Families of p-Divisible Groups with Constant Newton Polygon

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Abstract. Let X be a p-divisible group with constant Newton polygon over a normal Noetherian scheme S. We prove that there exists an isogeny $X \to Y$ such that Y admits a slope filtration. In case S is regular this was proved by N. Katz for dim $S = 1$ and by T. Zink for dim $S \geq 1$.

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Introduction

In this paper we work over base fields, and over base schemes over $\mathbb{F}_p$, i.e. we work entirely in characteristic p. We study p-divisible groups X over a base scheme S (and, colloquially, a p-divisible group over a base scheme of positive dimension will be called a "family of p-divisible groups"), such that the Newton polygon of a fiber X_s is independent of $s \in S$. We call X a p-divisible group with constant Newton polygon.

A p-divisible group over a field has a slope filtration, see [Z1], Corollary 13; for the definition of a slope filtration, see Definition 1.1. Over a base of positive dimension a slope filtration can only exist if the Newton polygon is constant. In Example 4.1 we show that even in this case there are p-divisible groups which do not admit a slope filtration.

The main result of this paper has as a corollary that *for a p-divisible group with constant Newton polygon over a normal base up to isogeny a slope filtration does exist*, see Corollary 2.2.

We have access to this kind of questions by the definition of a *completely slope divisible* p-divisible group, see Definition 1.2, which implies a structure finer than a slope filtration. The main theorem of this paper, Theorem 2.1, says that over a *normal* base this structure on a p-divisible group exists up to isogeny.

In [Z1], Theorem 7, this was shown to be true over a regular base. In 4.2 we show that without the condition “normal” the conclusion of the theorem does not hold.

Here is a motivation for this kind definition and of results:

- A p-divisible group over an algebraically closed field is isogeneous with a p-divisible group which can be defined over a finite field.
- A p-divisible group over an algebraically closed field is completely slope divisible, if and only if it is isomorphic with a direct sum of isoclinic p-divisible groups which can be defined over a finite field, see 1.5.

We see that a completely slope divisible p-divisible group comes “as close as possible” to a constant one, in fact up to extensions of p-divisible groups annihilated by an inseparable extension of the base, and up to monodromy.

From Theorem 2.1 we deduce constancy results which generalize results of Katz [K] and more recently of de Jong and Oort [JO]. In particular we prove, Corollary 3.4 below:

Let R be a Henselian local ring with residue field k. Let h be a natural number. Then there exists a constant c with the following property. Let X and Y be isoclinic p-divisible groups over $S = \operatorname{Spec} R$ whose heights are smaller than h. Let $\psi : X_k \to Y_k$ be a homomorphism. Then $p^c\psi$ lifts to a homomorphism $X \to Y$.

1 Completely slope divisible p-divisible groups

In this section we present basic definitions and methods already used in the introduction.

Let S be a scheme over $\mathbb{F}_p$. Let $\mathrm{Frob} : S \to S$ be the absolute Frobenius morphism. For a scheme G/S we write:

$$G^{(p)} \quad = \quad G \times_{S,\mathrm{Frob}} S.$$

We denote by $\mathrm{Fr} = \mathrm{Fr}_G : G \to G^{(p)}$ the Frobenius morphism relative to S. If G is a finite locally free commutative group scheme we write $\mathrm{Ver} = \mathrm{Ver}_G : G^{(p)} \to G$ for the “Verschiebung”.

Let X be a p-divisible group over S. We denote by $X(n)$ the kernel of the multiplication by $p^n : X \to X$. This is a finite, locally free group scheme which has rank p^{nh} if X is of height h.

Let $s = \operatorname{Spec} k$ the spectrum of a field of characteristic p. Let X be a p-divisible group over s. Let $\lambda \geq 0$ be a rational number. We call X isoclinic of slope λ, if there exists integers $r \geq 0$, $s > 0$ such that $\lambda = r/s$, and a p-divisible group Y over S, which is isogeneous to X such that

$$p^{-r}\,\mathrm{Fr}^s : Y \to Y^{(p^s)}$$

is an isomorphism.

A p-divisible group X over S is called isoclinic of slope λ, if for each point $s \in S$ the group X_s is isoclinic of slope λ.

1.1 Definition. *Let X/S be a p-divisible group over a scheme S. A filtration*

$$0 = X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_m = X$$

consisting of p-divisible groups contained in X is called a slope filtration *of X if there exists rational numbers $\lambda_1, \ldots, \lambda_m$ satisfying $1 \geq \lambda_1 > \ldots > \lambda_m \geq 0$ such that every subquotient X_i/X_{i-1}, $1 < i \leq m$, is isoclinic of slope λ_i.*

A p-divisible group X over a field admits a slope filtration, see [Z1], Corollary 13. The slopes λ_i and the heights of X_i/X_{i-1} depend only on X. The height of X_i/X_{i-1} is called the multiplicity of λ_i.

Over connected base scheme S of positive dimension a slope filtration of X can only exist if the slopes of X_s and their multiplicities are independent of $s \in S$. In this case we say that X is a family of p-divisible groups with constant Newton polygon. Even if the Newton polygon is constant a slope filtration in general does not exist, see Example 4.1 below.

1.2 Definition. *Let $s > 0$ and $r_1, \ldots, r_m$ be integers such that $s \geq r_1 > r_2 > \ldots > r_m \geq 0$. A p-divisible group Y over a scheme S is said to be* completely slope divisible *with respect to these integers if Y has a filtration by p-divisible subgroups:*

$$0 = Y_0 \subset Y_1 \subset \ldots \subset Y_m = Y$$

such that the following properties hold:

- *The quasi-isogenies*

$$p^{-r_i} \operatorname{Fr}^s : Y_i \to Y_i^{(p^s)}$$

  are isogenies for $i = 1, \ldots, m$.

- *The induced morphisms:*

$$p^{-r_i} \operatorname{Fr}^s : Y_i/Y_{i-1} \to (Y_i/Y_{i-1})^{(p^s)}$$

  are isomorphisms.

Note that the last condition implies that Y_i/Y_{i-1} is isoclinic of slope $\lambda_i := r_i/s$. A filtration described in this definition is a slope filtration in the sense of the previous definition.

Remark. Note that we do not require s and r_i to be relatively prime. If Y is as in the definition, and $t \in \mathbb{Z}_{>0}$, it is also completely slope divisible with respect to $t{\cdot}s \geq t{\cdot}r_1 > t{\cdot}r_2 > \ldots > t{\cdot}r_m \geq 0$.

We note that the filtration Y_i of Y is uniquely determined, if it exists. Indeed, consider the isogeny $\Phi = p^{-r_m}\,\mathrm{Fr}^s : Y \to Y^{(p^s)}$. Then Y_m/Y_{m-1} is necessarily the Φ-étale part Y^Φ of Y, see [Z1] respectively 1.6 below. This proves the uniqueness by induction.

