# On the Uniqueness Problem of Bivariant Chern Classes

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Abstract. In this paper we show that the bivariant Chern class $\gamma : \mathbb{F} \to \mathbb{H}$ for morphisms from possibly singular varieties to nonsingular varieties are uniquely determined, which therefore implies that the Brasselet bivariant Chern class is unique for cellular morphisms with nonsingular target varieties. Similarly we can see that the Grothendieck transformation $\tau : \mathbb{K}_{\mathrm{alg}} \to \mathbb{H}_{\mathbb{Q}}$ constructed by Fulton and MacPherson is also unique for morphisms with nonsingular target varieties.

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## §1 Introduction

In [FM, Part I] W. Fulton and R. MacPherson developed the so-called *Bivariant Theories*, which are simultaneous generalizations of covariant functors and contravariant functors. They are equipped with three operations of *product, pushforward, pullback*, and they are supposed to satisfy seven kinds of axioms. A transformation from one bivariant theory to another bivariant theory, preserving these three operations, is called a *Grothendieck transformation*, which is a generalization of ordinary natural transformations.

The *Chern-Schwartz-MacPherson class* is the unique natural transformation $c_* : F \to H_*$ from the covariant functor $F$ of constructible functions to the integral homology covariant functor $H_*$, satisfying the normalization condition that the value $c_*(1\!\!1_X)$ of the characteristic function $1\!\!1_X$ of a nonsingular variety $X$ is equal to the Poincaré dual of the total Chern class $c(TX)$ of the tangent

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bundle $TX$ of $X$. The existence of this transformation was conjectured by Deligne and Grothendieck, and was proved by MacPherson (see [M] and also [BS], [Sc]).

In [FM, Part I, §10.4] Fulton and MacPherson conjectured (or posed as a question) the existence of a bivariant Chern class, i.e., a Grothendieck transformation $\gamma : \mathbb{F} \to \mathbb{H}$ from the bivariant theory $\mathbb{F}$ of constructible functions to the bivariant homology theory $\mathbb{H}$, satisfying the normalization condition that for a morphism from a nonsingular variety $X$ to a point the value $\gamma(\mathbb{1}_X)$ of the characteristic function $\mathbb{1}_X$ of $X$ is equal to the Poincaré dual of the total Chern class of $X$. The bivariant Chern class specializes to the original Chern-Schwartz-MacPherson class, i.e., when restricted to morphisms to a point it becomes the Chern-Schwartz-MacPherson class. As applications of the bivariant Chern class, for example, one obtains the Verdier-Riemann-Roch for Chern class and the Verdier's specialization of Chern classes [V].

In [B] J.-P. Brasselet has solved the conjecture affirmatively in the category of complex analytic varieties and cellular analytic maps. Any analytic map is "conjecturally" cellular and indeed no example of a non-cellular analytic map has been found so far. In this sense the condition of "cellularness" could be dropped. For example, it follows from a result of Teissier [T] that an analytic map to a smooth curve is cellular (see [Z2, 2.2.5 Lemme]). In [S] C. Sabbah gave another construction of bivariant Chern classes, using the notions of *bivariant cycle, relative local Euler obstruction, morphisme sans éclatement en codimension 0* (see [S] or [Z1, Z2] for more details). And in [Z1] (and [Z2]) J. Zhou showed that for a morphism from a variety to a smooth curve these two bivariant Chern classes due to Brasselet and Sabbah are identical. However, the uniqueness of bivariant Chern classes still remains as an open problem.

In [FM, Part II] Fulton and MacPherson constructed a Grothendieck transformation $\tau : \mathbb{K}_{\mathrm{alg}} \to \mathbb{H}_{\mathbb{Q}}$, which is a bivariant-theoretic version of Baum-Fulton-MacPherson's Riemann-Roch $\tau^{\mathrm{BFM}} : \mathbf{K}_0 \to H_{*\mathbb{Q}}$ constructed in [BFM]. The uniqueness problem of this Grothendieck transformation remains open.

