On the Γ-Factors of Motives II

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Abstract. Using an idea of C. Simpson we describe Serre's local Γ-factors in terms of a complex of sheaves on a simple dynamical system. This geometrizes our earlier construction of the Γ-factors. Relations of this approach with the ε-factors are also studied.

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1 Introduction

In [S] Serre defined local Euler factors $L_{\mathfrak{p}}(H^n(X), s)$ for the "motives" $H^n(X)$ where X is a smooth projective variety over a number field k. The definition at the finite places $\mathfrak{p}$ involves the Galois action on the l-adic cohomology groups $H^n_{\text{ét}}(X \otimes \overline{k}_{\mathfrak{p}}, \mathbb{Q}_l)$. At the infinite places the local Euler factor is a product of Gamma factors determined by the real Hodge structure on the singular cohomology $H^n_B(X \otimes \overline{k}_{\mathfrak{p}}, \mathbb{R})$. If $\mathfrak{p}$ is real then the Galois action induced by complex conjugation on $\overline{k}_{\mathfrak{p}}$ has to be taken into account as well.
Serre also conjectured a functional equation for the completed L-series, defined as the product over all places of the local Euler factors.
In his definitions and conjectures Serre was guided by a small number of examples and by the analogy with the case of varieties over function fields which is quite well understood. Since then many more examples over number fields notably from the theory of Shimura varieties have confirmed Serre's suggestions. The analogy between l-adic cohomology with its Galois action and singular cohomology with its Hodge structure is well established and the definition of the local Euler factors fits well into this philosophy. However in order to prove the functional equation in general, a deeper understanding than the one

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provided by an analogy is needed. First steps towards a uniform description of the local Euler factors were made in [D1], [D2], [D3]. There we constructed infinite dimensional complex vector spaces $\mathcal{F}_{\mathfrak{p}}(H^n(X))$ with a linear flow such that for all places:

$$L_{\mathfrak{p}}(H^n(X), s) = \det{}_{\infty}\Big(\frac{1}{2\pi}(s \cdot \mathrm{id} - \Theta) \,|\, \mathcal{F}_{\mathfrak{p}}(H^n(X))\Big)^{-1} . \tag{1}$$

Here Θ is the infinitesimal generator of the flow and $\det_{\infty}$ is the zeta-regularized determinant. Unfortunately the construction of the spaces $\mathcal{F}_{\mathfrak{p}}(H^n(X))$ was not really geometric. They were obtained by formal constructions from étale cohomology with its Galois action and from singular cohomology with its Hodge structure.

C. Consani [C] later developed a new infinite-dimensional cohomology theory $H^n_{\mathrm{Cons}}(Y)$ with operators N and Θ for varieties Y over $\mathbb{R}$ or $\mathbb{C}$ such that for infinite places $\mathfrak{p}$:

$$(\mathcal{F}_{\mathfrak{p}}(H^n(X)), \Theta) \cong (H^n_{\mathrm{Cons}}(X \otimes k_{\mathfrak{p}})^{N=0}, \Theta) . \tag{2}$$

Her constructions are inspired by the theory of degenerations of Hodge structure and her N has to be viewed as a monodromy operator. The formula for the archimedean local factors obtained by combining (1) and (2) is analogous to the expression for $L_{\mathfrak{p}}(H^n(X), s)$ at a prime $\mathfrak{p}$ of semistable reduction in terms of log-crystalline cohomology.

The conjectural approach to motivic L-functions outlined in [D7] suggests the following: It should be possible to obtain the spaces $\mathcal{F}_{\mathfrak{p}}(H^n(X))$ for archimedian $\mathfrak{p}$ together with their linear flow directly by some natural homological construction on a suitable non-linear dynamical system. Clearly, forming the intersection of the Hodge filtration with its complex conjugate and running the resulting filtration through a Rees module construction as in our first construction of $\mathcal{F}_{\mathfrak{p}}(H^n(X))$ in [D1] is not yet what we want: In this construction the linear flow appears only a posteriori on cohomology but it is not induced from a flow on some underlying space by passing to cohomology.

In the present paper in Theorems 4.2, 4.3, 4.4 we make a step towards this goal of a more direct dynamical description of the archimedian Gamma-factors. The approach is based on a result of Simpson which roughly speaking replaces the consideration of the Hodge filtration by looking at a relative de Rham complex with a deformed differential.

In our case, instead of the Hodge filtration $F^{\bullet}$ we require the non-algebraic filtration $F^{\bullet} \cap \overline{F^{\bullet}}$. This forces us to work in a real analytic context even for complex $\mathfrak{p}$. It seems difficult to carry Simpson's method over to this new context. However this is not necessary. By a small miracle – the splitting of a certain long exact sequence – his result can be brought to bear directly on our more complicated situation.

In the appendix to section 4 we explain a relation between Simpson's deformed complex and a relative de Rham complex on the deformation of X to the normal bundle of a base point. This observation probably holds the key for a complete dynamical understanding of the Gamma-factor.

Our main construction also provides a C^ω-vector bundle on $\mathbb{R}$ with a flow. Its fibre at zero can be used for a "dynamical" description of the contribution from $\mathfrak{p} \mid \infty$ in the motivic "explicit formulas" of analytic number theory.

In our investigation we encounter a torsion sheaf whose dimension is to some extent related to the ε-factor at $\mathfrak{p}$ of $H^n(X)$.

Using forms with logarithmic singularities one can probably deal more generally with the motives $H^n(X)$ where X is only smooth and quasiprojective.

It would also be of interest to give a construction for Consani's cohomology theory using the methods of the present paper.

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2 Preliminaries on the algebraic Rees sheaf

In this section we recall and expand upon a simple construction which to any filtered complex vector space attaches a sheaf on $\mathbb{A}^1 = \mathbb{A}^1_{\mathbb{C}}$ with a $\mathbb{G}_m$-action. We had used it in earlier work on the Γ-factors [D1], [D3] § 5. Later Simpson [Si] gave a more elegant treatment and proved some further properties. Most importantly for us he proved Theorem 5.1 below which was the starting point for the present paper. In the following we also extend his results to a variant of the construction where one starts from a filtered vector space with an involution. This is necessary later to deal not only with the complex places but with the real places as well.

Let $\mathcal{F}il_{\mathbb{C}}$ be the category of finite dimensional complex vector spaces V with a descending filtration $\mathrm{Fil}^r V$ such that $\mathrm{Fil}^{r_1} V = 0, \mathrm{Fil}^{r_2} V = V$ for some integers r_1, r_2. Let $\mathcal{F}il^{\pm}_{\mathbb{R}}$ be the category of finite dimensional complex vector spaces with a filtration as above and with an involution F_∞ which respects the filtration. Finally let $\mathcal{F}il_{\mathbb{R}}$ be the full subcategory of $\mathcal{F}il^{\pm}_{\mathbb{R}}$ consisting of objects where F_∞ induces multiplication by $(-1)^\bullet$ on $\mathrm{Gr}^\bullet V$.

These additive categories have $\otimes$-products and internal Hom's. We define Tate twists for every integer n by

$$(V, \mathrm{Fil}^\bullet V)(n) = (V, \mathrm{Fil}^{\bullet+n} V) \quad \text{in } \mathcal{F}il_{\mathbb{C}}$$

and by

$$(V, \mathrm{Fil}^\bullet V, F_\infty)(n) = (V, \mathrm{Fil}^{\bullet+n} V, (-1)^n F_\infty) \quad \text{in } \mathcal{F}il^{\pm}_{\mathbb{R}} \text{ and } \mathcal{F}il_{\mathbb{R}} \; .$$

Note that the full embedding:

$$i : \mathcal{F}il_{\mathbb{R}} \hookrightarrow \mathcal{F}il^{\pm}_{\mathbb{R}}$$

is split by the functor

$$s : \mathcal{F}il_{\mathbb{R}}^{\pm} \longrightarrow \mathcal{F}il_{\mathbb{R}}$$

which sends $(V, \widetilde{\mathrm{Fil}}^r V, F_\infty)$ to

$$(V, \mathrm{Fil}^r V = (\widetilde{\mathrm{Fil}}^r V)^{(-1)^r} + (\widetilde{\mathrm{Fil}}^{r+1} V)^{(-1)^{r+1}}, F_\infty)$$

i.e. $s \circ i = \mathrm{id}$. Here $W^{\pm 1}$ denotes the ± 1 eigenspace of F_∞ on W. For V in $\mathcal{F}il_{\mathbb{C}}$ following [Si] § 5 define a locally free sheaf $\xi_{\mathbb{C}}(V) = \xi_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V)$ over $\mathbb{A}^1$ with action of $\mathbb{G}_m$ by

$$\xi_{\mathbb{C}}(V) = \sum_p \mathrm{Fil}^p V \otimes z^{-p} \mathcal{O}_{\mathbb{A}^1} \subset V \otimes j_* \mathcal{O}_{\mathbb{G}_m} \; .$$

Here $j : \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ is the inclusion and z denotes a coordinate on $\mathbb{A}^1$ determined up to a scalar in $\mathbb{C}^*$. Unless stated otherwise the constructions in this paper are independent of z. The global sections of the "Rees sheaf" $\xi_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V)$ form the "Rees module" over $\mathbb{C}[z]$:

$$\mathrm{Fil}^0(V \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]) = \sum_p \mathrm{Fil}^p V \otimes z^{-p} \mathbb{C}[z] \subset V \otimes \mathbb{C}[z, z^{-1}]$$

where $\mathrm{Fil}^p \mathbb{C}[z, z^{-1}] = z^p \mathbb{C}[z]$ for $p \in \mathbb{Z}$. The natural action of $\mathbb{G}_m$ on $\mathbb{A}^1$ induces a $\mathbb{G}_m$-action on $\xi_{\mathbb{C}}$ by pullback

$$\lambda^* : (\lambda)^{-1} \xi_{\mathbb{C}} \longrightarrow \xi_{\mathbb{C}} \; , \; v \otimes g(z) \longmapsto v \otimes g(\lambda z) \; . \tag{3}$$

Here $(\lambda)^{-1}\xi_{\mathbb{C}}$ denotes the inverse image of $\xi_{\mathbb{C}}$ under the multiplication by $\lambda \in \mathbb{C}^*$ map.

Let $sq : \mathbb{A}^1 \to \mathbb{A}^1$ be the squaring map $sq(z) = z^2$ and define $F_\infty : \mathbb{A}^1 \to \mathbb{A}^1$ as $F_\infty = -\mathrm{id}$. For V in $\mathcal{F}il_{\mathbb{R}}^{\pm}$ the actions of F_∞ on V and $\mathbb{G}_m \subset \mathbb{A}^1$ combine to an action

$$F_\infty^* : F_\infty^{-1}(V \otimes j_* \mathcal{O}_{\mathbb{G}_m}) \longrightarrow V \otimes j_* \mathcal{O}_{\mathbb{G}_m} \; .$$

Thus we get an involution F_∞ on the sheaf $sq_*(V \otimes j_* \mathbb{G}_m)$ and we define a locally free sheaf on $\mathbb{A}^1$ by:

$$\xi_{\mathbb{R}}(V) = \xi_{\mathbb{R}}(V, \mathrm{Fil}^\bullet V, F_\infty) = (sq_* \xi_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V))^{F_\infty} \; .$$

The $\mathbb{G}_m$-action on $\xi_{\mathbb{C}}$ leads to an action

$$\lambda^* : (\lambda^2)^{-1} \xi_{\mathbb{R}} \longrightarrow \xi_{\mathbb{R}} \; .$$

The global sections of $\xi_{\mathbb{R}}(V, \mathrm{Fil}^\bullet V, F_\infty)$ are given by

$$\Big(\sum_p \mathrm{Fil}^p V \otimes z^{-p} \mathbb{C}[z] \Big)^{F_\infty} \subset V \otimes \mathbb{C}[z, z^{-1}]$$

viewed as a $\mathbb{C}[z^2]$-module.

Remark 2.1 The global sections of ξ – with the action of the Lie algebra of $\mathbb{G}_m$ – were also considered in [D3] § 5

$$\begin{aligned} \Gamma(\mathbb{A}^1, \xi_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V)) &= \mathbb{D}^+(V, \mathrm{Fil}^\bullet V) \\ \Gamma(\mathbb{A}^1, \xi_{\mathbb{R}}(V, \mathrm{Fil}^\bullet V, F_\infty)) &= \mathbb{D}^+(V, \mathrm{Fil}^\bullet V, F_\infty) \end{aligned}$$

in the notation of that paper.