We will say that a p-divisible group is completely slope divisible if it is completely slope divisible with respect to some set integers and inequalities $s \geq r_1 > r_2 > \ldots > r_m \geq 0$.

Remark. A p-divisible group Y over a field K is completely slope divisible iff $Y \otimes_K L$ is completely slope divisible for some field $L \supset K$. - Proof. The slope filtration on Y/K exists. We have $\mathrm{Ker}(p^r_Y) \subset \mathrm{Ker}(\mathrm{Fr}^s_Y)$ iff $\mathrm{Ker}(p^r_{Y_L}) \subset \mathrm{Ker}(\mathrm{Fr}^s_{Y_L})$, and the same for equalities. This proves that the conditions in the definition for completely slope divisibility hold over K iff they hold over $L \supset K$.

1.3 Proposition. *Let Y be a completely slope divisible p-divisible group over a perfect scheme S. Then Y is isomorphic to a direct sum of isoclinic and completely slope divisible p-divisible groups.*

Proof. With the notation of Definition 1.2 we set $\Phi = p^{-r_m}\,\mathrm{Fr}^s$. Let $Y(n) = \mathrm{Spec}\,\mathcal{A}(n)$ and let $Y(n)^\Phi = \mathrm{Spec}\,\mathcal{L}(n)$ be the Φ-étale part (see Corollary 1.7 below). Then Φ induces a Frob^s-linear endomorphism Φ^* of $\mathcal{A}(n)$ and $\mathcal{L}(n)$ which is by definition bijective on $\mathcal{L}(n)$ and nilpotent on the quotient $\mathcal{A}(n)/\mathcal{L}(n)$. One verifies (compare [Z1], page 84) that there is a unique Φ^*-equivariant section of the inclusion $\mathcal{L}(n) \to \mathcal{A}(n)$. This shows that Y_m/Y_{m-1} is a direct factor of $Y = Y_m$. The result follows by induction on m. *Q.E.D.*

Although not needed, we give a characterization of completely slope divisible p-divisible groups over a field. If X is a p-divisible group over a field K, and $k \supset K$ is an algebraic closure, then X is completely slope divisible if and only if X_k is completely slope divisible; hence it suffices to give a characterization over an algebraically closed field.

Convention: We will work with the covariant Dieudonné module of a p-divisible group over a perfect field ([Z2], [Me]). We write V, respectively F for Verschiebung, respectively Frobenius on Dieudonné modules. Let K be a perfect field, and let $W(K)$ be its ring of Witt vectors. A Dieudonné module M over K is the Dieudonné module of an isoclinic p-divisible group of slope r/s, iff there exists a $W(K)$-submodule $M' \subset M$ such that M/M' is annihilated by a power of p, and such that $p^{-r}V^s(M') = M'$.

For later use we introduce the p-divisible group $G_{m,n}$ for coprime positive integers m and n. Its Dieudonné module is generated by one element, which is stable under $F^m - V^n$. $G_{m,n}$ is isoclinic of slope $\lambda = m/(m+n)$ (in the terminology of this paper). The height of $G_{m,n}$ is $h = m + n$, and this p-divisible group is completely slope divisible with respect to $h = m + n \geq m$. The group $G_{m,n}$ has dimension m, and its Serre dual has dimension n.

We have $G_{1,0} = \mu_{p^\infty}$, and $G_{0,1} = \underline{\mathbb{Q}_p/\mathbb{Z}_p}$.

1.4 Proposition. *Let k be an algebraically closed field. An isoclinic p-divisible group Y over k is completely slope divisible iff it can be defined over a finite field; i.e. iff there exists a p-divisible group Y' over some $\mathbb{F}_q$ and an isomorphism $Y \cong Y' \otimes_{\mathbb{F}_q} k$.*

Proof. Assume that Y is slope divisible with respect to $s \geq r \geq 0$. Let M be the covariant Dieudonné module of Y. We set $\Phi = p^{-r}V^s$. By assumption this is a semilinear automorphism of M. By a theorem of Dieudonné (see 1.6 below) M has a basis of Φ-invariant vectors. Hence $M = M_0 \otimes_{W(\mathbb{F}_{p^s})} W(k)$ where $M_0 \subset M$ is the subgroup of Φ-invariant vectors. Then M_0 is the Dieudonné module of a p-divisible group over $\mathbb{F}_{p^s}$ such that $Y \cong Y' \otimes_{\mathbb{F}_q} k$.

Conversely assume that Y is isoclinic over a finite field k of slope r/s. Let M be the Dieudonné module of Y. By definition there is a finitely generated $W(k)$-submodule $M' \subset M \otimes \mathbb{Q}$ such that $p^m M' \subset M \subset M'$ for some natural number m, and such that $p^{-r}V^s(M') = M'$. Then $\Phi = p^{-r}V^s$ is an automorphism of the finite set $M'/p^m M'$. Hence some power Φ^t acts trivially on this set. This implies that $\Phi^t(M) = M$. We obtain that $p^{-rt}F^{st}$ induces an automorphism of Y. Therefore Y is completely slope divisible. *Q.E.D.*

1.5 Corollary. *Let Y be a p-divisible group over an algebraically closed field k. This p-divisible group is completely slope divisible iff $Y \cong \oplus Y_i$ such that every Y_i is isoclinic, and can be defined over a finite field.*

Proof. Indeed this follows from 1.3 and 1.4. *Q.E.D.*

1.6 The Φ-étale part.

For further use, we recall a notion explained and used in [Z1], Section 2. This method goes back to Hasse and Witt, see [HW], and to Dieudonné, see [D], Proposition 5 on page 233. It can be formulated and proved for locally free sheaves, and it has a corollary for finite flat group schemes.

Let V be a finite dimensional vector space over a separably closed field k of characteristic p. Let $f : V \to V$ be a Frob^s-linear endomorphism. The set $C_V = \{x \in V \mid f(x) = x\}$ is a vector space over $\mathbb{F}_{p^s}$. Then $V^f = C_V \otimes_{\mathbb{F}_{p^s}} k$ is a subspace of V. The endomorphism f acts as a Frob^s-linear automorphism on V^f and acts nilpotently on the quotient V/V^f. This follows essentially form Dieudonné loc.cit.. Moreover if k is any field of characteristic p we have still unique exact sequence of k-vector spaces

$$0 \to V^f \to V \to V/V^f \to 0$$

such that f acts as a a Frob^s-linear automorphism on V^f and acts nilpotently on the quotient V/V^f.