As remarked in [FM, Part I, §10.9: Uniqueness Questions], there are few uniqueness theorems available concerning Grothendieck transformations.

In this paper we show that the bivariant Chern class for morphisms with nonsingular target varieties is unique if it exists. Therefore it follows that the Brasselet bivariant Chern class is unique for cellular morphisms with nonsingular target varieties, thus it gives another proof of Zhou's result mentioned above. Our method also implies that the above Grothendieck transformation $\tau : \mathbb{K}_{\mathrm{alg}} \to \mathbb{H}_{\mathbb{Q}}$ constructed by Fulton and MacPherson is unique for morphisms with nonsingular target varieties.

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## §2 Bivariant constructible functions and bivariant homology theory

For a general reference for the bivariant theory, see Fulton-MacPherson's book [FM]. In this section, we recall some basic ingredients, needed in this paper, of the bivariant theory of constructible functions and bivariant homology theory. For a morphism $f : X \to Y$ the bivariant theory $\mathbb{F}(X \xrightarrow{f} Y)$ of constructible functions consists of all the constructible functions on $X$ which satisfy the *local Euler condition with respect to* $f$, i.e., the condition that for any point $x \in X$ and for any local embedding $(X, x) \to (\mathbf{C}^N, 0)$ the following equality holds

$$\alpha(x) = \chi\bigl(B_\epsilon \cap f^{-1}(z); \alpha\bigr),$$

where $B_\epsilon$ is a sufficiently small open ball of the origin 0 with radius $\epsilon$ and $z$ is any point close to $f(x)$ (see [B], [FM], [S], [Z1]). The three operations on $\mathbb{F}$ are defined as follows:

(i): the product operation

$$\bullet : \mathbb{F}(X \xrightarrow{f} Y) \otimes \mathbb{F}(Y \xrightarrow{g} Z) \to \mathbb{F}(X \xrightarrow{gf} Z)$$

is defined by:

$$\alpha \bullet \beta := \alpha \cdot f^*\beta.$$

(ii): the pushforward operation

$$f_* : \mathbb{F}(X \xrightarrow{gf} Z) \to \mathbb{F}(Y \xrightarrow{g} Z)$$

is the pushforward

$$(f_*\alpha)(y) := \chi(f^{-1}(y); \alpha) = \int_{f^{-1}(y)} c_*(\alpha|_{f^{-1}(y)}).$$

(iii): For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y, \end{array}$$

the pullback operation

$$g^* : \mathbb{F}(X \xrightarrow{f} Y) \to \mathbb{F}(X' \xrightarrow{f'} Y')$$

is the functional pullback

$$({g'}^*\alpha)(x') := \alpha(g'(x')).$$

These operations satisfy the seven axioms listed in [FM, Part I, §2.2] and it is also known that these three operations are well-defined (e.g., see [BY], [FM], [S], [Z1]). Note that $\mathbb{F}(X \xrightarrow{\mathrm{id}_X} X)$ consists of all locally constant functions and $\mathbb{F}(X \to pt) = F(X)$.

Let $\mathbb{H}$ be the Fulton-MacPherson bivariant homology theory, constructed from the cohomology theory. For a morphism $f : X \to Y$, choose a morphism $\phi : X \to M$ to a smooth manifold $M$ of real dimension $n$ such that $\Phi := (f, \phi) : X \to Y \times M$ is a closed embedding. Of course, the morphism $\phi : X \to M$ can be already an embedding. Then the $i$-th bivariant homology group $\mathbb{H}^i(X \xrightarrow{f} Y)$ is defined by

$$\mathbb{H}^i(X \xrightarrow{f} Y) := H^{i+n}(Y \times M, (Y \times M) \setminus X_\phi),$$

where $X_\phi$ is defined to be the image of the morphism $\Phi = (f, \phi)$. The definition is independent of the choice of $\phi$, i.e., for any other morphism $\phi' : X \to M'$ to a smooth manifold $M'$ of real dimension $n'$ there is an isomorphism

$$H^{i+n}(Y \times M, (Y \times M) \setminus X_\phi) \cong H^{i+n'}(Y \times M', (Y \times M') \setminus X_{\phi'}).$$

See [FM, §3.1] for more details of $\mathbb{H}$.

