Reconstruction Phases for Hamiltonian Systems on Cotangent Bundles

Anthony D. Blaom

Received: October 11, 2000
Revised: August 5, 2002

Communicated by Bernold Fiedler

Abstract. Reconstruction phases describe the motions experienced by dynamical systems whose symmetry-reduced variables are undergoing periodic motion. A well known example is the non-trivial rotation experienced by a free rigid body after one period of oscillation of the body angular momentum vector.

Here reconstruction phases are derived for a general class of Hamiltonians on a cotangent bundle T^*Q possessing a group of symmetries G, and in particular for mechanical systems. These results are presented as a synthesis of the known special cases $Q = G$ and G Abelian, which are reviewed in detail.

2000 Mathematics Subject Classification: 70H33, 53D20.
Keywords and Phrases: mechanical system with symmetry, geometric phase, dynamic phase, reconstruction phase, Berry phase, cotangent bundle.

Contents

Summary of selected notation

Numbers in parentheses refer to the relevant subsection.

$p_\mu : \mathfrak{g}^* \to \mathfrak{g}_\mu^*$	natural projection (dual of inclusion $\mathfrak{g}_\mu \to \mathfrak{g}$) (2.2)
$\mathrm{pr}_\mu : \mathfrak{g} \to \mathfrak{g}_\mu$	orthogonal projection (2.8)
$(\,\cdot\,)_Q$	associated form (3.2, 7.2)
$i_\mathcal{O} \in \Omega^0(\mathcal{O}, \mathfrak{g}^*)$	inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$ (3.3)
$\rho^\circ : (\ker \mathrm{T}\rho)^\circ \to \mathrm{T}^*(Q/G)$	map sending $d_q(f \circ \rho)$ to $d_{\rho(q)} f$ (4.1)
$\mathbf{A}' : \mathrm{T}^*Q \to \mathbf{J}^{-1}(0)$	projection along $(\ker \mathbf{A})^\circ$ (4.2)
$\mathrm{T}^*_\mathbf{A}\rho : \mathrm{T}^*Q \to \mathrm{T}^*(Q/G)$	Hamiltonian analogue of $\mathrm{T}\rho : \mathrm{T}Q \to \mathrm{T}(Q/G)$ (4.2)
$i_\mu : \mathbf{J}^{-1}(\mu) \hookrightarrow P$	inclusion (5.1)
$E(\mu) \subset \mathrm{T}_\mu \mathfrak{g}^*$	space orthogonal to $\mathrm{T}_\mu(G \cdot \mu)$ (6.1)
$\mathrm{forg}\, E(\mu) \subset \mathfrak{g}^*$	image of $E(\mu)$ under identification $\mathrm{T}_\mu \mathfrak{g}^* \cong \mathfrak{g}^*$ (6.1)
$\iota_\mu : [\mathfrak{g}, \mathfrak{g}_\mu]^\circ \to \mathfrak{g}_\mu^*$	restriction of projection $p_\mu : \mathfrak{g}^* \to \mathfrak{g}_\mu^*$ (6.1)
$\mathrm{id}_{\mathfrak{g}^*} \in \Omega^0(\mathfrak{g}^*, \mathfrak{g}^*)$	the identity map $\mathfrak{g}^* \to \mathfrak{g}^*$ (6.4)

1 Introduction

When the body angular momentum of a free rigid body undergoes one period of oscillation the body itself undergoes some overall rotation in the inertial frame of reference. This rotation is an example of a *reconstruction phase*, a notion one may formulate for an arbitrary dynamical system possessing symmetry, whenever the symmetry-reduced variables are undergoing periodic motion. Interest in reconstruction phases stems from problems as diverse as the control of artificial satellites [8] and wave phenomena [3, 2].

This paper studies reconstruction phases in the context of holonomic mechanical systems, from the Hamiltonian point of view. Our results are quite general in the sense that *non-Abelian* symmetries are included; however certain singularities must be avoided. We focus on so-called *simple* mechanical systems (Hamiltonian='kinetic energy' + 'potential energy') but our results are relevant to other Hamiltonian systems on cotangent bundles T^*Q. The primary prerequisite is invariance of the Hamiltonian with respect to the cotangent lift of a free and proper action on the configuration space Q by the symmetry group G. Our results are deduced as a special case of those in [6].

We do not study phases in the context of mechanical control systems and locomotion generation, as in [17] and [15]; nor do we discuss Hanay-Berry phases for 'moving' mechanical systems (such as Foucault's pendulum), as in [16]. Nevertheless, these problems share many features with those studied here and our results may be relevant to generalizations of the cited works.

1.1 Limiting cases

The free rigid body is a prototype for an important class of simple mechanical systems, namely those for which $Q = G$. Those systems whose symmetry group G is *Abelian* constitute another important class, of which the heavy top is a prototype. Reconstruction phases in these two general classes have been studied before [16], [6]. Our general results are essentially a synthesis of these two cases, but because the synthesis is rather sophisticated, detailed results are formulated after reviewing the special cases in Section 2. This introduction describes the new results informally after pointing out key features of the two prototypes. A detailed outline of the paper appears in 1.5 below.

1.2 The free rigid body

In the free (Euler-Poinsot) rigid body reconstruction phases are given by an elegant formula due to Montgomery [23]. Both the configuration space Q and symmetry group G of the free rigid body can be identified with the rotation group SO(3) (see, e.g., [18, Chapter 15]); here we are viewing the body from an inertial reference frame centered on the mass center. Associated with each state x is a spatial angular momentum $\mathbf{J}(x)$ which is conserved. The *body* representation of angular momentum $\nu \in \mathbb{R}^3$ of a state x with configuration

$q \in \mathrm{SO}(3)$ is

(1) $$\nu = q^{-1}\mathbf{J}(x) \ .$$

The body angular momentum ν evolves according to well known equations of Euler which, in particular, constrain solutions to a sphere $\mathcal{O}$ centered at the origin and having radius $\|\mu_0\|$, where $\mu_0 = \mathbf{J}(x_0)$ is the initial spatial angular momentum. This sphere has a well known interpretation as a co-adjoint orbit of SO(3).

Solutions to Euler's equations are intersections with $\mathcal{O}$ of level sets of the *reduced Hamiltonian* $h : \mathbb{R}^3 \to \mathbb{R}$, given by $h(\nu) \equiv \frac{1}{2}\nu \cdot \mathbb{I}^{-1}\nu$. Here $\mathbb{I}$ denotes the body inertia tensor; see Fig. 1. Typically, a solution $\nu_t \in \mathcal{O}$ is periodic, in

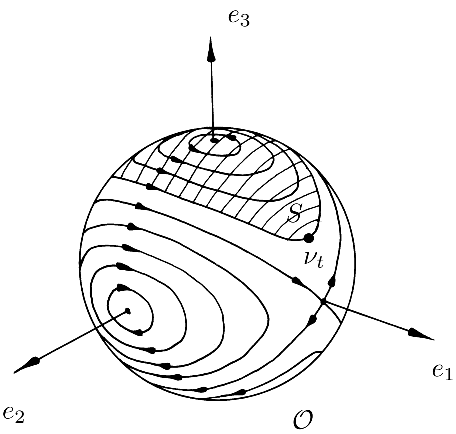

Figure 1: The dynamics of body angular momentum in the free rigid body.

which case (1) implies that $q_T\mu_0 = q_0\mu_0$, where T is the period. This means $q_T = gq_0$ for some rotation $g \in \mathrm{SO}(3)$ about the μ_0-axis. According to [23], the angle $\Delta\theta$ of rotation is given by

(2) $$\Delta\theta = \frac{2Th(\nu_0)}{\|\mu_0\|} - \frac{1}{\|\mu_0\|^2}\int_S dA_{\mathcal{O}} \ ,$$

where $S \subset \mathcal{O}$ denotes the region bounded by the curve ν_t (see figure) and $dA_{\mathcal{O}}$ denotes the standard area form on the sphere $\mathcal{O} \subset \mathbb{R}^3$.

Astonishingly, it seems that (2) was unknown to 19th century mathematicians, a vindication of the 'bundle picture' of mechanics promoted in Montgomery's thesis [22].

1.3 The heavy top

Consider a rigid body free to rotate about a point O fixed to the earth (Fig. 2). The configuration space is $Q \equiv \mathrm{SO}(3)$ but full SO(3) spatial symmetry is broken by gravity (unless O and the center of mass coincide). A residual symmetry group $G \equiv S^1$ acts on Q according to $\theta \cdot q \equiv R^3_\theta q$ ($\theta \in S^1$); here R^3_θ denotes a rotation about the vertical axis e_3 through angle θ.

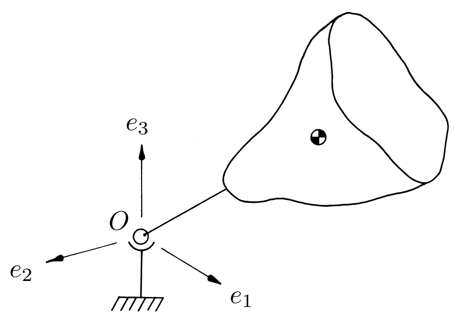

Figure 2: The heavy top.

The quotient space Q/G, known more generally as the *shape space*, is here identifiable with the unit sphere S^2: for a configuration $q \in \mathrm{SO}(3)$ the corresponding 'shape' $r \in S^2$ is the position of the vertical axis viewed in body coordinates:

$$r = q^{-1} e_3 \quad . \tag{1}$$

In place of Euler's rigid body equations one considers the Euler-Poisson heavy top equations [5, (10) & (11), Chapter 1], [18, §15.10]. In the special Lagrange top case these equations are integrable (see, e.g., [4, §30]), but more generally they admit chaotic solutions. In any case, a *periodic* solution to the Euler-Poisson equations determines a periodic solution $r_t \in S^2$ in shape space but the corresponding motion of the body $q_t \in \mathrm{SO}(3)$ need *not* be periodic. However, if T is the period of the given solution to the Euler-Poisson equations, then (1) implies $q_T = R^3_{\Delta\theta} q_0$, for some angle $\Delta\theta$. Assume $r_t \in S^2$ is an embedded curve having T as its *minimal* period. Then

$$\Delta\theta = \int_0^T \frac{dt}{r_t \cdot \mathbb{I} r_t} - \int_S f \, dA_{S^2} \qquad , \tag{2}$$
$$\text{where} \qquad f(r) \equiv \frac{\mathrm{Trace}\, \mathbb{I}}{r \cdot \mathbb{I} r} - \frac{2\mathbb{I}r \cdot \mathbb{I}r}{(r \cdot \mathbb{I}r)^2} \quad .$$

Here $S \subset S^2$ denotes the region bounded by the curve r_t, dA_{S^2} denotes the standard area form on S^2, and $\mathbb{I}$ denotes the inertia tensor, about O, of the body in its reference configuration ($q = \mathrm{id}$). Equation (2) follows, for instance, from results reviewed in 2.6 and 2.7, together with a curvature calculation along the lines of [16, pp. 48–50].