This can be applied in the following situation: Let S be a scheme over $\mathbb{F}_p$. Let G be a locally free group scheme over S endowed with a homomorphism

$$\Phi : G \to G^{(p^s)}.$$

In case $S = \mathrm{Spec}(K)$, where K is a field we consider the affine algebra A of G. The Φ induces a Frob^s-linear endomorphism $f : A \to A$. The vector subspace A^f inherits the structure of a bigebra. We obtain a finite group scheme $G^{\Phi} = \mathrm{Spec}\, A^f$, which is called the Φ-étale part of G. Moreover we have an exact sequence of group schemes:

$$0 \to G^{\Phi-\mathrm{nil}} \longrightarrow G \longrightarrow G^{\Phi} \to 0.$$

The morphism Φ induces an isomorphism $G^{\Phi} \to (G^{\Phi})^{(p^s)}$, and acts nilpotently on the kernel $G^{\Phi-\mathrm{nil}}$.

Let now S be an arbitrary scheme over $\mathbb{F}_p$. Then we can expect a Φ-étale part only in the case where the rank of G_x^{Φ} is independent of $x \in S$:

1.7 Corollary. *Let $G \to S$ be a finite, locally free group scheme; let $\Phi : G \to G^{(p^t)}$ be a homomorphism. Assume that the function*

$$S \to \mathbb{Z}, \quad \textit{defined by} \quad x \mapsto \mathrm{rank}((G_x)^{\Phi_x})$$

is constant. Then there exists an exact sequence

$$0 \to G^{\Phi-\mathrm{nil}} \longrightarrow G \longrightarrow G^{\Phi} \to 0$$

such that Φ is nilpotent on $G^{\Phi-\mathrm{nil}}$ and an isomorphism on the Φ-étale part G^{Φ}.

The prove is based on another proposition which we use in section 3. Let $\mathcal{M}$ be a finitely generated, locally free $\mathcal{O}_S$-module. Let $t \in \mathbb{Z}_{>0}$

$$f : \mathcal{O}_S \otimes_{\mathrm{Frob}_S^t, S} \mathcal{M} = \mathcal{M}^{(p^t)} \to \mathcal{M}$$

be a morphism of $\mathcal{O}_S$-modules. To every morphism $T \to S$ we associate

$$C_{\mathcal{M}}(T) = \{x \in \Gamma(T, \mathcal{M}_T) \mid f(1 \otimes x) = x\}.$$

1.8 Proposition *(*see [Z1], Proposition 3*). The functor $C_{\mathcal{M}}$ is represented by a scheme that is étale and affine over S. Suppose S to be connected; the scheme $C_{\mathcal{M}}$ is finite over S iff for each geometric point $\eta \to S$ the cardinality of $C_{\mathcal{M}}(\eta)$ is the same.*

Let X be a p-divisible group over a field K. Suppose $\Phi : X \to X^{(p^t)}$ is a homomorphism. Then the Φ-étale part X^{Φ} is the inductive limit of $X(n)^{\Phi}$. This is a p-divisible group.

1.9 Corollary. *Let X be a p-divisible group over S. Assume that for each geometric point $\eta \to S$ the height of the Φ-étale part of X_{η} is the same. Then a p-divisible group X^{Φ} exists and commutes with arbitrary base change. There is an exact sequence of p-divisible groups:*

$$0 \to X^{\Phi-\mathrm{nil}} \to X \to X^{\Phi} \to 0.$$

The following proposition can be deduced from proposition 1.8.

1.10 Corollary. *Assume that $G \to S$ is a finite, locally free group scheme over a connected base scheme S. Let $\Phi : G \xrightarrow{\sim} G^{(q)}$, $q = p^s$ be an isomorphism. Then there exists a finite étale morphism $T \to S$, and a morphism $T \to \mathrm{Spec}(\mathbb{F}_q)$, such that G_T is obtained by base change from a finite group scheme H over $\mathbb{F}_q$:*

$$H \otimes_{\mathrm{Spec}\, \mathbb{F}_q} T \xrightarrow{\sim} G_T.$$

Moreover Φ is induced from the identity on H.

Remark. If S is a scheme over $\bar{\mathbb{F}}_p$ the Corollary says in particular that G_T is obtained by base change from a finite group scheme over $\bar{\mathbb{F}}_p$. In this case we call G_T constant (compare [K], (2.7)). This should not be confused with the étale group scheme associated to a finite Abelian group A. We will discuss "constant" p-divisible groups, see Section 3 below.

2 The main result: slope filtrations

In this section we show:

2.1 Theorem. *Let h be a natural number. Then there exists a natural number $N(h)$ with the following property. Let S be an integral, normal Noetherian scheme. Let X be a p-divisible group over S of height h with constant Newton polygon. Then there is a completely slope divisible p-divisible group Y over S, and an isogeny:*

$$\varphi : X \to Y \quad \textit{over} \quad S \quad \textit{with} \quad \deg(\varphi) \leq N(h).$$

In Example 4.2 we see that the condition "S is normal" is essential. By this theorem we see that a slope filtration exists up to isogeny:

2.2 Corollary. *Let X be a p-divisible group with constant Newton polygon over an integral, normal Noetherian scheme S. There exists an isogeny φ : $X \to Y$, such that Y over S admits a slope filtration.*

Q.E.D.

2.3 Proposition. *Let S be an integral scheme with function field $K = \kappa(S)$. Let X be a p-divisible group over S with constant Newton polygon, such that X_K is completely slope divisible with respect to the integers $s \geq r_1 > r_2 > \ldots > r_m \geq 0$. Then X is completely slope divisible with respect to the same integers.*

Proof. The quasi-isogeny $\Phi = p^{-r_m}\,\mathrm{Fr}^s : X \to X^{(p^s)}$ is an isogeny, because this is true over the general point. Over any geometric point $\eta \to S$ the Φ-étale part of X_η has the same height by constancy of the Newton polygon. Hence the Φ-étale part of X exists by Corollary 1.9. We obtain an exact sequence:

$$0 \to X^{\Phi-\mathrm{nil}} \to X \to X^{\Phi} \to 0.$$

Assuming an induction hypothesis on $X^{\Phi-\mathrm{nil}}$ gives the result. *Q.E.D.*

A basic tool in the following proofs is the moduli scheme of isogenies of degree d of a p-divisible group (compare [RZ] 2.22): Let X be a p-divisible group over a scheme S, and let d be a natural number. Then we define the following functor $\mathcal{M}$ on the category of S-schemes T. A point of $\mathcal{M}(T)$ consists of a p-divisible group Z over T and an isogeny $\alpha : X_T \to Z$ of degree d up to isomorphism. *The functor $\mathcal{M}$ is representable by a projective scheme over S.* Indeed, to each finite, locally free subgroup scheme $G \subset X_T$ there is a unique isogeny α with kernel G. Let n be a natural number such that $p^n \geq d$. Then G is a finite, locally free subgroup scheme on $X(n)_T$. We set $X(n) = \operatorname{Spec}_S \mathcal{A}$. The affine algebra of G is a quotient of the locally free sheaf $\mathcal{A}_T$. Hence we obtain a point of the Grassmannian of $\mathcal{A}$. This proves that $\mathcal{M}$ is representable as a closed subscheme of this Grassmannian.

