Products of Harmonic Forms and Rational Curves

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Abstract. In general, the product of harmonic forms is not harmonic. We study the top exterior power of harmonic two-forms on compact Kähler manifolds. The non-harmonicity in this case is related to the geometry of the manifold and to the existence of rational curves in particular. K3 surfaces and hyperkähler manifolds are discussed in detail.

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The product of closed forms is closed again. The analogous statement for harmonic forms, however, fails. A priori, there is no reason why the product of harmonic forms should be harmonic again. This phenomenon was recently studied by Merkulov [8]. He shows that it leads to a natural A_∞-structure on a Kähler manifold. In the context of mirror symmetry Polishchuk made use of (a twisted version of) this A_∞-structure on elliptic curves to confirm Kontsevich's homological version of mirror symmetry in this case [9].

In the present paper we show that this failure of harmonicity in fact happens quite frequently. It usually is related to certain geometric properties of the manifold and to the existence of rational curves in particular. In fact, we are only interested in the product of harmonic $(1,1)$-forms, as this is the geometric relevant case. We wish to emphasize that the interplay between harmonicity and geometry is by far not completely understood. The results of this paper just seem to indicate that there is in fact a relationship. As the failure of harmonicity is related to the shape of the *'non-linear Kähler cone'* (cf. 1.2, 2.3), the results of this paper can roughly be phrased by saying that the geometry of the manifold forces the non-linear Kähler cone to be curved. For a deeper understanding of the situation one will have to study the shape of the non-linear Kähler cone or, equivalently, the non-harmonicity of products of harmonic forms.

Let us briefly indicate the main results for the special case of compact Ricci-flat Kähler manifolds. For a Kähler class $\omega \in H^2(X,\mathbb{R})$ on such a manifold there exists a unique Ricci-flat Kähler form $\tilde{\omega}$ representing it. Let $\mathcal{H}^{1,1}(\tilde{\omega})$ denote the space of $(1,1)$-forms harmonic with respect to $\tilde{\omega}$. Of course, for a different Kähler class ω' and the representing Ricci-flat Kähler form $\tilde{\omega}'$ this space might be different.

The main technical result (Prop. 2.3) says that $\mathcal{H}^{1,1}(\tilde{\omega})$ is independent of ω if and only if the top exterior power of any harmonic form $\alpha \in \mathcal{H}^{1,1}(\tilde{\omega})$ is again harmonic. This can be used to interpret the failure of harmonicity of the top exterior power geometrically. Proposition 3.2 asserts that there always exist harmonic $(1,1)$-forms with non-harmonic top exterior power, whenever the Kähler cone (or ample cone) does not form a connected component of the (integral) cone of all classes $\alpha \in H^{1,1}(X,\mathbb{R})$ with $\int_X \alpha^N > 0$.

Note that there are many instances where the Kähler cone is strictly smaller. E.g. this is the case for any Calabi-Yau manifold that is birational to a non-isomorphic Calabi-Yau manifold (Prop. 6.1).

In Section 4 we apply the result for K3 surfaces. One finds that on any K3 surfaces containing a rational curve there exists a harmonic $(1,1)$-form α such that α^2 is not harmonic. This can be extended to arbitrary K3 surfaces by using the existence of rational curves on nearby K3 surfaces.

1 PREPARATIONS

Let X be a compact Kähler manifold. Then $\mathcal{K}_X \subset H^{1,1}(X,\mathbb{R})$ denotes the Kähler cone, i.e. the open set of all Kähler classes on X. For a class $\alpha \in H^{1,1}(X,\mathbb{R})$ we usually denote by $\tilde{\alpha} \in \mathcal{A}^{1,1}(X)_{\mathbb{R}}$ a closed real $(1,1)$-form representing α. Let us recall the Calabi-Yau theorem [3].

THEOREM 1.1 — *Let X be an N-dimensional compact Kähler manifold with a given volume form* vol $\in \mathcal{A}^{N,N}(X)_{\mathbb{R}}$. *For any Kähler class $\omega \in \mathcal{K}_X$ there exists a unique Kähler form $\tilde{\omega} \in \mathcal{A}^{1,1}(X)_{\mathbb{R}}$ representing ω, such that $\tilde{\omega}^N = c \cdot \mathrm{vol}$, with $c \in \mathbb{R}$.*

Since $\tilde{\omega}^N$ is harmonic with respect to $\tilde{\omega}$, this can be equivalently expressed by saying that any Kähler class ω can uniquely be represented by a Kähler form $\tilde{\omega}$ with respect to which the given volume form is harmonic. Note that the constant c can be computed as $c = \int_X \omega^N/\mathrm{vol}(X)$.

DEFINITION 1.2 — *For a given volume form* vol $\in \mathcal{A}^{N.N}(X)_{\mathbb{R}}$ *we let $\tilde{\mathcal{K}}_X \subset \mathcal{A}^{1,1}(X)_{\mathbb{R}}$ be the set of Kähler forms $\tilde{\omega}$ with respect to which* vol *is harmonic.*

