Publ. RIMS, Kyoto Univ.
8 (1972/73), 295-310

On Cohomology Theories of Infinite CW-complexes, I

By

Zen-ichi YOSIMURA*

In [2], § 3, we discussed some convergence conditions of spectral sequences associated with an additive cohomology theory h. In this note we give a criterion for strong convergence of the spectral sequences (Theorem 5) and prove that the spectral sequences are strongly convergent under some finiteness assumption on h (Theorem 6).

In § 1 we study some basic results (Theorems 1 and 2) on inverse limit functor and its derived functor. In § 2 we construct a certain five term exact sequence (Theorem 3) and discuss convergence conditions of the spectral sequences.

1. Inverse Limit Functor

1.1. Let I be a partially ordered set. As in [2], we associate with I a semi-simplicial complex $I_*=\{I_n\}_{n\geqq 0}$ equipped with the following structure: an n-simplex is a sequence

$$\sigma=\{\alpha_0, \alpha_1, \cdots, \alpha_n\}, \alpha_0\leqq\alpha_1\leqq\cdots\leqq\alpha_n;$$

i-th faces $F_i\sigma$ and i-th degeneracies $D_i\sigma$, $0\leqq i\leqq n$, of n-simplex σ are defined by

$$F_i\sigma=\{\alpha_0, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_n\}$$

Received May 19, 1972.
Communicated by N. Shimada.
* Department of Mathematics, Osaka City University, Osaka, Japan.

and

$$D_i\sigma=\{\alpha_0, \cdots, \alpha_i, \alpha_i, \cdots, \alpha_n\}.$$

Let I'_n denote the set of all non-degenerate n-simplexes of I, i.e.,

$$I'_n=\{\sigma=\{\alpha_0, \alpha_1, \cdots, \alpha_n\}; \alpha_0<\alpha_1<\cdots>\alpha_n\}.$$

1.2. Let Λ be a ring and $\mathscr{A}=\{A_\alpha, g_\alpha^\beta\}$ an inverse system of (left) Λ-modules and Λ-homomorphisms indexed by I.

We define n-cochain groups $C^n(I; \mathscr{A})$ for $n\geqq 0$ by

$$C^n(I; \mathscr{A})=\prod_{\sigma\in I'_n} A_\sigma, \tag{1.1}$$

where $A_\sigma=A_{\alpha_0}$ and α_0 is the leading vertex of σ for each $\sigma=\{\alpha_0, \cdots, \alpha_n\}\in I'_n$, and coboundary homomorphisms

$$\delta^{n-1}: C^{n-1}(I; \mathscr{A})\longrightarrow C^n(I; \mathscr{A}) \qquad \text{for } n\geqq 1$$

by

$$p_\sigma\delta^{n-1}=\sum_{i=0}^{n}(-1)^i\psi_{i,\sigma}p_{F_i\sigma} \tag{1.2}$$

for each $\sigma\in I'_n$ where $\psi_{i,\sigma}: A_{F_i\sigma}\to A_\sigma$, $0\leqq i\leqq n$, are defined by

$$\psi_{0,\sigma}=g_{\alpha_0}^{\alpha_1} \text{ and } \psi_{i,\sigma}=id \qquad \text{for } 1\leqq i\leqq n, \tag{1.3}$$

and $p_\tau: C^m(I; \mathscr{A})\to A_\tau$ is the projection for each $\tau\in I'_m$. Then we obtain a cochain complex $\{C^*(I; \mathscr{A}), \delta^*\}$ and

$$\varprojlim \mathscr{A}=\varprojlim_\alpha A_\alpha=H^0(C^*(I: \mathscr{A}), \delta^*).$$

The "n-th derived functor" $\varprojlim^n$, $n\geqq 1$, of inverse limit functor $\varprojlim$ are defined by

$$\varprojlim{}^n \mathscr{A}=\varprojlim_\alpha{}^n A_\alpha=H^n(C^*(I; \mathscr{A}), \delta^*) \tag{1.4}$$

(see [7], [8] and also [2]).

Let $\mathscr{A}=\{A_\alpha, g_\alpha^\beta\}$ be an inverse system indexed by I. For each

$\alpha \in I$ Λ-modules $\bar{A}_\alpha$ and A'_α are defined by

(1.5) $$\bar{A}_\alpha = \prod_{\gamma \leqq \alpha} A_\gamma \quad \text{and} \quad A'_\alpha = \prod_{\gamma' < \alpha} A_{\gamma'}$$

and for each $\alpha < \beta$ Λ-homomorphisms

(1.6) $$\bar{g}^\beta_\alpha : \bar{A}_\beta \to \bar{A}_\alpha \quad \text{and} \quad g'^\beta_\alpha : A'_\beta \to A'_\alpha$$

by

$$p_\gamma \bar{g}^\beta_\alpha = p_\gamma \qquad \text{for } \gamma \leqq \alpha < \beta$$

and

$$p_{\gamma'} g'^\beta_\alpha = p_{\gamma'} - g^\alpha_{\gamma'} p_\alpha \qquad \text{for } \gamma' < \alpha < \beta$$

where p_ε is the projection onto the ε-factor A_ε for each $\varepsilon \in I$. We can easily prove that

$$\bar{\mathscr{A}} = \{\bar{A}_\alpha, \bar{g}^\beta_\alpha\} \quad \text{and} \quad \mathscr{A}' = \{A'_\alpha, g'^\beta_\alpha\}$$

are inverse systems indexed by I.