A bivariant Chern class is a Grothendieck transformation from the bivariant theory $\mathbb{F}$ of constructible functions to the bivariant homology theory $\mathbb{H}$

$$\gamma : \mathbb{F} \to \mathbb{H}$$

satisfying the normalization condition that for a nonsingular variety $X$ and for the map $\pi : X \to pt$ to a point $pt$

$$\gamma(\mathbb{1}_\pi) = c(TX) \cap [X]$$

where $\mathbb{1}_\pi = \mathbb{1}_X \in F(X) = \mathbb{F}(X \xrightarrow{\pi} pt)$. Here the Grothendieck transformation $\gamma : \mathbb{F} \to \mathbb{H}$ preserves the three operations of product, pushforwad and pullback, i.e.,

(i) $\gamma(\alpha \bullet \beta) = \gamma(\alpha) \bullet \gamma(\beta)$,
(ii) $\gamma(f_*\alpha) = f_*\gamma(\alpha)$ and
(iii) $\gamma(g^*\alpha) = g^*\gamma(\alpha)$.

Theorem (2.1). *(Brasselet's Theorem [B]) For the category of analytic varieties with cellular morphisms there exists a bivariant Chern class* $\gamma : \mathbb{F} \to \mathbb{H}$.

## §3 Uniqueness of the bivariant Chern class

First we take a bit closer look at the definition of the bivariant homology theory. As seen above, for *any* morphism $\phi : X \to M$ such that $(f, \phi) : X \to Y \times M$ is a closed embedding (or simply, for any closed embedding $\phi : X \to M$), we have the isomorphism

$$\mathbb{H}(X \xrightarrow{f} Y) \cong H^*(Y \times M, (Y \times M) \setminus X).$$

This isomorphism is thought to be a "realization isomorphism with respect to the embedding $X \to Y \times M$" of the group $\mathbb{H}(X \xrightarrow{f} Y)$. We denote this isomorphism by $\Re_{X \hookrightarrow Y \times M}$, emphasizing the embedding $\Phi : X \to Y \times M$. In particular, for a morphism $f : X \to pt$ to a point pt, the bivariant homology group $\mathbb{H}(X \xrightarrow{f} pt)$ is considered to be the homology group $H_*(X)$, since for any embedding of $X$ into any manifold $N$ we have the Alexander duality isomorphism

$$H^*(N, N \setminus X) \cong H_*(X),$$

which shall be denoted by $\mathcal{A}_{X \hookrightarrow N}$, again indicating the embedding $X \hookrightarrow N$. Note that the Alexander isomorphism is given by taking the cap product with the fundamental class, i.e.,$\mathcal{A}_{X \hookrightarrow N}(a) = a \cap [N]$. Therefore $\mathbb{H}(X \xrightarrow{f} pt) = H_*(X)$ and $\Re_{X \hookrightarrow N} = \big(\mathcal{A}_{X \hookrightarrow N}\big)^{-1}$. In particular, if $X$ is nonsingular, the Alexander duality isomorphism is the Poincaré duality isomorphism via Thom isomorphism, denoted by $\mathcal{P}_X$:

$$\mathcal{A}_{X \hookrightarrow X} = \mathcal{P}_X : H^*(X) \cong H_*(X).$$

With these notation, it follows from the definition of the bivariant product in $\mathbb{H}$ [FM, Part I, §3.1.7] that the bivariant product

$$\bullet : \mathbb{H}(X \xrightarrow{f} Y) \otimes \mathbb{H}(Y \xrightarrow{g} Z) \to \mathbb{H}(X \xrightarrow{gf} Z)$$

is described as follows: consider the following commutative diagram where the rows are closed embedding and the verticals are the projections with $M$ and $N$ being manifolds:

$$\begin{array}{ccccc} X & \longrightarrow & Y \times M & \longrightarrow & Z \times N \times M \\ & & \downarrow & & \downarrow p \\ & & Y & \longrightarrow & Z \times N \\ & & & & \downarrow \\ & & & & Z \end{array}$$