We denote by $E = E(V, \mathrm{Fil}^\bullet V)$ resp. $E = E(V, \mathrm{Fil}^\bullet V, F_\infty)$ the vector bundle on $\mathbb{A}^1$ corresponding to the locally free sheaf ξ. It has a contravariant $\mathbb{G}_m$-action with respect to the action of $\mathbb{G}_m$ on $\mathbb{A}^1$ by

$$\mathbb{G}_m \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1 \ , \ (\lambda, a) \longmapsto \lambda^{e_K} a \tag{4}$$

where $e_{\mathbb{C}} = 1$ and $e_{\mathbb{R}} = 2$.
For $K = \mathbb{C}$ resp. $\mathbb{R}$ let $\mathcal{D}_K$ be the category of locally free $\mathcal{O}_{\mathbb{A}^1}$-modules of finite rank with contravariant action by $\mathbb{G}_m$ with respect to the action (4) on $\mathbb{A}^1$. The category $\mathcal{D}_K$ has $\otimes$-products and internal Homs. For $\mathcal{M} \in \mathcal{D}_K$ set $\mathcal{M}(n) = z^n \mathcal{M}$ for any integer n. Thus $\mathcal{M}(n)$ is isomorphic to $\mathcal{M}$ as an $\mathcal{O}_{\mathbb{A}^1}$-module but with $\mathbb{G}_m$-action twisted as follows:

$$\lambda^*_{\mathcal{M}(n)} = \lambda^n \cdot \lambda^*_{\mathcal{M}} \ .$$

The following construction provides inverses to $\xi_{\mathbb{C}}$ and $\xi_{\mathbb{R}}$. For $\mathcal{M}$ in $\mathcal{D}_K$ set

$$\eta_K(\mathcal{M}) = \Gamma(\mathbb{G}_m, j^*\mathcal{M})^{\mathbb{G}_m} = (\Gamma(\mathbb{A}^1, \mathcal{M}) \otimes_{\mathbb{C}[z^{e_K}]} \mathbb{C}[z, z^{-1}])^{\mathbb{G}_m}$$

with the filtration (and in case $K = \mathbb{R}$ the involution) coming from the one on $\mathbb{C}[z, z^{-1}]$.
The main properties of ξ and η are contained in the following proposition.
Recall that a map $\varphi : V \to W$ of filtered vector spaces is called strict if $\varphi^{-1}\mathrm{Fil}^i W = \mathrm{Fil}^i V$ for all i.

Proposition 2.2 *a) The functor $\xi_K : \mathcal{F}il_K \to \mathcal{D}_K$ is an equivalence of additive categories with quasi-inverse η_K. It commutes with $\otimes$-products and internal Homs and we have that*

$$\dim V = \mathrm{rk} \xi_K(V) \quad \text{and} \quad \dim \eta_K(\mathcal{M}) = \mathrm{rk}\mathcal{M}$$

for all V in $\mathcal{F}il_K$ and $\mathcal{M}$ in $\mathcal{D}_K$.
The functors $\xi_{\mathbb{C}}$ and $\xi_{\mathbb{R}} : \mathcal{F}il^{\pm}_{\mathbb{R}} \to \mathcal{D}_{\mathbb{R}}$ commute with Tate twists.
For $(V, \mathrm{Fil}^\bullet V, F_\infty) \in \mathcal{F}il^{\pm}_{\mathbb{R}}$ there is a canonical isomorphism:

$$\xi_{\mathbb{R}}(V, \mathrm{Fil}^\bullet V, F_\infty)^* = \xi_{\mathbb{R}}(V^*, \mathrm{Fil}^{\bullet -1} V^*, F^*_\infty) \ .$$

Here $\mathrm{Fil}^p V^* := (\mathrm{Fil}^{1-p} V)^\perp$ *in* V^*.
b) The diagrams

are commutative.
c) If $\varphi : U \to V$ *is a morphism in* $\mathcal{F}il_{\mathbb{C}}$ *resp.* $\mathcal{F}il_{\mathbb{R}}^{\pm}$ *then*
i) $\xi(\ker \varphi) = \ker(\xi(U) \to \xi(V))$
and
ii) $\xi(\operatorname{coker} \varphi) = \operatorname{coker}(\xi(U) \to \xi(V))/T$
where T *is the subsheaf of torsion elements. We have* $T = 0$ *if and only if* φ *resp.* $s(\varphi)$ *is strict.*

Proof a) is shown in [Si] § 5 for $K = \mathbb{C}$. Every object in $\mathcal{F}il_{\mathbb{R}}^{\pm}$ is the direct sum of objects $\mathbb{C}(n)^{\pm}$ defined as follows: The underlying vector space of $\mathbb{C}(n)^{\pm}$ is $\mathbb{C}$, the filtration is given by $\mathrm{Fil}^p\mathbb{C}(n)^{\pm} = \mathbb{C}(n)^{\pm}$ if $p \le -n$ and $= 0$ if $p > -n$. Finally F_∞ acts on $\mathbb{C}(n)^{\pm}$ by multiplication with ± 1. The objects of $\mathcal{F}il_{\mathbb{R}}$ are direct sums of objects $\mathbb{C}(n)^{(-1)^n}$ and we have that

$$s(\mathbb{C}(n)^{(-1)^n}) = \mathbb{C}(n)^{(-1)^n} \quad \text{and} \quad s(\mathbb{C}(n)^{(-1)^{n+1}}) = \mathbb{C}(n+1)^{(-1)^{n+1}} .$$

Using decompositions into $\mathbb{C}(n)^{(-1)^n}$'s, one checks that the natural maps

$$\xi_{\mathbb{R}}(V) \otimes \xi_{\mathbb{R}}(W) \longrightarrow \xi_{\mathbb{R}}(V \otimes W)$$

and

$$\underline{\mathrm{Hom}}(\xi_{\mathbb{R}}(V), \xi_{\mathbb{R}}(W)) \longrightarrow \xi_{\mathbb{R}}(\underline{\mathrm{Hom}}(V, W))$$

are isomorphisms for all V, W in $\mathcal{F}il_{\mathbb{R}}$. Moreover the rank assertions in a) follow. Commutation with Tate twists follows immediately from the definitions. The final isomorphism follows from the above and the first diagram in b) since a short calculation gives that:

$$(V, s\mathrm{Fil}^{\bullet} V, F_\infty)^* = (V^*, s(\mathrm{Fil}^{\bullet -1} V^*), F_\infty^*) .$$

The commutativities in b) can be seen using decompositions into $\mathbb{C}(n)^{\pm}$'s. In particular $\eta_{\mathbb{R}} \circ \xi_{\mathbb{R}} = \mathrm{id}$ for $\xi_{\mathbb{R}} : \mathcal{F}il_{\mathbb{R}} \to \mathcal{D}_{\mathbb{R}}$. The opposite isomorphism $\xi_{\mathbb{R}} \circ \eta_{\mathbb{R}} = \mathrm{id}$ follows as in Simpson [Si] § 5. Finally c) is stated in loc. cit. for $K = \mathbb{C}$ and remains true for $K = \mathbb{R}$. Part i) is straightforward. As for ii), by functoriality of ξ and the fact that $\xi(\operatorname{coker} \varphi)$ is torsion-free one is reduced to proving that the kernel of the natural surjection

$$\operatorname{coker}(\xi(U) \to \xi(V)) \twoheadrightarrow \xi(\operatorname{coker} \varphi)$$

is torsion. This can be checked using a suitable splitting of φ. □

The following facts about the structure of the Rees bundle were noted for $K = \mathbb{C}$ in [Si] § 5.

Proposition 2.3 *i) For all* V *in* $\mathcal{F}il_{\mathbb{C}}$ *resp.* $\mathcal{F}il_{\mathbb{R}}^{\pm}$ *there are canonical isomorphisms of vector bundles over* $\mathbb{G}_m$

$$j^* E(V, \mathrm{Fil}^{\bullet} V) \xrightarrow{\sim} V \times \mathbb{G}_m$$

resp.

$$sq^*j^*E(V, \mathrm{Fil}^\bullet V, F_\infty) \xrightarrow{\sim} V \times \mathbb{G}_m$$

functorial in V and compatible with the (contravariant) $\mathbb{G}_m$-action. Thus the local systems $F_\mathbb{C}$ and $sq^*F_\mathbb{R}$ are trivialized functorially in V. Under the isomorphism

$$E(V, \mathrm{Fil}^\bullet V, F_\infty)_1 = (sq^*E(V, \mathrm{Fil}^\bullet V, F_\infty))_1 \xrightarrow{\sim} V$$

the monodromy representation of $\pi_1(\mathbb{C}^*, 1) \cong \mathbb{Z}$ maps n to F_∞^n.
ii) For V in $\mathcal{F}il_\mathbb{C}$ resp. $\mathcal{F}il_\mathbb{R}^\pm$ there are isomorphisms depending on the choice of a coordinate z on $\mathbb{A}^1$:

$$E(V, \mathrm{Fil}^\bullet V)_0 \xrightarrow{\sim} \mathrm{Gr}^\bullet V$$

resp.

$$E(V, \mathrm{Fil}^\bullet V, F_\infty)_0 \xrightarrow{\sim} \mathrm{Gr}^\bullet(sV)$$

functorial in V. They are compatible with the $\mathbb{G}_m$-action if $\mathbb{G}_m$ acts on $\mathrm{Gr}^p V$ resp. $\mathrm{Gr}^p(sV)$ by the character z^{-p}.

Proof i) We treat the case $K = \mathbb{R}$. It suffices to check that

$$sq^*j^*\xi(V, \mathrm{Fil}^\bullet V, F_\infty) \xrightarrow{\sim} V \otimes \mathcal{O}_{\mathbb{G}_m}$$

compatibly with the $\mathbb{G}_m$-action and functorially in V. This can be verified on global sections. The required maps

$$A = \Big(\sum_p \mathrm{Fil}^p V \otimes z^{-p}\mathbb{C}[z]\Big)^{F_\infty} \otimes_{\mathbb{C}[z^2]} \mathbb{C}[z, z^{-1}] \longrightarrow V \otimes \mathbb{C}[z, z^{-1}]$$

are obtained by composition:

$$\begin{aligned} A &\longrightarrow (V \otimes \mathbb{C}[z, z^{-1}]) \otimes_{\mathbb{C}[z^2]} \mathbb{C}[z, z^{-1}] \\ &\longrightarrow (V \otimes \mathbb{C}[z, z^{-1}]) \otimes_{\mathbb{C}[z]} \mathbb{C}[z, z^{-1}] = V \otimes \mathbb{C}[z, z^{-1}]\,. \end{aligned}$$

That they are isomorphisms needs to be checked on the generators $\mathbb{C}(n)^{(-1)^n}$ of $\mathcal{F}il_\mathbb{R}$ only. As for the second assertion it suffices to show that the diagram:

$$\begin{array}{ccc} (sq^*F_\mathbb{R}(V))_1 = & F_\mathbb{R}(V)_1 & = (sq^*F_\mathbb{R}(V))_{-1} \\ \alpha_1 \downarrow \wr & & \alpha_{-1} \downarrow \wr \\ V & \xrightarrow{F_\infty} & V \end{array}$$

is commutative where the vertical arrows come from the above trivialization. They are given by setting $z = 1$ on the left and $z = -1$ on the right. The value of a global section of the form

$$\frac{1}{2}(v \otimes z^p + F_\infty(v \otimes z^p)) = \frac{1}{2}(v + (-1)^p F_\infty(v)) \otimes z^p$$

in $sq^*F_{\mathbb{R}}(V)_{\pm 1}$ is mapped by $\alpha_{\pm 1}$ to $(-1)^p \frac{1}{2}(v + (-1)^p F_\infty(v))$. Hence the composition $\alpha_{-1} \circ (\alpha_1)^{-1}$ maps $w = \frac{1}{2}(v + (-1)^p F_\infty(v))$ to $(-1)^p w = F_\infty(w)$. Since the w's generate V we have $\alpha_{-1} \circ (\alpha_1)^{-1} = F_\infty$ as claimed.
ii) This is a special case of Proposition 6.1. □

3 A real-analytic version of the Rees sheaf

A real structure on an object V of $\mathcal{F}il_{\mathbb{C}}$ leads to a real structure in the algebraic sense on the vector bundle $E_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V)$ – it is then defined over $\mathbb{A}^1_{\mathbb{R}}$. For reasons explained in section 3 we are interested however in obtaining a real structure in the *topological* sense on the Rees bundle. There does not seem to be a natural real Rees bundle over $\mathbb{C} = \mathbb{A}^1(\mathbb{C})$. However over $\mathbb{R} \subset \mathbb{C}$ a suitable topologically real bundle can be constructed, and its properties will be important in section 4. We now proceed with the details.
Let $\mathcal{F}il^{\mathrm{real}}_{\mathbb{C}}$ etc. be categories defined as before but using real instead of complex vector spaces. Let $\mathcal{A}_Y$ denote the sheaf of real valued real-analytic functions on a real C^ω-manifold or more generally orbifold Y. For V in $\mathcal{F}il^{\mathrm{real}}_{\mathbb{C}}$ we set

$$\xi^\omega_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V) = \sum_p \mathrm{Fil}^p V \otimes r^{-p} \mathcal{A}_{\mathbb{R}} \subset V \otimes j_* \mathcal{A}_{\mathbb{R}^*}$$

where r denotes the coordinate on $\mathbb{R}$ and $j : \mathbb{R}^* \hookrightarrow \mathbb{R}$ is the inclusion. This is a free $\mathcal{A}_{\mathbb{R}}$-module. With respect to the flow $\phi^t_{\mathbb{C}}(r) = re^{-t}$ on $\mathbb{R}$ it is equipped with an action

$$\psi^t : (\phi^t_{\mathbb{C}})^{-1} \xi^\omega_{\mathbb{C}} \longrightarrow \xi^\omega_{\mathbb{C}}$$

which is induced by the pullback action:

$$\psi^t = (\phi^t_{\mathbb{C}})^* : (\phi^t_{\mathbb{C}})^{-1} j_* \mathcal{A}_{\mathbb{R}^*} \longrightarrow j_* \mathcal{A}_{\mathbb{R}^*} \; .$$

Let $sq : \mathbb{R} \to \mathbb{R}^{\geq 0}$ be the squaring map $sq(r) = r^2$ and consider the action of $\mu_2 = \{\pm 1\}$ on $\mathbb{R}$ by multiplication. Let $\rho : \mathbb{R} \to \mathbb{R}/\mu_2$ be the natural projection. If we view $\mathbb{R}^{\geq 0}$ as a C^ω-orbifold via the isomorphism

$$\overline{sq} : \mathbb{R}/\mu_2 \xrightarrow{\sim} \mathbb{R}^{\geq 0} \; , \; [r] \longmapsto r^2$$

we have

$$\mathcal{A}_{\mathbb{R}^{\geq 0}} = \overline{sq}_* (\rho_* \mathcal{A}_{\mathbb{R}})^{\mu_2} = (sq_* \mathcal{A}_{\mathbb{R}})^{\mu_2} \; .$$

In the previous situation over $\mathbb{C}$ the adjunction map:

$$sq^* : \mathcal{O}_{\mathbb{C}} \longrightarrow (sq_* \mathcal{O}_{\mathbb{C}})^{\mu_2} \; , \; f \longmapsto (z \mapsto f(z^2))$$

was an isomorphism and we used it to view $\xi_{\mathbb{R}} = (sq_* \xi_{\mathbb{C}})^{F_\infty}$ as an $\mathcal{O}_{\mathbb{C}}$-module. Over $\mathbb{R}$ however the corresponding map is not an isomorphism since $sq : \mathbb{R} \to \mathbb{R}$ is not even surjective and we will have to work with $\mathcal{A}_{\mathbb{R}^{\geq 0}}$ in the following.