Moreover for each $\alpha \in I$ we define Λ-homomorphisms

$$\mu_\alpha : A_\alpha \to \bar{A}_\alpha \quad \text{and} \quad \nu_\alpha : \bar{A}_\alpha \to A'_\alpha$$

by

$$p_\gamma \mu_\alpha = g^\alpha_\gamma \qquad \text{for } \gamma \leqq \alpha$$

and

$$p_{\gamma'} \nu_\alpha = p_{\gamma'} - g^\alpha_{\gamma'} p_\alpha \qquad \text{for } \gamma' < \alpha.$$

By a routine computation we see that

$$\mu = \{\mu_\alpha\} : \mathscr{A} \to \bar{\mathscr{A}} \quad \text{and} \quad \nu = \{\nu_\alpha\} : \bar{\mathscr{A}} \to \mathscr{A}'$$

are morphisms of inverse systems. Hence we get an exact sequence of inverse systems

(1.7) $$0 \to \{A_\alpha, g^\beta_\alpha\} \xrightarrow{\mu} \{\bar{A}_\alpha, \bar{g}^\beta_\alpha\} \xrightarrow{\nu} \{A'_\alpha, g'^\beta_\alpha\} \to 0.$$

The following proposition is essentially contained in Nöbeling [7].

Proposition 1. $\varprojlim^p \bar{A}_\alpha = 0$ *for* $p \geqq 1$.

Proof. For each $\gamma \in I$ we define an inverse system ${}_\gamma\bar{\mathscr{A}} = \{{}_\gamma A_\alpha, {}_\gamma\sigma_\alpha^\beta\}$ as follows:

$$ {}_\gamma A_\alpha = \begin{cases} A_\gamma & \gamma \leqq \alpha \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad {}_\gamma\sigma_\alpha^\beta = \begin{cases} id & \gamma \leqq \alpha \leqq \beta \\ 0 & \text{otherwise} \end{cases}. $$

Then we define a cochain contraction ${}_\gamma s^* = \{{}_\gamma s^n\colon C^n(I; {}_\gamma\bar{\mathscr{A}}) \to C^{n-1}(I; {}_\gamma\bar{\mathscr{A}})\}$ for each $\gamma \in I$ by

$$ p_\sigma \cdot {}_\gamma s^n = \begin{cases} p_{\sigma(\gamma)} & \gamma < \alpha_0 \\ 0 & \text{otherwise} \end{cases} $$

where $\sigma(\gamma) = \{\gamma, \alpha_0, \cdots, \alpha_{n-1}\} \in I_n$ for each $\sigma = \{\alpha_0, \cdots, \alpha_{n-1}\} \in I'_{n-1}$. Hence we get

$$ \varprojlim^p {}_\gamma\bar{\mathscr{A}} = 0 \qquad \text{for } p \geqq 1. $$

On the other hand we see

$$ \bar{A}_\alpha = \prod_\gamma {}_\gamma A_\alpha \quad \text{and} \quad \bar{g}_\alpha^\beta = \prod_\gamma {}_\gamma\sigma_\alpha^\beta, $$

i.e., $\bar{\mathscr{A}} = \prod_\gamma {}_\gamma\bar{\mathscr{A}}$. Therefore

$$ \varprojlim^p \bar{\mathscr{A}} = \prod_\gamma \varprojlim^p {}_\gamma\bar{\mathscr{A}} = 0 \qquad \text{for } p \geqq 1. \qquad \text{Q.E.D.} $$

Combining Proposition 1 with (1.7) we obtain

Corollary 2. *There are an exact sequence*

$$ 0 \to \varprojlim_\alpha A_\alpha \to \varprojlim_\alpha \bar{A}_\alpha \to \varprojlim_\alpha A'_\alpha \to \varprojlim_\alpha{}^1 A_\alpha \to 0 $$

and isomorphisms

$$ \varprojlim_\alpha{}^p A_\alpha \cong \varprojlim_\alpha{}^{p-1} A'_\alpha \qquad \textit{for } p \geqq 2. $$

1.3. Let I and J be partially ordered sets and $\mathscr{A} = \{A_{\alpha,\beta}\}$ an inverse system indexed by $I\times J$. In the preceding subsection we defined the cochain complex $\{C^*(I\times J;\mathscr{A}),\delta^*\}$. Here we construct another cochain complex $\{\bar{C}^*(I\times J;\mathscr{A}),d^*\}$ for the inverse system $\mathscr{A}$ over the double index $I\times J$.