1.4 General characteristics of reconstruction phases

In both 1.2(2) and 1.3(2) the angle $\Delta\theta$ splits into two parts known as the *dynamic* and *geometric phases*. The dynamic phase amounts to a *time* integral involving the inertia tensor.[1] The geometric phase is a *surface* integral, the integrand depending on the inertia tensor in the case of the heavy top but being independent of system parameters in the case of the free rigid body. Apart from this, an important difference is the *space* in which the phase calculations occur. In the heavy top this is shape space (which is just a point in the free rigid body). In the free rigid body one computes on momentum spheres, i.e., on co-adjoint orbits (which are trivial for the symmetry group S^1 of the heavy top).

As we will show, phases in general mechanical systems are computed in 'twisted products' of shape space Q/G and co-adjoint orbits $\mathcal{O}$, and geometric phases have both a 'shape' and 'momentum' contribution. The source of geometric phases is *curvature*. The 'shape' contribution comes from curvature of a connection $\mathbf{A}$ on Q, bundled over shape space Q/G, constructed using the kinetic energy. This is the so-called *mechanical connection*. The 'momentum' contribution to geometric phases comes from curvature of a connection α_{μ_0} on G, bundled over a co-adjoint orbit $\mathcal{O}$, constructed using an Ad-invariant inner product on the Lie algebra $\mathfrak{g}$ of G. We tentatively refer to this as a *momentum connection*. The mechanical connection depends on the Hamiltonian; the momentum connection is a purely Lie-theoretic object . This explains why system parameters appear explicitly in geometric phases for the heavy top but not in the free rigid body.

In arbitrary simple mechanical systems the dynamic phase is a time integral involving the so-called *locked inertia tensor* $\mathbb{I}$. Roughly speaking, this tensor represents the contribution to the kinetic energy metric coming from symmetry variables. In a system of coupled rigid bodies moving freely through space, it is the inertia tensor about the instantaneous mass center of the rigid body obtained by locking all coupling joints [14, §3.3]

1.5 Paper outline

The new results of this paper are Theorems 3.4 and 3.5 (Section 3). These theorems contain formulas for geometric and dynamic phases in general Hamiltonian systems on cotangent bundles, and in particular for simple mechanical systems. These results are derived as a special case of [6], of which Section 2 is mostly a review. Specifically, Section 2 gives the abstract definition of reconstruction phases, presents a phase formula for systems on arbitrary symplectic manifolds, and surveys the special limiting cases relevant to cotangent bundles. The mechanical connection $\mathbf{A}$, the momentum connection α_{μ_0}, and limiting cases of the locked inertia tensor $\mathbb{I}$ are also defined.

[1] In the free rigid body one has $2Th(\nu_0) = 2Th(\nu_t) = \int_0^T h(\nu_t)\, dt = 2\int_0^T \nu_t \cdot \mathbb{I}^{-1}\nu_t\, dt$.

Section 3 begins by showing how the curvatures of $\mathbf{A}$ and α_{μ_0} can be respectively lifted and extended to structures $\Omega_{\mathbf{A}}$ and Ω_{μ_0} on 'twisted products' of shape space Q/G and co-adjoint orbits $\mathcal{O}$. On these products we also introduce the *inverted locked inertia function* $\xi_{\mathbb{I}}$.

The remainder of the paper is devoted to a proof of Theorems 3.4 and 3.5. Sections 4 and 5 review relevant aspects of cotangent bundle reduction, culminating in an intrinsic formula for symplectic structures on leaves of the Poisson-reduced space $(\mathrm{T}^*Q)/G$. Section 6 builds a natural 'connection' on the symplectic stratification of $(\mathrm{T}^*Q)/G$, and Sections 7 and 8 provide the detailed derivations of dynamic and geometric phases. Appendix A describes the covariant exterior calculus of bundle-valued differential forms, from the point of view of associated bundles.

1.6 Connections to other work

Above what is explicitly cited here, our project owes much to [16]. Additionally, we make crucial use of Cendra, Holm, Marsden and Ratiu's description of reduced spaces in mechanical systems as certain fiber bundle products [9].

In independent work, carried out from the Lagrangian point of view, Marsden, Ratiu amd Scheurle [19] obtain reconstruction phases in mechanical systems with a possibly non-Abelian symmetry group by directly solving appropriate reconstruction equations. Rather than identify separate geometric and dynamic phases, however, their formulas express the phase as a single time integral (no surface integral appears). This integral is along an implicitly defined curve in Q, whereas our formula expresses the phase in terms of 'fully reduced' objects.

The author thanks Matthew Perlmutter for helpful discussions and for making a preliminary version of [24] available.

2 Review

In the setting of Hamiltonian systems on a general symplectic manifold P, reconstruction phases can be expressed by an elegant formula involving derivatives of leaf symplectic structures and the reduced Hamiltonian, these derivatives being computed *transverse* to the symplectic leaves of the Poisson-reduced phase space P/G [6]. This formula, recalled in Theorem 2.3 below, grew out of a desire to 'Poisson reduce' the earlier scheme of Marsden et al. [16, §2A], in which geometric phases were identified with holonomy in an appropriate principal bundle equipped with a connection. Familiarity with this holonomy interpretation is not a prerequisite for understanding and applying Theorem 2.3.

We are ultimately concerned with the special case of cotangent bundles $P = \mathrm{T}^*Q$, and in particular with simple mechanical systems, which are introduced in 2.4. After recalling the definition of the mechanical connection $\mathbf{A}$ in 2.5 we recall the formula for phases in the case of G Abelian (Theorem 2.6 & Addendum 2.7). After introducing the momentum connection α_μ in 2.8 we write down phase formulas for the other limiting case, $Q = G$ (Theorem & Addendum 2.9).

2.1 An abstract setting for reconstruction phases

Assume G is a connected Lie group acting symplectically from the left on a smooth (C^∞) symplectic manifold (P, ω), and assume the existence of an Ad^*-equivariant momentum map $\mathbf{J} : P \to \mathfrak{g}^*$. (For relevant background, see [14, 1, 18].) Here $\mathfrak{g}$ denotes the Lie algebra of G. Assume G acts freely and properly, and that the fibers of $\mathbf{J}$ are connected. All these hypotheses hold in the case $P = \mathrm{T}^*Q$ when we take G to act by cotangent-lifting a free and proper action on Q and assume Q is connected; details will be recalled in Section 3.
In general, P/G is not a symplectic manifold but merely a Poisson manifold, i.e., a space stratified by lower dimensional symplectic manifolds called *symplectic leaves*; see opi cited. In the free rigid body, for example, one has $P = \mathrm{T}^* \mathrm{SO}(3)$, $G = \mathrm{SO}(3)$, and $P/G \cong \mathfrak{so}(3)^* \cong \mathbb{R}^3$. The symplectic leaves are the co-adjoint orbits, i.e., the spheres centered on the origin.
Let x_t denote an integral curve of the Hamiltonian vector field X_H on P corresponding to some G-invariant Hamiltonian H. Restrict attention to the case that the image curve y_t under the projection $\pi : P \to P/G$ is T-periodic ($T > 0$). Then the associated *reconstruction phase* is the unique $g_{\mathrm{rec}} \in G$ such that $x_T = g_{\mathrm{rec}} \cdot x_0$; see Fig. 3.

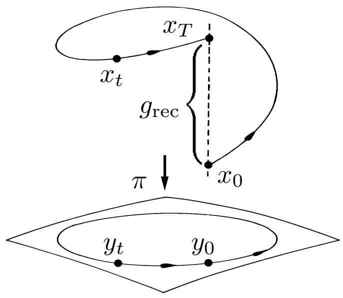

Figure 3: The definition of the reconstruction phase g_{rec}.

Noether's theorem ($\mathbf{J}(x_t) =$ constant) implies that y_t, which is called the *reduced solution*, lies in the *reduced space* P_{μ_0} (see the figure), where

$$P_\mu \equiv \pi(\mathbf{J}^{-1}(\mu)) \subset P/G \qquad (\mu \in \mathfrak{g}^*) \ ,$$

and where $\mu_0 \equiv \mathbf{J}(x_0)$ is the initial momentum. In fact, P_{μ_0} is a symplectic leaf of P/G (see Theorem 5.1) and the Ad^*-equivariance of $\mathbf{J}$ implies $g_{\mathrm{rec}} \in G_{\mu_0}$, where G_{μ_0} is the isotropy of the co-adjoint action at $\mu_0 \in \mathfrak{g}^*$. Invariance of H means $H = h \circ \pi$ for some $h : P/G \to \mathbb{R}$ called the *reduced Hamiltonian*; the reduced solution $y_t \in P_{\mu_0}$ is an integral curve of the Hamiltonian vector field $X_{h_{\mu_0}}$ corresponding to Hamiltonian $h_{\mu_0} \equiv h|P_{\mu_0}$.

2.2 Differentiating across symplectic leaves

We wish to define a kind of derivative in P/G *transverse* to symplectic leaves; these derivatives occur in the phase formula for general Hamiltonian systems to be recalled in 2.3 below. For this we require a notion of infinitesimal transverse. Specifically, if C denotes the characteristic distribution on P/G (the distribution tangent to the symplectic leaves), then a *connection on the symplectic stratification of* P/G is a distribution D on P/G complementary to C: $\mathrm{T}P = C \oplus D$. In that case there is a *canonical two-form* ω_D on P/G determined by D, whose restriction to a symplectic leaf delivers that leaf's symplectic structure, and whose kernel is precisely D.

Below we concern ourselves exclusively with connections D defined in a neighborhood of a nondegenerate symplectic leaf, assuming D to be smooth in the usual sense of constant rank distributions. Then ω_D is smooth also.

Fix a leaf P_μ and assume $D(y)$ is defined for all $y \in P_\mu$. Then at each $y \in P_\mu$ there is, according to the Lemma below, a natural identification of the infinitesimal transverse $D(y)$ with $\mathfrak{g}_\mu^*$, denoted $L(D, \mu, y) : \mathfrak{g}_\mu^* \xrightarrow{\sim} D(y)$.