By the Calabi-Yau theorem the natural projection $\tilde{\mathcal{K}}_X \to \mathcal{K}_X$ is bijective. The Kähler cone $\mathcal{K}_X$ is an open subset of $H^{1,1}(X,\mathbb{R})$, whereas $\tilde{\mathcal{K}}_X$ is in general not

contained as an open subset in a linear subspace of $\mathcal{A}^{1,1}(X)$ (cf. 2.1). Thus it might be appropriate to call $\tilde{\mathcal{K}}_X$ the *non-linear Kähler cone*.
Let $\tilde{\omega} \in \tilde{\mathcal{K}}_X$ and $c = \int_X \omega^N/\mathrm{vol}(X)$. The tangent space of $\tilde{\mathcal{K}}_X$ at $\tilde{\omega}$ can be computed as follows. Firstly, we may write $\tilde{\mathcal{K}}_X = \mathbb{R}_+ \times \tilde{\mathcal{K}}_X^c$, where $\tilde{\mathcal{K}}_X^c = \{\tilde{\omega} \in \tilde{\mathcal{K}}_X | \tilde{\omega}^N = c \cdot \mathrm{vol}\}$. Secondly, the infinitesimal deformations of $\tilde{\omega}$ in the direction of $\tilde{\mathcal{K}}_X^c$ are of the form $\tilde{\omega} + \varepsilon\tilde{v}$, where $\tilde{v}$ is a closed real $(1,1)$-form and such that $(\tilde{\omega} + \varepsilon\tilde{v})^N = \tilde{\omega}^N$. The latter condition gives $\tilde{\omega}^N + N\varepsilon\tilde{\omega}^{N-1}\tilde{v} = \tilde{\omega}^N$, i.e. $\tilde{v}$ is primitive. As any closed primitive $(1,1)$-form is harmonic, this shows that the tangent space of $\tilde{\mathcal{K}}_X^c$ at $\tilde{\omega}$ is the space $\mathcal{H}^{1,1}(\omega)_{\mathbb{R},prim}$ of real $\tilde{\omega}$-primitive $\tilde{\omega}$-harmonic $(1,1)$-forms. Thirdly, the $\mathbb{R}_+$-direction corresponds to the scaling of $\tilde{\omega}$ and this tangent direction is therefore canonically identified with $\mathbb{R}\tilde{\omega}$. Altogether, one obtains that $T_{\tilde{\omega}}\tilde{\mathcal{K}}_X = \mathcal{H}^{1,1}(\tilde{\omega})_{\mathbb{R}}$ is the space of real $\tilde{\omega}$-harmonic $(1,1)$-forms. In particular, $\tilde{\mathcal{K}}_X$ is a smooth connected submanifold of $\mathcal{A}^{1,1}(X)_{\mathbb{R}}$.
To make this approach rigorous, one first completes $\mathcal{A}^{1,1}(X)$ in the L_k^2-topology, where $k > N$. The Sobolev embedding theorem then shows that $L_k^2(\Lambda_{\mathbb{R}}^{1,1})_{cl} \to L_k^2(\Lambda_{\mathbb{R}}^{N,N})$ given by $\alpha \mapsto \alpha^N$ is a well-defined continuous multi-linear map and hence differentiable. Then, $\tilde{\mathcal{K}}_X$ is contained as an open subset in the fibre over vol. It inherits the differentiable structure and the above calculation then shows that it is smooth. Also note that the projection from the closed L_k^2 forms onto cohomology is differentiable. Hence, $\tilde{\mathcal{K}}_X \to \mathcal{K}_X$ is a differentiable map. Moreover, again due to the description of the tangent space, this map is in fact a diffeomorphism. In particular, the bijection $\tilde{\mathcal{K}}_X \to \mathcal{K}_X$ yields a differentiable map $\mathcal{K}_X \to \mathcal{A}^2(X)$ (in the L_k^2-topology). This fact is used in 2.2.

DEFINITION 1.3 — *Let X be a compact Kähler manifold with a given volume form. Then one associates to a given Kähler class $\omega \in \mathcal{K}_X$ the space $\mathcal{H}^{p,q}(\omega) := \mathcal{H}^{p,q}(\tilde{\omega})$ of (p,q)-forms that are harmonic with respect to the unique $\tilde{\omega} \in \tilde{\mathcal{K}}_X$ representing ω.*

Note that two different Kähler forms $\tilde{\omega}_1$ and $\tilde{\omega}_2$ representing the same Kähler class $\omega_1 = \omega_2$ always have different spaces of harmonic $(1,1)$-forms. Indeed, $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are $\tilde{\omega}_1$-harmonic respectively $\tilde{\omega}_2$-harmonic. Since any class, in particular $\omega_1 = \omega_2$, is represented by a unique harmonic form and $\tilde{\omega}_1 \neq \tilde{\omega}_2$, this yields $\mathcal{H}^{1,1}(\tilde{\omega}_1) \neq \mathcal{H}^{1,1}(\tilde{\omega}_2)$. But one might ask whether $\mathcal{H}^{1,1}(\tilde{\omega}_1)$ and $\mathcal{H}^{1,1}(\tilde{\omega}_2)$ can be equal for two Kähler forms $\tilde{\omega}_1, \tilde{\omega}_2$ not representing the same class, e.g. $\tilde{\omega}_1, \tilde{\omega}_2 \in \tilde{\mathcal{K}}_X$. It is quite interesting to observe that the dependence of $\mathcal{H}^{1,1}(\tilde{\omega})$ on the Kähler class ω is related to the problem discussed in the introduction. This is explained in the next section.

2 How 'harmonic' depends on the Kähler form

As before, we consider a compact Kähler manifold X with a fixed volume form and we let $\tilde{\mathcal{K}}_X$ be the associated non-linear Kähler cone. Let us begin with the following fact which relates the shape of $\tilde{\mathcal{K}}_X$ to the dependence of $\mathcal{H}^{1,1}(\omega)$ on ω.