Let $\bar{C}^{p,q}(I\times J;\mathscr{A})$, $p,q\geqq 0$, be Λ-modules defined by

$$\bar{C}^{p,q}(I\times J;\mathscr{A})=\prod_{\sigma\in I'_p,\,\tau\in J'_q}A_{\sigma,\tau} \tag{1.8}$$

where $A_{\sigma,\tau}=A_{\alpha_0,\beta_0}$ and α_0 and β_0 are leading vertecies of σ and τ, and

$$d_1^{p-1,q}:\ \bar{C}^{p-1,q}(I\times J;\mathscr{A})\to\bar{C}^{p,q}(I\times J;\mathscr{A})$$

and

$$d_2^{p,q-1}:\ \bar{C}^{p,q-1}(I\times J;\mathscr{A})\to\bar{C}^{p,q}(I\times J;\mathscr{A})$$

be Λ-homomorphisms defined by

$$p_{\sigma,\tau}d_1^{p-1,q}=\sum_{i=0}^{p}(-1)^i\psi'_{i,(\sigma,\tau)}p_{F_i\sigma,\tau}$$

and

$$p_{\sigma,\tau}d_2^{p,q-1}=\sum_{j=0}^{q}(-1)^{p+j}\psi''_{j,(\sigma,\tau)}p_{\sigma,F_j\tau}$$

for $\sigma\in I'_p$ and $\tau\in J'_q$ where $\psi'_{i,(\sigma,\tau)}:\ A_{F_i\sigma,\tau}\to A_{\sigma,\tau}$ and $\psi''_{j,(\sigma,\tau)}:A_{\sigma,F_j\tau}\to A_{\sigma,\tau}$ are defined like (1.3). Then $\{\bar{C}^{*,*}(I\times J;\mathscr{A}),d_1,d_2\}$ is a double complex and the associated cochain complex $\{\bar{C}^*(I\times J;\mathscr{A}),d^*\}$ with the total differential d^* is given by

$$\bar{C}^n(I\times J;\mathscr{A})=\prod_{\sigma\in I'_p,\tau\in J'_{n-p}}A_{\sigma,\tau}\quad\text{and}\quad d^*=d_1+d_2. \tag{1.9}$$

We have a cochain map $\rho=\{\rho^n\}_{n\geqq 0}$ with

$$\rho^n:\ \bar{C}^n(I\times J;\mathscr{A})\to C^n(I\times J;\mathscr{A})$$

defined by

$$p_{\sigma\times\tau}\rho^n=\sum_{j=0}^{n}g^{(\alpha_0,\beta_j)}_{(\alpha_0,\beta_0)}p_{F_{j+1}\cdots F_n\sigma,F_0\cdots F_{j-1}\tau}$$

for $\sigma=(\alpha_0, \cdots, \alpha_n)\in I_n$ and $\tau=(\beta_0, \cdots, \beta_n)\in J_n$ such that $\sigma\times\tau\in(I\times J)'_n$. In fact, putting $F_0^j=F_0\cdots F_{j-1}$ and $F_n^j=F_{j+1}\cdots F_n$ we have

$$F_0^jF_i=F_{i-j}F_0^j,\ F_{n-1}^jF_i=F_n^j \qquad \text{for } 0\leqq j\leqq i\leqq n$$

and

$$F_0^jF_i=F_0^{j+1},\ F_{n-1}^jF_i=F_iF_n^{j+1} \qquad \text{for } 0\leqq i\leqq j\leqq n-1.$$

Using these relations we see easily that

$$p_{\sigma\times\tau}\delta^{n-1}\rho^{n-1}=p_{\sigma\times\tau}\rho^n d^{n-1}$$

for each $\sigma\times\tau\in(I\times J)'_n$.

Lemma 3. *The cochain map ρ induces isomorphisms*

$$H^n(\rho):\ H^n(\bar{C}^*(I\times J;\mathscr{A}),\ d^*)\cong \varprojlim{}^n\,\mathscr{A}$$

for all $n\geqq 0$.