Now let λ be an arbitrary $\mathbb{R}$-valued p-form on P/G, defined in a neighborhood of P_μ . Then we declare the *transverse derivative* $D_\mu\lambda$ of λ to be the $\mathfrak{g}_\mu$-valued p-form on P_μ defined through

$$\langle \nu, D_\mu\lambda(v_1, \dots, v_p)\rangle = d\lambda\big(L(D, \mu, y)(\nu), v_1, \dots, v_p \big)$$
$$\text{where } \nu \in \mathfrak{g}_\mu^*,\ v_1, \dots, v_p \in \mathrm{T}_y P_\mu \text{ and } y \in P_\mu.$$

Lemma and Definition. *Let $p_\mu : \mathfrak{g}^* \to \mathfrak{g}_\mu^*$ denote the natural projection, and define $\mathrm{T}_{\mathbf{J}^{-1}(\mu)}P \equiv \cup_{x \in \mathbf{J}^{-1}(\mu)} \mathrm{T}_x P$. Fix $y \in P_\mu$ and let $v \in D(y)$ be arbitrary. Then for all $w \in \mathrm{T}_{\mathbf{J}^{-1}(\mu)}P$ such that $\mathrm{T}\pi \cdot w = v$, the value of $p_\mu\langle d\mathbf{J}, w\rangle \in \mathfrak{g}_\mu^*$ is the same. Moreover, the induced map $v \mapsto p_\mu\langle d\mathbf{J}, w\rangle : D(y) \to \mathfrak{g}_\mu^*$ is an isomorphism. The inverse of this isomorphism (which depends on D, μ and y) is denoted by $L(D, \mu, y) : \mathfrak{g}_\mu^* \xrightarrow{\sim} D(y)$.*

We remark that the definition of $L(D, \mu, y)$ is considerably simpler in the case of Abelian G; see [6].

2.3 Reconstruction phases for general Hamiltonian systems

Let $\mathfrak{g}^*_{\mathrm{reg}} \subset \mathfrak{g}^*$ denote the set of regular points of the co-adjoint action, i.e., the set of points lying on co-adjoint orbits of maximal dimension (which fill an open dense subset). If $\mu_0 \in \mathfrak{g}^*_{\mathrm{reg}}$ then $\mathfrak{g}_{\mu_0}$ is Abelian; see Appendix B. In that case G_{μ_0} is Abelian if it is connected.

Now suppose, in the scenario described earlier, that a reduced solution $y_t \in P_{\mu_0}$ bounds a compact oriented surface $\Sigma \subset P_{\mu_0}$.

Theorem (Blaom [6]). *If $\mu_0 \in \mathfrak{g}^*_{\mathrm{reg}}$ and G_{μ_0} is Abelian, then the reconstruc-*

tion phase associated with the periodic solution $y_t \in \partial\Sigma$ is

$$g_{\mathrm{rec}} = g_{\mathrm{dyn}} g_{\mathrm{geom}} \ , \qquad \textit{where:}$$
$$g_{\mathrm{dyn}} = \exp \int_0^T D_{\mu_0} h(y_t)\, dt \ , \qquad g_{\mathrm{geom}} = \exp \int_\Sigma D_{\mu_0} \omega_D \ .$$

Here h denotes the reduced Hamiltonian, D denotes an arbitrary connection on the symplectic stratification of P/G, ω_D denotes the canonical two-form on P/G determined by$D$, and D_{μ_0} denotes the transverse derivative operator determined by D as described above.
The Theorem states that dynamic phases are time integrals of transverse derivatives of the reduced Hamiltonian while geometric phases are surface integrals of transverse derivatives of leaf symplectic structures.
We emphasize that while g_{dyn} and g_{geom} depend on the choice of D, the total phase g_{rec} is, by definition, independent of any such choice.
For the application of the above to non-free actions see [6].

2.4 Simple mechanical systems

Suppose a connected Lie group G acts freely and properly on a connected manifold Q. All actions in this paper are understood to be *left* actions. A Hamiltonian $H : \mathrm{T}^*Q \to \mathbb{R}$ is said to enjoy *G-symmetry* if it is invariant with respect to the cotangent-lifted action of G on T^*Q (see [1, p. 283] for the definition of this action). This action admits an Ad^*-equivariant momentum map $\mathbf{J} : \mathrm{T}^*Q \to \mathfrak{g}^*$ defined through

$$(1) \qquad \langle \mathbf{J}(x), \xi \rangle \equiv \langle x, \xi^Q(q) \rangle \qquad (x \in \mathrm{T}^*_q Q,\ q \in Q,\ \xi \in \mathfrak{g}) \ ,$$

where ξ^Q denotes the infinitesimal generator on Q corresponding to ξ. A *simple mechanical system* is a Hamiltonian $H : \mathrm{T}^*Q \to \mathbb{R}$ of the form

$$H(x) = \frac{1}{2} \langle\!\langle x, x \rangle\!\rangle_Q^* + V(q) \qquad (x \in \mathrm{T}^*_q Q) \ .$$

Here $\langle\!\langle \cdot\, , \cdot \rangle\!\rangle_Q^*$ denotes the symmetric contravariant two-tensor on Q determined by some prescribed Riemannian metric $\langle\!\langle \cdot\, , \cdot \rangle\!\rangle_Q$ on Q (the *kinetic energy metric*), and V is some prescribed G-invariant function on Q (the *potential energy*). To ensure G-symmetry we are supposing that G acts on Q by $\langle\!\langle \cdot\, , \cdot \rangle\!\rangle_Q$-isometries.

2.5 Mechanical connections

In general, the configuration space Q is bundled in a topologically non-trivial way over shape space Q/G, i.e., there is no global way to separate shape variables from symmetry variables. However, fixing a connection on the bundle allows one to split individual motions. In the case of simple mechanical systems such a connection is determined by the kinetic energy, but in general there is no canonical choice. All the phase formulas we shall present assume some choice has been made.

Under our free and properness assumptions, the projection $\rho : Q \to Q/G$ is a principal G-bundle. So we will universally require that this bundle be equipped with a connection one-form $\mathbf{A} \in \Omega^1(Q, \mathfrak{g})$. If a G-invariant Riemannian metric on Q is prescribed (e.g., the kinetic energy in the case of simple mechanical systems) a connection $\mathbf{A}$ is determined by requiring that the corresponding distribution of horizontal spaces hor $\equiv \ker \mathbf{A}$ are orthogonal to the ρ-fibers (G-orbits). In this context, $\mathbf{A}$ is called the *mechanical connection*; its history is described in [14, §3.3]

As we shall recall in 4.2, a connection $\mathbf{A}$ on $\rho : Q \to Q/G$ allows one to construct a Hamiltonian analogue $\mathrm{T}^*_{\mathbf{A}}\rho : \mathrm{T}^*Q \to \mathrm{T}^*(Q/G)$ for the tangent map $\mathrm{T}\rho : \mathrm{T}Q \to \mathrm{T}(Q/G)$. Thus for a state $x \in \mathrm{T}^*_q Q$ one may speak of the 'generalized momentum' $\mathrm{T}^*_{\mathbf{A}}\rho \cdot x \in \mathrm{T}^*_r(Q/G)$ of the corresponding shape $r = \rho(q) \in Q/G$.

2.6 Phases for Abelian symmetries

Let $H : \mathrm{T}^*Q \to \mathbb{R}$ be an arbitrary Hamiltonian enjoying G-symmetry. When G is Abelian it is known that each reduced space P_μ ($\mu \in \mathfrak{g}^*$, $P = \mathrm{T}^*Q$) is isomorphic to $\mathrm{T}^*(Q/G)$ equipped with the symplectic structure

$$\omega_\mu = \omega_{Q/G} - \langle \mu, (\tau^*_{Q/G})^* \operatorname{curv} \mathbf{A} \rangle \ .$$

It should be emphasized that the identification $P_\mu \cong \mathrm{T}^*(Q/G)$ depends on the choice of connection $\mathbf{A}$. See, e.g., [6] for the details. In the above equation $\omega_{Q/G}$ denotes the canonical symplectic structure on $\mathrm{T}^*(Q/G)$ and $\tau^*_{Q/G} : \mathrm{T}^*(Q/G) \to Q/G$ is the usual projection; $\operatorname{curv} \mathbf{A}$ denotes the curvature of $\mathbf{A}$, viewed as a $\mathfrak{g}$-valued two-form on Q/G (see, e.g., [16, §4]). The value of the reduced Hamiltonian $h_\mu : \mathrm{T}^*(Q/G) \to \mathbb{R}$ at a point $y \in \mathrm{T}^*(Q/G)$ is $H(x)$ where $x \in \mathrm{T}^*Q$ is any point satisfying $\mathbf{J}(x) = \mu$ and $\mathrm{T}^*_{\mathbf{A}}\rho \cdot x = y$.

The Theorem below is implicit in [6]. The special case in Addendum 2.7 is due to Marsden et al [16] (explicitly appearing in [6]).

Theorem. *Let $y_t \in P_{\mu_0} \cong \mathrm{T}^*(Q/G)$ be a periodic reduced solution curve. Let $r_t \equiv \tau^*_{Q/G}(y_t) \in Q/G$ denote the corresponding curve in shape space. Assume $t \mapsto r_t$ bounds a compact oriented surface $S \subset Q/G$. Assume r_t and y_t have the same minimal period T. Then the reconstruction phase associated with y_t is*

$$g_{\mathrm{rec}} = g_{\mathrm{dyn}} g_{\mathrm{geom}} \ , \qquad \textit{where:}$$

$$g_{\mathrm{dyn}} = \exp \int_0^T \frac{\partial h}{\partial \mu}(\mu_0, y_t)\, dt \ , \qquad g_{\mathrm{geom}} = \exp\left(-\int_S \operatorname{curv} \mathbf{A} \right) \ ,$$

and where $\partial h/\partial \mu\,(\mu', y') \in \mathfrak{g}$ is defined through

$$\langle \nu, \frac{\partial h}{\partial \mu}(\mu', y') \rangle = \frac{d}{dt}\, h_{\mu' + t\nu}(y') \Big|_{t=0} \qquad (\, \nu, \mu' \in \mathfrak{g}^*,\ y' \in \mathrm{T}^*(Q/G)\,) \ .$$

Here $\mathbf{A}$ *denotes an arbitrary connection on* $Q \to Q/G$.

2.7 Locked inertia tensor (Abelian case)

In the special case of a simple mechanical system one may be explicit about the dynamic phase. To this end, define for each $q \in Q$ a map $\hat{\mathbb{I}}(q) : \mathfrak{g} \to \mathfrak{g}^*$ through

$$\langle \hat{\mathbb{I}}(q)(\xi), \eta \rangle = \langle\langle \xi^Q(q), \eta^Q(q) \rangle\rangle_Q \qquad (\xi, \eta \in \mathfrak{g}) \ ,$$

where ξ^Q denotes the infinitesimal generator on Q corresponding to ξ. Varying over all $q \in Q$, one obtains a function $\hat{\mathbb{I}} : Q \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}^*)$. When G is Abelian, $\hat{\mathbb{I}}$ is G-invariant, dropping to a function $\mathbb{I} : Q/G \to \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}^*)$ called the *locked inertia tensor* (terminology explained in 1.4). As G acts freely on Q, $\hat{\mathbb{I}}(q) : \mathfrak{g} \to \mathfrak{g}^*$ has an inverse $\hat{\mathbb{I}}(q)^{-1} : \mathfrak{g}^* \to \mathfrak{g}$ leading to functions $\hat{\mathbb{I}}^{-1} : Q \to \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g})$ and $\mathbb{I}^{-1} : Q/G \to \mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g})$.