PROPOSITION 2.1 — *The subspace $\mathcal{H}^{1,1}(\omega) \subset \mathcal{A}^{1,1}(X)$ is independent of ω if and only if $\tilde{\mathcal{K}}_X$ spans an $\mathbb{R}$-linear subspace of dimension $h^{1,1}(X)$.*

Proof. Let $\mathcal{H}^{1,1}(\omega) \subset \mathcal{A}^{1,1}(X)$ be independent of $\omega \in \mathcal{K}_X$. Since for any $\omega \in \mathcal{K}_X$ the unique $\tilde{\omega} \in \tilde{\mathcal{K}}_X$ representing it is $\tilde{\omega}$-harmonic, the assumption immediately yields $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega)_{\mathbb{R}}$ for any $\omega \in \mathcal{K}_X$.
Conversely, if $\tilde{\mathcal{K}}_X$ spans an $\mathbb{R}$-linear subspace of dimension $h^{1,1}(X)$, then this subspace coincides with the tangent space of $\tilde{\mathcal{K}}_X$ at every point $\tilde{\omega} \in \tilde{\mathcal{K}}_X$. But the latter was identified with $\mathcal{H}^{1,1}(\omega)_{\mathbb{R}}$. Hence, the linear subspace equals $\mathcal{H}^{1,1}(\omega)_{\mathbb{R}}$ for any $\omega \in \mathcal{K}_X$ and $\mathcal{H}^{1,1}(\omega)$, therefore, does not depend on ω. □

REMARK 2.2 — The assertion might be rephrased from a slightly different point of view as follows. Use the differentiable map $\mathcal{K}_X \to \mathcal{A}^2(X)$. The proposition then just says that this map is linear if and only if the Gauss map is constant. It might be instructive to rephrase some of the results later on in this spirit, e.g. Proposition 3.2.

The next proposition states that the 'global' change of $\mathcal{H}^{1,1}(\omega)$ for $\omega \in \mathcal{K}_X$ is determined by the 'harmonic' behavior with respect to a single $\omega \in \mathcal{K}_X$.

PROPOSITION 2.3 — *Let X be a compact Kähler manifold of dimension N with a fixed Kähler form $\tilde{\omega}_0$ and volume form $\tilde{\omega}_0^N/N!$. Then the following statements are equivalent:*
i) The linear subspace $\mathcal{H}^{1,1}(\omega)_{\mathbb{R}} \subset \mathcal{A}^{1,1}(X)_{\mathbb{R}}$ does not depend on $\omega \in \mathcal{K}_X$.
ii) For all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ one has $\alpha^N \in \mathcal{H}^{N,N}(\omega_0)$.

Proof. Let us assume *i)*. By the previous proposition the lifted Kähler cone $\tilde{\mathcal{K}}_X$ spans the $\mathbb{C}$-linear subspace $\mathcal{H}^{1,1}(\omega_0)$. Since $\tilde{\mathcal{K}}_X$ is open in $\mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$ and all $\alpha \in \tilde{\mathcal{K}}_X$ satisfy the $\mathbb{C}$-linear equation

$$\alpha^N = (\int_X \alpha^N / \int_X \omega_0^n) \cdot \omega_0^N \tag{1}$$

which is an algebraic condition, in fact all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ satisfy (1). Hence, for all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ the top exterior power α^N is harmonic, i.e. *ii)* holds true.
Let us now assume *ii)*. If $\alpha \in \mathcal{H}^{1,1}(\omega_0)$, such that its cohomology class $\omega := [\alpha]$ is a Kähler class, let $\tilde{\omega} \in \tilde{\mathcal{K}}_X$ denote the distinguished Kähler form representing ω. If α itself is strictly positive definite, then the unicity of $\tilde{\omega}$ and *ii)* imply $\alpha = \tilde{\omega}$. Thus, the intersection of the closed subset $\mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$ with the open cone of strictly positive definite real $(1,1)$-forms is contained in $\tilde{\mathcal{K}}_X$. This intersection is non-empty, as it contains $\tilde{\omega}_0$. Since $\tilde{\mathcal{K}}_X$ is a closed connected subset of this open cone of the same dimension as $\mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$ this yields $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega_0)_{\mathbb{R}}$. By Prop. 2.1 one concludes that $\mathcal{H}^{1,1}(\omega)$ does not depend on $\omega \in \mathcal{K}_X$. □

3 The positive cone

The next proposition is a first step towards a geometric understanding of the failure of harmonicity of α^N for a harmonic form α. To state it we recall the following notation.

Definition 3.1 — *For a compact Kähler manifold X the positive cone $\mathcal{C}_X \subset H^{1,1}(X,\mathbb{R})$ is the connected component of $\{\alpha \in H^{1,1}(X,\mathbb{R}) \mid \int_X \alpha^N > 0\}$ that contains the Kähler cone.*

Note that by definition $\mathcal{K}_X \subset \mathcal{C}_X$. Also note that the positive cone $\mathcal{C}_X$ might not be convex. However, for hyperkähler manifolds also $\mathcal{C}_X$ is convex, as it coincides with the cone defined by the Beauville-Bogomolov quadratic form (see Sect. 5).