For V in $\mathcal{F}il_{\mathbb{R}}^{\pm\mathrm{real}}$ we set

$$\xi_{\mathbb{R}}^{\omega}(V, \mathrm{Fil}^{\bullet} V, F_{\infty}) = (sq_* \xi_{\mathbb{C}}^{\omega}(V, \mathrm{Fil}^{\bullet} V))^{F_{\infty}} \subset sq_*(V \otimes j_* \mathcal{A}_{\mathbb{R}^*})$$

viewed as a free $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-module on the orbifold $\mathbb{R}^{\geq 0}$. With respect to the flow $\phi_{\mathbb{R}}^t(r') = r' e^{-2t}$ on $\mathbb{R}^{\geq 0}$ where $r' = r^2$ we have an action

$$\psi^t : (\phi_{\mathbb{R}}^t)^{-1} \xi_{\mathbb{R}}^{\omega} \longrightarrow \xi_{\mathbb{R}}^{\omega}$$

induced by the action ψ^t on $\xi_{\mathbb{C}}^{\omega}$.
Let $\mathcal{D}_K^{\omega}$ be the category of locally free $\mathcal{A}_{\mathbb{R}}$- resp. $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-modules $\mathcal{M}$ with an action

$$\psi^t : (\phi_K^t)^{-1} \mathcal{M} \longrightarrow \mathcal{M} .$$

Then $\mathcal{D}_K^{\omega}$ has $\otimes$-products and internal Hom's and we define the Tate twist by an integer n as $\mathcal{M}(n) = r^n \mathcal{M}$. Then $\mathcal{M}(n)$ is canonically isomorphic to $\mathcal{M}$ as a module but equipped with the twisted action:

$$\psi_{\mathcal{M}(n)}^t = e^{-tn} \psi_{\mathcal{M}}^t .$$

As before $\xi_{\mathbb{C}}^{\omega}$ and $\xi_{\mathbb{R}}^{\omega} : \mathcal{F}il_{\mathbb{R}}^{\pm\mathrm{real}} \to \mathcal{D}_{\mathbb{R}}^{\omega}$ commute with Tate twists.
The relation with the previous algebraic construction is the following. For Y as above set $\mathcal{O}_Y = \mathcal{A}_Y \otimes_{\mathbb{R}} \mathbb{C}$. Let $i : \mathbb{R} \hookrightarrow \mathbb{C}$ denote the inclusion. Then we have $\mathcal{O}_{\mathbb{R}} = i^{-1} \mathcal{O}_{\mathbb{C}}$ and $\mathcal{O}_{\mathbb{R}^{\geq 0}} = (sq_* \mathcal{O}_{\mathbb{R}})^{\mu_2} = i^{-1}(sq_* \mathcal{O}_{\mathbb{C}})^{\mu_2}$. Moreover:

$$\xi_K^{\omega}(V) \otimes_{\mathbb{R}} \mathbb{C} = i^{-1} \xi_K^{\mathrm{an}}(V \otimes \mathbb{C}) . \tag{5}$$

Here $\xi_K^{\mathrm{an}}(V \otimes \mathbb{C})$ is obtained from $\xi_K(V \otimes \mathbb{C})$ by analytification. It carries a natural involution J coming from the real structures V of $V \otimes \mathbb{C}$ and $\mathbb{R}[z, z^{-1}]$ of $\mathbb{C}[z, z^{-1}]$. The involution $\mathrm{id} \otimes c$ on the left of (5), where c is complex conjugation corresponds to $i^{-1}(J)$ on the right.
These facts can be used to see that the analytic version ξ_K^{ω} over $\mathbb{R}$ resp. $\mathbb{R}^{\geq 0}$ of ξ_K has analogous properties as the algebraic ξ_K on $\mathbb{A}_{\mathbb{C}}^1$.
An object of $\mathcal{F}il_{\mathbb{C}}$ resp. $\mathcal{F}il_{\mathbb{R}}^{\pm}$ may be viewed as an object of $\mathcal{F}il_{\mathbb{C}}^{\mathrm{real}}$ resp. $\mathcal{F}il_{\mathbb{R}}^{\pm\mathrm{real}}$ by considering the underlying $\mathbb{R}$-vector space. We write this functor as $V \mapsto V_{\mathbb{R}}$. It is clear from the definitions that

$$\xi_K^{\omega}(V_{\mathbb{R}}) = i^{-1} \xi_K^{\mathrm{an}}(V) \tag{6}$$

as $\mathcal{A}_{\mathbb{R}}$- resp. $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-modules.
Looking at associated C^{ω}-vector bundles we get:

Corollary 3.1 *To every V in $\mathcal{F}il_{\mathbb{C}}^{\mathrm{real}}$ resp. $\mathcal{F}il_{\mathbb{R}}^{\pm\mathrm{real}}$ there is functorially attached a real C^{ω}-bundle E^{ω} over $\mathbb{R}$ resp. $\mathbb{R}^{\geq 0}$ together with a C^{ω}-action*

$$\psi^t : \phi_K^{t*} E^{\omega} \longrightarrow E^{\omega} .$$

The rank of E^{ω} equals the dimension of V and there are functorial isomorphisms:

$$E^{\omega}(V, \mathrm{Fil}^{\bullet} V)_0 \xrightarrow{\sim} \mathrm{Gr}^{\bullet} V \quad \textit{resp. } E^{\omega}(V, \mathrm{Fil}^{\bullet} V, F_{\infty})_0 \xrightarrow{\sim} \mathrm{Gr}^{\bullet}(sV)$$

such that ψ_0^t corresponds to $e^{\bullet t}$.

4 The relation of Rees sheaves and Rees bundles with archimedian invariants of motives

In this section we first recall briefly the definition of the spaces $\mathcal{F}_{\mathfrak{p}}(M)$ for archimedian primes using Rees sheaves. We then describe the local contribution from $\mathfrak{p}$ in the motivic "explicit formulas" of analytic number theory in terms of a suitable Rees bundle. This formula is new. We also explain the motivation for considering C^{ω}-Rees bundles over $\mathbb{R}$ or $\mathbb{R}^{\geq 0}$ in the preceeding section.
Consider the category of (mixed) motives $\mathcal{M}_k$ over a number field k, for example in the sense of Deligne [De2] or Jannsen [J].
For an infinite place $\mathfrak{p}$ let $M_{\mathfrak{p}}$ be the real Hodge structure of $M \otimes_k k_{\mathfrak{p}}$. In case $\mathfrak{p}$ is real $M_{\mathfrak{p}}$ carries the action of an $\mathbb{R}$-linear involution $F_{\mathfrak{p}}$ which maps the Hodge filtration $F^{\bullet} M_{\mathfrak{p},\mathbb{C}}$ on $M_{\mathfrak{p},\mathbb{C}} = M_{\mathfrak{p}} \otimes_{\mathbb{R}} \mathbb{C}$ to $\overline{F}^{\bullet} M_{\mathfrak{p},\mathbb{C}}$. Consider the descending filtration

$$\gamma^{\nu} M_{\mathfrak{p}} = M_{\mathfrak{p}} \cap F^{\nu} M_{\mathfrak{p},\mathbb{C}} = M_{\mathfrak{p}} \cap F^{\nu} M_{\mathfrak{p},\mathbb{C}} \cap \overline{F}^{\nu} M_{\mathfrak{p},\mathbb{C}}$$

on $M_{\mathfrak{p}}$ and set

$$n_{\nu}(M_{\mathfrak{p}}) = \dim \mathrm{Gr}^{\nu}_{\gamma} M_{\mathfrak{p}} \ .$$

For real $\mathfrak{p}$ write

$$n^{\pm}_{\nu}(M_{\mathfrak{p}}) = \dim (\mathrm{Gr}^{\nu}_{\gamma} M_{\mathfrak{p}})^{\pm}$$

where $\pm$ denotes the ± 1 eigenspace of $F_{\mathfrak{p}}$.
Set $\mathcal{V}^{\nu} M_{\mathfrak{p}} = \gamma^{\nu} M_{\mathfrak{p}}$ if $\mathfrak{p}$ is complex and

$$\mathcal{V}^{\nu} M_{\mathfrak{p}} = (F^{\nu} M_{\mathfrak{p},\mathbb{C}} \cap M_{\mathfrak{p}})^{(-1)^{\nu}} \oplus (F^{\nu+1} M_{\mathfrak{p},\mathbb{C}} \cap M_{\mathfrak{p}})^{(-1)^{\nu+1}}$$

if $\mathfrak{p}$ is real. In other words:

$$(M_{\mathfrak{p}}, \mathcal{V}^{\bullet} M_{\mathfrak{p}}, F_{\infty}) = s(M_{\mathfrak{p}}, \gamma^{\bullet} M_{\mathfrak{p}}, F_{\infty}) \ .$$

In the real case there is an exact sequence

$$0 \longrightarrow (\mathrm{Gr}^{\nu+1}_{\gamma} M_{\mathfrak{p}})^{(-1)^{\nu}} \longrightarrow \mathrm{Gr}^{\nu}_{\mathcal{V}} M_{\mathfrak{p}} \longrightarrow (\mathrm{Gr}^{\nu}_{\gamma} M_{\mathfrak{p}})^{(-1)^{\nu}} \longrightarrow 0 \ .$$

We set $d_{\nu}(M_{\mathfrak{p}}) = \dim \mathrm{Gr}^{\nu}_{\mathcal{V}} M_{\mathfrak{p}}$ and $\Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$ and $\Gamma_{\mathbb{R}}(s) = 2^{-1/2}\pi^{-s/2}\Gamma(s/2)$.
In [F-PR] the local Euler factors of M for the infinite places were defined as follows:

$$L_{\mathfrak{p}}(M,s) = \prod_{\nu} \Gamma_{\mathbb{C}}(s-\nu)^{n_{\nu}(M_{\mathfrak{p}})} \quad \text{if } \mathfrak{p} \text{ is complex}$$

and

$$L_{\mathfrak{p}}(M,s) = \prod_{\nu} \Gamma_{\mathbb{R}}(s+\varepsilon_{\nu}-\nu)^{n^{+}_{\nu}(M_{\mathfrak{p}})} \Gamma_{\mathbb{R}}(s+1-\varepsilon_{\nu}-\nu)^{n^{-}_{\nu}(M_{\mathfrak{p}})}$$

if $\mathfrak{p}$ is real. Here $\varepsilon_\nu \in \{0,1\}$ is determined by $\varepsilon_\nu \equiv \nu \bmod 2$.
Using the above exact sequence we get an alternative formula for real $\mathfrak{p}$:

$$L_{\mathfrak{p}}(M,s) = \prod_\nu \Gamma_{\mathbb{R}}(s-\nu)^{d_\nu(M_{\mathfrak{p}})} .$$

See also [D4] for background. It follows from remark 2.1 that for $\mathfrak{p} \,|\, \infty$ the space $\mathcal{F}_{\mathfrak{p}}(M)$ of [D3] § 5 is given as follows:

$$\mathcal{F}_{\mathfrak{p}}(M) = \Gamma(\mathbb{A}^1, \xi_{\mathbb{C}}(M_{\mathfrak{p}\mathbb{C}}, \gamma^\bullet M_{\mathfrak{p}\mathbb{C}})) \quad \text{if } \mathfrak{p} \text{ is complex}$$

and

$$\begin{aligned} \mathcal{F}_{\mathfrak{p}}(M) &= \Gamma(\mathbb{A}^1, \xi_{\mathbb{R}}(M_{\mathfrak{p}\mathbb{C}}, \gamma^\bullet M_{\mathfrak{p}\mathbb{C}}, F_\infty)) \\ &= \Gamma(\mathbb{A}^1, \xi_{\mathbb{R}}(M_{\mathfrak{p}\mathbb{C}}, \mathcal{V}^\bullet M_{\mathfrak{p}\mathbb{C}}, F_\infty)) \quad \text{if } \mathfrak{p} \text{ is real.} \end{aligned}$$

According to [D3] Cor. 6.5 we have:

$$L_{\mathfrak{p}}(M,s) = \det{}_\infty\Big(\frac{1}{2\pi}(s\cdot \mathrm{id} - \Theta) \,|\, \mathcal{F}_{\mathfrak{p}}(M)\Big)^{-1} .$$