Proof. In Proposition 1 we proved

$$\varprojlim{}^n\,\bar{\mathscr{A}}=0 \qquad \text{for } n>0.$$

Similarly we can show

$$H^n(\bar{C}^*(I\times J;\bar{\mathscr{A}}))=0 \qquad \text{for } n>0.$$

Indeed we define a cochain contraction ${}_{(\gamma,\varepsilon)}\bar{s}^*=\{{}_{(\gamma,\varepsilon)}\bar{s}^n\colon \bar{C}^n(I\times J;{}_{(\gamma,\varepsilon)}\bar{\mathscr{A}})\to\bar{C}^{n-1}(I\times J;{}_{(\gamma,\varepsilon)}\bar{\mathscr{A}})\}$ for each $(\gamma,\varepsilon)\in I\times J$ by

$$p_{\sigma,\tau}\cdot{}_{(\gamma,\varepsilon)}\bar{s}^n=\begin{cases} p_{\sigma(\gamma),\tau} & \gamma<\alpha_0\\ 0 & \text{otherwise}\end{cases}$$

for $\sigma\in I'_p$ and $\tau\in J'_{n-p-1}$. Hence (1.7) induces a commutative diagram

$$\begin{array}{ccccccc} H^n(\bar{C}^*(I\times J;\bar{\mathscr{A}})) & \to & H^n(\bar{C}^*(I\times J;\mathscr{A}')) & \to & H^{n+1}(\bar{C}^*(I\times J;\mathscr{A})) & \to & 0\\ \downarrow & & \downarrow & & \downarrow & & \\ \varprojlim{}^n\,\bar{\mathscr{A}} & \longrightarrow & \varprojlim{}^n\,\mathscr{A}' & \longrightarrow & \varprojlim{}^{n+1}\mathscr{A} & \longrightarrow & 0\end{array}$$

for $n\geqq 0$ in which rows are exact. It is obvious that

$$H^0(\rho):\ H^0(\bar{C}^*(I\times J;\mathscr{A}))\cong \varprojlim \mathscr{A}.$$

Applying "five lemma" in the above diagram we obtain

$$H^n(\bar{C}^*(I\times J;\mathscr{A}))\cong \varprojlim{}^n \mathscr{A} \qquad \text{for } n\geqq 0$$

by an induction on n. Q.E.D.

By the above Lemma 3 and standard arguments about the spectral sequence associated with a double complex we get the following Theorem, which is originally given by Roos [8].

Theorem 1 (Roos). *Let I and J be partially ordered sets and $\mathscr{A}=\{A_{\alpha,\beta}\}$ be an inverse system of Λ-modules indexed by $I\times J$. There exist two strongly convergent spectral sequences $\{E_r\}$ and $\{\bar{E}_r\}$ associated with $\varprojlim^*_{\alpha,\beta} A_{\alpha,\beta}$ by suitable filtrations such that*

$$E_2^{p,q}=\varprojlim_{\alpha}{}^p \varprojlim_{\beta}{}^q A_{\alpha,\beta} \quad \textit{and} \quad \bar{E}_2^{p,q}=\varprojlim_{\beta}{}^p \varprojlim_{\alpha}{}^q A_{\alpha,\beta}.$$

1.4. Here we shall restrict our interest to the category of inverse systems of *compact Hausdorff abelian groups* and *continuous homomorphisms* indexed by I. Further we suppose that the index set I is *directed.*

Proposition 4. *The inverse limit functor on the category of inverse systems of compact Hausdorff abelian groups and continuous homomorphisms indexed by a directed set I is an exact functor*. (Cf., [3], p. 523).

Proof. Let $0\longrightarrow\{A_\alpha, g_\alpha^\beta\}\xrightarrow{\{\varphi_\alpha\}}\{B_\alpha, h_\alpha^\beta\}\xrightarrow{\{\psi_\alpha\}}\{C_\alpha, f_\alpha^\beta\}\longrightarrow 0$ be an exact sequence of inverse systems. Since we have an exact sequence

$$0\to\varprojlim_\alpha A_\alpha\xrightarrow{\varphi}\varprojlim_\alpha B_\alpha\xrightarrow{\psi}\varprojlim_\alpha C_\alpha$$

it is sufficient to show that

$$\psi:\ \varprojlim_\alpha B_\alpha\longrightarrow\varprojlim_\alpha C_\alpha$$

is an epimorphism.

Take any $z=\{z_\alpha\}\in\varprojlim_\alpha C_\alpha$, i.e., $f^\beta_\alpha z_\beta=z_\alpha$. Putting $E_\alpha=\psi_\alpha^{-1}z_\alpha$ for each $\alpha\in I$, E_α is a nonvacuous compact Hausdorff subspace of B_α and $h^\beta_\alpha E_\beta\subset E_\alpha$ for $\beta\geqq\alpha$. Hence $\{E_\alpha, h^\beta_\alpha\}$ is an inverse system of nonvacuous compact Hausdorff spaces. According to [5], Theorem 3.6 of VIII, $\varprojlim_\alpha E_\alpha$ is also nonvacuous. Thus there exists $y=\{y_\alpha\}\in\varprojlim_\alpha E_\alpha\subset\varprojlim_\alpha B_\alpha$ such that $\psi(y)=z$. This implies that ψ is an epimorphism. Q.E.D.