Proposition 3.2 — *If X is a compact Kähler manifold such that $\mathcal{K}_X$ is strictly smaller than $\mathcal{C}_X$, then for any Kähler form $\tilde\omega$ there exists a $\tilde\omega$-harmonic $(1,1)$-form α such that α^N is not $\tilde\omega$-harmonic.*

Proof. Assume that there exists a Kähler form $\tilde\omega_0$ such that for all $\alpha \in \mathcal{H}^{1,1}(\tilde\omega_0)$ also α^N is $\tilde\omega_0$-harmonic. We endow X with the volume form $\tilde\omega_0^N/N!$. By Prop. 2.3 the lifted Kähler cone $\tilde{\mathcal{K}}_X$ is contained in $\mathcal{H}^{1,1}(\tilde\omega_0)$. Since $\mathcal{K}_X$ is strictly smaller than $\mathcal{C}_X$ there exists a sequence $\omega_t \in \mathcal{K}_X$ converging towards a $\omega \in \mathcal{C}_X \setminus \mathcal{K}_X$. As $\tilde{\mathcal{K}}_X$ is contained in the finite-dimensional space $\mathcal{H}^{1,1}(\tilde\omega_0)$ the lifted Kähler forms $\tilde\omega_t \in \tilde{\mathcal{K}}_X$ will converge towards a form (!) and just only a current $\tilde\omega \in \mathcal{H}^{1,1}(\tilde\omega_0) \setminus \tilde{\mathcal{K}}_X$. As a limit of strictly positive definite forms $\tilde\omega$ is still semi-positive definite. Moreover, $\tilde\omega$ is strictly positive definite at $x \in X$ if and only if $\tilde\omega^N$ does not vanish at x. By assumption $\tilde\omega^N = c \cdot \tilde\omega_0^N$ with $c = \int_X \omega^N / \int_X \omega_0^N$. Since $\omega \in \mathcal{C}_X$, the scalar factor c is strictly positive. Hence, $\tilde\omega^N$ is everywhere non-trivial. Thus $\tilde\omega$ is strictly positive definite. This yields the contradiction. $\square$

The interesting thing here is that the proposition in particular can be used to determine the positivity of a class with positive top exterior power just by studying the space of harmonic forms with respect to a single given, often very special Kähler form:

Corollary 3.3 — *Let X be a compact Kähler manifold with a given Kähler form $\tilde\omega_0$. If for all $\tilde\omega_0$-harmonic $(1,1)$-forms α the top exterior power α^N is also $\tilde\omega_0$-harmonic, then any class $\omega \in \mathcal{C}_X$ is a Kähler class.* $\square$

Here is another version of the same corollary in a more algebraic spirit.

Corollary 3.4 — *Let X be a compact Kähler manifold with a given Kähler form $\tilde\omega_0$, such that for every $\tilde\omega_0$-harmonic $(1,1)$-form α the top exterior power α^N is also $\tilde\omega_0$-harmonic. Then, a line bundle L on X is ample if and only if $c_1(L) \in \mathcal{C}_X$.* $\square$

We conclude this section with a few examples, where the assumption of the corollary is met *a priori*. In the later sections we will discuss examples where $\mathcal{K}_X$ is strictly smaller than $\mathcal{C}_X$ and where Prop. 3.2 can be used to conclude the 'failure' of harmonicity.

Examples 3.5 — *i)* If X is a complex torus and ω is a flat Kähler form, then harmonic forms are constant forms and their products are again constant, hence harmonic. In particular, one recovers the fact that on a torus the Kähler cone and the positive cone coincide.
ii) If for two Kähler manifolds $(X,\tilde{\omega})$ and $(X',\tilde{\omega}')$ with $b_1(X)\cdot b_1(X')=0$ the top exterior power of any harmonic $(1,1)$-forms on X or on X' is again harmonic, then the same holds for the product $(X\times X',\tilde{\omega}\times\tilde{\omega}')$. The additional assumption on the Betti-numbers is necessary as the product of two curves of genus at least two shows. Indeed, any $\varphi\in H^{1,0}(X)$, for a curve X, is harmonic, but $\varphi\wedge\bar{\varphi}$ is not. Hence, $\alpha=\varphi\times\bar{\varphi}+\bar{\varphi}\times\varphi$ is a harmonic $(1,1)$-form on $X\times X'$ with non-harmonic α^2.
iii) Let X be a Kähler manifold and $X\to X'$ a smooth finite quotient. Consider the non-linear Kähler cone on X with respect to the pull-back of a volume form on X'. Then $\mathcal{H}^{1,1}(\omega)$ with $\omega\in\mathcal{K}_X$ does not depend on ω if and only if the same holds true for X'.
iv) For hermitian symmetric spaces of compact type it is known that the space of harmonic forms equals the space of forms invariant under the real form. As the latter space is invariant under products, the Kähler cone of an irreducible hermitian symmetric space coincides with the positive cone.

4 K3 surfaces

As indicated earlier the behavior of the Kähler cone is closely related to the geometry of the manifold. We shall study this in more detail for K3 surfaces. The next proposition follows directly from the well-known description of the Kähler cone of a K3 surface.

Proposition 4.1 — *Let X be a K3 surface containing a smooth rational curve. Then for any Kähler form $\tilde{\omega}$ there exists an $\tilde{\omega}$-harmonic form $(1,1)$-form α such that α^2 is not harmonic.*

Proof. If X contains a smooth rational curve, then $\mathcal{K}_X$ is strictly smaller than $\mathcal{C}_X$ and we apply Prop. 3.2. Indeed, a smooth rational curve $C\subset X$ determines a (-2)-class $[C]$, whose perpendicular hyperplane $[C]^\perp$ cuts $\mathcal{C}_X$ into two parts and $\mathcal{K}_X$ is contained in the part that is positive on C. □

If the harmonicity of the top exterior powers fails for a Kähler manifold with a given Kähler form $(X,\tilde{\omega})$ then it should do so for any small deformation of

$(X, \tilde{\omega})$. For a Ricci-flat Kähler structure on a K3 surface the argument can be reversed and one can use the existence of rational curves on arbitrarily near deformations to prove the assertion in the above proposition on any K3 surface with respect to a Ricci-flat Kähler form.

