a]) \cong \bar{\mathcal{M}}^{ASD}(E, a).$$

The last equalities hold according to Claim 3 and Claim 4. Let further Y_1 be the border $\bar{\mathcal{M}}^{ASD}(E, a) \setminus \mathcal{M}^{ASD}(E, a)$ of the Uhlenbeck compactification and Z_1 be the locus $\mathcal{M}^{st}(r, L, c_2) \setminus \mathcal{M}^{st}(E, L)$ of singular stable sheaves in Spl_X. Z is smooth by Claim 1 and Hausdorff, Y_1 is locally non-disconnecting by Proposition 4.2, $\overset{\circ}{Z}_1 = \emptyset$ by Claim 3 and $\pi\,|_{Z \setminus Z_1, Y \setminus Y_1}$ is a homeomorphism by Theorem 4.1. In order to be able to apply Lemma 4.4 and thus close the proof we only need to check that Z_1 is compact. We want to reduce this to the compactness of $\mathcal{M}^{st}(r, L, c_2 - 1)$ which is ensured by the induction hypothesis. We consider a finite open covering (T_i) of $\mathcal{M}^{st}(r, L, c_2 - 1)$ such that over each $X \times T_i$ an universal family $\mathcal{E}_i$ exists. The relative Douady space D_i parameterizing quotients of length one in the fibers of $\mathcal{E}_i$ is proper over (T_i). In fact it was shown in [11] that $D_i \cong \mathbb{P}(\mathcal{E}_i)$. If $\pi_i : D_i \longrightarrow T_i$ are the projections, we have universal quotients $\mathcal{Q}_i$ of $\pi^*\mathcal{E}_i$ and $\mathcal{F}_i := \mathrm{Ker}(\pi^*\mathcal{E}_i \longrightarrow \mathcal{Q}_i)$ are flat over D_i. This induces canonical morphisms $D_i \longrightarrow Z_1$. It is enough to notice that their images cover Z_1, or equivalently, that any singular stable sheaf F over X sits in an exact sequence of coherent sheaves

$$0 \longrightarrow F \longrightarrow E \longrightarrow Q \longrightarrow 0$$

with $\mathrm{length}\, Q = 1$ and E torsion-free. Such an extension is induced from

$$0 \longrightarrow F \longrightarrow F^{\vee\vee} \longrightarrow F^{\vee\vee}/F \longrightarrow 0$$

by any submodule Q of length one of $F^{\vee\vee}/F$. (To see that such Q exist recall that $(F^{\vee\vee}/F)_x$ is artinian over $\mathcal{O}_{X,x}$ and use Nakayama's Lemma). The Theorem is proved. □

Remark 4.5 As a consequence of this theorem we get that when X is a 2-dimensional complex torus or a primary Kodaira surface and (r, L, c_2) is chosen in the stable irreducible range as in [24], [23] or [1], then $\mathcal{M}^{st}(r, L, c_2)$ is a holomorphically symplectic compact complex manifold.

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