Theorem 2. *Let $\mathscr{A}=\{A_\alpha, g^\beta_\alpha\}$ be an inverse system of compact Hausdorff abelian groups and continuous homomorphisms indexed by a directed set I. Then*

$$\varprojlim_\alpha{}^p A_\alpha=0 \qquad \textit{for } p\geqq 1.$$

Proof. In the exact sequence

$$0\to\{A_\alpha, g^\beta_\alpha\}\to\{\bar{A}_\alpha, \bar{g}^\beta_\alpha\}\to\{A'_\alpha, g'^\beta_\alpha\}\to 0$$

of inverse systems given in (1.7), A_α, $\bar{A}_\alpha$ and A'_α are compact Hausdorff and g^β_α, $\bar{g}^\beta_\alpha$ and g'^β_α are continuous. By Corollary 2 and Proposition 4 we see

$$\varprojlim_\alpha{}^1 A_\alpha=0 \quad\text{and}\quad \varprojlim_\alpha{}^{p+1} A_\alpha\simeq\varprojlim_\alpha{}^p A'_\alpha \qquad \text{for } p\geqq 1.$$

Therefore we obtain

$$\varprojlim_\alpha{}^p A_\alpha=0 \qquad \text{for } p\geqq 1$$

by an induction on p. Q.E.D.

As an immediate corollary of the above Theorem we have

Corollary 5. *Let $\mathscr{A}=\{A_\alpha\}$ be inverse system of finite abelian groups indexed by a directed set I. Then*

$$\varprojlim_\alpha{}^p A_\alpha=0 \qquad \textit{for } p\geqq 1.$$

2. Convergence Conditions of Spectral Sequences

2.1. Let h be a (general reduced) cohomology theory defined on the category of based CW-complexes and X be a based CW-complex with an increasing filtration $\{X_p\}_{p\geqq 0}$, $X=X_\infty=\cup X_p$, by subcomplexes. We define a decreasing filtration of $h^n(X)$ by

$$F^p h^n(X)=\operatorname{Ker}\{h^n(X)\to h^n(X_{p-1})\} \qquad \text{for } p\geqq 0.$$

According to [4] we obtain the spectral sequence $\{E_r\}_{r\geqq 1}$ of h associated with the filtration $\{X_p\}$ of X such that

$$E_1^{p,q}=h^{p+q}(X_p/X_{p-1}) \quad \text{and} \quad E_\infty^{p,q}\cong F^p h^{p+q}(X)/F^{p+1}h^{p+q}(X)$$

(see [2], §3).

In this case there is an exact sequence

$$0\to E_\infty^{p,q}\to E_r^{p,q}\to Z_r^{p,q}/Z_\infty^{p,q}\to 0$$

for each $r>p$ as $B_{p+1}^{p,q}=\cdots=B_\infty^{p,q}$. For each p, q this yields an exact sequence

$$(2.1) \qquad 0\to E_\infty^{p,q}\to \varprojlim_{r>p} E_r^{p,q}\to \varprojlim_r (Z_r^{p,q}/Z_\infty^{p,q})\to 0$$

and an isomorphism

$$(2.2) \qquad \varprojlim_{r>p}{}^1 E_r^{p,q}\cong \varprojlim_r{}^1 (Z_\infty^{p,q}/Z_\infty^{p,q}).$$

We define groups $W_r^{p,n}$ by

$$(2.3) \qquad W_r^{p,n}=\operatorname{Im}\{h^n(X_{p+r-1}/X_{p-1})\to h^{n+1}(X/X_{p+r-1})\}$$

for each r, $0\leqq r<\infty$. Obviously we have

$$(2.4) \qquad W_r^{p,n}\supset W_{r-1}^{p+1,n}\supset\cdots\supset W_0^{p+r,n}=0.$$

Consider the following commutative diagram

$$\begin{array}{ccccc}
 & & & & h^n(X_{p+r-1}/X_p) \\
 & & & \swarrow & \downarrow \psi' \\
 & & h^n(X_{p+r-1}/X_{p-1}) & \xrightarrow{\psi} & h^{n+1}(X/X_{p+r-1}) \\
 & \nearrow & \varphi \downarrow & \overset{\eta}{\searrow} & \downarrow \psi'' \\
h^n(X/X_{p-1}) & \xrightarrow[\varphi']{} & h^n(X_p/X_{p-1}) & \xrightarrow[\varphi'']{} & h^{n+1}(X/X_p)
\end{array}$$

in which the bottom row and the right column are exact and $\mathrm{Im}\,\psi = W_r^{p,n}$, $\mathrm{Im}\,\psi' = W_{r-1}^{p+1,n}$, $\mathrm{Im}\,\varphi = Z_r^{p,n-p}$ and $\mathrm{Im}\varphi' = Z_\infty^{p,n-p}$. By chasing the above diagram we get

$$Z_r^{p,n-p}/Z_\infty^{p,n-p} \cong W_r^{p,n}/W_{r-1}^{p+1,n}. \tag{2.5}$$

2.2. Let h be a cohomology theory and X a based CW-complex with a filtration $\{X_p\}_{p\geqq 0}$. We suppose that h is *additive* [6], i.e., h^n satisfies the wedge axiom for each degree n. Then Milnor [6] established a short exact sequence

$$0 \to \varprojlim_p{}^1 h^{n-1}(X_p) \to h^n(X) \to \varprojlim_p h^n(X_p) \to 0 \tag{2.6}$$

for each n which is important in the present discussions.