Then for $\alpha \in \mathbb{H}(X \xrightarrow{f} Y)$ and $\beta \in \mathbb{H}(Y \xrightarrow{g} Z)$

$$\alpha \bullet \beta := \Big(\Re_{X \hookrightarrow Z \times N \times M}\Big)^{-1} \Bigg(\Re_{X \hookrightarrow Y \times M}(\alpha) \cdot p^* \Re_{Y \hookrightarrow Z \times N}(\beta)\Bigg), \tag{3.1}$$

where the center dot $\cdot$ is the product defined by [FM, §3.1.7 (1), p.36]. The well-definedness of the bivariant homology product given in Fulton-MacPherson's book [FM] means that the above description (3.1) is independent of the choices of $M$ and $N$, i.e., the realization isomorphisms. This viewpoint becomes a crucial one in our proof.

*Remark (3.2).* Suppose that $\gamma : \mathbb{F} \to \mathbb{H}$ is a bivariant Chern class. For a morphism $f : X \to Y$, we denote the homomorphim $\gamma : \mathbb{F}(X \xrightarrow{f} Y) \to \mathbb{H}(X \xrightarrow{f} Y)$ by $\gamma_{X\to Y}$. Then, for any variety $X$ we see that the homomorphism $\gamma_{X\to pt} : F(X) = \mathbb{F}(X \to pt) \to \mathbb{H}(X \to pt) = H_*(X)$ is nothing but the Chern-Schwartz-MacPherson class homomorphism $c_* : F(X) \to H_*(X)$, because $\gamma_{X\to pt}$ is a natural transformation satisfying the normalization condtion and thus it has to be the Chern-Schwartz-MacPherson class $c_* : F(X) \to H_*(X)$ since it is unique.

Let $\gamma : \mathbb{F} \to \mathbb{H}$ be a bivariant Chern class and let $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$. Then we have

$$\gamma_{X\to pt}(\alpha) = \gamma_{X\to Y}(\alpha) \bullet \gamma_{Y\to pt}(1\!\!1_Y),$$

Therefore it follows from Remark (3.2) that we have

$$c_*(\alpha) = \gamma_{X\to Y}(\alpha) \bullet c_*(1\!\!1_Y). \tag{3.3}$$

Furthermore, for any constructible function $\beta \in F(Y)$, we have

$$c_*(\alpha \cdot f^*\beta) = \gamma_{X\to Y}(\alpha) \bullet c_*(\beta).$$

The uniqueness of $\gamma : \mathbb{F} \to \mathbb{H}$, therefore, follows if we can show that $\omega \in \mathbb{H}(X \xrightarrow{f} Y)$ and $\omega \bullet c_*(\beta) = 0$ for any $\beta \in F(Y)$ automatically implies that $\omega = 0$.

Heuristically or very loosely speaking, the bivariant Chern class $\gamma_{X\to Y}(\alpha)$ of the bivariant constructible function $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ could be or should be "described" as a "quotient"

$$\gamma_{X\to Y}(\alpha) := \frac{c_*(\alpha)}{c_*(1\!\!1_Y)}$$

in a reasonable way. Otherwise it would be an interesting problem to see if there is a reasonable bivariant homology theory so that this "quotient" is well-defined. We hope to come back to this problem in a different paper.

However, in the case of morphisms whose target varieties are nonsingular, the above argument gives us the uniqueness of the bivariant Chern class and furthermore we can describe the above "quotient" $\frac{c_*(\alpha)}{c_*(\mathbf{1}_Y)}$ explicitly.