Here Θ is the infinitesimal generator of the $\mathbb{G}_m$-action of $\mathcal{F}_{\mathfrak{p}}(M)$, i.e. the induced action by $1 \in \mathbb{C} = \mathrm{Lie}\,\mathbb{G}_m$.
We define the real analytic version of $\mathcal{F}_{\mathfrak{p}}(M)$ as follows:

$$\mathcal{F}^\omega_{\mathfrak{p}}(M) = \Gamma(\mathbb{R}, \xi^\omega_{\mathbb{C}}(M_{\mathfrak{p}}, \gamma^\bullet M_{\mathfrak{p}})) \quad \text{if } \mathfrak{p} \text{ is complex}$$

and

$$\begin{aligned} \mathcal{F}^\omega_{\mathfrak{p}}(M) &= \Gamma(\mathbb{R}^{\ge 0}, \xi^\omega_{\mathbb{R}}(M_{\mathfrak{p}}, \gamma^\bullet M_{\mathfrak{p}}, F_\infty)) \\ &= \Gamma(\mathbb{R}^{\ge 0}, \xi^\omega_{\mathbb{R}}(M_{\mathfrak{p}}, \mathcal{V}^\bullet M_{\mathfrak{p}}, F_\infty)) \quad \text{if } \mathfrak{p} \text{ is real.} \end{aligned}$$

It follows from the above formula for $L_{\mathfrak{p}}(M,s)$ in terms of $\mathcal{F}_{\mathfrak{p}}(M)$ and the relation between ξ^ω_K and ξ_K that we have for all $\mathfrak{p} \,|\, \infty$:

$$L_{\mathfrak{p}}(M,s) = \det{}_\infty\Big(\frac{1}{2\pi}(s\cdot \mathrm{id} - \Theta) \,|\, \mathcal{F}^\omega_{\mathfrak{p}}(M)\Big)^{-1} . \tag{7}$$

Here Θ denotes the infinitesimal generator of the flow ψ^{t*} induced on $\mathcal{F}^\omega_{\mathfrak{p}}(M)$ by the actions ψ^t and $\phi^t_{k_{\mathfrak{p}}}$ which were defined in section 2.
In the next section we will express $\mathcal{F}^\omega_{\mathfrak{p}}(H^n(X))$ for smooth projective varieties X/k in “dynamical” terms. Via formula (7) we then get formulas for the archimedian L-factors $L_{\mathfrak{p}}(H^n(X),s)$ which come from the geometry of a simple dynamical system.
Let us now turn to the motivic “explicit formulas” of analytic number theory. To every motive M in $\mathcal{M}_k$ one can attach local Euler factors $L_{\mathfrak{p}}(M,s)$ for all the places $\mathfrak{p}$ in k and global L-functions:

$$L(M,s) = \prod_{\mathfrak{p}\nmid\infty} L_{\mathfrak{p}}(M,s) \quad \text{and} \quad \hat{L}(M,s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(M,s) ,$$

c.f. [F-PR], [D4]. Assuming standard conjectures about the analytical behaviour of $L(M,s)$ and $L(M^*,s)$ proved in many interesting cases the following explicit formula in the analytic number theory of motives holds for every φ in $\mathcal{D}(\mathbb{R}^+) = C_0^\infty(\mathbb{R}^+)$ c.f. [D-Sch] (2.2.1):

$$-\sum_\rho \Phi(\rho) \mathrm{ord}_{s=\rho} \hat{L}(M,s) = \sum_{\mathfrak{p}} W_{\mathfrak{p}}(\varphi)\;. \tag{8}$$

Here $\Phi(s) = \int_{\mathbb{R}} \varphi(t) e^{ts}\, dt$ and $\mathfrak{p}$ runs over all places of k.
For finite $\mathfrak{p}$ we have

$$W_{\mathfrak{p}}(\varphi) = \log N\mathfrak{p} \sum_{k=1}^\infty \mathrm{Tr}(\mathrm{Fr}_{\mathfrak{p}}^k \mid M_l^{I_{\mathfrak{p}}}) \varphi(k \log N\mathfrak{p}) \tag{9}$$

where $\mathrm{Fr}_{\mathfrak{p}}$ denotes a geometric Frobenius at $\mathfrak{p}$ and $M_l^{I_{\mathfrak{p}}}$ is the fixed module under inertia of the l-adic realization of M with $\mathfrak{p} \nmid l$.

The terms $W_{\mathfrak{p}}$ for the infinite places are given as follows: For complex $\mathfrak{p}$ we have:

$$W_{\mathfrak{p}}(\varphi) = \sum_\nu n_\nu(M_{\mathfrak{p}}) \int_0^\infty \varphi(t) \frac{e^{\nu t}}{1-e^{-t}}\, dt \tag{10}$$

whereas for real $\mathfrak{p}$:

$$W_{\mathfrak{p}}(\varphi) = \sum_\nu d_\nu(M_{\mathfrak{p}}) \int_0^\infty \varphi(t) \frac{e^{\nu t}}{1-e^{-2t}}\, dt\;. \tag{11}$$

The distributions $W_{\mathfrak{p}}$ for $\mathfrak{p} \mid \infty$ can be rewritten as follows:

$$W_{\mathfrak{p}} = \frac{\mathrm{Tr}(e^{\bullet t} \mid \mathrm{Gr}_{\mathcal{V}}^\bullet M_{\mathfrak{p}})}{1 - e^{-\kappa_{\mathfrak{p}} t}} \tag{12}$$

where $e^{\bullet t}$ is the map $e^{\nu t}$ on Gr^ν and $\kappa_{\mathfrak{p}} = 2$ resp. 1 according to whether $\mathfrak{p}$ is real or complex.
In terms of our conjectural cohomology theory c.f. [D3] § 7, equation (8) can thus be reformulated as an equality of distributions on $\mathbb{R}^+$:

$$\begin{aligned} &\sum_i (-1)^i \mathrm{Tr}(\psi^* \mid H^i(\text{“}\overline{\mathrm{spec}\, \mathfrak{o}_k}\text{”}, \mathcal{F}(M)))_{\mathrm{dis}} \\ &= \sum_{\mathfrak{p} \nmid \infty} \log N\mathfrak{p} \sum_{k=1}^\infty \mathrm{Tr}(\mathrm{Fr}_{\mathfrak{p}}^k \mid M_l^{I_{\mathfrak{p}}}) \delta_{k \log N\mathfrak{p}} + \sum_{\mathfrak{p} \mid \infty} \frac{\mathrm{Tr}(e^{\bullet t} \mid \mathrm{Gr}_{\mathcal{V}}^\bullet M_{\mathfrak{p}})}{1 - e^{-\kappa_{\mathfrak{p}} t}}\;. \end{aligned} \tag{13}$$

Compare [D-Sch] (3.1.1) for the elementary notion of distributional trace used on cohomology here. In the rest of this section we will be concerned with a deeper understanding of the function $\mathrm{Tr}(e^{\bullet t} \mid \mathrm{Gr}_{\mathcal{V}}^\bullet M_{\mathfrak{p}})$.

Certain dynamical trace formulas for vector bundles E over a manifold X with a flow ϕ^t and an action $\psi^t : \phi^{t*}E \to E$ involve local contributions at the fixed points x of the form:

$$\frac{\mathrm{Tr}(\psi^t_x \mid E_x)}{1 - e^{-\kappa_x t}} \qquad \text{some } \kappa_x > 0 \; .$$

This is explained in [D7] § 4. These formulas bear a striking resemblance to the "explicit formulas" and they suggest that infinite places correspond to fixed points of a flow. Incidentially the finite places would correspond to the periodic orbits. This analogy suggests that for the infinite places it should be possible to attach to M a real vector bundle E in the topological sense over a dynamical system with fixed points. If 0 denotes the fixed point corresponding to $\mathfrak{p}$, we should have:

$$\mathrm{Tr}(\psi^t_0 \mid E_0) = \mathrm{Tr}(e^{\bullet t} \mid \mathrm{Gr}^{\bullet}_{\mathcal{V}} M_{\mathfrak{p}}) \; .$$

At least over one flowline this is achieved by Corollary 3.1 as follows.
Define as follows a real C^{ω}-bundle $E^{\omega}_{\mathfrak{p}}(M)$ over $\mathbb{R}$ resp. $\mathbb{R}^{\geq 0}$ together with a C^{ω}-action

$$\psi^t : \phi^{t*}_K E^{\omega}_{\mathfrak{p}}(M) \longrightarrow E^{\omega}_{\mathfrak{p}}(M) \; .$$

Set

$$E^{\omega}_{\mathfrak{p}}(M) = E^{\omega}(M_{\mathfrak{p}}, \gamma^{\bullet} M_{\mathfrak{p}}) \quad \text{if } \mathfrak{p} \text{ is complex}$$

and

$$\begin{aligned} E^{\omega}_{\mathfrak{p}}(M) &= E^{\omega}(M_{\mathfrak{p}}, \gamma^{\bullet} M_{\mathfrak{p}}, F_{\infty}) \\ &= E^{\omega}(M_{\mathfrak{p}}, \mathcal{V}^{\bullet} M_{\mathfrak{p}}, F_{\infty}) \quad \text{if } \mathfrak{p} \text{ is real.} \end{aligned}$$

Note that this is just the C^{ω}-bundle corresponding to the locally free sheaf $\mathcal{F}^{\omega}_{\mathfrak{p}}(M)$ defined earlier. According to Corollary 3.1 we then have:

Proposition 4.1 *There are functorial isomorphisms*

$$E^{\omega}_{\mathfrak{p}}(M)_0 \xrightarrow{\sim} \mathrm{Gr}^{\bullet}_{\mathcal{V}} M_{\mathfrak{p}}$$

for all $\mathfrak{p} \mid \infty$ such that ψ^t_0 corresponds to $e^{\bullet t}$. In particular we find:

$$\begin{aligned} \mathrm{Tr}(\psi^t_0 \mid E^{\omega}_{\mathfrak{p}}(M)_0) &= \mathrm{Tr}(e^{\bullet t} \mid \mathrm{Gr}^{\bullet}_{\mathcal{V}} M_{\mathfrak{p}}) \\ &= (1 - e^{-\kappa_{\mathfrak{p}} t}) W_{\mathfrak{p}} \; . \end{aligned}$$

5 A geometrical construction of $\mathcal{F}^{\omega}_{\mathfrak{p}}(M)$ and $E^{\omega}_{\mathfrak{p}}(M)$ for $M = H^n(X)$

In this section we express the locally free sheaf $\mathcal{F}^{\omega}_{\mathfrak{p}}(H^n(X))$ over $\mathbb{R}$ resp. $\mathbb{R}^{\geq 0}$ of section 3 in terms of higher direct image sheaves modulo torsion. The construction is based on the following result of Simpson [Si] Prop. 5.1, 5.2. For a variety $X/\mathbb{C}$ we write X^{an} for the associated complex space.

Theorem 5.1 (Simpson) *Let $X/\mathbb{C}$ be a smooth proper variety and let $F^\bullet$ be the Hodge filtration on $H^n(X^{\mathrm{an}}, \mathbb{C})$. Then we have:*

$$\xi_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{C}), F^\bullet) = R^n\pi_*(\Omega^\bullet_{X\times\mathbb{A}^1/\mathbb{A}^1}, zd)$$

where $\pi : X \times \mathbb{A}^1 \to \mathbb{A}^1$ denotes the projection.

Remarks (1) The $\mathbb{G}_m$-action on the deformed complex $(\Omega^\bullet_{X\times\mathbb{A}^1/\mathbb{A}^1}, zd)$ given by sending a homogenous form ω to $\lambda^{-\deg\omega} \cdot \lambda^*(\omega)$ for $\lambda \in \mathbb{G}_m$ induces a $\mathbb{G}_m$-action on $R^n\pi_*(\Omega^\bullet_{X\times\mathbb{A}^1/\mathbb{A}^1}, zd)$. Under the isomorphism of the theorem it corresponds to the $\mathbb{G}_m$-action on $\xi_{\mathbb{C}}(H^n(X, \mathbb{C}), F^\bullet)$ defined by formula (4).
(2) In the appendix to this section we relate the complex $(\Omega^\bullet_{X\times\mathbb{A}^1/\mathbb{A}^1}, zd)$ to the complex of relative differential forms on a suitable deformation of $X \times \mathbb{A}^1$.