COROLLARY 4.2 — *Let X be an arbitrary K3 surface. If $\tilde{\omega}$ is any hyperkähler form on X, then there exists an $\tilde{\omega}$-harmonic $(1,1)$-form α such that α^2 is not $\tilde{\omega}$-harmonic.*

Proof. Let $H^0(X, \Omega_X^2) = \mathbb{C}\sigma$. Then

$$\begin{array}{rcl} \mathcal{H}^2(\tilde{\omega}) & = & \mathcal{H}^{1,1}(\tilde{\omega}) \oplus \mathcal{H}^{2,0}(\tilde{\omega}) \oplus \mathcal{H}^{0,2}(\tilde{\omega}) \\ & = & \mathcal{H}^{1,1}(\tilde{\omega}) \oplus \mathbb{C}\sigma \oplus \mathbb{C}\bar{\sigma} \end{array}$$

As the space of harmonic forms only depends on the underlying hyperkähler metric g, the space $\mathcal{H}^{1,1}(\tilde{\omega}) \oplus \mathbb{C}\sigma \oplus \mathbb{C}\bar{\sigma}$ contains $\mathcal{H}^{1,1}(\tilde{\omega}_{aI+bJ+cK})$ for all $(a,b,c) \in S^2$. Here, I, J, and K are the three complex structures associated with the hyperkähler metric g (cf. Sect. 5).
Assume α^2 is g-harmonic for all $\alpha \in \mathcal{H}^{1,1}(\tilde{\omega})$. Since $\sigma = \tilde{\omega}_J + i\tilde{\omega}_K$ (up to a scalar factor) and since the product of a harmonic form with the Kähler form is again harmonic, also $\sigma\bar{\sigma}$ is harmonic. This implies that α^2 is harmonic for all $\alpha \in \mathcal{H}^2(\tilde{\omega})$, as $\sigma^2 = \bar{\sigma}^2 = \alpha\sigma = \alpha\bar{\sigma} = 0$ for $\alpha \in \mathcal{H}^{1,1}(\tilde{\omega})$. Thus, α^2 is g-harmonic for all $\alpha \in \mathcal{H}^{1,1}(\tilde{\omega}_{aI+bJ+cK})$ and all $(a,b,c) \in S^2$. On the other hand, it is well-known that for a non-empty (dense) subset of S^2 the K3 surface $(X, aI + bJ + cK)$ contains a smooth rational curve. Indeed, if $e \in H^2(X, \mathbb{Z})$ is any (-2)-class, then the subset of the moduli space of marked K3 surfaces for which e is of type $(1,1)$ is a hyperplane section. This hyperplane section, necessarily, cuts the complete curve given by the base $\mathbb{P}^1 = S^2$ of the twistor family. Hence, on one of the K3 surfaces $(X, aI + bJ + cK)$ the class e is a (-2)-class of type $(1,1)$ and, thus, X contains a smooth rational curve. This yields a contradiction to Prop. 4.1. $\square$

REMARK 4.3 — What are the bad harmonic $(1,1)$-forms? Certainly $\tilde{\omega}^2$ is harmonic and for any harmonic form α also $\tilde{\omega}\alpha$ is harmonic. So, if there is any bad harmonic $(1,1)$-form there must be also one that is $\tilde{\omega}$-primitive. Most likely, it is even true that the square of any primitive harmonic form is not harmonic. The proof of it should closely follow the arguments in the proof of Proposition 3.2, but there is a slight subtlety concerning the existence of sufficiently many (-2)-classes, that I do not know how to handle. We sketch the rough idea:
Assume there exists a $\tilde{\omega}$-harmonic $\tilde{\omega}$-primitive real $(1,1)$-form α such that α^2 is $\tilde{\omega}$-harmonic. As an $\tilde{\omega}$-harmonic $\tilde{\omega}$-primitive $(1,1)$-form, the form α is also of type $(1,1)$ with respect to any complex structure $\lambda = aI + bJ + cK$ induced by

the hyperkähler metric corresponding to $\tilde{\omega}$ (see Prop. 7.5 [6]). Moreover, α is also primitive with respect to all Kähler forms $\tilde{\omega}_\lambda$.
Assume that there exists a complex structure $\lambda \in S^2$, such that $\mathcal{C}_X \cap \mathbb{R}[\alpha] \oplus \mathbb{R}\omega_\lambda$ is not contained in $\mathcal{K}_X$. This condition can be easily rephrased in terms of (-2)-classes and thus becomes a question on the lattice $3U \oplus 2(-E_8)$. It looks rather harmless, but for the time being I do not know a complete proof of it. Under this assumption, we may even assume that in fact $\lambda = I$. Since α^2 is harmonic, in fact β^2 is harmonic for all $\beta \in \mathbb{R}\alpha \oplus \mathbb{R}\tilde{\omega} \subset \mathcal{H}^{1,1}(\omega)$. Going back to the proof of Prop. 3.2, we see that the second part of it can be adapted to this situation and shows that $\psi^{-1}(\mathcal{K}_X \cap \mathbb{R}[\alpha] \oplus \mathbb{R}\omega) \subset \mathbb{R}\alpha \oplus \mathbb{R}\tilde{\omega}$, where $\psi : \tilde{\mathcal{K}}_X \to \mathcal{K}_X$. The space $\psi^{-1}(\mathcal{K}_X \cap \mathbb{R}[\alpha] \oplus \mathbb{R}\omega)$ is the space of the distinguished Kähler forms whose classes are linear combinations of $[\alpha]$ and ω. Therefore, all these forms are harmonic and linear combinations of α and $\tilde{\omega}$ themselves. To conclude, we imitate the proof of Prop. 3.2 and choose a sequence $\omega_t \in \mathcal{K}_X \cap \mathbb{R}[\alpha] \oplus \mathbb{R}\omega$ converging towards $\omega' \in \mathcal{C}_X \setminus \mathcal{K}_X$. The corresponding sequence $\tilde{\omega}_t \in \tilde{\mathcal{K}}_X$ is contained in $\mathbb{R}\alpha \oplus \mathbb{R}\tilde{\omega}$ and converges towards a form $\tilde{\omega}'$. As in the proof of Prop. 3.2 this leads to a contradiction.