Theorem (3.4). *Let $\gamma : \mathbb{F} \to \mathbb{H}$ be a bivariant Chern class. Then it is unique, when restricted to morphisms whose target varieties are nonsingular.*

*Explicitly, for a morphism $f : X \to Y$ with $Y$ being nonsingular and for any bivariant constructible function $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ the bivariant Chern class $\gamma_{X\to Y}(\alpha)$ is expressed by*

$$\gamma_{X\to Y}(\alpha) = f^*s(TY) \cap c_*(\alpha)$$

*where $s(TY)$ is the total Segre class of the tangent bundle $TY$, i.e., $s(TY) = c(TY)^{-1}$ the inverse of the total Chern class $c(TY)$.*

Corollary (3.5). *The Brasselet bivariant Chern classes, defined on cellular morphisms with nonsingular target varieties, are unique.*

Thus in particular, we get the following

Corollary (3.6). *(Zhou's theorem [Z1, Z2]) For a morphism $f : X \to S$ with $S$ being a smooth curve, Brasselet's bivariant Chern class and Sabbah's bivariant Chern class are the same.*

*Proof of Theorem (3.4).* First, the hypothesis that the target variety $Y$ is nonsingular implies that we have

$$\mathbb{H}(X \xrightarrow{f} Y) = \mathbb{H}(X \to pt) = H_*(X).$$

This turns out to be a key fact. Let $\gamma : \mathbb{F} \to \mathbb{H}$ be a bivariant Chern class. Then it follows from (3.3) that we have

$$c_*(\alpha) = \gamma_{X \to Y}(\alpha) \bullet c_*(1\!\!1_Y).$$

To consider the product, we look at the following commutative diagram with $j : X \to Y \times M$ being an embedding and $p : Y \times M \to Y$ the projection such that $f = p \circ j$:

$$\begin{array}{ccccc} X & \xrightarrow{j} & Y \times M & \xrightarrow{\text{id}} & Y \times M \\ & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ & & Y & \xrightarrow{\text{id}} & Y \\ & & & & \downarrow \\ & & & & pt \end{array}$$

Hence we have, via the realization isomorphisms $\Re_{X \hookrightarrow Y \times M}$, that

$$c_*(\alpha) = \Big(\Re_{X \hookrightarrow Y \times M}\Big)^{-1} \Bigg(\Re_{X \hookrightarrow Y \times M}(\gamma_{X \to Y}(\alpha)) \cdot p^* \Re_{Y \hookrightarrow Y}(c_*(Y))\Bigg).$$

Here it should be noted that the realization isomorphism $\Re_{X \hookrightarrow Y \times M}$ functions as two kinds of realization isomorphism: the first one is

$$\Re_{X \hookrightarrow Y \times M} : \mathbb{H}(X \to pt) = H_*(X) \cong H^*(Y \times M, (Y \times M) \setminus X)$$

and the second one is

$$\Re_{X \hookrightarrow Y \times M} : \mathbb{H}(X \xrightarrow{f} Y) = H_*(X) \cong H^*(Y \times M, (Y \times M) \setminus X).$$

Since $\text{id} : Y \times M \to Y \times M$ is the identity, it follows from the definition of the product $\cdot$ that it is nothing but the usual cup product, thus we have

$$c_*(\alpha) = \Big(\Re_{X \hookrightarrow Y \times M}\Big)^{-1} \Bigg(\Re_{X \hookrightarrow Y \times M}(\gamma_{X \to Y}(\alpha)) \cup p^* \Re_{Y \hookrightarrow Y}(c_*(Y))\Bigg).$$

Since $Y$ is nonsingular, $c_*(Y) = c(TY) \cap [Y]$ and $\Re_{Y \hookrightarrow Y} : H_*(Y) = \mathbb{H}(Y \to pt) \cong H^*(Y)$ which is the inverse of the Poincaré duality isomorphism $\mathcal{P}_Y$, we have $\Re_{Y \hookrightarrow Y}(c_*(Y)) = c(TY)$. Therefore we get that

$$c_*(\alpha) = \Big(\Re_{X \hookrightarrow Y \times M}\Big)^{-1} \Bigg(\Re_{X \hookrightarrow Y \times M}(\gamma_{X \to Y}(\alpha)) \cup p^* c(TY)\Bigg).$$