In the situation of the theorem consider the natural morphism from the spectral sequence:

$$E_1^{pq} = (R^q\pi_*(\Omega^p_{X\times\mathbb{A}^1/\mathbb{A}^1}))^{\mathrm{an}} \Longrightarrow (R^n\pi_*(\Omega^\bullet_{X\times\mathbb{A}^1/\mathbb{A}^1}, zd))^{\mathrm{an}}$$

to the spectral sequence

$$E_1^{pq} = R^q\pi_*(\Omega^p_{X^{\mathrm{an}}\times\mathbb{C}/\mathbb{C}}) \Longrightarrow R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{C}/\mathbb{C}}, zd) \; .$$

By *GAGA* it is an isomorphism on the E_1-terms and hence on the end terms as well. Thus we get a natural isomorphism of locally free $\mathcal{O}_{\mathbb{C}}$-modules:

$$\xi^{\mathrm{an}}_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{C}), F^\bullet) = R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{C}/\mathbb{C}}, zd) \; . \tag{14}$$

Let $\mathrm{id} \times i : X^{\mathrm{an}} \times \mathbb{R} \hookrightarrow X^{\mathrm{an}} \times \mathbb{C}$ be the inclusion and set

$$\Omega^p_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}} = (\mathrm{id} \times i)^{-1}\Omega^p_{X^{\mathrm{an}}\times\mathbb{C}/\mathbb{C}} \; .$$

It is the subsheaf of $\mathbb{C}$-valued smooth relative differential forms on $X^{\mathrm{an}} \times \mathbb{R}/\mathbb{R}$ which are holomorphic in the X^{an}-coordinates and real analytic in the $\mathbb{R}$-variable. We then have an equality of complexes

$$(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd) = (\mathrm{id} \times i)^{-1}(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{C}/\mathbb{C}}, zd) \; .$$

We define an action ψ^t of $\mathbb{R}$ on $(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd)$ by sending a homogenous form ω to $e^{t\deg\omega} \cdot (\mathrm{id} \times \phi^t_{\mathbb{C}})^*\omega$:

$$\psi^t : (\mathrm{id} \times \phi^t_{\mathbb{C}})^{-1}(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd) \longrightarrow (\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd) \; .$$

This induces an action:

$$\psi^t : (\phi^t_{\mathbb{C}})^{-1}R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd) \longrightarrow R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd)$$

and hence an $\mathcal{A}_{\mathbb{R}}$-linear action

$$\psi^t : \phi_{\mathbb{C}}^{t*} R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd) \longrightarrow R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd) \ .$$

By proper base change we obtain from (14) that

$$i^{-1}\xi_{\mathbb{C}}^{\mathrm{an}}(H^n(X^{\mathrm{an}},\mathbb{C}), F^\bullet) = R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd) \ . \tag{15}$$

According to (6) this gives an isomorphism

$$\xi_{\mathbb{C}}^{\omega}((H^n(X^{\mathrm{an}},\mathbb{C}), F^\bullet)_{\mathbb{R}}) = R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd) \tag{16}$$

of locally free $\mathcal{A}_{\mathbb{R}}$-modules which is compatible with the action ψ^t relative to $\phi^t_{\mathbb{C}}$.
Let $\mathcal{DR}_{X/\mathbb{C}}$ be the cokernel of the natural inclusion of complexes of $\pi^{-1}\mathcal{A}_{\mathbb{R}}$-modules on $X^{\mathrm{an}}\times\mathbb{R}$ with action ψ^t

$$\pi^{-1}\mathcal{A}_{\mathbb{R}} \longrightarrow (\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}}, rd) \ .$$

Here $\pi^{-1}\mathcal{A}_{\mathbb{R}}$ is viewed as a complex concentrated in degree zero and on it ψ^t acts by pullback via $\mathrm{id}\times\phi^t_{\mathbb{C}}$. The projection formula gives us

$$R^n\pi_*(\pi^{-1}\mathcal{A}_{\mathbb{R}}) = H^n(X^{\mathrm{an}},\mathbb{R})\otimes\mathcal{A}_{\mathbb{R}} = \xi_{\mathbb{C}}^{\omega}(H^n(X^{\mathrm{an}},\mathbb{R}), \mathrm{Fil}_0^\bullet) \tag{17}$$

where

$$\mathrm{Fil}_0^p H^n(X^{\mathrm{an}},\mathbb{R}) = H^n(X^{\mathrm{an}},\mathbb{R})$$

for $p \le 0$ and $\mathrm{Fil}_0^p = 0$ for $p > 0$. We thus get a long exact ψ^t-equivariant sequence of coherent $\mathcal{A}_{\mathbb{R}}$-modules:

$$\begin{array}{ccccccc} \ldots & \to & \xi_{\mathbb{C}}^{\omega}(H^n(X^{\mathrm{an}},\mathbb{R}),\mathrm{Fil}_0^\bullet) & \to & \xi_{\mathbb{C}}^{\omega}((H^n(X^{\mathrm{an}},\mathbb{C}),F^\bullet)_{\mathbb{R}}) & \to & \\ & \to & R^n\pi_*\mathcal{DR}_{X/\mathbb{C}} & \to & \xi_{\mathbb{C}}^{\omega}(H^{n+1}(X^{\mathrm{an}},\mathbb{R}),\mathrm{Fil}_0^\bullet) & \to & \ldots \end{array} \tag{18}$$

For any n the natural map

$$\xi_{\mathbb{C}}^{\omega}(H^n(X^{\mathrm{an}},\mathbb{R}),\mathrm{Fil}_0^\bullet) \longrightarrow \xi_{\mathbb{C}}^{\omega}((H^n(X^{\mathrm{an}},\mathbb{C}),F^\bullet)_{\mathbb{R}})$$

is injective by the $\xi_{\mathbb{C}}^{\omega}$-analogue of Prop. 2.2 c), part i) since it is induced by the inclusion of objects in $\mathcal{F}il_{\mathbb{C}}^{\mathrm{real}}$:

$$(H^n(X^{\mathrm{an}},\mathbb{R}),\mathrm{Fil}_0^\bullet) \hookrightarrow (H^n(X^{\mathrm{an}},\mathbb{C}),F^\bullet)_{\mathbb{R}} \ . \tag{19}$$

The injectivity can also be seen by noting that the fibres of the associated C^ω-vector bundles for $r\in\mathbb{R}^*$ are naturally isomorphic to $H^n(X^{\mathrm{an}},\mathbb{R})$ resp. $H^n(X^{\mathrm{an}},\mathbb{C})$, the map being the inclusion c.f. the $\xi_{\mathbb{C}}^{\omega}$-analogue of Proposition 2.3 i).

Therefore the long exact sequence (18) splits into the short exact sequences:

$$0 \to \xi^\omega_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}), \mathrm{Fil}^\bullet_0) \to \xi^\omega_{\mathbb{C}}((H^n(X^{\mathrm{an}}, \mathbb{C}), F^\bullet)_{\mathbb{R}}) \overset{\alpha}{\to} R^n\pi_*\mathcal{DR}_{X/\mathbb{C}} \to 0 \; . \tag{20}$$

Using the $\xi^\omega_{\mathbb{C}}$-version of Proposition 3.1 c) ii) we therefore get a ψ^t-equivariant isomorphism of $\mathcal{A}_{\mathbb{R}}$-modules:

$$R^n\pi_*(\mathcal{DR}_{X/\mathbb{C}})/\mathcal{A}_{\mathbb{R}}\text{-torsion} \xrightarrow{\sim} \xi^\omega_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^\bullet)) \; . \tag{21}$$

Here we have used the exact sequence:

$$0 \longrightarrow H^n(X^{\mathrm{an}}, \mathbb{R}) \longrightarrow H^n(X^{\mathrm{an}}, \mathbb{C}) \xrightarrow{\pi_1} H^n(X^{\mathrm{an}}, \mathbb{R}(1)) \longrightarrow 0 \; ,$$

where $\pi_1(f) = \frac{1}{2}(f - \overline{f})$.
Let us now indicate the necessary amendments for the case $K = \mathbb{R}$. We consider a smooth and proper variety $X/\mathbb{R}$. Its associated complex manifold X^{an} is equipped with an antiholomorphic involution F_∞, which in turn gives rise to an involution $\overline{F}^*_\infty$ of $H^n(X^{\mathrm{an}}, \mathbb{R}(1))$ which maps the filtration $\pi_1(F^\bullet)$ to itself. By definition of $\xi^\omega_{\mathbb{R}}$ we have

$$\xi^\omega_{\mathbb{R}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^\bullet), \overline{F}^*_\infty) = (sq_*\xi^\omega_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^\bullet)))^{F_\infty} \tag{22}$$

where $F_\infty \mathrel{\widehat{=}} \overline{F}^*_\infty \otimes (-\mathrm{id})^*$.
To deal with the other side of (21) consider the μ_2-action on $X^{\mathrm{an}} \times \mathbb{R}$ by $F_\infty \times (-\mathrm{id})$ and let

$$\lambda : X^{\mathrm{an}} \times \mathbb{R} \longrightarrow X^{\mathrm{an}} \times_{\mu_2} \mathbb{R} = (X^{\mathrm{an}} \times \mathbb{R})/\mu_2$$

be the canonical projection.
The map

$$\lambda_*(\pi^{-1}\mathcal{A}_{\mathbb{R}}) \longrightarrow \lambda_*(\Omega^\bullet_{X^{\mathrm{an}} \times \mathbb{R}/\mathbb{R}}, rd)$$

becomes μ_2-equivariant if $-1 \in \mu_2$ acts by $(F_\infty \times (-\mathrm{id}))^*$ on the left and by sending a homogenous form ω to $(-1)^{\deg \omega}(F_\infty \times (-\mathrm{id}))^*\overline{\omega}$ on the right. We set

$$\Omega^\bullet_{X^{\mathrm{an}} \times_{\mu_2} \mathbb{R}/(\mathbb{R}/\mu_2)} = \Big(\lambda_*(\Omega^\bullet_{X^{\mathrm{an}} \times \mathbb{R}/\mathbb{R}}, rd)\Big)^{\mu_2}$$

and

$$\mathcal{DR}_{X/\mathbb{R}} = (\lambda_*\mathcal{DR}_{X/\mathbb{C}})^{\mu_2} \; .$$

Let π be the composed map

$$\pi : X^{\mathrm{an}} \times_{\mu_2} \mathbb{R} \longrightarrow \mathbb{R}/\mu_2 \xrightarrow[\sim]{\overline{sq}} \mathbb{R}^{\geq 0} \; .$$

Combining the isomorphisms (21) and (22) we obtain an isomorphism of free $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-modules on $\mathbb{R}^{\geq 0}$:

$$R^n\pi_*(\mathcal{DR}_{X/\mathbb{R}})/\mathcal{A}_{\mathbb{R}^{\geq 0}}\text{-torsion} \xrightarrow{\sim} \xi^\omega_{\mathbb{R}}(H^n(X^{\mathrm{an}},\mathbb{R}(1)),\pi_1(F^\bullet),\overline{F}^*_\infty)\ . \tag{23}$$

The left hand side carries a natural action ψ^t with respect to the flow $\phi^t_{\mathbb{R}}$ on $\mathbb{R}^{\geq 0}$ and the isomorphism (23) is ψ^t-equivariant.
As before we have a short exact sequence:

$$\begin{array}{rcl} 0 \to \xi^\omega_{\mathbb{R}}(H^n(X^{\mathrm{an}},\mathbb{R}),\mathrm{Fil}^\bullet_0,F^*_\infty) & \to & \xi^\omega_{\mathbb{R}}((H^n(X^{\mathrm{an}},\mathbb{C}),F^\bullet,\overline{F}^*_\infty)_{\mathbb{R}}) \\ & \xrightarrow{\alpha} & R^n\pi_*\mathcal{DR}_{X/\mathbb{R}} \to 0\ . \end{array} \tag{24}$$

The first main result of this section is the following:

Theorem 5.2 *Fix a smooth and proper variety X/K of dimension d where $K = \mathbb{C}$ or $\mathbb{R}$. Assume that $n+m = 2d$. Then we have natural isomorphisms:*

1) $\xi_{\mathbb{C}}(H^m(X^{\mathrm{an}},\mathbb{R}),\gamma^\bullet) = (2\pi i)^{1-d}\underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(R^n\pi_*\mathcal{DR}_{X/\mathbb{C}},\mathcal{A}_{\mathbb{R}}(-d))$
in case $K = \mathbb{C}$ and

2) $\xi_{\mathbb{R}}(H^m(X^{\mathrm{an}},\mathbb{R}),\mathcal{V}^\bullet,F_\infty)$
$= (2\pi i)^{1-d}\underline{\mathrm{Hom}}_{\mathcal{A}^{\geq 0}_{\mathbb{R}}}(R^n\pi_*\mathcal{DR}_{X/\mathbb{R}},\mathcal{A}_{\mathbb{R}^{\geq 0}}(1-d))$
if $K = \mathbb{R}$.

These isomorphisms respect the $\mathcal{A}_{\mathbb{R}}$-resp. $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-module structure and the flow ψ^t.

Proof Consider the perfect pairing of $\mathbb{R}$-Hodge structures:

$$\langle,\rangle : H^n(X^{\mathrm{an}}) \times H^m(X^{\mathrm{an}}) \xrightarrow{\cup} H^{2d}(X^{\mathrm{an}}) \overset{\mathrm{tr}}{\xrightarrow{\sim}} \mathbb{R}(-d) \tag{25}$$

given by $\cup$-product followed by the trace isomorphism

$$\mathrm{tr}(c) = \frac{1}{(2\pi i)^d}\int_{X^{\mathrm{an}}} c\ .$$

It says in particular that

$$F^iH^n(X^{\mathrm{an}},\mathbb{C})^\perp = F^{d+1-i}H^m(X^{\mathrm{an}},\mathbb{C})\ . \tag{26}$$

Moreover it leads to a perfect pairing of $\mathbb{R}$-vector spaces:

$$\langle,\rangle : H^n(X^{\mathrm{an}},\mathbb{R}(1)) \times H^m(X^{\mathrm{an}},\mathbb{R}(d-1)) \longrightarrow \mathbb{R}\ . \tag{27}$$

Now according to the ω-version of Proposition 2.2 a) we have:

$$\begin{array}{rl} & \underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(\xi^\omega_{\mathbb{C}}(H^n(X^{\mathrm{an}},\mathbb{R}(1)),\pi_1(F^\bullet)),\mathcal{A}_{\mathbb{R}}) \\ = & \xi^\omega_{\mathbb{C}}\left(H^n(X^{\mathrm{an}},\mathbb{R}(1))^*,\pi_1(F^{1-\bullet})^\perp\right) \\ \overset{(27)}{=} & \xi^\omega_{\mathbb{C}}\left(H^m(X^{\mathrm{an}},\mathbb{R}(d-1)),\mathrm{Fil}^\bullet\right) \end{array}$$

where Fil^p consists of those elements $u \in H^m(X^{\mathrm{an}}, \mathbb{R}(d-1))$ with

$$\langle \pi_1(F^{1-p}), u\rangle = \pi_d \langle F^{1-p}, u\rangle = 0$$

i.e. with

$$\langle F^{1-p}, u\rangle = 0 \; .$$

Using (26) we find:

$$\begin{aligned}\mathrm{Fil}^p &= H^m(X^{\mathrm{an}}, \mathbb{R}(d-1)) \cap F^{p+d} H^m(X^{\mathrm{an}}, \mathbb{C}) \\ &= (2\pi i)^{d-1} \gamma^{p+d} H^m(X^{\mathrm{an}}, \mathbb{R})\end{aligned}$$

and therefore:

$$\begin{aligned}&\underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(\xi^\omega_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^\bullet)), \mathcal{A}_{\mathbb{R}}) \\ &\quad = (2\pi i)^{d-1} \xi^\omega_{\mathbb{C}}((H^m(X^{\mathrm{an}}, \mathbb{R}), \gamma^\bullet)(d)) \\ &\quad = (2\pi i)^{d-1} \xi^\omega_{\mathbb{C}}(H^m(X^{\mathrm{an}}, \mathbb{R}), \gamma^\bullet)(d) \; .\end{aligned}$$

