On the Derived Category of Sheaves on a Manifold

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Abstract. Let M be a non–compact, connected manifold of dimension ≥ 1. Let $D(\text{sheaves}/M)$ be the unbounded derived category of chain complexes of sheaves of abelian groups on M. We prove that $D(\text{sheaves}/M)$ is not a compactly generated triangulated category, but is well generated.

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0. Introduction

We remind the reader. Let $\mathcal{T}$ be a triangulated category in which arbitrary coproducts exist. For example, we may take $\mathcal{T} = D(\text{sheaves}/M)$, the unbounded derived category of chain complexes of sheaves of abelian groups on a topological space M. An object $c \in \mathcal{T}$ is called *compact* if, for every collection $\{t_\lambda \mid \lambda \in \Lambda\}$ of objects in $\mathcal{T}$,

$$\mathcal{T}\left(c, \coprod_{\lambda\in\Lambda} t_\lambda\right) \;=\; \bigoplus_{\lambda\in\Lambda} \mathcal{T}(c, t_\lambda).$$

A triangulated category $\mathcal{T}$ is called *compactly generated* if arbitrary coproducts exist in $\mathcal{T}$, and there are plenty of compact objects. [The precise definition is that every non–zero object in $\mathcal{T}$ admits a non–zero map from a compact object, and there is a set of isomorphism classes of compact objects.] In this article, we will see the following

Theorem 0.1. *Let M be a non–compact, connected manifold of dimension ≥ 1. Let $D(\text{sheaves}/M)$ be the unbounded derived category of chain complexes of sheaves of abelian groups on M. Then the only compact object in $D(\text{sheaves}/M)$ is the zero object.*

In a recent book [6], the author defined a generalisation of compactly generated categories, the well generated triangulated categories. We will not repeat the definition here. Assuming the reader is familiar with the definitions, we state our next result.

Theorem 0.2. *Let M and $D(\text{sheaves}/M)$ be as in Thoerem 0.1. Then the category $D(\text{sheaves}/M)$ is well generated. More generally, for any Grothendieck abelian category $\mathcal{A}$, the derived category $D(\mathcal{A})$ is well generated.*

The two theorems above should perhaps explain the point of the book [6]. Surely the category $D(\text{sheaves}/M)$ is natural enough, we want to prove something about it. The fact that it is not compactly generated says that the old theorems of the subject cannot be applied. Well generated triangulated categories is an attempt to pick out a class of triangulated categories which includes $D(\text{sheaves}/M)$, but is restrictive enough so that we can prove good theorems, for example Brown representability. For much more information about well generated triangulated categories the reader is referred both to the book [6], as well as several beautiful insights in Krause's paper [4].

In Section 1 we give a fairly detailed and self contained account of the proof of Theorem 0.1. Compactly generated triangulated categories have been around for many years. The people who have worked with them might understandably want a good account of why $D(\text{sheaves}/M)$ is not compactly generated. It seems only right to make the presentation easily accessible. In Section 2 we give a very terse proof of Theorem 0.2. The argument makes no attempt to be self-contained. The proof relies heavily on results from Alonso, Jeremías and Souto's [1], and from [6].

I would like to thank Marco Schlichting, who asked me to provide proofs for the two theorems above. In the case of Theorem 0.2 I know several proofs, and the one presented here was chosen mostly because it is only a paragraph long. In the case of Theorem 0.1, until Schlichting prompted me I had carelessly assumed it was a true fact, without ever checking the details.

1. The proof of Theorem 0.1

In this section we give a fairly detailed and complete proof of Theorem 0.1. The proof attempts to be reasonably self-contained. We will, however, assume that the reader is familiar with the six gluing functors. Let M be a topological space. Suppose $M = U \cup Z$ is the disjoint union of an open set U and its complement Z. Let $i : Z \longrightarrow M$, $j : U \longrightarrow M$ be the inclusions. Then there are six functors on chain complexes of sheaves of abelian groups, denoted $j_!$, j^*, j_*, i^*, i_* and $i^!$, which allow us to glue complexes of sheaves on U and Z to form complexes of sheaves on M. There are many excellent accounts of this in the literature, for example in Beilinson, Bernstein and Deligne's [2].