2.4 LEMMA. *For every $h \in \mathbb{Z}_{>0}$ there exists a number $N(h) \in \mathbb{Z}$ with the following property. Let S be an integral Noetherian scheme. Let X be a p-divisible group of height h over S with constant Newton polygon. There is a non-empty open subset $U \subset S$, and a projective morphism $\pi : S^{\sim} \to S$ of integral schemes which induces an isomorphism $\pi : \pi^{-1}(U) \to U$ such that there exist a completely slope divisible p-divisible group Y over $S^{\sim}$, and an isogeny $X_{S^{\sim}} \to Y$, whose degree is bounded by $N(h)$.*

PROOF. Let K be the function field of S. We know by [Z1], Prop. 12, that there is a completely slope divisible p-divisible group Y^0 over K, and an isogeny $\beta^0 : X_K \to Y^0$, whose degree is bounded by a constant which depends only on the height of X. The kernel of this isogeny is a finite group scheme $G^0 \subset X_K(n)$, for some n. Let $\bar{G}$ be the scheme-theoretic image of G^0 in $X(n)$, see EGA, I.9.5.3. Then $\bar{G}$ is flat over some nonempty open set $U \subset S$, and inherits there the structure of a finite, locally free group scheme $G \subset X(n)_U$. We form the p-divisible group $Z = X_U/G$. By construction there are integers $s \geq r_m$, such that

$$p^{-r_m} \operatorname{Fr}^s : Z_K \to Z_K^{(p^s)}$$

is an isogeny, and r_m/s is a smallest slope in the Newton polygon of X. Therefore $\Phi = p^{-r_m} \operatorname{Fr}^s : Z \to Z^{(p^s)}$ is an isogeny too. As in the proof of the last proposition the constancy of the Newton polygon implies that the Φ-étale part Z^{Φ} exists. We obtain an exact sequence of p-divisible groups on U:

$$0 \to Z^{\Phi-\mathrm{nil}} \to Z \to Z^{\Phi} \to 0.$$

By induction we find a non-empty open subset $V \subset U$ and a completely slope divisible p-divisible group Y_{m-1} which is isogeneous to $Z_V^{\Phi-\mathrm{nil}}$. Taking the push-out of the last exact sequence by the isogeny $Z_V^{\Phi-\mathrm{nil}} \to Y_{m-1}$ we find a completely slope divisible p-divisible group Y over V which is isogeneous to X_V.

Let d be the degree of the isogeny $\rho : X_V \to Y$. We consider the moduli scheme $\mathcal{M}$ of isogenies of degree d of X defined above. The isogeny $X_V \to Y$ defines an S-morphism $V \to \mathcal{M}$. The scheme-theoretic image $S^\sim$ of V is an integral scheme, which is projective over S. Moreover the morphism $\pi : S^\sim \to S$ induces an isomorphism $\pi^{-1}(V) \to V$. The closed immersion $S^\sim \to \mathcal{M}$ corresponds to an isogeny $\rho^\sim : X_{S^\sim} \to Y^\sim$ to a p-divisible group $Y^\sim$ on $S^\sim$. Moreover the restriction of $\rho^\sim$ to V is ρ. Since $Y^\sim$ has constant Newton polygon, and since $Y^\sim$ is completely slope divisible in the generic point of $S^\sim$ it is completely slope divisible by Proposition 2.3. *Q.E.D.*

2.5 Lemma. *Let k be an algebraically closed field of characteristic p. Let $s \geq r_1 > r_2 > \ldots > r_m \geq 0$ and $d > 0$ be integers. Let X be a p-divisible group over k. Then there are up to isomorphism only finitely many isogenies $X \to Z$ of degree d to a p-divisible group Z, which is completely slope divisible with respect to $s \geq r_1 > r_2 > \ldots > r_m \geq 0$.*

Proof. It suffices to show this in case also X is completely slope divisible with respect to $s \geq r_1 > r_2 > \ldots > r_m \geq 0$. Then X and Z are a direct product of isoclinic slope divisible groups. Therefore we assume that we are in the isoclinic case $m = 1$.

In this case we consider the contravariant Dieudonné modules M of X, and N of Z. Let σ be the Frobenius on $W(k)$. Then $N \subset M$ is a submodule such that $\operatorname{length} M/N = \log_p d$. By assumption $\Phi = p^{-r_1}F^s$ induces a σ^s-linear automorphism of M respectively N. Let C_N respectively C_M be the invariants of Φ acting of N respectively M. Hence C_N is a $W(\mathbb{F}_{p^s})$-submodule of N such that $W(k) \otimes_{W(\mathbb{F}_{p^s})} C_N = N$ (e.g. [Z2] 6.26). The same holds for M. We see that C_N is a $W(\mathbb{F}_{p^s})$-submodule of C_M, such that $\operatorname{length} C_M/C_N = \log_p d$. Since there are only finitely many such submodules, the assertion follows. *Q.E.D.*

2.6 Lemma. *Let $f : T \to S$ be a proper morphism of schemes such that $f_*\mathcal{O}_T = \mathcal{O}_S$. Let $g : T \to M$ be a morphism of schemes. We assume that for any point $\xi \in S$ the set-theoretic image of the fiber T_ξ by g is a single point in M. Then there is a unique morphism $h : S \to M$ such that $hf = g$.*

Proof. For $\xi \in S$ we set $h(\xi) = f(T_\xi)$. This defines a set-theoretic map $h : S \to M$. If $U \subset M$ is an open neighborhood of $h(\xi)$ then $g^{-1}(U)$ is an open neighborhood of T_ξ. Since f is closed we find an open neighborhood V of ξ with $f^{-1}(V) \subset g^{-1}(U)$. Hence $h(V) \subset U$; we see that h is continuous. Then $h_*\mathcal{O}_S = h_*f_*\mathcal{O}_T = g_*\mathcal{O}_T$. We obtain a morphism of ringed spaces $\mathcal{O}_M \to g_*\mathcal{O}_T = h_*\mathcal{O}_S$. *Q.E.D.*

Theorem 2.1 follows from the following technical variant which is useful if we do not know that the normalization is finite. We will need that later on.

2.7 Proposition. *Let h be a natural number. Then there exists a natural number $N(h)$ with the following property. Let S be an integral Noetherian scheme. Let X be a p-divisible group over S of height h with constant Newton* *polygon. Then there is a finite birational morphism $T \to S$, a completely slope divisible p-divisible group Y over T, and an isogeny:*

$$X_T \to Y$$

over T whose degree is smaller than $N(h)$.