Which implies that

$$\Re_{X \hookrightarrow Y \times M}(c_*(\alpha)) = \Re_{X \hookrightarrow Y \times M}(\gamma_{X \to Y}(\alpha)) \cup p^* c(TY).$$

Thus we get that

$$\Re_{X \hookrightarrow Y \times M}(\gamma_{X \to Y}(\alpha)) = \Re_{X \hookrightarrow Y \times M}(c_*(\alpha)) \cup p^* s(TY),$$

which implies that

$$\gamma_{X \to Y}(\alpha) = \Big(\Re_{X \hookrightarrow Y \times M}\Big)^{-1} \Bigg(\Re_{X \hookrightarrow Y \times M}(c_*(\alpha)) \cup p^* s(TY)\Bigg).$$

Furthermore this can be simplified more as follows. Since, as we observe in the previous section,

$$\Big(\Re_{X \hookrightarrow Y \times M}\Big)^{-1}(a) = a \cap [Y \times M],$$

we get

$$\gamma_{X \to Y}(\alpha) = \Bigg(\Re_{X \hookrightarrow Y \times M}(c_*(\alpha)) \cup p^* s(TY)\Bigg) \cap [Y \times M].$$

Then it follows from the equation [F, §19.1, (8), p.371] that we get the following:

$$\begin{aligned} \gamma_{X \to Y}(\alpha) &= \Bigg(\Re_{X \hookrightarrow Y \times M}(c_*(\alpha)) \cup p^* s(TY)\Bigg) \cap [Y \times M] \\ &= j^* p^* s(TY) \cap \Bigg(\Re_{X \hookrightarrow Y \times M}(c_*(\alpha)) \cap [Y \times M]\Bigg) \\ &= j^* p^* s(TY) \cap c_*(\alpha) \\ &= f^* s(TY) \cap c_*(\alpha). \end{aligned}$$

□

By the same argument as above, we can show the following:

Theorem (3.7). *The Grothendieck transformation*

$$\tau : \mathbb{K}_{\mathrm{alg}} \to \mathbb{H}_{\mathbb{Q}}$$

*constructed in [FM, Part II] is unique on morphisms with nonsingular target varieties. And the bivariant class* $\tau_{X\to Y}(\alpha)$ *for a bivariant coherent sheaf* $\alpha \in \mathbb{K}_{\mathrm{alg}}(X \to Y)$ *is given by*

$$\tau_{X\to Y}(\alpha) = \frac{1}{f^*td(TY)} \cap \tau^{\mathrm{BFM}}(\alpha)$$

*where* $\tau^{\mathrm{BFM}} : \mathbf{K}_0 \to H_{*\mathbb{Q}}$ *is the Baum-Fulton-MacPherson's Riemann-Roch and* $td(TY)$ *is the total Todd class of the tangent bundle.*

*Remark (3.8).* In the case when the target variety Y is singular, the above argument does not work at all. Thus the target variety being nonsingular is essential (cf. [Y]). However, "modulo resolution" the uniqueness holds. Namely, by taking any resolution of singularities $\pi : \widetilde{Y} \to Y$, for any bivariant constructible function $\alpha \in \mathbb{F}(X \to Y)$, the pullback $\pi^*\gamma(\alpha)$ is uniquely determined; i.e., suppose that we have two bivariant Chern classes $\gamma, \gamma' : \mathbb{F} \to \mathbb{H}$, then for any resolution $\pi : \widetilde{Y} \to Y$ we have

$$\pi^*\gamma(\alpha) = \pi^*\gamma'(\alpha).$$

It is the same for the Grothendieck transformation $\tau : \mathbb{K}_{\mathrm{alg}} \to \mathbb{H}_{\mathbb{Q}}$, i.e.,

$$\pi^*\tau(\alpha) = \pi^*\tau'(\alpha).$$

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