Addendum. *When* $H : \mathrm{T}^*Q \to \mathbb{R}$ *is a simple mechanical system and* $\mathbf{A}$ *is the mechanical connection, then the dynamic phase appearing in the preceding Theorem is given by*

$$g_{\mathrm{dyn}} = \int_0^T \mathbb{I}^{-1}(r_t)\mu_0 \, dt \ .$$

In particular, the reconstruction phase g_{rec} *is computed entirely in the shape space* Q/G.

2.8 Momentum connections

In the rigid body example discussed in 1.2 ($G = \mathrm{SO}(3)$), the angle $\Delta\theta$ may be identified with an element of $\mathfrak{g}_{\mu_0}$, where $\mu_0 \in \mathfrak{g}^* \cong \mathbb{R}^3$ is the initial spatial angular momentum. This angle is the logarithm of the reconstruction phase $g_{\mathrm{rec}} \in G_{\mu_0}$, there denoted g. Let $\omega_{\mathcal{O}}^-$ denote the 'minus' version of the symplectic structure on $\mathcal{O}$, viewed as co-adjoint orbit (see below). Then Equation 1.2(2) may alternatively be written

$$(1) \qquad \langle \mu_0, \log g_{\mathrm{rec}} \rangle = 2Th(\nu_0) + \int_S \omega_{\mathcal{O}}^- \ .$$

As we shall see, this generalizes to arbitrary groups G, but it refers only to the μ_0-component of the log phase. This engenders the following question, answered in the Proposition below: Of what $\mathfrak{g}_{\mu_0}$-valued two-form on $\mathcal{O}$ is $\omega_{\mathcal{O}}^-$ the μ_0-component?

For an arbitrary connected Lie group G equip $\mathfrak{g}^*$ with the 'minus' Lie-Poisson structure (see, e.g., [14, §2.8]). The symplectic leaves are the co-adjoint orbits; the symplectic structure on an orbit $\mathcal{O} = G \cdot \mu_0$ is $\omega_{\mathcal{O}}^-$, where $\omega_{\mathcal{O}}^-$ is given implicitly by

$$(2) \qquad \omega_{\mathcal{O}}^- \left(\frac{d}{dt} \exp(t\xi_1) \cdot \mu \Big|_{t=0}, \frac{d}{dt} \exp(t\xi_2) \cdot \mu \Big|_{t=0} \right) = -\langle \mu, [\xi_1, \xi_2] \rangle \ ,$$

for arbitrary $\mu \in \mathcal{O}$ and $\xi_1, \xi_2 \in \mathfrak{g}$. The map $\tau_{\mu_0} : G \to \mathcal{O}$ sending g to $g^{-1} \cdot \mu_0$ is a principal G_{μ_0}-bundle. If we denote by $\theta_G \in \Omega^1(G, \mathfrak{g})$ the right-invariant Mauer-Cartan form on G, then (2) may be succinctly written

$$\tau_{\mu_0}^* \omega_{\mathcal{O}}^- = -\langle \mu_0, \frac{1}{2} \theta_G \wedge \theta_G \rangle \ . \tag{3}$$

Assuming $\mathfrak{g}$ admits an Ad-invariant inner product, the bundle $\tau_{\mu_0} : G \to \mathcal{O} \cong G/G_{\mu_0}$ comes equipped with a connection one-form $\alpha_{\mu_0} \equiv \langle \mathrm{pr}_{\mu_0}, \theta_G \rangle$; here $\mathrm{pr}_{\mu_0} : \mathfrak{g} \to \mathfrak{g}_{\mu_0}$ denotes the orthogonal projection. We shall refer to α_{μ_0} as the *momentum connection* on $G \to \mathcal{O} \cong G/G_{\mu_0}$.
For simplicity, assume that μ_0 lies in $\mathfrak{g}^*_{\mathrm{reg}}$ and that G_{μ_0} is Abelian, as in 2.3. Then the curvature of α_{μ_0} may be identified with a $\mathfrak{g}_{\mu_0}$-valued two-form on $\mathcal{O} = G \cdot \mu_0$ denoted $\mathrm{curv}\, \alpha_{\mu_0}$.

Proposition. *Under the above conditions*

$$(\mathrm{curv}\, \alpha_{\mu_0}) \left(\frac{d}{dt} \exp(t\xi_1\,) \cdot \mu \Big|_{t=0}, \frac{d}{dt} \exp(t\xi_2\,) \cdot \mu \Big|_{t=0} \right) = \mathrm{pr}_{\mu_0}\, g^{-1} \cdot [\xi_1, \xi_2] \ ,$$

where g is any element of G such that $\mu = g \cdot \mu_0$, and $\xi_1, \xi_2 \in \mathfrak{g}$ are arbitrary. In particular, $\omega_{\mathcal{O}}^-$ is a component of curvature: $\omega_{\mathcal{O}}^- = -\langle \mu_0, \mathrm{curv}\, \alpha_{\mu_0} \rangle$.

Proof. Because G_{μ_0} is assumed Abelian, we have $\tau_{\mu_0}^* \mathrm{curv}\, \alpha_{\mu_0} = d\alpha_{\mu_0} = \langle \mathrm{pr}_{\mu_0}, d\theta_G \rangle$. Applying the Mauer-Cartan identity $d\theta_G = \frac{1}{2}\theta_G \wedge \theta_G$, we obtain $\tau_{\mu_0}^* \mathrm{curv}\, \alpha_{\mu_0} = \langle \mathrm{pr}_{\mu_0}, \frac{1}{2}\theta_G \wedge \theta_G \rangle$, which implies both the first part of the Proposition and the identity $\tau_{\mu_0}^* \langle \mu_0, \mathrm{curv}\, \alpha_{\mu_0} \rangle = \langle \mu_0 \circ \mathrm{pr}_{\mu_0}, \frac{1}{2}\theta_G \wedge \theta_G \rangle$. But $\mu_0 \in \mathfrak{g}^*_{\mathrm{reg}}$ implies that the space $\mathfrak{g}_{\mu_0}^\perp$ orthogonal to $\mathfrak{g}_{\mu_0}$ coincides with $[\mathfrak{g}, \mathfrak{g}_{\mu_0}]$ (see Appendix B), implying $\langle \mu_0, \mathrm{pr}_{\mu_0} \xi \rangle = \langle \mu_0, \xi \rangle$ for all $\xi \in \mathfrak{g}$. So $\tau_{\mu_0}^* \langle \mu_0, \mathrm{curv}\, \alpha_{\mu_0} \rangle = \langle \mu_0, \frac{1}{2}\theta_G \wedge \theta_G \rangle\rangle = -\tau_{\mu_0}^* \omega_{\mathcal{O}}^-$, by (3). This implies $\omega_{\mathcal{O}}^- = -\langle \mu_0, \mathrm{curv}\, \alpha_{\mu_0} \rangle$. □

2.9 Phases for $Q = G$

When $Q = G$, the Poisson manifold $P/G = (\mathrm{T}^*G)/G$ is identifiable with $\mathfrak{g}^*$ and the reduced space P_{μ_0} is the co-adjoint orbit $\mathcal{O} \equiv G \cdot \mu_0$, equipped with the symplectic structure $\omega_{\mathcal{O}}^-$ discussed above. Continue to assume that $\mathfrak{g}$ admits an Ad-invariant inner product. As we will show in Proposition 6.1, the restriction $\iota_{\mu_0} : [\mathfrak{g}, \mathfrak{g}_{\mu_0}]^\circ \to \mathfrak{g}^*_{\mu_0}$ of the natural projection $p_{\mu_0} : \mathfrak{g}^* \to \mathfrak{g}^*_{\mu_0}$ is then an isomorphism, assuming $\mu_0 \in \mathfrak{g}^*_{\mathrm{reg}}$. Here $^\circ$ denotes annihilator. The following result is implicit in [6].

Theorem. *Assume $\mu_0 \in \mathfrak{g}^*_{\mathrm{reg}}$ and G_{μ_0} is Abelian. Let $\nu_t \in P_{\mu_0} \cong \mathcal{O} \equiv G \cdot \mu_0$ be a periodic reduced solution curve bounding a compact oriented surface $S \subset \mathcal{O}$. Let $g_t \in G$ be any curve such that $\nu_t = g_t \cdot \mu_0 \equiv \mathrm{Ad}^*_{g_t^{-1}} \mu_0$. Then the* *reconstruction phase associated with* ν_t *is given by*

$$g_{\rm rec} = g_{\rm dyn} g_{\rm geom} \ , \qquad \textit{where:}$$
$$g_{\rm dyn} = \exp \int_0^T w(t)\, dt \ , \qquad g_{\rm geom} = \exp\left(- \int_S {\rm curv}\, \alpha_{\mu_0} \right) \ ,$$

and where $w(t) \in \mathfrak{g}_{\mu_0}$ *is defined through*

$$\langle \lambda, w(t) \rangle = \frac{d}{d\tau} h\Big(g_t \cdot (\, \mu_0 + \tau \iota_{\mu_0}^{-1}(\lambda)\,)\Big)\Big|_{\tau=0} \qquad (\lambda \in \mathfrak{g}_{\mu_0}^*) \ .$$

Here α_{μ_0} *denotes the momentum connection on* $G \to \mathcal{O} \cong G/G_{\mu_0}$.

For a simple mechanical system on T^*G the reduced Hamiltonian $h : \mathfrak{g}^* \to \mathbb{R}$ is of the form

$$h(\nu) = \frac{1}{2} \langle \nu, \mathbb{I}^{-1} \nu \rangle \ , \qquad (\nu \in \mathfrak{g}^*)$$

for some isomorphism $\mathbb{I} : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$, the *inertia tensor*, which we may suppose is symmetric as an element of $\mathfrak{g}^* \otimes \mathfrak{g}^*$.

Addendum ([6]). *Let* G *act on* $\mathrm{Hom}(\mathfrak{g}^*, \mathfrak{g})$ *via conjugation, so that* $g \cdot \mathbb{I}^{-1} = \mathrm{Ad}_{g^{-1}} \circ \mathbb{I}^{-1} \circ \mathrm{Ad}_{g^{-1}}^*$ $(g \in G)$. *Then for a simple mechanical system one has*

$$w(t) = \mathrm{pr}_{\mu_0}\Big((g_t^{-1} \cdot \mathbb{I}^{-1})(\mu_0) \Big) \ ,$$

where $\mathrm{pr}_{\mu_0} : \mathfrak{g} \to \mathfrak{g}_{\mu_0}$ *is the orthogonal projection. Moreover the generalization* 2.8(1) *of Montgomery's rigid body formula holds.*

3 Formulation of new results

According to known results reviewed in the preceding section, phases for simple mechanical systems are computed in shape space Q/G when G is Abelian, and on a co-adjoint orbit $\mathcal{O} = G \cdot \mu_0$ when $Q = G$. For the general case, G non-Abelian and $Q \neq G$, we need to introduce the concepts of associated bundles and forms, and the locked inertia tensor for non-Abelian groups (3.1–3.3). In 3.4 and 3.5 we present the main results of the paper, namely explicit formulas for geometric and dynamic phases in Hamiltonian systems on cotangent bundles.