Proof. Consider the proper birational map $\pi : S^\sim \to S$, and the isogeny $\rho : X_{S^\sim} \to Y$ given by Lemma 2.4. Take the Stein factorization $S^\sim \to T \to S$. It is enough to find over T an isogeny to a completely slope divisible p-divisible group. Therefore we assume $S = T$, i.e. $\pi_* \mathcal{O}_{S^\sim} = \mathcal{O}_S$.

Let $\mathcal{M} \to S$ be the moduli scheme of isogenies of X of degree $d =$ degree ρ. We will show that the S-morphism $g : S^\sim \to \mathcal{M}$ defined by Y factors through $S \to \mathcal{M}$.

Let $\xi \in S$. We write $S^\sim_\xi = S^\sim \times_S \operatorname{Spec}(\kappa(\xi))$. By Lemma 2.6 it is suffices to show that the set-theoretic image of $S^\sim_\xi$ by g is a single point of $\mathcal{M}$. Clearly $\mathcal{M}_\xi$ classifies isogenies starting at X_ξ of degree d. Over the algebraic closure $\bar{\xi}$, by Lemma 2.5, there are only finitely many isogenies of $X_{\bar{\xi}} \to Z$ of degree d to a completely slope divisible group (for fixed s and r_i). This shows that the image of $S^\sim_\xi(\bar{\xi}) \to \mathcal{M}_\xi(\bar{\xi})$ is finite. Since $S^\sim_\xi$ is connected, see [EGA], III^1.4.3.1, the image of $S^\sim_\xi$ is a single point of $\mathcal{M}_\xi$.

Hence we have the desired factorization $S \to \mathcal{M}$. It defines an isogeny $X \to Z$ over S. Finally Z is completely slope divisible since it is completely slope divisible in the general point of S, and because its Newton polygon is constant.

Q.E.D. 2.7 & 2.1

3 Constancy results

Let T be a scheme over $\bar{\mathbb{F}}_p$. We study the question if a p-divisible group X over T is constant up to isogeny, i.e. there exists a p-divisible group Y over $\bar{\mathbb{F}}_p$ such that X is isogeneous to $Y \times_{\bar{\mathbb{F}}_p} T$.

3.1 Proposition. *Let S be a Noetherian integral normal scheme over $\bar{\mathbb{F}}_p$. Let K be the function field of S and let $\bar{K}$ be an algebraic closure of K. We denote by $L \subset \bar{K}$ the maximal unramified extension of K with respect to S. Let T be the normalization of S in L.*

Let X be an isoclinic p-divisible group over S. Then there is a p-divisible group X_0 over $\bar{\mathbb{F}}_p$ and an isogeny $X \times_S T \to X_0 \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} T$ such that the degree of this isogeny is smaller than an integer which depends only on the height of X.

Proof: We use Theorem 2.1: there exists an isogeny $\varphi : X \to Y$, where Y over S is completely slope divisible. There are natural numbers r and s, such that

$$\Phi = p^{-r} \operatorname{Fr}^s : Y \to Y^{(p^s)}$$

is an isomorphism. Applying Corollary 1.10 to $Y(n)$ and Φ we obtain finite group schemes $X_0(n)$ over $\mathbb{F}_{p^s}$ and isomorphisms

$$Y(n)_T \cong X_0(n) \times_{\mathbb{F}_{p^s}} T$$

The inductive limit of the group schemes $X_0(n)$ is a p-divisible group X_0 over $\mathbb{F}_{p^s}$. It is isogeneous to X over T. *Q.E.D.*

3.2 Corollary. *Let S and T be as in the proposition. Let $T^{\mathrm{perf}} \to T$ be the perfect hull of T. Let X be a p-divisible group over S with constant Newton polygon. Then there is a p-divisible group X_0 over $\bar{\mathbb{F}}_p$ and an isogeny $X_0 \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} T^{\mathrm{perf}} \to X \times_S T^{\mathrm{perf}}$, whose degree is smaller than an integer which depends only on the height of X.*

Proof. This follows using Proposition 1.3. *Q.E.D.*

Finally we prove constancy results without the normality condition.

3.3 Proposition. *Let R be a strictly Henselian reduced local ring over $\bar{\mathbb{F}}_p$. Let X be an isoclinic p-divisible group over $S = \operatorname{Spec} R$. Then there is a p-divisible group X_0 over $\bar{\mathbb{F}}_p$ and an isogeny $X_0 \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} S \to X$, whose degree is smaller than an integer which depends only on the height of X.*

3.4 Corollary. *To each natural number h there is a natural number c with the following property: Let R be a Henselian reduced local ring over $\mathbb{F}_p$ with residue field k. Let X and Y be isoclinic p-divisible groups over $S = \operatorname{Spec} R$ whose heights are smaller than h. Let $\psi : X_k \to Y_k$ be a homomorphism. Then $p^c\psi$ lifts to a homomorphism $X \to Y$.*

A proof of the proposition, and of the corollary will be given later.

Remark. In case the R considered in the previous proposition, or in the previous corollary, is not reduced, but satisfies all other properties, the conclusions still hold, except that the integer bounding the degree of the isogeny, respectively the integer c, depend on h and on R.

If R is strictly Henselian the corollary follows from Proposition 3.3. Indeed, assume that X and Y are isogeneous to constant p-divisible groups X_0 and Y_0 by isogenies which are bounded by a constant which depends only on h. The corollary follows because:

$$\operatorname{Hom}((X_0)_k, (Y_0)_k) = \operatorname{Hom}((X_0)_R, (Y_0)_R).$$

Conversely the corollary implies the proposition since by Proposition 3.1 over a separably closed field an isoclinic p-divisible group is isogeneous to a constant p-divisible group.

Remark. Assume Corollary 3.4. An isoclinic slope divisible p-divisible group Y over k can be lifted to an isoclinic slope divisible p-divisible group over R. Indeed the étale schemes associated by 1.8 to the affine algebra of $Y(n)$ and the isomorphism $p^{-r}\operatorname{Fr}^s$ lift to R. Hence the categories of isoclinic p-divisible groups up to isogeny over R respectively k are equivalent.

3.5 Lemma. *Consider a commutative diagram of rings over $\mathbb{F}_p$:*

$$\begin{array}{ccc} R & \to & A \\ \downarrow & & \downarrow \\ R_0 & \to & A_0. \end{array}$$

Assume that $R \to R_0$ is a surjection with nilpotent kernel $\mathfrak{a}$, and that $A \to A_0$ is a surjection with nilpotent kernel $\mathfrak{b}$. Moreover let $R \to A$ be a monomorphism. Let X and Y be p-divisible groups over R. Let $\varphi_0 : X_{R_0} \to Y_{R_0}$ be a morphism of the p-divisible groups obtained by base change. Applying base change with respect to $R_0 \to A_0$ we obtain a morphism $\psi_0 : X_{A_0} \to Y_{A_0}$.
If ψ_0 lifts to a morphism $\psi : X_A \to Y_A$, then φ_0 lifts to a morphism $\varphi : X \to Y$.