5 Hyperkähler manifolds

We will try to improve upon Proposition 3.2 in the case of hyperkähler manifolds. In particular, we will replace the question whether the top exterior power α^N of an harmonic form α is harmonic by the corresponding question for the square of α. The motivation for doing so stems from the general philosophy that hyperkähler manifolds should be treated in almost complete analogy to K3 surfaces, whereby the top intersection pairing should be replaced by the Beauville-Bogomolov form [2], which is the higher dimensional analogue of the intersection pairing on a K3 surface.
Let us begin by recalling some notations and basic facts. By a compact hyperkähler manifold X we understand a simply-connected compact Kähler manifold, such that $H^0(X, \Omega^2) = \mathbb{C}\sigma$, where σ is an everywhere non-degenerate holomorphic two-form. A Ricci-flat Kähler form $\tilde{\omega}$ turns out to be a hyperkähler form (cf. [2]), i.e. there exists a metric g and three complex structures I, J, and $K := IJ$, such that the corresponding Kähler forms $\tilde{\omega}_{aI+bJ+cK}$ are closed for all $(a, b, c) \in S^2$, such that I is the complex structure defining X, and such that $\tilde{\omega} = \tilde{\omega}_I$. One may renormalize σ, such that $\sigma = \tilde{\omega}_J + i\tilde{\omega}_K$. In particular, multiplying with σ maps harmonic forms to harmonic forms, for this holds true for the Kähler forms $\tilde{\omega}_J$ and $\tilde{\omega}_K$.
The positive cone $\mathcal{C}_X \subset H^{1,1}(X, \mathbb{R})$ is a connected component of $\{\alpha \in H^{1,1}(X, \mathbb{R}) \mid q_X(\alpha) > 0\}$, where q_X is the Beauville-Bogomolov form (cf. [2]).

Proposition 5.1 — *Let X be a $2n$-dimensional compact hyperkähler manifold with a fixed hyperkähler form $\tilde{\omega}_0$ and the unique holomorphic two-form σ. Then,*

$\alpha^2(\sigma\bar\sigma)^{n-1}$ is harmonic for all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ if and only if the linear subspace $\mathcal{H}^{1,1}(\omega) \subset \mathcal{A}^{1,1}(X)$ does not depend on $\omega \in \mathcal{K}_X$.

Proof. Assume that for all $\alpha \in \mathcal{H}^{1,1}(\omega_0)$ also $\alpha^2(\sigma\bar\sigma)^{n-1}$ is harmonic. If α is in addition strictly positive definite and $\tilde\omega \in \tilde{\mathcal{K}}_X$ with $[\alpha] = \omega$, then $\alpha^2(\sigma\bar\sigma)^{n-1} = \tilde\omega^2(\sigma\bar\sigma)^{n-1}$. We adapt Calabi's classical argument to deduce that in this case $\alpha = \tilde\omega$: If $\alpha^2(\sigma\bar\sigma)^{n-1} = \tilde\omega^2(\sigma\bar\sigma)^{n-1}$, then $(\alpha-\tilde\omega)(\alpha+\tilde\omega)(\sigma\bar\sigma)^{n-1} = 0$. Since α and $\tilde\omega$ are strictly positive definite, also $(\alpha + \tilde\omega)$ is strictly positive definite. It can be shown that also $(\alpha + \tilde\omega)(\sigma\bar\sigma)^{n-1}$ is strictly positive. As $[\alpha] = \omega = [\tilde\omega]$, the difference $\alpha-\tilde\omega$ can be written as $dd^c\varphi$ for some real function φ. But by the maximum principle the equation $(\alpha+\tilde\omega)(\sigma\bar\sigma)^{n-1}dd^c\varphi = 0$ implies $\varphi \equiv const$. Hence, $\alpha = \tilde\omega$.
As in the proof of Proposition 3.2 this shows that the intersection of the closed subset $\mathcal{H}^{1,1}(\omega_0)_\mathbb{R}$ with the open cone of strictly positive definite forms in $\mathcal{A}^{1,1}(X)_\mathbb{R}$ is contained in $\tilde{\mathcal{K}}_X$ and one concludes that $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega_0)_\mathbb{R}$.
Hence, $\tilde{\mathcal{K}}_X$ spans a linear subspace of the same dimension and, by Lemma 2.1 this shows that $\mathcal{H}^{1,1}(\omega)$ is independent of $\omega \in \mathcal{K}_X$.
Conversely, let $\mathcal{H}^{1,1}(\omega)$ be independent of $\omega \in \mathcal{K}_X$. Then $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega)_\mathbb{R}$ for any $\omega \in \mathcal{K}_X$. Therefore, $\alpha^2(\sigma\bar\sigma)^{n-1} = c(\sigma\bar\sigma)^n$ with $c \in \mathbb{R}$ for α in the Zariski-dense open subset $\tilde{\mathcal{K}}_X \subset \mathcal{H}^{1,1}(\omega)_\mathbb{R}$. Hence, $\alpha^2(\sigma\bar\sigma)^{n-1}$ is harmonic for any $\alpha \in \mathcal{H}^{1,1}(\omega)$ (cf. the proof of Prop. 2.3). □