3.1 Associated bundles

Given an arbitrary principal bundle $\rho : Q \to Q/G$ and manifold $\mathcal{O}$ on which G acts, we denote the quotient of $Q \times \mathcal{O}$ under the diagonal action of G by $\mathcal{O}_Q$. This is the total space of a bundle $\rho_{\mathcal{O}} : \mathcal{O}_Q \to Q/G : [q, \nu]_G \mapsto [q]_G$ known as the *associated bundle* for $\mathcal{O}$. As its fibers are diffeomorphic to $\mathcal{O}$, it may be regarded as a 'twisted product' of Q/G and $\mathcal{O}$.

Here the important examples will be the *co-adjoint bundle* $\mathfrak{g}_Q^*$ and the *co-adjoint orbit bundle* $\mathcal{O}_Q \subset \mathfrak{g}_Q^*$, where $\mathcal{O} \subset \mathfrak{g}^*$ is a co-adjoint orbit.

We have seen that log geometric phases are surface integrals of the curvature $\operatorname{curv} \mathbf{A} \in \Omega^2(Q/G, \mathfrak{g})$ of the mechanical connection $\mathbf{A}$, when G is Abelian, and of the curvature $\operatorname{curv} \alpha_{\mu_0} \in \Omega^2(\mathcal{O}, \mathfrak{g}_{\mu_0})$ of the momentum connection α_{μ_0}, when $Q = G$. For simple mechanical systems the log dynamic phase is a time integral of an inverted inertia tensor $\mathbb{I}^{-1}$ in both cases. To elaborate on the claims regarding the general case made in 1.4, we need to see how $\operatorname{curv} \mathbf{A}$, $\operatorname{curv} \alpha_{\mu_0}$ and $\mathbb{I}^{-1}$ can be viewed as objects on $\mathcal{O}_Q$.
A non-Abelian G forces us to regard $\operatorname{curv} \mathbf{A}$ as an element of $\Omega^2(Q/G, \mathfrak{g}_Q)$, i.e., as *bundle*-valued. See, e.g., Note A.6 and A.2(1) for the definition. The pull-back $\rho_{\mathcal{O}}^* \operatorname{curv} \mathbf{A}$ is then a two-form on $\mathcal{O}_Q$, but with values in the pull-back bundle $\rho_{\mathcal{O}}^* \mathfrak{g}_Q$. Pull-backs of bundles and forms are briefly reviewed in Appendix A.
On the other hand, $\operatorname{curv} \alpha_{\mu_0}$ is *vector*-valued because $\mathfrak{g}_{\mu_0}$ is Abelian under the hypothesis $\mu_0 \in \mathfrak{g}^*_{\mathrm{reg}}$. It is a two-form on the model space $\mathcal{O}$ of the fibers of $\rho_{\mathcal{O}} : \mathcal{O}_Q \to Q/G$. Its natural 'extension' to a two-form on $\mathcal{O}_Q$ is the *associated form* $(\operatorname{curv} \alpha_{\mu_0})_Q \in \Omega^2(\mathcal{O}_Q, \mathfrak{g}_{\mu_0})$, which we now define more generally.

3.2 Associated forms

Let $\rho : Q \to Q/G$ be a principal bundle equipped with a connection $\mathbf{A}$, and let $\mathcal{O}$ be a manifold on which G acts. If λ is a G-invariant, $\mathbb{R}$-valued k-form on $\mathcal{O}$ then the *associated form* λ_Q is the $\mathbb{R}$-valued k-form on $\mathcal{O}_Q$ defined as follows: For arbitrary $u_1, \ldots, u_k \in \mathrm{T}_{[q,\nu]_G} \mathcal{O}_Q$, there exist $\mathbf{A}$-horizontal curves $t \mapsto q_i^{\mathrm{hor}}(t) \in Q$ through q, and curves $t \mapsto \nu_i(t) \in Q$ through ν, such that

$$u_i = \frac{d}{dt} [q_i^{\mathrm{hor}}(t), \nu_i(t)]_G \Big|_{t=0} ,$$

in which case λ_Q is well defined by

$$\lambda_Q(u_1, \ldots, u_k) = \lambda \left(\frac{d}{dt} \nu_1(t) \Big|_{t=0}, \ldots, \frac{d}{dt} \nu_k(t) \Big|_{t=0} \right) . \tag{1}$$

When $\mathbb{R}$ is replaced by a general vector space V on which G acts linearly, then the *associated form* λ_Q of a *G-equivariant*, V-valued k-form λ is a certain k-form on $\mathcal{O}_Q$ taking values in the pull-back bundle $\rho_{\mathcal{O}}^* V_Q$. Its definition is postponed to 7.2. In symbols, we have a map

$$\begin{gathered} \lambda \mapsto \lambda_Q \\ \Omega^k_G(\mathcal{O}, V) \to \Omega^k(\mathcal{O}_Q, \rho_{\mathcal{O}}^* V_Q) \ . \end{gathered}$$

The identity $(\lambda \wedge \mu)_Q = \lambda_Q \wedge \mu_Q$ holds. If G acts trivially on V (e.g., $V = \mathbb{R}$ or $\mathfrak{g}_{\mu_0}$), then $\rho_{\mathcal{O}}^* V_Q \cong \mathcal{O}_Q \times V$ and we identify λ_Q with a V-valued form on

$\mathcal{O}_Q$ and (1) holds.

This last remark applies, in particular, to $\operatorname{curv}\alpha_{\mu_0}$.

3.3 Locked inertia tensor (general case)

When G is non-Abelian the map $\hat{\mathbb{I}} : Q \to \operatorname{Hom}(\mathfrak{g}, \mathfrak{g}^*)$ defined in 2.7 is G-equivariant if G acts on $\operatorname{Hom}(\mathfrak{g}, \mathfrak{g}^*)$ via conjugation. It therefore drops to a (bundle-valued) function $\mathbb{I} \in \Omega^0(Q/G, \operatorname{Hom}(\mathfrak{g}, \mathfrak{g}^*)_Q)$, the *locked inertia tensor*:

$$\mathbb{I}([q]_G) \equiv [q, \hat{\mathbb{I}}(q)]_G \ .$$

The inverse $\mathbb{I}^{-1} \in \Omega^0(Q/G, \operatorname{Hom}(\mathfrak{g}^*, \mathfrak{g})_Q)$ is defined similarly.
View the inclusion $i_{\mathcal{O}} : \mathcal{O} \hookrightarrow \mathfrak{g}^*$ as an element of $\Omega^0(\mathcal{O}, \mathfrak{g}^*)$. Then with the help of the associated form $(i_{\mathcal{O}})_Q \in \Omega^0(\mathcal{O}_Q, \rho_{\mathcal{O}}^* \mathfrak{g}_Q^*)$ one obtains a function $\rho_{\mathcal{O}}^* \mathbb{I}^{-1} \wedge (i_{\mathcal{O}})_Q$ on $\mathcal{O}_Q$ taking values in $\rho_{\mathcal{O}}^* \mathfrak{g}_Q$. (Under the canonical identification $\rho_{\mathcal{O}}^* \mathfrak{g}_Q^* \cong \mathcal{O}_Q \oplus \mathfrak{g}_Q^*$, one has $(i_{\mathcal{O}})_Q(\eta) = \eta \oplus \eta$.) Here the wedge $\wedge$ implies a contraction $\operatorname{Hom}(\mathfrak{g}^*, \mathfrak{g}) \otimes \mathfrak{g}^* \to \mathfrak{g}$.

3.4 Phases for simple mechanical systems

Before stating our new results, let us summarize with a few definitions. Put

$\Omega_{\mathbf{A}} \equiv \rho_{\mathcal{O}}^* \operatorname{curv} \mathbf{A}$:	the *mechanical curvature,*
$\Omega_{\mu_0} \equiv (\operatorname{curv}\alpha_{\mu_0})_Q$:	the *momentum curvature,*
$\xi_{\mathbb{I}} \equiv \rho_{\mathcal{O}}^* \mathbb{I}^{-1} \wedge (i_{\mathcal{O}})_Q$:	the *inverted locked inertia function.*

Recall here that $\mathbf{A}$ denotes a connection on $Q \to Q/G$ (the mechanical connection if H is a simple mechanical system), α_{μ_0} denotes the momentum connection on $G \to \mathcal{O} \cong G/G_{\mu_0}$, $\rho_{\mathcal{O}} : \mathcal{O}_Q \to Q/G$ denotes the associated bundle projection and $i_{\mathcal{O}} \in \Omega^0(\mathcal{O}, \mathfrak{g}^*)$ denotes the inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$.
By construction, $\Omega_{\mathbf{A}}$, Ω_{μ_0} and $\xi_{\mathbb{I}}$ are all differential forms on $\mathcal{O}_Q$. The momentum curvature Ω_{μ_0} is $\mathfrak{g}_{\mu_0}$-valued, and can therefore be integrated over surfaces $S \subset \mathcal{O}_Q$; the forms $\Omega_{\mathbf{A}}$ and $\xi_{\mathbb{I}}$ are $\rho_{\mathcal{O}}^* \mathfrak{g}_Q$-valued. To make them $\mathfrak{g}_{\mu_0}$-valued requires an appropriate projection:

Definition. Let G act on $\operatorname{Hom}(\mathfrak{g}, \mathfrak{g}_{\mu_0})$ via $g \cdot \sigma \equiv \operatorname{Ad}_g \circ \sigma$ and let $\operatorname{Pr}_{\mu_0} \in \Omega^0(\mathcal{O}, \operatorname{Hom}(\mathfrak{g}, \mathfrak{g}_{\mu_0}))$ denote the unique equivariant zero-form whose value at μ_0 is the orthogonal projection $\operatorname{pr}_{\mu_0} : \mathfrak{g} \to \mathfrak{g}_{\mu_0}$.