Proof. By rigidity, liftings of homomorphisms of p-divisible groups are unique. Therefore we may replace R_0 by its image in A_0 and assume that $R_0 \to A_0$ is injective. Then we obtain $\mathfrak{a} = \mathfrak{b} \cap R$.

Let n be a natural number such that $\mathfrak{b}^n = 0$. We argue by induction on n. If $n = 0$, we have $\mathfrak{b} = 0$ and therefore $\mathfrak{a} = 0$. In this case there is nothing to prove. If $n > 0$ we consider the commutative diagram:

$$\begin{array}{ccc} R & \to & A \\ \downarrow & & \downarrow \\ R/(\mathfrak{b}^{n-1} \cap R) & \to & A/\mathfrak{b}^{n-1} \\ \downarrow & & \downarrow \\ R_0 & \to & A_0. \end{array}$$

We apply the induction hypothesis to the lower square. Hence it is enough to show the lemma for the upper square. We assume therefore without loss of generality that $\mathfrak{a}^2 = 0,\ \mathfrak{b}^2 = 0$.

Let D_X and D_Y be the crystals associated to X and Y by Messing [Me]. The values $D_X(R)$ respectively $D_Y(R)$ are finitely generated projective R-modules which are endowed with the Hodge filtration $Fil_X \subset D_X(R)$ respectively $Fil_Y \subset D_Y(R)$. We put on $\mathfrak{a}$ respectively $\mathfrak{b}$ the trivial divided power structure. Then φ_0 induces a map $D_X(R) \to D_Y(R)$. By the criterion of Grothendieck and Messing φ_0 lifts to a homomorphism over R, iff $D(\varphi_0)(Fil_X) \subset Fil_Y$.

Since the construction of the crystal commutes with base change, see [Me], Chapt. IV, 2.4.4, we have canonical isomorphisms:

$$\begin{aligned} D_{X_A}(A) &= A \otimes_R D_X(R), & Fil_{X_A} &= A \otimes_R Fil_X, \\ D_{Y_A}(A) &= A \otimes_R D_Y(R), & Fil_{Y_A} &= A \otimes_R Fil_Y. \end{aligned}$$

Since ψ_0 lifts we have $id_A \otimes D(\varphi)(A \otimes_R Fil_X) \subset A \otimes_R Fil_Y$. Since $R \to A$ is injective this implies $D(\varphi_0)(Fil_X) \subset Fil_Y$. *Q.E.D.*

Proof of Proposition 3.3. We begin with the case where R is an integral domain. By Proposition 2.7 there is a finite ring extension $R \to A$ such that A is contained in the quotient field of R, and such that there is an isogeny

$X_A \to Y$ to a completely slope divisible p-divisible group Y over A. The degree of this isogeny is smaller than a constant which depends only on the height of X. Since A is a product of local rings we may assume without loss of generality that A is local. The ring A is a strictly Henselian local ring, see [EGA] IV 18.5.10, and has therefore no non-trivial finite étale coverings. The argument of the proof of Proposition 3.1 shows that Y is obtained by base change from a p-divisible group X_0 over $\bar{\mathbb{F}}_p$. Therefore we find an isogeny

$$\varphi : (X_0)_A \to X_A$$

Let us denote by k the common residue field of A and R. Then φ induces an isogeny $\varphi : (X_0)_k \to X_k$. The last lemma shows that φ_0 lifts to an isogeny $\hat{\varphi} : (X_0)_{\hat{R}} \to X_{\hat{R}}$ over the completion $\hat{R}$ of R.

We apply the following fact:

Claim. Consider a fiber product of rings:

$$\begin{array}{ccc} R & \to & A_1 \\ \downarrow & & \downarrow \\ A_2 & \to & B. \end{array}$$

Let X and Y be p-divisible groups over R. Let $\psi_i : X_{A_i} \to Y_{A_i}$ for $i = 1, 2$ be two homomorphisms of p-divisible groups which agree over B. Then there is a unique homomorphism $\psi : X \to Y$ which induces ψ_1 and ψ_2.

In our concrete situation we consider the diagram:

$$\begin{array}{ccc} R & \to & A \\ \downarrow & & \downarrow \\ \hat{R} & \to & \hat{A} = \hat{R} \otimes_R A. \end{array}$$

The morphisms $\hat{\varphi}$ and φ agree over $\hat{A}$ because they agree over the residue field k. This proves the case of an integral domain R.

In particular we have shown the Corollary 3.4 in the case where R is a strictly Henselian integral domain. To show the corollary in the reduced case we consider the minimal prime ideals $\mathfrak{p}_1, \ldots \mathfrak{p}_s$ of R. Let $\psi : X_k \to Y_k$ be a homomorphism. Then $p^c\psi$ lifts to a homomorphims over each of the rings $R/\mathfrak{p}_i$, for $i = 1, \ldots, s$. But then we obtain a homomorphism over R using the Claim above. This proves the Corollary 3.4 and hence the Proposition 3.3 in the case where R is reduced and strictly Henselian.

If R is not reduced one applies standard deformation theory to $R \to R_{\mathrm{red}}$, [Z2], 4.47. *Q.E.D.*

Proof of Corollary 3.4. Consider the diagram:

$$\begin{array}{ccc} R & \to & R^{\mathrm{sh}} \\ \downarrow & & \downarrow \\ \hat{R} & \to & \hat{R}^{\mathrm{sh}}. \end{array}$$

The upper index "sh" denotes the strict henselization. Using the fact that the categories of finite étale coverings of R, k, respectively $\hat{R}$ are equivalent it is easy to see that the last diagram is a fiber product. We have already proved that $p^c\psi$ lifts to a homomorphism over R^{sh}. Applying Lemma 3.5 to the following diagram we see that $p^c\psi$ lifts to $\hat{R}$. This is enough to prove the corollary (compare the Claim above).

$$\begin{array}{ccc} R/\mathfrak{m}^n & \rightarrow & R^{\mathrm{sh}}/\mathfrak{m}^n R^{\mathrm{sh}} \\ \downarrow & & \downarrow \\ R/\mathfrak{m} & \rightarrow & R^{\mathrm{sh}}/\mathfrak{m} R^{\mathrm{sh}} \end{array}$$

In this diagram n is a positive integer and $\mathfrak{m}$ is the maximal ideal of R. *Q.E.D.*