Similar to Proposition 3.2 one has

COROLLARY 5.2 — *Let X be a $2n$-dimensional compact hyperkähler manifold. If the positive cone $\mathcal{C}_X$ is strictly smaller than the Kähler cone, then for any hyperkähler form $\tilde\omega_0$ there exists a harmonic form $\alpha \in \mathcal{H}^{1,1}(\tilde\omega_0)$, such that $\alpha^2(\sigma\bar\sigma)^{n-1}$ is not harmonic.* □

Of course, one expects that $\mathcal{H}^{1,1}(\omega)$ does in fact depend on ω, as it is the case for K3 surfaces. This would again follow from the existence of rational curves in nearby hyperkähler manifolds in the twistor space. In fact, for the two main series of examples of higher dimensional hyperkähler manifolds, i.e. Hilbert schemes of points on K3 surfaces and generalized Kummer varieties, this trivially holds true, since in these cases $\mathcal{C}_X$ is strictly bigger that $\mathcal{K}_X$ and so Corollary 5.2 applies. But already for global deformations of these examples the situation is not clear. One might speculate that Hilbert schemes respectively Kummer varieties are dense in their deformation spaces, so that arguments similar to those in the proof of Corollary 4.2 could be applied. But an understanding of the global deformations of Hilbert schemes respectively Kummer varieties seems difficult.
Actually, it would be more interesting to reverse the argument: Assume that X is a hyperkähler manifold, such that for any small deformation X' of X the Kähler cone $\mathcal{K}_{X'}$ equals $\mathcal{C}_{X'}$. I expect that this is equivalent to saying that

$\mathcal{H}^{1,1}(\omega)$ does not depend on ω. If for some other reason than the existence of rational curves as used in the K3 surface case this can be excluded, then one could conclude that there always is a nearby deformation X' for which $\mathcal{K}_{X'}$ is strictly smaller than $\mathcal{C}_{X'}$. The latter implies the existence of rational curves on X' (cf. [6]). Along these lines one could try to attack the Kobayashi conjecture, as the existence of rational curves on nearby deformations would say that X itself cannot be hyperbolic. Unfortunately, I cannot carry this through even for K3 surface.

6 Various other examples

Here we collect a few examples where algebraic geometry predicts the failure of harmonicity of the top exterior power of harmonic $(1,1)$-forms. In all examples this is linked to the existence of rational curves.

Varieties of general type. Let X be a non-minimal smooth variety of general type. As I learned from Keiji Oguiso this immediately implies that the Kähler cone is strictly smaller than the positive cone. His proof goes as follows: By definition the canonical divisor K_X is big and by the Kodaira Lemma (cf. [7]) it can therefore be written as the sum $K_X = H + E$ of an ample divisor H and an effective divisor E (with rational coefficients). Consider the segment $H_t := H + tE$ with $t \in [0,1)$. If all H_t were contained in the positive cone $\mathcal{C}_X$, then K_X would be in the closure of $\mathcal{C}_X$. If the Kähler cone coincided with the positive cone $\mathcal{C}_X$, then K_X would be nef, contradicting the hypothesis that X is not minimal. Hence $t_0 := \sup\{t|H_t \in \mathcal{C}_X\} \in (0,1)$. If H_{t_0} is not nef, then $\mathcal{K}_X$ is strictly smaller than $\mathcal{C}_X$. Thus, it suffices to show that H_{t_0} is not nef. If H_{t_0} were nef then all expressions of the form $H_{t_0}^{N-i}.H^{i-1}.E$ would be non-negative. Then $0 = H_{t_0}^N = H_{t_0}^{N-1}(H + t_0E) = H_{t_0}^{N-1}.H + t_0H_{t_0}^{N-1}.E$, so both summands must vanish. In particular, $0 = H_{t_0}^{N-1}.H = H^2.H_{t_0}^{N-2} + t_0H.H_{t_0}^{N-2}.E$. Again this yields the vanishing of both terms and in particular $0 = H^2.H_{t_0}^{N-2}$. By induction we eventually obtain $0 = H^{N-1}.H_{t_0}$ and, furthermore, $0 = H^{N-1}.H_{t_0} = H^N + t_0H^{N-1}.E$. But this time $H^N > 0$ yields the contradiction. Therefore, for a non-minimal variety of general type one has $\mathcal{K}_X \neq \mathcal{C}_X$ and hence there exist harmonic (with respect to any Kähler metric) $(1,1)$-forms with non-harmonic top exterior power. Note that a non-minimal variety contains rational curves. As the reader will notice, the above proof goes through on any manifold X that admits a big, but not nef line bundle L (replacing the canonical divisor). Also in this case the positive cone and the Kähler cone differ.

For a Calabi-Yau manifold X the following proposition shows that if X admits a 'special' Kähler form in the sense that the top power of any harmonic $(1,1)$-form is harmonic, then X is a unique birational model.