Before plunging into the proof, we remind the reader of a key property of compact objects.

Reminder 1.1. Let $\mathcal{A}$ be a Grothendieck abelian category, $\mathcal{T} = D(\mathcal{A})$ its derived category. Suppose c is compact in $D(\mathcal{A})$, and suppose we are given a sequence of chain complexes

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots$$

with $\underrightarrow{\text{colim}} X_i = X$. Then $\mathcal{T}(c, X) = \underrightarrow{\text{colim}} \mathcal{T}(c, X_i)$.

Proof. The proof may be found in Lemma 2.8 of [5] combined with Remark 2.2 of [3]. □

Lemma 1.2. *Suppose M is a manifold, and that $c \in D(\text{sheaves}/M)$ is a compact object. Then there is a compact set $K \subset M$, so that c is acyclic outside K.*

Proof. Choose an increasing sequence of open subsets $U_\ell \subset M$, $\ell \in \mathbb{N}$ so that

(i) $M = \cup U_\ell$ is the union of the U_ℓ.
(ii) The closure $\overline{U}_\ell$ of U_ℓ is compact, and $\overline{U}_\ell \subset U_{\ell+1}$.

Let $j_\ell : U_\ell \longrightarrow M$ be the inclusion. Because the U_ℓ are increasing, we have a sequence of chain complexes of sheaves

$$\{j_1\}_! j_1^* c \longrightarrow \{j_2\}_! j_2^* c \longrightarrow \{j_3\}_! j_3^* c \longrightarrow \cdots$$

with direct limit c. The identity map $c \longrightarrow c$ is a map from a compact object c to a direct limit. By Remark 1.1, it must factor through $\{j_\ell\}_! j_\ell^* c$ for some ℓ. And the complex $\{j_\ell\}_! j_\ell^* c$ is acyclic outside the compact set $\overline{U}_\ell$. □

Definition 1.3. *The* support *of a chain complex c of sheaves on M is the set of all points $p \in M$ so that the stalk at p of c is not acyclic.*

Lemma 1.4. *Suppose M is a manifold, and that $c \in D(\text{sheaves}/M)$ is a compact object. Then the support of c is compact.*

Proof. Let p be point outside the support of c; that is, the stalk of c at p is acyclic. Let $U = M - \{p\}$ be the complement of p. Let $j : U \longrightarrow M$ be the inclusion of the open set, $i : \{p\} \longrightarrow M$ the inclusion of its complement. Consider now the triangle

$$j_! j^* c \longrightarrow c \longrightarrow i_* i^* c \longrightarrow \Sigma j_! j^* c.$$

Since the stalk of c at p vanishes, we know that $i^* c = 0$. Thus c is quasi-isomorphic to $j_! j^* c$. In particular $j_! j^* c$ is compact. We assert that $j^* c$ must also be compact. This is because

$$\begin{aligned}
&\mathrm{Hom}\left(j^*c\,,\ \coprod_{\lambda\in\Lambda} t_\lambda\right) && \\
&= \mathrm{Hom}\left(j^*c\,,\ j^*j_! \coprod_{\lambda\in\Lambda} t_\lambda\right) && \text{since } j^*j_! = 1\\
&= \mathrm{Hom}\left(j_!j^*c\,,\ j_! \coprod_{\lambda\in\Lambda} t_\lambda\right) && \text{by adjunction}\\
&= \mathrm{Hom}\left(j_!j^*c\,,\ \coprod_{\lambda\in\Lambda} j_! t_\lambda\right) && \text{as } j_! \text{ respects coproducts}\\
&= \bigoplus_{\lambda\in\Lambda} \mathcal{T}(j_!j^*c\,,\, j_!t_\lambda) && \text{since } c = j_!j^*c \text{ is compact}\\
&= \bigoplus_{\lambda\in\Lambda} \mathcal{T}(j^*c\,,\, j^*j_!t_\lambda) && \text{by adjunction}\\
&= \bigoplus_{\lambda\in\Lambda} \mathcal{T}(j^*c\,,\, t_\lambda) && \text{since } j^*j_! = 1.
\end{aligned}$$