With the help of the associated form $(\operatorname{Pr}_{\mu_0})_Q$ and an implied contraction $\operatorname{Hom}(\mathfrak{g}, \mathfrak{g}_{\mu_0}) \otimes \mathfrak{g} \to \mathfrak{g}_{\mu_0}$, we obtain $\rho_{\mathcal{O}}^*(\mathfrak{g}_{\mu_0})_Q$-valued forms $(\operatorname{Pr}_{\mu_0})_Q \wedge \Omega_{\mathbf{A}}$ and $(\operatorname{Pr}_{\mu_0})_Q \wedge \xi_{\mathbb{I}}$. As we declare G to act trivially on $\mathfrak{g}_{\mu_0}$, these forms are in fact identifiable with $\mathfrak{g}_{\mu_0}$-valued forms as required.
For $P = \mathrm{T}^*Q$ and G non-Abelian the reduced space P_{μ_0} can be identified with $\mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q$, where $\mathcal{O} \equiv G \cdot \mu_0$. Here $\oplus$ denotes product in the category of

fiber bundles over Q/G (see Notation in 4.2). This observation was first made in the Lagrangian setting by Cendra et al. [9]. We recall details in 4.2 and Proposition 5.1. A formula for the symplectic structure on P_{μ_0} has been given by Perlmutter [24]. We derive the form of it we will require in 5.2. The value of the reduced Hamiltonian $h_{\mu_0} : P_{\mu_0} \to \mathbb{R}$ at $z \oplus [q,\mu]_G \in \mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q$ is $H(x)$, where $x \in \mathrm{T}^*_q Q$ is any point satisfying $\mathrm{T}^*_{\mathbf{A}}\rho \cdot x = z$ and $\mathbf{J}(x) = \mu$.
In the case of simple mechanical systems one has

$$(1) \qquad h_{\mu_0}(z \oplus [q,\mu]_G) = \frac{1}{2}\langle\!\langle z,z\rangle\!\rangle^*_{Q/G} + \frac{1}{2}\langle \mu, \hat{\mathbb{I}}^{-1}(q)\mu\rangle + V_{Q/G}(\rho(q)) \ .$$

Here $V_{Q/G}$ denotes the function on Q/G to which the potential V drops on account of its G-invariance, and $\langle\!\langle \cdot\,,\,\cdot\rangle\!\rangle^*_{Q/G}$ denotes the symmetric contravariant two-tensor on Q/G determined by the Riemannian metric $\langle\!\langle \cdot\,,\,\cdot\rangle\!\rangle_{Q/G}$ that Q/G inherits from the G-invariant metric $\langle\!\langle \cdot\,,\,\cdot\rangle\!\rangle_Q$ on Q. (The second term above may be written intrinsically as $1/2\ ((\mathrm{id}_{\mathfrak{g}^*})_Q \wedge (\rho^*_{\mathfrak{g}^*}\mathbb{I}^{-1} \wedge (\mathrm{id}_{\mathfrak{g}^*})_Q))([q,\mu]_G)$, where $(\mathrm{id}_{\mathfrak{g}^*})_Q$ is defined in 6.4.) The formula (1) is derived in 7.1.

Theorem. *Let $H : \mathrm{T}^*Q \to \mathbb{R}$ be a simple mechanical system, as defined in 2.4. Assume $\mu_0 \in \mathfrak{g}^*_{\mathrm{reg}}$, G_{μ_0} is Abelian, and let $z_t \oplus \eta_t \in P_{\mu_0} \cong \mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q$ ($\mathcal{O} = G \cdot \mu_0$) denote a periodic reduced solution curve. Assume $z_t \oplus \eta_t$ and η_t have the same minimal period T and assume $t \mapsto \eta_t$ bounds a compact oriented surface $S \subset \mathcal{O}_Q$. Then the corresponding reconstruction phase is*

$$g_{\mathrm{rec}} = g_{\mathrm{dyn}} g_{\mathrm{geom}} \ , \qquad \textit{where}$$

$$g_{\mathrm{dyn}} = \exp \int_0^T (\mathrm{Pr}_{\mu_0})_Q \wedge \xi_{\mathbb{I}}\,(\eta_t)\,dt \ ,$$

$$g_{\mathrm{geom}} = \exp\left(- \int_S (\,\Omega_{\mu_0} + (\mathrm{Pr}_{\mu_0})_Q \wedge \Omega_{\mathbf{A}}\,) \right) \ .$$

Here $\Omega_{\mathbf{A}}$ is the mechanical curvature, Ω_{μ_0} the momentum curvature, and $\xi_{\mathbb{I}}$ the inverted locked inertia function, as defined above; $\mathbf{A}$ denotes the mechanical connection.

Notice that the phase g_{rec} does not depend on the z_t part of the reduced solution curve (z_t, η_t), i.e., is computed exclusively in the space $\mathcal{O}_Q$.

3.5 Phases for arbitrary systems on cotangent bundles

We now turn to the case of general Hamiltonian functions on T^*Q (not necessarily simple mechanical systems). To formulate results in this case, we need the fact, recalled in Theorem 4.2, that $(\mathrm{T}^*Q)/G$ is isomorphic to $\mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q$, where $\oplus$ denotes product in the category of fiber bundles over Q/G (see Notation 4.2). This isomorphism depends on the choice of connection $\mathbf{A}$ on $\rho : Q \to Q/G$.

Theorem. *Let $H : \mathrm{T}^*Q \to \mathbb{R}$ be an arbitrary G-invariant Hamiltonian and $h : \mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q \to \mathbb{R}$ the corresponding reduced Hamiltonian. Consider a periodic reduced solution curve $z_t \oplus \eta_t \in P_{\mu_0} \cong \mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q$, as in the Theorem above. Then the conclusion of that Theorem holds, with the dynamic phase now given by*

$$g_{\mathrm{dyn}} = \exp \int_0^T D_{\mu_0} h(z_t \oplus \eta_t)\, dt \ ,$$

where $D_{\mu_0} h(\,\cdot\,) \in \mathfrak{g}_{\mu_0}$ is defined through

$$(1)\ \ \langle \nu, D_{\mu_0} h(z \oplus [q, \mu_0]_G)\rangle = \frac{d}{dt} h(z \oplus [q, \mu_0 + t\iota_{\mu_0}^{-1}(\nu)]_G)\Big|_{t=0} \qquad (\nu \in \mathfrak{g}^*_{\mu_0}) \ .$$

Here $\iota_{\mu_0} : [\mathfrak{g}, \mathfrak{g}_{\mu_0}]^\circ \xrightarrow{\sim} \mathfrak{g}^*_{\mu_0}$ is the isomorphism defined in 2.9.

Theorems 3.4 and 3.5 will be proved in Sections 7 and 8.

4 Symmetry reduction of cotangent bundles

In this section and the next, we revisit the process of reduction in cotangent bundles by describing the symplectic leaves in the associated Poisson-reduced space. For an alternative treatment and a brief history of cotangent bundle reduction, see Perlmutter [24, Chapter 3].

In the sequel G denotes a connected Lie group acting freely and properly on a connected manifold Q, and hence on T^*Q; $\mathbf{J} : \mathrm{T}^*Q \to \mathfrak{g}^*$ denotes the momentum map defined in 2.4(1); $\mathbf{A}$ denotes an arbitrary connection one-form on the principal bundle $\rho : Q \to Q/G$.

4.1 The zero momentum symplectic leaf

The form of an arbitrary symplectic leaf P_μ of $(\mathrm{T}^*Q)/G$ will be described in Section 5.1 using a concrete model for the abstract quotient $(\mathrm{T}^*Q)/G$ described in 4.2 below. However, the structure of the particular leaf $P_0 = \mathbf{J}^{-1}(0)/G$ can be described directly. Moreover, we shall need this description to relate symplectic structures on T^*Q and $\mathrm{T}^*(Q/G)$ (Corollary 4.3).

Since $\rho : Q \to Q/G$ is a submersion, it determines a natural vector bundle morphism $\rho^\circ : (\ker \mathrm{T}\rho)^\circ \to \mathrm{T}^*(Q/G)$ sending $d_q(f \circ \rho)$ to $d_{\rho(q)} f$, for each locally defined function f on Q/G. Here $(\ker \mathrm{T}\rho)^\circ$ denotes the annihilator of $\ker \mathrm{T}\rho$. In fact, 2.4(1) implies that $(\ker \mathrm{T}\rho)^\circ = \mathbf{J}^{-1}(0)$, so that $\mathbf{J}^{-1}(0)$ is a vector bundle over Q, and we have the commutative diagram

$$\begin{array}{ccc} \mathbf{J}^{-1}(0) & \xrightarrow{\rho^\circ} & \mathrm{T}^*(Q/G) \\ \downarrow & & \downarrow \\ Q & \xrightarrow[\rho]{} & Q/G \end{array} .$$

Notation. We will write $\mathbf{J}^{-1}(0)_q \equiv \mathbf{J}^{-1}(0) \cap \mathrm{T}^*_q Q = (\ker \mathrm{T}_q\rho)^\circ$ for the fiber of $\mathbf{J}^{-1}(0)$ over $q \in Q$.

From the definition of ρ°, it follows that ρ° maps $\mathbf{J}^{-1}(0)_q$ isomorphically onto $\mathrm{T}^*_{\rho(q)}(Q/G)$. In particular, ρ° is surjective.
It is readily demonstrated that the fibers of ρ° are G-orbits so that ρ° determines a diffeomorphism between $\mathrm{T}^*(Q/G)$ and $P_0 = \mathbf{J}^{-1}(0)/G$. Moreover, if $\omega_{Q/G}$ denotes the canonical symplectic structure on $\mathrm{T}^*(Q/G)$ and $i_0 : \mathbf{J}^{-1}(0) \hookrightarrow \mathrm{T}^*Q$ the inclusion, then we have

(1) $$(\rho^\circ)^*\omega_{Q/G} = i_0^*\omega \ .$$

This formula is verified by first checking the analogous statement for the canonical one-forms on T^*Q and $\mathrm{T}^*(Q/G)$.

4.2 A model for the Poisson-reduced space $(\mathrm{T}^*Q)/G$

Let hor $= \ker \mathbf{A}$ denote the distribution of horizontal spaces on Q determined by $\mathbf{A} \in \Omega^1(Q, \mathfrak{g})$. Then have the decomposition of vector bundles over Q

(1) $$\mathrm{T}Q = \mathrm{hor} \oplus \ker \mathrm{T}\rho \ ,$$

and the corresponding dual decomposition

(2) $$\mathrm{T}^*Q = \mathbf{J}^{-1}(0) \oplus \mathrm{hor}^\circ \ .$$

If $\mathbf{A}' : \mathrm{T}^*Q \to \mathbf{J}^{-1}(0)$ denotes the projection along hor°, then the composite

(3) $$\mathrm{T}^*_{\mathbf{A}}\rho \equiv \rho^\circ \circ \mathbf{A}' : \mathrm{T}^*Q \to \mathrm{T}^*(Q/G)$$

is a vector bundle morphism covering $\rho : Q \to Q/G$. It the Hamiltonian analogue of the tangent map $\mathrm{T}\rho : \mathrm{T}Q \to \mathrm{T}(Q/G)$.
The momentum map $\mathbf{J} : \mathrm{T}^*Q \to \mathfrak{g}^*$ determines a map $\mathbf{J}' : \mathrm{T}^*Q \to \mathfrak{g}^*_Q$ through

$$\mathbf{J}'(x) \equiv [q, \mathbf{J}(x)]_G \quad \text{for } x \in \mathrm{T}^*_q Q \text{ and } q \in Q \ .$$

Note that while $\mathbf{J}$ is equivariant, the map $\mathbf{J}'$ is G-*in*variant.