Combining this with the isomorphism (21) we get the first assertion. As for the second note that by Proposition 2.2 a) we have:

$$\begin{aligned}&\underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}\geq 0}}(\xi^\omega_{\mathbb{R}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^\bullet), \overline{F}^*_\infty), \mathcal{A}_{\mathbb{R}\geq 0}) \\ &\quad = \xi^\omega_{\mathbb{R}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1))^*, \pi_1(F^{2-\bullet})^\perp \; , \text{ dual of } \overline{F}^*_\infty) \; .\end{aligned}$$

Since for $X/\mathbb{R}$ the pairing (25) is $\overline{F}^*_\infty$-equivariant this equals

$$\xi^\omega_{\mathbb{R}}(H^m(X^{\mathrm{an}}, \mathbb{R}(d-1)), \mathrm{Fil}^\bullet, \overline{F}^*_\infty)$$

where Fil^p consists of those elements u with:

$$\langle \pi_1(F^{2-p}), u\rangle = 0 \; .$$

Thus

$$\mathrm{Fil}^p = (2\pi i)^{d-1} \gamma^{p+d-1} H^m(X^{\mathrm{an}}, \mathbb{R})$$

in $\mathcal{F}il^{\pm\mathrm{real}}_{\mathbb{R}}$. Hence:

$$\begin{aligned}&\underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}\geq 0}}(\xi^\omega_{\mathbb{R}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^\bullet), \overline{F}^*_\infty), \mathcal{A}_{\mathbb{R}\geq 0}) \\ &\quad = (2\pi i)^{d-1} \xi^\omega_{\mathbb{R}}((H^m(X^{\mathrm{an}}, \mathbb{R}), \gamma^\bullet, F^*_\infty)(d-1)) \\ &\quad = (2\pi i)^{d-1} \xi^\omega_{\mathbb{R}}(H^m(X^{\mathrm{an}}, \mathbb{R}), \gamma^\bullet, F^*_\infty)(d-1) \; .\end{aligned}$$

Since we can replace $\gamma^\bullet$ by $\mathcal{V}^\bullet = s\gamma^\bullet$ in the last expression the second formula of the theorem now follows by invoking the isomorphism (23). □

If X/K is projective, fixing a polarization defined over K the hard Lefschetz theorem together with Poincaré duality provides an isomorphism of $\mathbb{R}$-Hodge structures over K:

$$H^n(X^{\mathrm{an}})^* = H^n(X^{\mathrm{an}})(1) \; . \tag{28}$$

Similar arguments as before based on (28) instead of (25) then give the following result:

Corollary 5.3 *Fix a smooth projective variety X/K together with the class of a hyperplane section over K. There are canonical isomorphisms:*
1) $\xi_{\mathbb{C}}(H^n(X^{\mathrm{an}},\mathbb{R}),\gamma^\bullet) = \underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(R^n\pi_*\mathcal{DR}_{X/\mathbb{C}},\mathcal{A}_{\mathbb{R}}(-1))$
in case $K=\mathbb{C}$ and
2) $\xi_{\mathbb{R}}(H^n(X^{\mathrm{an}},\mathbb{R}),\mathcal{V}^\bullet,F_\infty^*) = \underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}^{\ge 0}}}(R^n\pi_*\mathcal{DR}_{X/\mathbb{R}},\mathcal{A}_{\mathbb{R}^{\ge 0}})$
if $K=\mathbb{R}$. These isomorphisms respect the $\mathcal{A}_{\mathbb{R}}$-resp. $\mathcal{A}_{\mathbb{R}^{\ge 0}}$-module structure and the action of the flow.

A consideration of the sequence

$$\begin{aligned} 0 \longrightarrow (H^n(X^{\mathrm{an}},\mathbb{R}),\gamma^\bullet) \;&\longrightarrow\; (H^n(X^{\mathrm{an}},\mathbb{C}),F^\bullet)_{\mathbb{R}} \\ &\xrightarrow{\pi_1}\; (H^n(X^{\mathrm{an}},\mathbb{R}(1)),\pi_1(F^\bullet)) \longrightarrow 0 \end{aligned}$$

in $\mathcal{F}il^{\mathrm{real}}_{\mathbb{C}}$ and of

$$\begin{aligned} 0 \longrightarrow (H^n(X^{\mathrm{an}},\mathbb{R}),\gamma^\bullet,F_\infty^*) \;&\longrightarrow\; (H^n(X^{\mathrm{an}},\mathbb{C}),F^\bullet,\overline{F}_\infty^*)_{\mathbb{R}} \\ &\xrightarrow{\pi_1}\; (H^n(X^{\mathrm{an}},\mathbb{R}(1)),\pi_1(F^\bullet),\overline{F}_\infty^*) \longrightarrow 0 \end{aligned}$$

in $\mathcal{F}il^{\pm\mathrm{real}}_{\mathbb{R}}$ leads to the following expressions for ξ_K of $(H^n(X^{\mathrm{an}},\mathbb{R}),\gamma^\bullet,(F_\infty^*))$ which are not based on duality:

Theorem 5.4 *Let X be a smooth and proper variety over K. Then we have for $K=\mathbb{C}$*

1) $\xi_{\mathbb{C}}(H^n(X^{\mathrm{an}},\mathbb{R}),\gamma^\bullet)$
$= \mathrm{Ker}\left(R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}},rd) \xrightarrow{\alpha} R^n\pi_*\mathcal{DR}_{X/\mathbb{C}}/\mathcal{A}_{\mathbb{R}}\text{-}\mathit{torsion}\right)$
= inverse image in $R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times\mathbb{R}/\mathbb{R}},rd)$ of the maximal $\mathcal{A}_{\mathbb{R}}$-submodule of $R^n\pi_*\mathcal{DR}_{X/\mathbb{C}}$ with support in $0\in\mathbb{R}$.

For $K=\mathbb{R}$ we find similarly:

2) $\xi_{\mathbb{R}}((H^n(X^{\mathrm{an}},\mathbb{R}),\mathcal{V}^\bullet,F_\infty^*)$
$= \mathrm{Ker}\left(R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times_{\mu_2}\mathbb{R}/(\mathbb{R}/\mu_2)},rd) \xrightarrow{\alpha} R^n\pi_*\mathcal{DR}_{X/\mathbb{R}}/\mathcal{A}_{\mathbb{R}^{\ge 0}}\text{-}\mathit{torsion}\right)$
= inverse image in $R^n\pi_*(\Omega^\bullet_{X^{\mathrm{an}}\times_{\mu_2}\mathbb{R}/(\mathbb{R}/\mu_2)},rd)$ of the maximal $\mathcal{A}_{\mathbb{R}^{\ge 0}}$-submodule of $R^n\pi_*\mathcal{DR}_{X/\mathbb{R}}$ with support in $0\in\mathbb{R}^{\ge 0}$.

By passing to the associated C^ω-vector bundles over $\mathbb{R}$ resp. $\mathbb{R}^{\ge 0}$ the preceeding theorems and corollary give a geometric construction of the C^ω-bundle $E^\omega_{\mathfrak{p}}(M)$ attached to a motive M in section 3. The Hodge theoretic notions previously required for its definition have been replaced by using suitably deformed complexes and their dynamics.

Appendix

In this appendix we relate the deformed complex $(\Omega^\bullet_{X\times\mathbb{A}^1/\mathbb{A}^1}, zd)$ in Simpson's theorem 4.1 to the ordinary complex of relative differential forms on a suitable space.
Let X be a variety over a field k. For a closed subvariety $Y \subset X$ let $M = M(Y,X)$ denote the deformation to the normal bundle c.f. [V1] § 2. Let $I \subset \mathcal{O}_X$ be the ideal corresponding to Y. Filtering $\mathcal{O}_X$ by the powers I^i for $i \in \mathbb{Z}$ with $I^i = \mathcal{O}_X$ for $i \le 0$ we have:

$$\begin{aligned} M &= \operatorname{spec} \mathrm{Fil}^0(k[z,z^{-1}] \otimes_k \mathcal{O}_X) \\ &= \operatorname{spec} \left(\bigoplus_{i\in\mathbb{Z}} z^{-i} I^i \right) . \end{aligned}$$

Here spec denotes the spectrum of a quasi-coherent $\mathcal{O}_X$-algebra. By construction M is equipped with a flat map

$$\pi_M : M \longrightarrow \mathbb{A}^1$$

and an affine map

$$\rho : M \longrightarrow X .$$

They combine to a map:

$$h = (\rho, \pi_M) : M \longrightarrow X \times \mathbb{A}^1$$

such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\quad h \quad} & X \times \mathbb{A}^1 \\ & {\scriptstyle \pi_M} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & \mathbb{A}^1 & \end{array}$$

commutes.
The map π_M is equivariant with respect to the natural $\mathbb{G}_m$-actions on M and $\mathbb{A}^1$ defined by $\lambda \cdot z = \lambda z$ for $\lambda \in \mathbb{G}_m$. The map h becomes equivariant if $\mathbb{G}_m$ acts on $X \times \mathbb{A}^1$ via the second factor.
It is immediate from the definitions that if $f : X' \to X$ is a flat map of varieties and $Y' = Y \times_X X'$ then

$$M(Y',X') = M(Y,X) \times_X X' . \qquad (29)$$

Moreover the diagram

$$\begin{array}{ccc} M(Y',X') & \xrightarrow{h} & X' \times \mathbb{A}^1 \\ {\scriptstyle f_M} \downarrow & & \downarrow {\scriptstyle f \times \mathrm{id}} \\ M(Y,X) & \xrightarrow{h} & X \times \mathbb{A}^1 \end{array} \qquad (30)$$

is commutative and cartesian.
From now on let X be a smooth variety over an algebraically closed field k and fix a base point $* \in X$. Set $M = M(*, X)$ and consider the natural map

$$h : M \longrightarrow X \times \mathbb{A}^1 \; .$$

Pullback of differential forms induces a map :

$$\mu : h^* \Omega^p_{X \times \mathbb{A}^1/\mathbb{A}^1} \longrightarrow \Omega^p_{M/\mathbb{A}^1} \; , \; \mu(\omega) = h^*(\omega) \; .$$

We can now formulate the main observation of this appendix:

Theorem 5.5 *For every $p \geq 0$ the sheaf $\Omega^p_{M/\mathbb{A}^1}$ has no z-torsion and we have that*

$$\operatorname{Im} \mu = z^p \Omega^p_{M/\mathbb{A}^1} \; .$$

The map of $\mathcal{O}_M$-modules:

$$\alpha : h^* \Omega^p_{X \times \mathbb{A}^1/\mathbb{A}^1} \longrightarrow \Omega^p_{M/\mathbb{A}^1} \; , \; \alpha(\omega) = z^{-p} h^*(\omega)$$

which is well defined by the preceeding assertions is an isomorphism. Hence we get an isomorphism of complexes:

$$\alpha : h^*(\Omega^\bullet_{X \times \mathbb{A}^1/\mathbb{A}^1}, zd) \xrightarrow{\sim} \Omega^\bullet_{M/\mathbb{A}^1} \; , \; \alpha(\omega) = z^{-\deg \omega} h^*(\omega) \; .$$

Remarks. 1) Under the isomorphism α the $\mathbb{G}_m$-action on the left, as defined after theorem 5.1, corresponds to the natural $\mathbb{G}_m$-action on $\Omega^\bullet_{M/\mathbb{A}^1}$ by pullback $\lambda \cdot \omega = \lambda^*(\omega)$.
2) By a slightly more sophisticated construction one can get rid of the choice of base point: The spaces $M(*, X)$ define a family $\mathcal{M} \to X$. The maps $h : M(*, X) \to X \times \mathbb{A}^1$ lead to a map $\mathcal{M} \to X \times X \times \mathbb{A}^1$. Replace M by the inverse image in $\mathcal{M}$ of $\Delta \times \mathbb{A}^1$ where $\Delta \subset X \times X$ is the diagonal. This is independent of the choice of base point.