Lemma 1.2, applied to the complex j^*c on $M - \{p\}$, now tells us that the support of j^*c is contained in a compact subset of $M - \{p\}$. Hence so is the support of $c \cong j_!j^*c$. What this proves is that any point p not in the support of c is in the interior of the complement of the support. The support is closed. By Lemma 1.2, the support is contained in a compact subset of M. Being a closed subset of a compact subset, the support of c must be compact. $\square$

NOTATION 1.5. From now on, let c be a compact object in $D(\text{sheaves}/M)$, with M a non-compact, connected manifold. Let K be the support of c. By Lemma 1.4 we know that K is compact. To prove Theorem 0.1 we must show that K is empty. In the rest of this section we assume $K \neq \emptyset$, and deduce a contradiction.

LEMMA 1.6. *Let the notation be as in Notation 1.5. There is a compact set $L \subset M$ so that*

(i) *The support of c is contained in L; that is, $K \subset L$.*
(ii) *L is a deformation retract of a neighbourhood.*
(iii) *The boundary of L contains a point in K.*

Proof. Choose a Morse function on the manifold M; that is, a proper function $\varphi : M \longrightarrow [0, \infty)$ with only non-degenerate critical points. Now $\varphi(K)$ is a compact subset of $\mathbb{R}$. Let $k \in \mathbb{R}$ be the maximum of $\varphi(K)$. By jiggling φ a little, we may assume that k is a regular value of φ. Now let $L = \varphi^{-1}[0, k]$. Because φ is proper, L must be compact. Because k is the maximum of φ on $K \subset M$, there must be a point $x \in K$ with $\varphi(x) = k$. Since φ is regular at x, x must be a boundary point of L. Now, for any Riemannian metric on M, the gradient flow along φ deformation retracts $\varphi^{-1}[0, k+\varepsilon]$ to $L = \varphi^{-1}[0, k]$. $\square$

THEOREM 1.7. *With the notation as above, the object c vanishes.*

Proof. With the notation as in the proof of Lemma 1.6, let $L = \varphi^{-1}[0, k]$, and let $\widehat{L} = \varphi^{-1}[0, k+\varepsilon]$. Denote by $i : L \longrightarrow M$ and $\widehat{\imath} : \widehat{L} \longrightarrow M$ the inclusions. Let $p : \widehat{L} \longrightarrow L$ be the retraction, and $j : U = \{M - L\} \longrightarrow M$ the open inclusion.
We wish to consider the complex $b = \widehat{\imath}_* p^* c$ on M. The fact that $M = L \cup U$ is a disjoint union of an open set and its complement gives a triangle

$$j_!j^*b \longrightarrow b \longrightarrow i_*i^*b \longrightarrow \Sigma j_!j^*b.$$

Now

$$\begin{aligned} i_*i^*b &= i_*i^*\widehat{\imath}_*p^*c && \text{by definition of } b \\ &= i_*i^*c && \text{since } p \text{ and } \widehat{\imath} \text{ are the identity on } L \\ &= c && \text{since } c \text{ is supported on } K \subset L. \end{aligned}$$