3.6 Corollary. *Let R be a strictly Henselian reduced local ring over $\bar{\mathbb{F}}_p$. Let R^{perf} be the perfect hull of R. Let X be a p-divisible group over $S = \operatorname{Spec} R$ with constant Newton polygon. We set $S^{\mathrm{perf}} = \operatorname{Spec}(R^{\mathrm{perf}})$. Then there is a p-divisible group X_0 over $\bar{\mathbb{F}}_p$ and an isogeny $X_0 \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} S^{\mathrm{perf}} \rightarrow X \times_S S^{\mathrm{perf}}$ such that the degree of this isogeny is bounded by an integer which depends only on the height of X.*

Proof: This follows using Proposition 1.3. *Q.E.D.*

4 Examples

In this section we use the p-divisible groups either $Z = G_{1,n}$, with $n \geq 1$, or $Z = G_{m,1}$, with $m \geq 1$ as building blocks for our examples. These have the property to be iso-simple, they are defined over $\mathbb{F}_p$, they contain a unique subgroup scheme $N \subset Z$ isomorphic with α_p, and $Z/N \cong Z$. Indeed, for $Z = G_{1,n}$ we have an exact sequence of sheaves:

$$0 \rightarrow \alpha_p \rightarrow Z \xrightarrow{\mathrm{Fr}} Z \rightarrow 0.$$

For $Z = G_{m,1}$ we have the exact sequence

$$0 \rightarrow \alpha_p \rightarrow Z \xrightarrow{\mathrm{Ver}} Z \rightarrow 0.$$

Moreover every such Z has the following property: if $Z_K \rightarrow Z'$ is an isogeny over some field K, and k an algebraic closed field containing K, then $Z_k \cong Z'_k$.

4.1 Example. *In this example we produce a p-divisible group X with constant Newton polygon over a regular base scheme which does not admit a slope filtration.*

Choose Z_1 and Z_2 as above, with $\operatorname{slope}(Z_1) = \lambda_1 > \lambda_2 = \operatorname{slope}(Z_2)$; e.g. $Z_1 = G_{1,1}$ and $Z_2 = G_{1,2}$. We choose $R = K[t]$, where K is a field. We write $S = \operatorname{Spec}(R)$ and $\mathcal{Z}_i = Z_i \times S$ for $i = 1, 2$. We define

$$(id, t) : \alpha_p \rightarrow \alpha_p \times \alpha_p \cong N_1 \times N_2; \quad \text{this defines} \quad \psi : \alpha_p \times S \rightarrow \mathcal{Z}_1 \times \mathcal{Z}_2.$$

Claim: *$\mathcal{X} := (\mathcal{Z}_1 \times \mathcal{Z}_2)/\psi(\alpha_p \times S)$ is a p-divisible group over S which does not admit a slope filtration.*

Indeed, for the generic point we do have slope filtration, where $X = \mathcal{X} \otimes K(t)$, and $0 \subset X_1 \subset X$ is given by: X_1 is the image of

$$\xi_K : (\mathcal{Z}_1 \otimes K(t) \to (\mathcal{Z}_1 \times \mathcal{Z}_2) \otimes K(t) \to X).$$

However the inclusion ξ_K extends uniquely a homomorphism $\xi : \mathcal{Z}_1 \to \mathcal{X}$, which is not injective at $t \mapsto 0$. This proves the claim.

4.2 Example. *In this example we construct a p-divisible group $\mathcal{X}$ with constant Newton polygon over a base scheme S* which is not normal*, such that there is no isogeny $\phi : \mathcal{X} \to \mathcal{Y}$ to a completely slope divisible p-divisible group. (i.e. we show the condition that S is* normal *in Theorem 2.1 is necessary).*

We start again with the exact sequences over $\mathbb{F}_p$:

$$\begin{array}{lllll} 0 & \to \alpha_p & \to G_{2,1} & \xrightarrow{\mathrm{Ver}} G_{2,1} & \to 0, \\ 0 & \to \alpha_p & \to G_{1,2} & \xrightarrow{\mathrm{Fr}} G_{1,2} & \to 0. \end{array} \tag{1}$$

We fix an algebraically closed field k. We write $T = \mathbb{P}^1_k$, and:

$$Z_1 = G_{2,1} \times_{\mathbb{F}_p} T, \quad Z_2 = G_{1,2} \times_{\mathbb{F}_p} T, \quad Z = Z_1 \times Z_2, \quad A = \alpha_p \times_{\mathbb{F}_p} T.$$

By base change we obtain sequences of sheaves on the projective line $T = \mathbb{P}^1_k$:

$$\begin{array}{lllll} 0 & \to A & \to Z_1 & \xrightarrow{\mathrm{Ver}} Z_1 & \to 0, \\ 0 & \to A & \to Z_2 & \xrightarrow{\mathrm{Fr}} Z_2 & \to 0. \end{array}$$

Lemma. *Consider $Z \to T = \mathbb{P}^1_k$ as above. Let $\beta : Z \to Y$ be an isogeny to a completely slope divisible p-divisible group Y over T. Then $Y = Y_1 \times Y_2$ is a product of two p-divisible groups and $\beta = \beta_1 \times \beta_2$ is the product of two isogenies $\beta_i : Z_i \to Y_i$.*

Proof. The statement is clear if we replace the base T by a perfect field, see Proposition 1.3. In our case we show first that the kernel of the morphisms $Z_i \to Y$, $i = 1, 2$ induced by β are representable by a finite, locally free group schemes G_i. Indeed, let $\mathcal{G}_i$ the kernel in the sense of f.p.p.f sheaves. Let us denote by G the kernel of the isogeny β. Choose a number n such that p^n annihilates G. Then p^n annihilates $\mathcal{G}_i$. Therefore $\mathcal{G}_i$ coincides with the kernel of the morphism of finite group schemes $Z_i(n) \to Y(n)$. Hence $\mathcal{G}_i$ is representable by a finite group scheme G_i. We prove that G_i is locally free. It suffices to verify that the rank of G_i in any geometric point η of T is the same. But we have seen that over η the p-divisible group Y splits into a product $Y_\eta = (Y_\eta)_1 \times (Y_\eta)_2$. This implies that

$$G_\eta = (G_1)_\eta \times (G_2)_\eta.$$

We conclude that the ranks of $(G_i)_\eta$ are independent of η since G is locally free.