Notation. If M_1, M_2 and B are smooth manifolds and there are maps $f_1 : M_1 \to B$ and $f_2 : M_2 \to B$, then one has the pullback manifold

$$\{(m_1, m_2) \in M_1 \times M_2 \mid f_1(m_1) = f_2(m_2)\} \ ,$$

which we will denote by $M_1 \oplus_B M_2$, or simply $M_1 \oplus M_2$. If f_1 and f_2 are fiber bundle projections then $M_1 \oplus M_2$ is a product in the category of fiber bundles over B. In particular, in the case of *vector* bundles, $M_1 \oplus M_2$ is the Whitney sum of M_1 and M_2. In any case, we write an element of $M_1 \oplus M_2$ as $m_1 \oplus m_2$ (rather than (m_1, m_2)).

Noting that $\mathrm{T}^*(Q/G)$ and $\mathfrak{g}^*_Q$ are both vector bundles over Q/G, we have the following result following from an unravelling of definitions:

Theorem. *The map* $\pi : \mathrm{T}^*Q \to \mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q$ *defined by* $\pi(x) \equiv \mathrm{T}^*_{\mathbf{A}}\rho \cdot x \oplus \mathbf{J}'(x)$ *is a surjective submersion whose fibers are the* G*-orbits in* T^*Q*. In other words,* $\mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q$ *is a realization of the abstract quotient* $(\mathrm{T}^*Q)/G$*, the map* $\pi : \mathrm{T}^*Q \to \mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q$ *being a realization of the natural projection* $\mathrm{T}^*Q \to (\mathrm{T}^*Q)/G$*.*

The above model of $(\mathrm{T}^*Q)/G$ is simply the dual of Cendra, Holm, Marsden and Ratiu's model of $(\mathrm{T}Q)/G$ [9].

4.3 Momentum shifting

Before attempting to describe the symplectic leaves of the Poisson-reduced space $(\mathrm{T}^*Q)/G \cong \mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q$, we should understand the projection $\pi : \mathrm{T}^*Q \xrightarrow{/G} \mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q$ better. In particular, we should understand the map $\mathrm{T}^*_{\mathbf{A}}\rho : \mathrm{T}^*Q \to \mathrm{T}^*(Q/G)$, which means first understanding the projection $\mathbf{A}' : \mathrm{T}^*Q \to \mathbf{J}^{-1}(0)$ along hor°.

Let $x \in \mathrm{T}^*_qQ$ be given and define $\mu \equiv \mathbf{J}(x)$. The restriction of $\mathbf{J}$ to T^*_qQ is a linear map onto $\mathfrak{g}^*$ (by 2.4(1)). The kernel of this restriction is $\mathbf{J}^{-1}(0)_q$ and $\mathbf{J}^{-1}(\mu)_q \equiv \mathbf{J}^{-1}(\mu) \cap \mathrm{T}^*_qQ$ is an affine subspace of T^*_qQ parallel to $\mathbf{J}^{-1}(0)_q$; see Fig. 4.

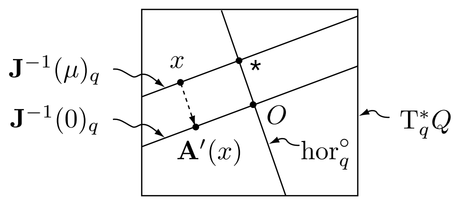

Figure 4: Describing the projection $x \mapsto \mathbf{A}'(x) : \mathrm{T}^*_qQ \to \mathbf{J}^{-1}(0)_q$ along hor°_q.

Since $\mathbf{J}^{-1}(0)_q$ and $\mathbf{J}^{-1}(\mu)_q$ are parallel, it follows from the decomposition 4.2(2) that $\mathbf{J}^{-1}(\mu)_q$ and hor°_q intersect in a single point $*$, as indicated in the figure.

We then have $\mathbf{A}'(x) = x - *$. Indeed, viewing the $\mathbb{R}$-valued one-form $\langle \mu, \mathbf{A} \rangle$ as a section of the cotangent bundle $\mathrm{T}^*Q \to Q$, one checks that the covector $\langle \mu, \mathbf{A} \rangle(q) \in \mathrm{T}^*_q Q$ belongs simultaneously to $\mathbf{J}^{-1}(\mu)$ and hor°, so that $* = \langle \mu, \mathbf{A} \rangle(q)$. We have therefore proven the following:

Lemma. *Define the* momentum shift $M_\mu : \mathrm{T}^*Q \to \mathrm{T}^*Q$, *which maps* $\mathbf{J}^{-1}(0)_q$ *onto to* $\mathbf{J}^{-1}(\mu)_q$, *by* $M_\mu(x) \equiv x + \langle \mu, \mathbf{A} \rangle(\tau^*_Q(x))$, *where* $\tau^*_Q : \mathrm{T}^*Q \to Q$ *denotes the cotangent bundle projection. Then*

$$\mathbf{A}'(x) = M^{-1}_{\mathbf{J}(x)}(x) \ .$$

If θ denotes the canonical one-form on T^*Q, then one readily computes $M^*_\mu \theta = \theta + \langle \mu, (\tau^*_Q)^* \mathbf{A} \rangle$. In particular, as $\omega = -d\theta$,

$$M^*_\mu \omega = \omega - \langle \mu, (\tau^*_Q)^* d\mathbf{A} \rangle \ .$$

This identity, Equation 4.1(1), and the above Lemma have the following important corollary, which relates the symplectic structures on the domain and range of the map $\mathrm{T}^*_{\mathbf{A}}\rho : \mathrm{T}^*Q \to \mathrm{T}^*(Q/G)$:

Corollary. *The two-forms* $(\mathrm{T}^*_{\mathbf{A}}\rho)^* \omega_{Q/G}$ *and* $\omega + \langle \mu, (\tau^*_Q)^* d\mathbf{A} \rangle$ *agree when restricted to* $\mathbf{J}^{-1}(\mu)$.

5 Symplectic leaves in Poisson reduced cotangent bundles

In this section we describe the symplectic leaves $P_\mu \subset (\mathrm{T}^*Q)/G$ as subsets of the model described in 4.2. We then describe explicitly their symplectic structures.

5.1 Reduced spaces as symplectic leaves

The following is a specialized version of the symplectic reduction theorem of Marsden, Weinstein and Meyer [20, 21], formulated such that the reduced spaces are realized as symplectic leaves (see, e.g., [7, Appendix E]).

Theorem. *Consider* P, ω, G, $\mathbf{J}$ *and* P_μ, *as defined in 2.1, where* $\mu \in \mathbf{J}(P)$ *is arbitrary. Then:*

(1) P_μ *is a symplectic leaf of* P/G *(which is a smooth Poisson manifold).*

(2) *The restriction* $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ *of* $\pi : P \to P/G$ *is a surjective submersion whose fibers are* G_μ*-orbits in* P, *i.e.,* P_μ *is a realization of the abstract quotient* $\mathbf{J}^{-1}(\mu)/G_\mu$.

(3) *If* ω_μ *is the leaf symplectic structure of* P_μ, *and* $i_\mu : \mathbf{J}^{-1}(\mu) \hookrightarrow P$ *the inclusion, then* $i^*_\mu \omega = \pi^*_\mu \omega_\mu$.

(4) $P_\mu \cap P_{\mu'} \neq \emptyset$ *if and only if* $P_\mu = P_{\mu'}$, *which is true if and only if* μ *and* μ' *lie on the same co-adjoint orbit. Also,* $P/G = \cup_{\mu \in \mathbf{J}(P)} P_\mu$.

(5) $\operatorname{codim} P_\mu = \operatorname{codim} G \cdot \mu$.

Proposition. *Fix $\mu \in \mathfrak{g}^*$. Then, taking $P \equiv \mathrm{T}^*Q$ and identifying P/G with $\mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q$ (Theorem 4.2), one obtains*

$$P_\mu = \mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q \ , \quad \textit{where} \quad \mathcal{O} \equiv G \cdot \mu \ .$$

Here $G \cdot \mu$ denotes the co-adjoint orbit through μ and the associated bundle $\mathcal{O}_Q$ is to be viewed as a fiber subbundle of $\mathfrak{g}^*_Q$ in the obvious way.

Proof. Under the given identification, the projection $P \to P/G$ is represented by the map $\pi : \mathrm{T}^*Q \to \mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q$ defined in Theorem 4.2. From this definition it easily follows that $P_\mu \equiv \pi(\mathbf{J}^{-1}(\mu))$ is contained in $\mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q$. We now prove the reverse inclusion $\mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q \subset \pi(\mathbf{J}^{-1}(\mu))$.
Let $z \oplus [q', \mu']_G$ be an arbitrary point in $\mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q$. Then $\mu' \in \mathcal{O}$, so that $\mu' = g \cdot \mu$ for some $g \in G$, giving us $z \oplus [q', \mu']_G = z \oplus [q, \mu]_G$, where $q \equiv g^{-1} \cdot q'$. Now z and $[q, \mu]_G$ necessarily have a common base point in Q/G, which means that $z \in \mathrm{T}^*_{\rho(q)}(Q/G)$. The map $\rho^\circ : \mathbf{J}^{-1}(0) \to \mathrm{T}^*(Q/G)$ of 4.1 maps $\mathbf{J}^{-1}(0)_q \equiv \mathbf{J}^{-1}(0) \cap \mathrm{T}^*_q Q$ isomorphically onto $\mathrm{T}^*_{\rho(q)}(Q/G)$. Therefore there exists $x_0 \in \mathbf{J}^{-1}(0)_q$ such that $\rho^\circ(x_0) = z$. Define $x \equiv M_\mu(x_0) \in \mathbf{J}^{-1}(\mu)$, where M_μ is the momentum shift of Lemma 4.3. Then $\mathrm{T}^*_{\mathbf{A}}\rho \cdot x = z$. We now compute

$$\pi(x) = \mathrm{T}^*_{\mathbf{A}}\rho \cdot x \oplus \mathbf{J}'(x) = z \oplus [\tau^*_Q(x), \mathbf{J}(x)]_G = z \oplus [q, \mu]_G = z \oplus [q', \mu']_G \ .$$

Since x lies in $\mathbf{J}^{-1}(\mu)$ and $z \oplus [q', \mu']_G$ was an arbitrary point of $\mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q$, this proves $\mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q \subset \pi(\mathbf{J}^{-1}(\mu))$. □

5.2 The leaf symplectic structures

The remainder of the section is devoted to the proof of the following key result, which is due (in a different form) to Perlmutter [24, Chapter 3]:

Theorem. *Let $\mathcal{O}$ denote the co-adjoint orbit through a point μ in the image of $\mathbf{J}$, let $\omega^-_{\mathcal{O}}$ denotes the 'minus' co-adjoint orbit symplectic structure on $\mathcal{O}$ (see 2.8), and let $i_{\mathcal{O}} \in \Omega^0(\mathcal{O}, \mathfrak{g}^*)$ denote the inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$. Let $(\omega^-_{\mathcal{O}})_Q \in \Omega^2(\mathcal{O}_Q)$ and $(i_{\mathcal{O}})_Q \in \Omega^0(\mathcal{O}_Q, \rho^*_{\mathcal{O}}\mathfrak{g}^*_Q)$ denote the corresponding associated forms; see 3.2. (Under the canonical identification $\rho^*_{\mathcal{O}}\mathfrak{g}^*_Q \cong \mathcal{O}_Q \oplus \mathfrak{g}^*_Q$, one has $(i_{\mathcal{O}})_Q(\eta) = \eta \oplus \eta$.) Then the symplectic structure of the leaf $P_\mu = \mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q$ is given by*

$$\omega_\mu = \mathrm{pr}_1^* \, \omega_{Q/G} + \mathrm{pr}_2^* \Big((\omega^-_{\mathcal{O}})_Q - (i_{\mathcal{O}})_Q \wedge \rho^*_{\mathcal{O}} \operatorname{curv} \mathbf{A} \Big) \ ,$$

where $\mathrm{pr}_1 : \mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q \to \mathrm{T}^*(Q/G)$ *and* $\mathrm{pr}_2 : \mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q \to \mathcal{O}_Q$ *denote the projections onto the first and second summands, and* $\operatorname{curv} \mathbf{A} \in \Omega^2(Q/G, \mathfrak{g}_Q)$ *denotes the curvature of* $\mathbf{A}$.