By the definition (2.3) of $W_r^{p,n}$ we have an exact sequence

$$\begin{aligned} 0 \to W_r^{p,n} \to h^{n+1}(X/X_{p+r-1}) \to h^{n+1}(X/X_{p-1}) \\ \to h^{n+1}(X_{p+r-1}/X_{p-1}) \to W_r^{p,n+1} \to 0. \end{aligned}$$

Since Milnor's exact sequence implies

$$\varprojlim_r h^{n+1}(X/X_{p+r-1}) = 0,$$

and obviously $\varprojlim_r{}^1 h^n(X/X_{p-1}) = 0$, we see

$$\varprojlim_r W_r^{p,n} = 0 \quad \text{and} \quad \varprojlim_r{}^1 W_r^{p,n} \cong \varprojlim_r{}^1 h^n(X_{p+r-1}/X_{p-1}) \tag{2.7}$$

(replacing $n+1$ by n in the above exact sequence). Then we obtain an exact sequence

$$0 \to \varprojlim_r (W_r^{p,n}/W_{r-1}^{p+1,n}) \to \varprojlim_r{}^1 h^n(X_{p+r-1}/X_p) \tag{2.8}$$

$$\to \varprojlim_r{}^1 h^n(X_{p+r-1}/X_{p-1}) \to \varprojlim_r{}^1 (W_r^{p,n}/W_{r-1}^{p+1,n}) \to 0$$

from the exact sequence

$$0 \to W_{r-1}^{p+1,n} \to W_r^{p,n} \to W_r^{p,n}/W_{r-1}^{p+1,n} \to 0.$$

By the aid of (2.1), (2.2), (2.5) and (2.8) we get a five term exact sequence as follows.

Theorem 3. *Let h be an additive cohomology theory and X be a based* CW*-complex with an increasing filtration $\{X_p\}_{p\geqq 0}$, $X=\cup X_p$, by subcomplexes. Let $\{E_r\}$ be the spectral sequence of h associated with the filtration $\{X_p\}$ of X. There exist exact sequences*

$$0 \to E_\infty^{p,q} \to \varprojlim_{r>p} E_r^{p,q} \to \varprojlim_r{}^1 h^{p+q}(X_{p+r-1}/X_p)$$

$$\to \varprojlim_r{}^1 h^{p+q}(X_{p+r-1}/X_{p-1}) \to \varprojlim_{r>p}{}^1 E_r^{p,q} \to 0$$

for all p and q.

Theorem 4. *Under the same situations as in the above Theorem we fix an integer n. The following three conditions are equivalent*:

i) $E_\infty^{p,n-p} \cong \varprojlim_{r>p} E_r^{p,n-p}$ *for* $p \geqq 0$ *and* $\varprojlim_r{}^1 h^n(X_r)=0$,

ii) $\varprojlim_r{}^1 h^n(X_{p+r}/X_{p-1})=0$ *for* $p \geqq 0$,

iii) $\varprojlim_{r>p}{}^1 E_r^{p,n-p}=0$ *for* $p \geqq 0$.

Proof. "ii)$\to$i)" and "ii)$\to$iii)" follow immediately from the above Theorem 3.

i)$\to$ii): In [2] we defined groups $C_r^{p,n-p}$ by

$$C_r^{p,n-p}=\operatorname{Im}\{h^n(X_{p+r-1}) \to h^n(X_p)\}$$

for each r, $1 \leqq r \leqq \infty$. We have an exact sequence

$$0 \to C_r^{p,n-p} \to h^n(X_p) \to h^{n+1}(X_{p+r-1}/X_p) \to h^{n+1}(X_{p+r-1}) \to C_r^{p,n-p+1} \to 0.$$

By the assumption that $\varprojlim_r{}^1 h^n(X_r)=0$ we get

$$(*)\qquad \varprojlim_r{}^1 C_r^{p,n-p}=0 \qquad \text{for } p\geqq 0$$

(replacing $n+1$ by n in the above exact sequence).