Proof of 4.5 We first check the assertions for the pair $(0, \mathbb{A}^n), n \geq 1$. In this case $M = M(0, \mathbb{A}^n)$ is the spectrum of the ring

$$B = k[z, x_1, \ldots, x_n, y_1, \ldots, y_n]/(zy_1 - x_1, \ldots, zy_n - x_n) \; .$$

The maps $\mathbb{A}^1 \xleftarrow{\pi_M} M \xrightarrow{\rho} \mathbb{A}^n$ are induced by the natural inclusions

$$k[z] \hookrightarrow B \hookleftarrow k[x_1, \ldots, x_n] \; .$$

The B-module $\Omega^1_{B/k[z]}$ is generated by $d\overline{x}_i, d\overline{y}_i$ for $1 \leq i \leq n$ modulo the relations $zd\overline{y}_i = d\overline{x}_i$. Hence it is freely generated by the $d\overline{y}_i$ and in particular z-torsion free. The B-module

$$\Omega^1_{k[z,x_1,\ldots,x_n]/k[z]} \otimes_{k[z,x_1,\ldots,x_n]} B$$

is free on the generators dx_i. The map μ corresponds to the natural inclusion of this free B-module into $\Omega^1_{B/k[z]}$ which sends dx_i to $d\overline{x}_i = zd\overline{y}_i$. The map α which sends dx_i to $d\overline{y}_i$ is an isomorphism. Hence the theorem for the pair $(0, \mathbb{A}^n)$.
In the general case choose an open subvariety $U \subset X$ containing $* \in U$ and an étale map

$$f : U \longrightarrow \mathbb{A}^n$$

such that $f^{-1}(0) = *$. By (29) and (30) we then have a cartesian diagram:

$$\begin{array}{ccccc} M(0,\mathbb{A}^n) \times_{\mathbb{A}^n} U & = & M(*,U) & \xrightarrow{h} & U \times \mathbb{A}^1 \\ & \searrow^{\text{proj.}} & \downarrow{\scriptstyle f_M} & & \downarrow{\scriptstyle f\times \mathrm{id}} \\ & & M(0,\mathbb{A}^n) & \xrightarrow{h} & \mathbb{A}^n \times \mathbb{A}^1 . \end{array}$$

Since $f \times \mathrm{id}$ and hence f_M are étale we know by [M] Theorem 25.1 (2) that

$$\Omega^p_{M(*,U)/\mathbb{A}^1} = f_M^* \Omega^p_{M(0,\mathbb{A}^n)/\mathbb{A}^1} \tag{31}$$

and

$$\Omega^p_{U\times\mathbb{A}^1/\mathbb{A}^1} = (f \times \mathrm{id})^* \Omega^p_{\mathbb{A}^n\times\mathbb{A}^1/\mathbb{A}^1} .$$

As we have seen, $\Omega^p_{M(0,\mathbb{A}^n)/\mathbb{A}^1}$ has no z-torsion. Since f_M is flat the same is true for $\Omega^p_{M(*,U)/\mathbb{A}^1}$ by (31). Applying f_M^* to the isomorphism

$$\alpha : h^* \Omega^p_{\mathbb{A}^n\times\mathbb{A}^1/\mathbb{A}^1} \xrightarrow{\sim} \Omega^p_{M(0,\mathbb{A}^n)/\mathbb{A}^1}$$

it follows from the above that

$$\alpha : h^* \Omega^p_{U\times\mathbb{A}^1/\mathbb{A}^1} \xrightarrow{\sim} \Omega^p_{M(*,U)/\mathbb{A}^1}$$

is an isomorphism as well.
We now choose an open subvariety $V \subset X$ not containing the point $*$ and such that $U \cup V = X$. Then $M(*,U)$ and $M(\emptyset, V)$ are open subvarieties of $M(*,X)$ and we have that

$$M(*,X) = M(*,U) \cup M(\emptyset, V) .$$

As we have seen the map α for $M(*,X)$ is an isomorphism over $M(*,U)$. Over $M(\emptyset, V)$ it is an isomorphism as well since

$$M(\emptyset, V) = V \times \mathbb{G}_m$$

canonically. Hence the theorem follows. □

6 The torsion of $\mathbf{R}^n\pi_*\mathcal{DR}_{X/K}$

In this section we describe the $\mathcal{A}_{\mathbb{R}}$-resp. $\mathcal{A}_{\mathbb{R}\geq 0}$-torsion $\mathcal{T}_{X/\mathbb{C}}$ resp. $\mathcal{T}_{X/\mathbb{R}}$ of the sheaves $R^n\pi_*\mathcal{DR}_{X/\mathbb{C}}$ resp. $R^n\pi_*\mathcal{DR}_{X/\mathbb{R}}$ which were introduced in the last section. For this we first have to extend Proposition 2.3 ii) somewhat.
For a filtered vector space $V \in \mathcal{F}il_{\mathbb{C}}$ and any $N \geq 1$ define a graded vector space by

$$^N\mathrm{Gr}^\bullet V = \bigoplus_{p\in\mathbb{Z}} \mathrm{Fil}^p V/\mathrm{Fil}^{p+N} V \; .$$

It becomes a $\mathbb{C}[z]/(z^N)$-module by letting z act as the one-shift to the left: For v in $\mathrm{Fil}^p V/\mathrm{Fil}^{p+N} V$ set

$$z \cdot v = \text{image of } v \text{ in } \mathrm{Fil}^{p-1} V/\mathrm{Fil}^{p+N-1} V \; .$$

This action depends on the choice of z. For $N = 1$ we have $^N\mathrm{Gr}^\bullet V = \mathrm{Gr}^\bullet V$.
To V in $\mathcal{F}il_{\mathbb{R}}^{\pm}, N \geq 1$ we attach the graded vector space:

$$^{2N}_{\mathbb{R}}\mathrm{Gr}^\bullet V := (\,^{2N}\mathrm{Gr}^\bullet V)^{F_\infty = (-1)^\bullet} \; .$$

It is a $\mathbb{C}[z^2]/(z^{2N})$-module and for $N = 1$ and V in $\mathcal{F}il_{\mathbb{R}}$ we have:

$$^{2}_{\mathbb{R}}\mathrm{Gr}^\bullet V = \mathrm{Gr}^\bullet V \; . \tag{32}$$

With these notations the following result holds:

Proposition 6.1 *a) For V in $\mathcal{F}il_{\mathbb{C}}, N \geq 1$ there are functorial isomorphisms of free $\mathbb{C}[z]/(z^N)$-modules:*

$$i_0^{-1}(\xi_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V) \otimes \mathcal{O}_{\mathbb{A}^1}/z^N\mathcal{O}_{\mathbb{A}^1}) = \,^N\mathrm{Gr}^\bullet V \; .$$

Here $i_0 : 0 \hookrightarrow \mathbb{A}^1$ denotes the inclusion of the origin.

b) For V in $\mathcal{F}il_{\mathbb{R}}^{\pm}, N \geq 1$ there are functorial isomorphisms of free $\mathbb{C}[z^2]/(z^{2N})$-modules:

$$i_0^{-1}(\xi_{\mathbb{R}}(V, \mathrm{Fil}^\bullet V, F_\infty) \otimes \mathcal{O}_{\underset{\sim}{\mathbb{A}}^1}/z^{2N}\mathcal{O}_{\underset{\sim}{\mathbb{A}}^1}) = \,^{2N}_{\mathbb{R}}\mathrm{Gr}^\bullet V \; .$$

Here, $\underset{\sim}{\mathbb{A}}^1 = \mathrm{spec}\,\mathbb{C}[z^2]$ and $i_0 : 0 \hookrightarrow \underset{\sim}{\mathbb{A}}^1$ is the inclusion.

The isomorphisms in a) and b) are compatible with the $\mathbb{G}_m$-action if $\mathbb{G}_m$ acts on the right in degree p by the character z^{-p}. They depend on the choice of z.

Proof For $V \in \mathcal{F}il_{\mathbb{C}}$ the map:

$$\mathrm{Fil}^p V/\mathrm{Fil}^{p+N} V \longrightarrow \Big(\sum_i \mathrm{Fil}^i V \otimes z^{-i}\mathbb{C}[z]\Big) \otimes \mathbb{C}[z]/(z^N)$$

sending $v + \mathrm{Fil}^{p+N} V$ to $v \otimes z^{-p}$ mod z^N is well defined. The induced map

$$ {}^N\mathrm{Gr}^\bullet V \longrightarrow \Gamma(\mathbb{A}^1, \xi_{\mathbb{C}}(V, \mathrm{Fil}^\bullet V)) \otimes \mathbb{C}[z]/(z^N) $$

is surjective and $\mathbb{C}[z]/(z^N)$-linear by construction. Since

$$ \dim \ {}^N\mathrm{Gr}^\bullet V = N \dim V $$

both sides have the same $\mathbb{C}$-dimension and hence a) follows.
Given $V \in \mathcal{F}il^{\pm}_{\mathbb{R}}$ we may view it as an object of $\mathcal{F}il_{\mathbb{C}}$ and we get an isomorphism of $\mathbb{C}[z]/(z^{2N})$-modules

$$ \begin{array}{rcl} {}^{2N}\mathrm{Gr}^\bullet V & \longrightarrow & \Big(\sum_i \mathrm{Fil}^i V \otimes z^{-i}\mathbb{C}[z]\Big) \otimes_{\mathbb{C}[z]} \mathbb{C}[z]/(z^{2N}) \\ & & \Vert \\ & & \Big(\sum_i \mathrm{Fil}^i V \otimes z^{-i}\mathbb{C}[z]\Big) \otimes_{\mathbb{C}[z^2]} \mathbb{C}[z^2]/(z^{2N}) \ . \end{array} $$

Passing to invariants under $F_\infty \otimes (-\mathrm{id})^*$ on the right corresponds to taking invariants under $(-1)^\bullet F_\infty$ on the left. Hence assertion b). The claim about the $\mathbb{G}_m$-action is clear. $\square$
As before there is an ω-version of this proposition over $\mathbb{R}$ resp. $\mathbb{R}^{\geq 0}$ which we will use in the sequel.
For a proper and smooth variety $X/\mathbb{C}$ consider the exact sequence of $\mathbb{R}$-vector spaces:

$$ 0 \longrightarrow H^n(X^{\mathrm{an}}, \mathbb{R}) \longrightarrow H^n(X^{\mathrm{an}}, \mathbb{C}) \xrightarrow{\pi_1} H^n(X^{\mathrm{an}}, \mathbb{R}(1)) \longrightarrow 0 \ . \tag{33} $$

It leads to a complex of $\mathbb{R}[r]/(r^N)$-modules:

$$ \begin{array}{rl} 0 \longrightarrow & {}^N\mathrm{Gr}^\bullet_{\mathrm{Fil}_0} H^n(X^{\mathrm{an}}, \mathbb{R}) \xrightarrow{\iota_N} {}^N\mathrm{Gr}^\bullet_F H^n(X^{\mathrm{an}}, \mathbb{C}) \\ \xrightarrow{\pi_1} & {}^N\mathrm{Gr}^\bullet_{\pi_1(F)} H^n(X^{\mathrm{an}}, \mathbb{R}(1)) \longrightarrow 0 \end{array} \tag{34} $$

which is right exact but not exact in the middle or on the left in general. Denote by ${}^N\mathcal{H}^\bullet_{X/\mathbb{C}}$ its middle cohomology.
For a proper and smooth variety $X/\mathbb{R}$ we obtain from (34) equipped with the action of $\overline{F}^*_\infty$ a complex of $\mathbb{R}[r^2]/(r^{2N})$-modules

$$ \begin{array}{rl} 0 \longrightarrow & {}^{2N}_{\ \mathbb{R}}\mathrm{Gr}^\bullet_{\mathrm{Fil}_0} H^n(X^{\mathrm{an}}, \mathbb{R}) \xrightarrow{\iota_{2N}} {}^{2N}_{\ \mathbb{R}}\mathrm{Gr}^\bullet_F H^n(X^{\mathrm{an}}, \mathbb{C}) \\ \xrightarrow{\pi_1} & {}^{2N}_{\ \mathbb{R}}\mathrm{Gr}^\bullet_{\pi_1(F)} H^n(X^{\mathrm{an}}, \mathbb{R}(1)) \longrightarrow 0 \ . \end{array} \tag{35} $$

It is again right exact and we denote its middle cohomology by ${}^{2N}\mathcal{H}^\bullet_{X/\mathbb{R}}$. As $\mathbb{R}$-vector spaces both ${}^N\mathcal{H}^\bullet_{X/\mathbb{C}}$ and ${}^{2N}\mathcal{H}^\bullet_{X/\mathbb{R}}$ are naturally graded.
We can now describe the torsion sheaves $\mathcal{T}_{X/K}$ for $K = \mathbb{C}, \mathbb{R}$:

Theorem 6.2 *For $N \gg 0$ the map α in (20) resp. (24) induces isomorphisms of $\mathcal{A}_{\mathbb{R}}$- resp. $\mathcal{A}_{\mathbb{R}^{\geq 0}}$-modules:*

$$\alpha_0 : i_{0*}({}^N\mathcal{H}^{\bullet}_{X/\mathbb{C}}) \xrightarrow{\sim} \mathcal{T}_{X/\mathbb{C}}$$

resp.

$$\alpha_0 : i_{0*}({}^{2N}\mathcal{H}^{\bullet}_{X/\mathbb{R}}) \longrightarrow \mathcal{T}_{X/\mathbb{R}} \; .$$

Here the operation ψ_0^t on $\mathcal{T}_K$ corresponds to multiplication by $e^{\bullet t}$ on the left.