It follows that the map

$$i_*i^*b \longrightarrow \Sigma j_!j^*b$$

is a map from a compact object $c = i_* i^* b$. Now write $U = M - L$ as an increasing union of $U_\ell \subset U$, with $\overline{U}_\ell$ compact and $\overline{U}_\ell \subset U_{\ell+1}$. As in the proof of Lemma 1.2, we write

$$j_! j^* b = \underrightarrow{\text{colim}} \{j_\ell\}_! j_\ell^* b$$

with $j_\ell : U_\ell \longrightarrow U$ the inclusion. The compactness of $i_* i^* b$ guarantees that the map

$$i_* i^* b \longrightarrow \Sigma j_! j^* b$$

must factor as

$$i_* i^* b \longrightarrow \Sigma \{j_\ell\}_! j_\ell^* b \longrightarrow \Sigma j_! j^* b.$$

But now $i_* i^* b$ is supported on L, while $\{j_\ell\}_! j_\ell^* b$ is supported on the compact set $\overline{U}_\ell \subset M - L$. The map between them must vanish. In the triangle

$$j_! j^* b \longrightarrow b \longrightarrow i_* i^* b \xrightarrow{\beta} \Sigma j_! j^* b.$$

we have shown that the map β vanishes. We conclude that there is an isomorphism in the derived category

$$b \cong i_* i^* b \oplus j_! j^* b.$$

Now let x be a point in K on the boundary of L; in the notation of the proof of Lemma 1.6 this means that $\varphi(x) = k$. Then $p^{-1}(x)$ is the interval $[k, k+\varepsilon]$. If we pull back the isomorphism $b \cong i_* i^* b \oplus j_! j^* b$ to $p^{-1}(x) = [k, k+\varepsilon]$, we have that b is the complex of constant sheaves on the interval $[k, k+\varepsilon]$, whose value is the stalk of c at x, which by hypothesis is not acyclic. This complex is quasi-isomorphic to a direct sum $b \cong i_* i^* b \oplus j_! j^* b$, where i is the inclusion of the endpoint k and j the inclusion of the complement. But this is absurd; it is easy to see that any map $b \longrightarrow j_! j^* b$ must vanish. □

2. The proof of Theorem 0.2

We need to prove that, for any Grothendieck abelian category $\mathcal{A}$, the derived category is well generated. By Proposition 5.1 in Alonso, Jeremías and Souto's paper [1], we know that there exists a ring R and a set of objects $L \subset D(R)$ for which the following is true.

Proposition 2.1. (=Proposition 5.1 in [1]) *Let $\mathcal{L}_{\mathcal{A}}$ be the smallest localising subcategory of $D(R)$ containing L. Then the derived category $D(\mathcal{A})$ of $\mathcal{A}$ is equivalent to the quotient $D(\mathcal{A}) \cong D(R)/\mathcal{L}_{\mathcal{A}}$.*

The category $D(R)$ is compactly generated, hence well generated. By Proposition 8.4.2 of [6] (more precisely by part (8.4.2.3) of the Proposition),

$$D(R) = \bigcup_\alpha \{D(R)\}^\alpha.$$

Since L is a set of objects, the coproduct of all the objects in L is an object of $D(R)$, and therefore must lie in $\{D(R)\}^\alpha$ for some regular cardinal α. Now apply Theorem 4.4.9 of [6]. We have that, for any regular cardinal $\beta \geq \alpha$, $\mathcal{L}_{\mathcal{A}}^\beta$ is just $\langle L\rangle^{\beta}$, the smallest β-localising subcategory containing L, while $\{D(R)\}^{\beta} = \langle R\rangle^{\beta}$ is the smallest β-localising subcategory containing R. And if $\beta > \aleph_0$, then $\{D(\mathcal{A})\}^{\beta} = \{D(R)\}^{\beta}/\mathcal{L}_{\mathcal{A}}^{\beta}$. The categories $\mathcal{L}_{\mathcal{A}}^{\beta}$, $\{D(R)\}^{\beta}$ and $\{D(\mathcal{A})\}^{\beta}$ are all essentially small, and generate $\mathcal{L}_{\mathcal{A}}$, $D(R)$ and $D(\mathcal{A})$ respectively. It follows that $D(\mathcal{A})$ is well generated.

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