Here we have the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \varprojlim_r{}^1 h^n(X_{p+r-1}/X_p) & \to & \varprojlim_r{}^1 h^n(X_{p+r-1}) \\
 & & & & \downarrow & & \downarrow \\
0\to C_\infty^{p,n-p} & \longrightarrow & h^n(X_p) & \longrightarrow & h^{n+1}(X/X_p) & \longrightarrow & h^{n+1}(X) \\
\downarrow & & \| & & \downarrow & & \downarrow \\
0\to \varprojlim_r C_r^{p,n-p} & \to & h^n(X_p) & \to & \varprojlim_r h^{n+1}(X_{p+r-1}/X_p) & \to & \varprojlim_r h^{n+1}(X_{p+r-1}) \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
\end{array}$$

involving Milnor's exact sequences (two columns). The upper row is obviously exact and the lower row is also exact because of (*). The assumptions that $E_\infty^{p,n-p}\cong\varprojlim_{r>p} E_r^{p,n-p}$ for $p\geqq 0$ and $\varprojlim_r{}^1 h^n(X_{p+r-1})=0$ yield that in the above diagram

$$C_\infty^{p,n-p}\cong\varprojlim_r C_r^{p,n-p} \qquad \text{for } p\geqq 0$$

and

$$h^{n+1}(X)\cong\varprojlim_r h^{n+1}(X_{p+r-1}),$$

using Lemma 7, iii) of [2]. With an application of "four lemma" we see

$$h^{n+1}(X/X_p)\cong\varprojlim_r h^{n+1}(X_{p+r-1}/X_p) \qquad \text{for } p\geqq 0,$$

i.e.,

$$\varprojlim_r{}^1 h^n(X_{p+r-1}/X_p)=0 \qquad \text{for } p\geqq 0.$$

iii)→ii): We put

$$A^p_{r,k}=W^{p,n}_r/W^{p+k,n}_{r-k}.$$

For each $p\geqq 0$ $\{A^p_{r,k}\}$ becomes an inverse system indexed by pairs (r, k). By Theorem 1 there exist two spectral sequences $\{\bar{E}_r\}$ and $\{\bar{\bar{E}}_r\}$ associated with $\varprojlim_{r,k}^* A^p_{r,k}$ such that

$$\bar{E}^{s,t}_2=\varprojlim_r{}^s \varprojlim_k{}^t A^p_{r,k} \quad \text{and} \quad \bar{\bar{E}}^{s,t}_2=\varprojlim_k{}^s \varprojlim_r{}^t A^p_{r,k}.$$

Here we calculate the $\bar{E}_2$- and $\bar{\bar{E}}_2$-terms. Remark that

$$\varprojlim_r{}^1 A^p_{r,1}=\varprojlim_r{}^1 (W^{p,n}_r/W^{p+1,n}_{r-1})\cong\varprojlim_{r>p}{}^1 E^{p,n-p}_r=0 \qquad \text{for } p\geqq 0,$$

by (2.2), (2.5) and our assumption iii). From the exact sequence

$$0\rightarrow A^{p+k-1}_{r-k+1,1}\rightarrow A^p_{r,k}\rightarrow A^p_{r,k-1}\rightarrow 0$$

we obtain an epimorphism $\varprojlim_r A^p_{r,k}\rightarrow\varprojlim_r A^p_{r,k-1}$ and an isomorphism $\varprojlim_r{}^1 A^p_{r,k}\cong\varprojlim_r{}^1 A^p_{r,k-1}$. Then by an induction on k we can show that

$$\varprojlim_r{}^1 A^p_{r,k}=0 \qquad \text{for } k\geqq 1$$

and in addition

$$\varprojlim_k{}^1 \varprojlim_r{}^0 A^p_{r,k}=0$$

(see [2], (2.6)). Therefore

$$\bar{\bar{E}}^{s,t}_2=0 \qquad \text{unless } (p, q)=(0, 0)$$

as $\bar{\bar{E}}^{s,t}_2=0$ for $s>1$ or $t>1$ (see [2], (2.4)). Thus

$$\varprojlim_{r,k}{}^m A^p_{r,k}=0 \qquad \text{for } m\geqq 1.$$

On the other hand, $\varprojlim_k A^p_{r,k}\cong W^{p,n}_r$ by (2.4). Hence we get

$$\varprojlim_r{}^1 W^{p,n}_r\cong\bar{E}^{1,0}_2=\bar{E}^{1,0}_\infty\cong\varprojlim_{r,k}{}^1 A^p_{r,k}=0.$$

Then (2.7) implies

$$\varprojlim_r{}^1 h^n(X_{p+r-1}/X_{p-1})=0 \qquad \text{for } p\geqq 0. \qquad \text{Q.E.D.}$$

As a corollary of the above Theorem we obtain

Theorem 5. *Let h be an additive cohomology theory and X be a based CW-complex with an increasing filtration $\{X_p\}_{p\geqq 0}$, $X=\cup X_p$, by subcomplexes. The spectral sequence $\{E_r\}$ of h associated with the filtration $\{X_p\}$ of X is strongly convergent if and only if $\varprojlim_r{}^1 h^n(X_{p+r}/X_{p-1})=0$ for all p and n.*

2.3. A topological abelian group is said to be *profinite* if it is an inverse limit of finite abelian groups with the inverse limit topology [3]. It is a trivial cosequence that

(2.9) *a profinite abelian group is compact Hausdorff.*

We call a cohomology theory h is *(F)-cohomology theory* when $h^n(S^0)$ is a finite abelian group for each degree n. Then $h^n(X)$ is a finite abelian group for any based finite CW-complex.