Proposition 6.1 — *If $\varphi : X - - \to X'$ is a birational map between two Calabi-Yau manifolds, then either φ can be extended to an isomorphism or $\mathcal{K}_X$ is strictly smaller than the positive cone $\mathcal{C}_X$. In the latter case, there exists for any Kähler form $\tilde{\omega}$ a $\tilde{\omega}$-harmonic $(1,1)$-form α with α^N not harmonic.*

Proof. The arguments are very similar to the one in the previous example. Let $H^{1,1}(X,\mathbb{R}) \cong H^{1,1}(X',\mathbb{R})$ be the natural isomorphism induced by the birational map. Let $\omega' \in H^{1,1}(X,\mathbb{R})$ correspond to a Kähler class on X'. By [5] the class ω' can be represented by a closed positive current. Furthermore, the birational map φ extends to an isomorphism if and only if $\omega' \in \mathcal{K}_X$. Assume this is not the case. Then $\omega' \notin \mathcal{K}_X$. If $\omega' \in \mathcal{C}_X$ one can apply Prop. 3.2 and we are done. If $\omega' \notin \mathcal{C}_X$ we may assume that it is also not in the boundary of $\mathcal{C}_X$, as we can change ω' slightly in the open cone $\mathcal{K}_{X'}$. If ω is a small enough Kähler class on X, then the difference $\alpha := \omega' - \omega$ can still be represented by a positive current. Let $t_0 := \sup\{t | \omega_t := \omega + t\alpha \in \mathcal{C}_X\}$. Then $t_0 \in (0,1)$. If $\mathcal{K}_X = \mathcal{C}_X$ then $\omega_{t_0}^{N-i}\omega^{i-1}\alpha \geq 0$ for all i. Then the above induction argument goes through and we eventually get $\omega^{N-1}\alpha = 0$. Since α is a positive current, this is only possible for $\alpha = 0$. Hence $\mathcal{K}_X \neq \mathcal{C}_X$. Note that for hyperkähler manifolds one knows that $\mathcal{K}_{X'} \subset \mathcal{C}_X$. This simplifies the argument. □

Remark 6.2 — *i)* Again, a non-trivial birational correspondence produces rational curves. We thus have another instance, where the special geometry of the variety is related to the non-harmonicity of products of harmonic forms.
ii) Most likely, finer information is encoded by Ricci-flat metrics. Those probably 'feel' contractible curves in small deformations. So, as for hyperkähler manifolds I would expect that $\mathcal{H}^{1,1}(\omega)$ depends on the Ricci-flat Kähler form representing ω.
iii) The arguments of the proof of 6.1 can be applied to the case of different birational minimal models (minimal models are not unique!). This shows that in the previous example the Kähler cone could be strictly smaller than the positive cone, even when K_X is nef or ample.

Blow-ups. This example is very much in the spirit of the previous two. Let $f : X \to Y$ be a non-trivial blow-up of a projective variety Y. Then $\mathcal{K}_X$ is strictly smaller than $\mathcal{C}_X$ and, therefore, for any Kähler structure on X there exist harmonic $(1,1)$-forms with non-harmonic maximal exterior power. Indeed, if L is an ample line bundle on Y then $f^*(L)$ is nef, but not ample, and it is contained in the positive cone. Hence, $f^*(L) \in \mathcal{C}_X \setminus \mathcal{K}_X$. Note that also the first example could be proved along these lines. By evoking the contraction theorem one shows that any non-minimal projective variety X admits a non-trivial contraction to a projective variety Y. The above argument then yields that $\mathcal{K}_X$ and $\mathcal{C}_X$ are different.

7 Chern forms

Let X be a compact Kähler manifold with a Ricci-flat Kähler form $\tilde{\omega}$. If F denotes the curvature of the Levi-Cevita connection ∇, then the Bianchi identity reads $\nabla F = 0$. The Kähler-Einstein condition implies $\Lambda_\omega F = 0$. The last equation can be expressed by saying that F is $\tilde{\omega}$-primitive. Analogously to the fact that any closed primitive $(1,1)$-form is in fact harmonic, one has that for F with $\nabla F = 0$ the primitivity condition $\Lambda_{\tilde{\omega}} F = 0$ is equivalent to the harmonicity condition $\nabla * F = 0$. As for untwisted harmonic $(1,1)$-forms one might ask for the harmonicity of the product F^m. Slightly less ambitious, one could ask whether the trace of this expression, an honest differential form, is harmonic. This trace is, in fact, a scalar multiple of the Chern character $ch_m(X, \tilde{\omega}) \in \mathcal{A}^{m,m}(X)_{\mathbb{R}}$.

Question. — Let $(X, \tilde{\omega})$ be a Ricci-flat Kähler manifold. Are the Chern forms $ch_m(X, \tilde{\omega})$ harmonic with respect to $\tilde{\omega}$?

By what was said about K3 surface we shall expect a negative answer to this question at least in this case:

Problem. — Let X be a K3 surface with a hyperkähler form $\tilde{\omega}$. Let $c_2 \in A^{2,2}(X)$ be the associated Chern form. Show that c_2 is not harmonic with respect to $\tilde{\omega}$!

So, this should be seen in analogy to the fact that α^2 is not harmonic for any primitive harmonic $(1,1)$-form α. Here, α is replaced by the curvature F and α^2 by trF^2. It is likely that the non-harmonicity of c_2 can be shown by standard methods in differential geometry, in particular by using the fact that c_2 is essentially $\|F\| \cdot \tilde{\omega}^2$ (see [3]), but I do not know how to do this.
Furthermore, it is not clear to me what the relation between the above question and the one treated in the previous sections is. I could imagine that the non-harmonicity of ch_m in fact implies the existence of harmonic $(1,1)$-forms with non-harmonic top exterior power.

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