Proof For any $N \geq 1$ the exact sequence:

$$0 \longrightarrow \mathcal{T}_{X/\mathbb{C}} \longrightarrow R^n\pi_*\mathcal{DR}_{X/\mathbb{C}} \xrightarrow{(21)} \xi^{\omega}_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^{\bullet})) \longrightarrow 0$$

remains exact after tensoring with $\mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}$ since $\xi^{\omega}_{\mathbb{C}}$ is $\mathcal{A}_{\mathbb{R}}$-torsion free. Together with the short exact sequence (34) and the ω-version of Proposition 6.1 a) we obtain the following exact and commutative diagram of $\mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}$-modules:

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
 & & & & i_0^{-1}(\mathcal{T}_{X/\mathbb{C}} \otimes \mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}) \\
 & & & & \downarrow \\
{}^N\mathrm{Gr}^{\bullet}_{\mathrm{Fil}_0} H^n(X^{\mathrm{an}}, \mathbb{R}) & \rightarrow & {}^N\mathrm{Gr}^{\bullet}_F H^n(X^{\mathrm{an}}, \mathbb{C}) & \xrightarrow{\alpha_0} & i_0^{-1}(R^n\pi_*\mathcal{DR}_{X/\mathbb{C}} \otimes \mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}) \rightarrow 0 \\
 & & \pi_1 \downarrow & & \downarrow \\
 & & {}^N\mathrm{Gr}^{\bullet}_{\pi_1(F)} H^n(X^{\mathrm{an}}, \mathbb{R}(1)) & = & i_0^{-1}(\xi^{\omega}_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}(1)), \pi_1(F^{\bullet})) \otimes \mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}) \\
 & & & & \downarrow \\
 & & & & 0 \; .
\end{array}$$

This shows that α_0 induces an isomorphism of $\mathcal{A}_{\mathbb{R}}$-modules

$$\alpha_0 : \; {}^N\mathcal{H}^{\bullet}_{X/\mathbb{C}} \xrightarrow{\sim} i_0^{-1}(\mathcal{T}_{X/\mathbb{C}} \otimes \mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}) \; .$$

Since $\mathcal{T}_{X/\mathbb{C}}$ is a coherent torsion sheaf with support in $0 \in \mathbb{R}$ we have

$$\mathcal{T}_{X/\mathbb{C}} = \mathcal{T}_{X/\mathbb{C}} \otimes \mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}$$

for $N \gg 0$ which gives the first assertion. The remark on ψ_0^t follows from Proposition 6.1 since the map α in the exact sequence (20) is ψ^t-equivariant. The assertion over $\mathbb{R}$ follows similarly. □

In the next result we will view $i_0^{-1}\mathcal{T}_{X/K}$ simply as a finite dimensional $\mathbb{R}$-vector space with a linear flow ψ_0^t. Let Θ be its infinitesimal generator i.e. $\psi_0^t = \exp t\Theta$ on $i_0^{-1}\mathcal{T}_{X/K}$.

Proposition 6.3 *The endomorphism Θ of $i_0^{-1}\mathcal{T}_{X/K}$ is diagonalizable over $\mathbb{R}$. For $\alpha = p \in \{1, \dots, n\}$ the dimension of its α-eigenspace is* $\dim \gamma^p H^n(X^{\mathrm{an}}, \mathbb{R})$

if $K = \mathbb{C}$ *and* $\dim(\gamma^p H^n(X^{\mathrm{an}}, \mathbb{R})^{(-1)^p})$ *if* $K = \mathbb{R}$*. For all other values of* α *the* α*-eigenspace is zero. In particular we have*

$$\begin{aligned} \det_{\mathbb{R}}(s - \Theta \,|\, i_0^{-1} \mathcal{T}_{X/\mathbb{C}}) &= \prod_{0<p\le n} (s-p)^{\dim \gamma^p} \\ \dim_{\mathbb{R}}(i_0^{-1} \mathcal{T}_{X/\mathbb{C}}) &= \sum_{p\in\mathbb{Z}} p \dim \mathrm{Gr}^p_{\gamma} H^n(X^{\mathrm{an}}, \mathbb{R}) \end{aligned}$$

and

$$\begin{aligned} \det_{\mathbb{R}}(s - \Theta \,|\, i_0^{-1} \mathcal{T}_{X/\mathbb{R}}) &= \prod_{0<p\le n} (s-p)^{\dim (\gamma^p)^{(-1)^p}} \\ \dim_{\mathbb{R}}(i_0^{-1} \mathcal{T}_{X/\mathbb{R}}) &= \frac{1}{2} \dim H^n(X^{\mathrm{an}}, \mathbb{R})^- + \frac{1}{2} \sum_{p\in\mathbb{Z}} p \dim \mathrm{Gr}^p_{\mathcal{V}} H^n(X^{\mathrm{an}}, \mathbb{R}) \;. \end{aligned}$$

Remark: According to the proposition the torsion $\mathcal{T}_{X/K}$ is zero iff $\gamma^1 = 0$ in case $K = \mathbb{C}$ and $(\gamma^1)^- = 0 = (\gamma^2)^+$ in case $K = \mathbb{R}$. These conditions are equivalent to the strictness of the inclusion (19) if $K = \mathbb{C}$ and to the strictness of

$$(H^n(X^{\mathrm{an}}, \mathbb{R}), s\mathrm{Fil}^{\bullet}_0) \hookrightarrow (H^n(X^{\mathrm{an}}, \mathbb{C}), sF^{\bullet})_{\mathbb{R}}$$

if $K = \mathbb{R}$. Here s is formed with respect to $\overline{F}^*_{\infty}$. This is as it must be according to proposition 2.2 c) ii). More explicitly $\mathcal{T}_{X/\mathbb{C}}$ is zero iff H^n has Hodge type $(n,0),(0,n)$ whereas $\mathcal{T}_{X/\mathbb{R}}$ is zero iff H^n has Hodge type either $(n,0),(0,n)$ or $(2,0),(1,1),(0,2)$ with F_{∞} acting trivially on H^{11}.

Proof of 6.3: We assume that $K = \mathbb{C}$, the case $K = \mathbb{R}$ being similar. According to theorem 6.2 the operator Θ is diagonalizable on $i_0^{-1}\mathcal{T}_{X/\mathbb{C}}$ the possible eigenvalues being integers. For $p \in \mathbb{Z}$ and $N \gg 0$ we have:

$$\begin{aligned} &\dim \mathrm{Ker}\,(p - \Theta \,|\, i_0^{-1}\mathcal{T}_{X/\mathbb{C}}) = \dim {}^N\mathcal{H}^p_{\mathbb{C}} \\ &\overset{(34)}{=} \quad \dim \mathrm{Ker}\, \iota^p_N - \dim {}^N\mathrm{Gr}^p_{\mathrm{Fil}_0} H^n(X^{\mathrm{an}}, \mathbb{R}) \\ &\qquad + \dim_{\mathbb{R}} {}^N\mathrm{Gr}^p_F H^n(X^{\mathrm{an}}, \mathbb{C}) - \dim {}^N\mathrm{Gr}^p_{\pi_1(F)} H^n(X^{\mathrm{an}}, \mathbb{R}(1)) \;. \end{aligned}$$

Using the exact sequence:

$$\begin{aligned} 0 \;\longrightarrow\;& {}^N\mathrm{Gr}^p_{\gamma} H^n(X^{\mathrm{an}}, \mathbb{R}) \longrightarrow {}^N\mathrm{Gr}^p_F H^n(X^{\mathrm{an}}, \mathbb{C}) \\ \xrightarrow{\pi_1}\;& {}^N\mathrm{Gr}^p_{\pi_1(F)} H^n(X^{\mathrm{an}}, \mathbb{R}(1)) \longrightarrow 0 \end{aligned}$$

we see that this is equal to:

$$\dim \mathrm{Ker}\, \iota^p_N + \dim {}^N\mathrm{Gr}^p_{\gamma} H^n(X^{\mathrm{an}}, \mathbb{R}) - \dim {}^N\mathrm{Gr}^p_{\mathrm{Fil}_0} H^n(X^{\mathrm{an}}, \mathbb{R}) \;.$$

Since

$$ {}^N\mathrm{Gr}^{\bullet}_{\mathrm{Fil}_0} H^n(X^{\mathrm{an}}, \mathbb{R}) = \bigoplus_{-N<p\le 0} H^n(X^{\mathrm{an}}, \mathbb{R}) $$

we find

$$ \begin{aligned} \operatorname{Ker} \iota_N &= \operatorname{Ker}\left(\bigoplus_{-N<p\le 0} H^n(X^{\mathrm{an}}, \mathbb{R}) \longrightarrow \bigoplus_{-N<p\le 0} F^p/F^{p+N} \right) \\ &= \bigoplus_{-N<p\le n-N} \gamma^{p+N} H^n(X^{\mathrm{an}}, \mathbb{R}) \end{aligned} $$

if $N \ge n$. A short calculation now gives the result. □

Remark. I cannot make the idea rigorous at present but it seems to me that the complexes $R^n\pi_*\mathcal{DR}_{X/\mathbb{C}}$ and $R^n\pi_*\mathcal{DR}_{X/\mathbb{R}}$ should have an interpretation in terms of a suitable perverse sheaf theory. Let us look at an analogy:
Consider a possibly singular variety Y over $\mathbb{F}_p$ and let $j : U \subset Y$ be a smooth open subvariety. If $\pi : X \to U$ is smooth and proper the intermediate extension $\mathcal{F} = j_{!*}R^n\pi_*\mathbb{Q}_l$ for $l \neq p$ is a pure perverse sheaf. We have the L-function

$$ \begin{aligned} L_Y(H^n(X), t) &:= \prod_{y\in|Y|} \det\nolimits_{\mathbb{Q}_l}(1 - t\mathrm{Fr}_{\overline{y}} \,|\, \mathcal{F}_{\overline{y}})^{-1} \\ &= \prod_i \det\nolimits_{\mathbb{Q}_l}(1 - t\mathrm{Fr}_p \,|\, H^i(Y \otimes \overline{\mathbb{F}}_p, \mathcal{F}))^{(-1)^{i+1}} . \end{aligned} $$

By perverse sheaf theory and Deligne's work on the Weil conjectures it satisfies a functional equation and the Riemann hypotheses.
For varieties over number fields Y corresponds to the "curve" $\overline{\operatorname{spec} \mathfrak{o}_k}$ and for U we can take e.g. $\operatorname{spec} \mathfrak{o}_k$. Hypothetically a better analogue for Y (or more precisely for $Y \otimes \overline{\mathbb{F}}_p$) is the dynamical system ("$\overline{\operatorname{spec} \mathfrak{o}_k}$", ϕ^t) whose existence is conjectured in [D7]. For U we would take the subsystem ("$\operatorname{spec} \mathfrak{o}_k$", ϕ^t) which has no fixed points of the flow i.e. singularities. This is one motivation for the above idea. Another comes from the discussion in sections 5 and 9 of [D5].
Incidentally the appendix to the preceeding section was motivated by the use of the deformation to the normal cone in perverse sheaf theory [V2].
We would also like to point out that there is an exact triangle in the derived category of $\mathcal{A}_{\mathbb{R}}$-modules with a flow:

$$ \xi_{\mathbb{C}}(H^n(X^{\mathrm{an}}, \mathbb{R}), \gamma^{\bullet}) \longrightarrow P \longrightarrow \mathcal{T}^*_{\mathbb{C}}(-1)[-1] \longrightarrow \dots $$

where

$$ P = R\underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(R^n\pi_*\mathcal{D}_{X/\mathbb{C}}, \mathcal{A}_{\mathbb{R}}(-1)) . $$

Here $\xi_{\mathbb{C}}$ sits in degree zero with one-dimensional support and $\mathcal{T}^*_{\mathbb{C}}(-1)[-1]$ sits in degree one with zero-dimensional support. This follows by applying $R\underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(\_, \mathcal{A}_{\mathbb{R}}(-1))$ to the exact sequence:

$$0 \longrightarrow \mathcal{T}_{X/\mathbb{C}} \longrightarrow R^n\pi_*\mathcal{D}_{X/\mathbb{C}} \longrightarrow R^n\pi_*(\mathcal{D}_{X/\mathbb{C}})/\mathcal{T}_{X/\mathbb{C}} \longrightarrow 0$$

and noting that

$$\begin{aligned} \underline{\mathrm{Ext}}^1_{\mathcal{A}_{\mathbb{R}}}(\mathcal{T}_{X/\mathbb{C}}, \mathcal{A}_{\mathbb{R}}(-1)) &= \mathcal{T}^*_{X/\mathbb{C}}(-1) \\ &:= \underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(\mathcal{T}_{X/\mathbb{C}}, \mathcal{A}_{\mathbb{R}}/r^N\mathcal{A}_{\mathbb{R}}(-1)) \quad \text{for } N \gg 0 \; . \end{aligned}$$

A similar exact triangle exists for $K = \mathbb{R}$ of course.

Remark. One may wonder whether the torsion $\mathcal{T}_{X/K}$ is also relevant for the L-function. It seems to be partly responsible for the ε-factor at infinity as follows: Let X/K be as usual a smooth and proper variety over $K = \mathbb{R}$ or $\mathbb{C}$. With normalizations as in [De1] 5.3 the ε-factor of $H^n(X)$ is given by:

$$\varepsilon = \exp(i\pi\mathcal{D}) \quad \text{where } \mathcal{D} = \frac{1}{e_K}\sum_{p\in\mathbb{Z}} p(h_p - d_p) \; .$$

Here:

$$h_p = \dim_{\mathbb{C}} \mathrm{Gr}^p_F H^n(X, \mathbb{C}) \quad \text{and} \quad d_p = \dim \mathrm{Gr}^p_{\mathcal{V}} H^n(X, \mathbb{R}) \; .$$

This description of the ε-factor can be checked directly. Alternatively it can be found in a more general context in the proof of [D6] Prop. 2.7.
With these notations we have by 6.3:

$$\dim(i_0^{-1}\mathcal{T}_{X/\mathbb{C}}) = \sum_{p\in\mathbb{Z}} pd_p$$

and

$$\dim(i_0^{-1}\mathcal{T}_{X/\mathbb{R}}) = \frac{1}{2}\sum_{p\in\mathbb{Z}} pd_p + \frac{1}{2}\dim H^n(X^{\mathrm{an}}, \mathbb{R})^- \; .$$

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