Let h be an additive (F)-cohomology theory, X a based CW-complex and $\mathfrak{U}(X)=\{X^\lambda\}$ be the set of all finite subcomplexes of X ordered by inclusions. $\mathfrak{U}(X)$ is a directed set. Since Corollary 5 implies $\varprojlim_\lambda{}^p h^n(X^\lambda)=0$ for $p\geqq 1$ we see that

$$h^n(X)\cong \varprojlim_\lambda h^n(X^\lambda) \qquad \text{for each } n,$$

using Corollary 12 of [2]. Thus $h^n(X)$ is a profinite abelian group for each n and hence compact Hausdorff.

Let $f\colon X\to Y$ be a continuous map of based CW-complexes. Since f induces a morphism $\mathfrak{U}(f)\colon \mathfrak{U}(X)\to\mathfrak{U}(Y)$ of partially ordered sets,

(2.10) *$f^*\colon h^n(Y)\to h^n(X)$ is a continuous homomorphism of compact Hausdorff abelian groups.*

Proposition 6. *Let h be an additive (F)-cohomology theory and X a based* CW*-complex. Let $\mathscr{C}=\{X_\alpha\}$ be a direct system of subcomplexes* *of X (by inclusions) with $X=\cup X_\alpha$ over a directed set I. Then*

$$h^n(X)\cong\varprojlim_\alpha h^n(X_\alpha) \quad \textit{and} \quad \varprojlim_\alpha{}^p h^n(X_\alpha)=0 \qquad \textit{for } p\geqq 1.$$

Proof. According to [2] we have a spectral sequence associated with $h^*(X)$ such that

$$E_2^{p,q}=\varprojlim_\alpha{}^p h^q(X_\alpha).$$

Using Theorem 2 and (2.10) we get

$$\varprojlim_\alpha{}^p h^n(X_\alpha)=0 \qquad \text{for } p\geqq 1.$$

Hence our spectral sequence collapses, and then it is strongly convergent by Proposition 9 of [2]. Therefore

$$\varprojlim_\alpha h^n(X_\alpha)=E_2^{0,n}=E_\infty^{0,n}\cong h^n(X). \qquad \text{Q.E.D.}$$

Putting Theorem 5 and Proposition 6 together we obtain the following

Theorem 6. *Let h be an additive (F)-cohomology theory and X be a based* CW*-complex with an increasing filtration $\{X_p\}_{p\geqq 0}$, $X=\cup X_p$, by subcomplexes. The spectral sequence $\{E_r\}$ of h associated with the filtration $\{X_p\}$ is strongly convergent.*

Let $h(\ ;Z_q)$ be the mod q cohomology theory [1] defined by

$$h^n(X;Z_q)=h^{n+2}(X\wedge M_q)$$

where M_q is a co-Moore space of type $(Z_q, 2)$. If h is additive and of finite type, i.e., $h^n(S^0)$ is a finitely generated abelian group for each degree n, then $h(\ ;Z_q)$ is an additive (F)-cohomology theory.

Corollary 7. *Let h be an additive cohomology theory of finite type and X be as in the above Theorem. The spectral sequence $\{E_r\}$ of $h(\ ;Z_q)$ associated with the filtration $\{X_p\}$ is strongly convergent.*

References

[1] Araki, S. and H. Toda, Multiplicative structures in mod q cohomology theories, I, *Osaka J. Math.* **2** (1965), 71-115.

[2] Araki, S. and Z. Yosimura, A spectral sequence associated with a cohomology theory of infinite CW-complexes, *Osaka J. Math.* to appear.

[3] Buhštaber, V.M. and A.S. Miščenko, K-theory in the category of infinite cell complexes, *Math. USSR Izv.* **2** (1968), 515-556, and **3** (1969), 227.

[4] Cartan, H. and S. Eilenberg, *Homological Algebra*, Princeton Univ. Press, 1956.

[5] Eilenberg, S. and N. Steenrod, *Foundation of Algebraic Topology*, Princeton Univ. Press, 1952.

[6] Milnor, J.W., On axiomatic homology theory, *Pacific J. Math.* **12** (1962), 337-341.

[7] Nöbeling, G., Über die derivierten des inversen und des directen Limes einer Modulfamilie, *Topology* **1** (1961), 47-61.

[8] Roos, J.-E., Sur les foncteurs derivés de $\varprojlim$. Applications, *C.R. Acad. Sci. Paris*, **252** (1961), 3702-3704.