We define p-divisible groups $Y_i = Z_i/G_i$. We obtain a homomorphism of p-divisible groups $Y_1 \times Y_2 \to Y$ which is an isomorphism over each geometric point η. Therefore this is an isomorphism. *Q.E.D.*

Next we construct a p-divisible group X on $T = \mathbb{P}^1_k$. Let $\mathcal{L}$ be a line bundle on $\mathbb{P}^1_k$. We consider the associated vector group

$$\mathbf{V}(\mathcal{L})(T') = \Gamma(T', \mathcal{L}'_T),$$

where $T' \to T = \mathbb{P}^1_k$ is a scheme and $\mathcal{L}'_T$ is the pull-back. The kernel of the Frobenius morphism $\mathrm{Fr} : \mathbf{V}(\mathcal{L}) \to \mathbf{V}(\mathcal{L})^{(p)}$ is a finite, locally free group scheme $\alpha_p(\mathcal{L})$ which is locally isomorphic to α_p. We set $A(-1) = \alpha_p(\mathcal{O}_{\mathbb{P}^1_k}(-1))$. There are up to multiplication by an element of k^* unique homomorphisms ι_0 respectively $\iota_\infty : \mathcal{O}_{\mathbb{P}^1_k}(-1) \to \mathcal{O}_{\mathbb{P}^1_k}$ whose unique zeroes are $0 \in \mathbb{P}^1_k$ respectively $\infty \in \mathbb{P}^1_k$. This induces homomorphisms of finite group schemes $\iota_0 : A(-1) \to A$ respectively $\iota_\infty : A(-1) \to A$ which are isomorphisms outside 0 respectively outside ∞. We consider the embeddings

$$A(-1) \quad \stackrel{(\iota_0,\iota_\infty)}{\longrightarrow} \quad A \times A \quad \subset \quad Z_1 \times Z_2 = Z.$$

We define $X = Z/A(-1)$:

$$\psi : Z \longrightarrow Z/A(-1) = X.$$

Note that

$$\psi_0 : G_{2,1} \times G_{1,2} = Z_0 \longrightarrow G_{2,1} \times (G_{1,2}/\alpha_p) = X_0,$$

and

$$\psi_\infty : G_{2,1} \times G_{1,2} = Z_\infty \longrightarrow (G_{2,1}/\alpha_p) \times G_{1,2} = X_\infty.$$

We consider the quotient space $S = \mathbb{P}^1_k/\{0, \infty\}$, by identifying 0 and ∞ into a normal crossing at $P \in S$, i.e.

$$\mathcal{O}_{S,P} = \{f \in \mathcal{O}_{T,0} \cap \mathcal{O}_{T,\infty} \mid f(0) = f(\infty)\};$$

S is a nodal curve, and

$$\mathbb{P}^1_k = T \longrightarrow S, \quad 0 \mapsto P, \quad \infty \mapsto P,$$

is the normalization morphism.

A finite, locally free scheme $\mathcal{G}$ over S is the same thing as a finite, locally free scheme G over $\mathbb{P}^1_k$ endowed with an isomorphism

$$G_0 \cong G_\infty.$$

It follows that the category of p-divisible groups $\mathcal{Y}$ over S is equivalent to the category of pairs $(Y, \gamma_{\mathcal{Y}})$, where Y is a p-divisible group on $\mathbb{P}^1_k$ and $\gamma_{\mathcal{Y}}$ is an isomorphism

$$\gamma_{\mathcal{Y}} : Y_0 \cong Y_\infty$$

of the fibers of Y over $0 \in \mathbb{P}^1_k$ and $\infty \in \mathbb{P}^1_k$. We call $\gamma_{\mathcal{Y}}$ the gluing datum of $\mathcal{Y}$. We construct a p-divisible $\mathcal{X}$ over S by defining a gluing datum on the p-divisible group X. In fact, the exact sequences in (1) give:

$$X_0 = G_{2,1} \times (G_{1,2}/\alpha_p) \quad \cong \quad (G_{2,1}/\alpha_p) \times G_{1,2} = X_\infty;$$

this gluing datum provides a p-divisible group $\mathcal{X}$ over S.

Claim: *This p-divisible group $\mathcal{X} \to S$ satisfies the property mentioned in the example.*

Let us assume that there exists an isogeny $\phi : \mathcal{X} \to \mathcal{Y}$ to a completely slope divisible p-divisible group $\mathcal{Y}$ over S. We set $Y = \mathcal{Y} \times_S T$ and consider the induced isogeny

$$\phi \times_S T = \varphi : X \to Y.$$

and the induced isogeny:

$$\beta = \varphi{\cdot}\psi : (Z \xrightarrow{\psi} X \xrightarrow{\varphi} Y).$$

By the lemma we see that

$$\beta = \beta_1 \times \beta_2 : Z \longrightarrow Y_1 \times Y_2 = Y.$$

Note that

$$\varphi_\infty = \phi_P = \varphi_0 : X_\infty = \mathcal{X}_P = X_0 \quad \longrightarrow \quad Y_\infty = \mathcal{Y}_P = Y_0;$$

these p-divisible groups both have a splitting into isoclinic summands:

$$X_\infty = \mathcal{X}_P = X_0 = X' \times X'', \quad Y_\infty = \mathcal{Y}_P = Y_0 = Y' \times Y'',$$

and

$$\phi_P = \varphi' \times \varphi'' : X' \times X'' \quad \longrightarrow \quad Y' \times Y''$$

is in diagonal form. On the one hand we conclude from

$$((\beta_1)_0 : (Z_1)_0 \longrightarrow (Y_1)_0) = \left((Z_1)_0 \xrightarrow{\sim} X' \to Y'\right)$$

that $\deg(\beta_1) = \deg(\beta_1)_0 = \deg(\varphi')$; on the other hand

$$((\beta_1)_\infty : (Z_1)_\infty \longrightarrow (Y_1)_\infty) = ((Z_1)_\infty \to (G_{2,1}/\alpha_p) = X' \to Y')\,;$$

hence

$$\deg(\beta_1) = \deg(\beta_1)_\infty = p{\cdot} \deg(\varphi').$$

We see that the assumption that the isogeny $\phi : \mathcal{X} \to \mathcal{Y}$ to a completely slope divisible $\mathcal{Y} \to S$ would exist leads to a contradiction. This finishes the description and the proof of Example 4.2. *Q.E.D.*

4.3 EXAMPLE. *For every positive integer d there exists a scheme S' of dimension d, a point $P' \in S'$ such that S' is regular outside P', and a p-divisible group $\mathcal{X}' \to S'$ which does not admit an isogeny to a completely slope divisible group over S'.*

This follows directly form the previous example. Indeed choose T as in the previous example, and let $T' \to T$ smooth and surjective with T' of dimension d. Pull back X/T to X'/T'; choose geometric points $0'$ and $\infty' \in T'$ above 0 and $\infty \in T$; construct S' by "identifying $0'$ and ∞'": outside $P' \in S'$, this scheme is $T' \backslash \{0', \infty'\}$, and the local ring of $P' \in S'$ is the set of pairs of elements in the local rings of $0'$ and ∞' having the same residue value. We can descend $X' \to T'$ to $\mathcal{X}' \to S'$, and this has the desired property.

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