Because the restriction $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ of $\pi : \mathrm{T}^*Q \to \mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q$ is a surjective submersion, by Theorem 5.1(2), to prove the above Theorem it suffices to verify the formula in 5.1(3). Appealing to the definition of π (Theorem 4.2) and Corollary 4.3, we compute

$$\pi_\mu^* \operatorname{pr}_1^* \omega_{Q/G} = i_\mu^* (\mathrm{T}^*_{\mathbf{A}}\rho)^* \omega_{Q/G} = i_\mu^* \omega + \langle \mu, i_\mu^* (\tau_Q^*)^* d\mathbf{A} \rangle \ . \tag{1}$$

For the next part of the proof we need the following technical result proven at the end:

Lemma. *If $u \in \mathrm{T}_x(\mathbf{J}^{-1}(\mu))$ is arbitrary, then*

$$\mathrm{T}\mathbf{J}' \cdot u = \frac{d}{dt} [q^{\mathrm{hor}}(t), \exp(t\xi) \cdot \mu]_G \Big|_{t=0} \ ,$$

for some $\mathbf{A}$-horizontal curve $t \mapsto q^{\mathrm{hor}}(t) \in Q$, where $\xi \equiv -\mathbf{A}(\mathrm{T}\tau_Q^* \cdot u)$.

Now $\pi_\mu^* \operatorname{pr}_2^* (\omega_{\mathcal{O}}^-)_Q = i_\mu^* (\mathbf{J}')^* (\omega_{\mathcal{O}}^-)_Q$ and, by definition,

$$\omega_{\mathcal{O}}^- \left(\frac{d}{dt} \exp(t\xi) \cdot \mu \Big|_{t=0}, \frac{d}{dt} \exp(t\eta) \cdot \mu \Big|_{t=0} \right) = -\langle \mu, [\xi, \eta] \rangle \qquad (\xi, \eta \in \mathfrak{g}) \ .$$

So it readily follows from the lemma that

$$\pi_\mu^* \operatorname{pr}_2^* (\omega_{\mathcal{O}}^-)_Q = -\frac{1}{2} \langle \mu, i_\mu^* (\tau_Q^*)^* (\mathbf{A} \wedge \mathbf{A}) \rangle \ . \tag{2}$$

A routine calculation of pullbacks shows that

$$(\pi_\mu^* \operatorname{pr}_2^* (i_{\mathcal{O}})_Q)(x) = [x \oplus \tau_Q^*(x), \mu]_G \in (\rho_{\mathcal{O}} \circ \operatorname{pr}_2 \circ \pi_\mu)^* \mathfrak{g}_Q^* \qquad (x \in \mathbf{J}^{-1}(\mu)) \tag{3}$$

and

$$(\pi_\mu^* \operatorname{pr}_2^* \rho_{\mathcal{O}}^* \operatorname{curv} \mathbf{A})(u_1, u_2) = [x \oplus \tau_Q^*(x), i_\mu^* (\tau_Q^*)^* \mathbf{DA}(u_1, u_2)]_G \in (\rho_{\mathcal{O}} \circ \operatorname{pr}_2 \circ \pi_\mu)^* \mathfrak{g}_Q \ , \tag{4}$$

for $u_1, u_2 \in \mathrm{T}_x(\mathbf{J}^{-1}(\mu))$, where $\mathbf{DA} \in \Omega^2(Q, \mathfrak{g})$ denotes the exterior covariant derivative of $\mathbf{A}$. In deriving (4) we have used the fact that $\rho_{\mathcal{O}} \circ \operatorname{pr}_2 \circ \pi_\mu = \rho \circ \tau_Q^* \circ i_\mu$ and that $\rho^* \operatorname{curv} \mathbf{A} \in \Omega^2(Q, \rho^* \mathfrak{g}_Q)$ satisfies the identity

$$(\rho^* \operatorname{curv} \mathbf{A})(v_1, v_2) = [q \oplus q, \mathbf{DA}(v_1, v_2)]_G \in \rho^* \mathfrak{g}_Q \qquad (v_1, v_2 \in \mathrm{T}_q Q) \ .$$

This identity simply states, in pullback jargon, that $\operatorname{curv} \mathbf{A}$ is the two-form $\mathbf{DA}$ on Q, viewed as a $\mathfrak{g}_Q$-valued form on the base Q/G.
Carrying out an implied contraction, Equations (3) and (4) deliver

$$\pi_\mu^* (\operatorname{pr}_2^* ((i_{\mathcal{O}})_Q \wedge \rho_{\mathcal{O}}^* \operatorname{curv} \mathbf{A})) = \langle \mu, i_\mu^* (\tau_Q^*)^* \mathbf{DA} \rangle \in \Omega^2(\mathbf{J}^{-1}(\mu)) \ . \tag{5}$$

From Equations (2), (5) and the Mauer-Cartan equation $d\mathbf{A} = \mathbf{DA} + \frac{1}{2} \mathbf{A} \wedge \mathbf{A}$, follows the formula

$$\pi_\mu^* \operatorname{pr}_2^* ((\omega_{\mathcal{O}}^-)_Q - (i_{\mathcal{O}})_Q \wedge \rho_{\mathcal{O}}^* \operatorname{curv} \mathbf{A}) = -\langle \mu, i_\mu^* (\tau_Q^*)^* d\mathbf{A} \rangle \ . \tag{6}$$

The formula in 5.1(3) follows from (6) and (1), which completes the proof of the theorem.

Proof of the Lemma. We have $u = d/dt\, x(t)\,|_{t=0}$ for some curve $t \mapsto x(t) \in \mathbf{J}^{-1}(\mu)$, in which case

$$\mathrm{T}\mathbf{J}' \cdot u = \frac{d}{dt} [q(t), \mu]_G \Big|_{t=0} \ ,$$

where $q(t) \equiv \tau_Q^*(x(t))$. We can write $q(t) = g(t) \cdot q^{\mathrm{hor}}(t)$ for some $\mathbf{A}$-horizontal curve $t \mapsto q^{\mathrm{hor}}(t) \in Q$ and some curve $t \mapsto g(t) \in G$ with $g(0) = \mathrm{id}$ and with

$$\frac{d}{dt} g(t) \Big|_{t=0} = \mathbf{A} \left(\frac{d}{dt} q(t) \Big|_{t=0} \right) = \mathbf{A}(\mathrm{T}\tau_Q^* \cdot u) = -\xi \ .$$

Then

$$\begin{aligned} \mathrm{T}\mathbf{J}' \cdot u &= \frac{d}{dt} [q^{\mathrm{hor}}(t), g(t)^{-1} \cdot \mu]_G \Big|_{t=0} \\ &= \frac{d}{dt} [q^{\mathrm{hor}}(t), \exp(t\xi) \cdot \mu]_G \Big|_{t=0} \ , \end{aligned}$$

as required. □

6 A connection on the Poisson-reduced phase space

To apply Theorem 2.3 to the case $P = \mathrm{T}^*Q$ we need to choose a connection D on the symplectic stratification of $P/G \cong \mathrm{T}^*(Q/G) \oplus \mathfrak{g}_Q^*$. Such connections were defined in 2.2. As we shall see, this more-or-less amounts to choosing an inner product on $\mathfrak{g}^*$ (or $\mathfrak{g}$). Life is made considerably easier if this choice is Ad-invariant. (For example, in the case $Q = G$, which we discuss first, one might be tempted to use the inertia tensor $\mathbb{I} \in \mathfrak{g}^* \otimes \mathfrak{g}^*$ to form an inner product. However, this seems to lead to intractable calculations of the phase. It also makes the geometric phase g_{geom} more 'dynamic' and less 'geometric.') Fortunately, we will see that the particular choice of invariant inner product is immaterial.

In 6.3 and 6.4 we discuss details needed to describe explicitly the transverse derivative operator D_μ, and we also compute the canonical two-form ω_D (both these depend on the choice of D). Recall that these will be needed to apply Theorem 2.3.

6.1 The limiting case $Q = G$

When $Q = G$, we have $P/G \cong \mathfrak{g}^*$ and the symplectic leaves are the co-adjoint orbits. A connection on the symplectic stratification of P/G is then distribution on $\mathfrak{g}^*$ furnishing a complement, at each point $\mu \in \mathfrak{g}^*$, for the space $\mathrm{T}_\mu(G \cdot \mu)$ tangent to the co-adjoint orbit $G \cdot \mu$ through μ. As a subspace of $\mathfrak{g}^*$ this tangent space is the annihilator $\mathfrak{g}_\mu^\circ$ of $\mathfrak{g}_\mu$.

Lemma. *Let G be a connected Lie group whose Lie algebra $\mathfrak{g}$ admits an* Ad-*invariant inner product. Then for all $\mu \in \mathfrak{g}^*_{\rm reg}$ one has*

$$\mathfrak{g}_\mu^\perp = [\mathfrak{g}, \mathfrak{g}_\mu] \ .$$

Here $\mathfrak{g}^*_{\rm reg}$ denotes the set of regular points of the co-adjoint action

Proof. See Appendix B. □

The following proposition constructs a connection E on the symplectic stratification of $\mathfrak{g}^*$.

Proposition. *Let G be a connected Lie group whose Lie algebra $\mathfrak{g}$ admits an* Ad-*invariant inner product and equip $\mathfrak{g}^*$ with the corresponding* Ad^*-*invariant inner product. Let E denote the connection on the symplectic stratification of $\mathfrak{g}^*$ obtained by orthogonalizing the distribution tangent to the co-adjoint orbits:*

$$E(\mu) \equiv \Big(\mathrm{T}_\mu (G \cdot \mu) \Big)^\perp \ .$$

Let $\operatorname{forg} E(\mu)$ *denotes the image of $E(\mu)$ under the canonical identification* $\mathrm{T}_\mu \mathfrak{g}^* \cong \mathfrak{g}^*$*, i.e.,* $\operatorname{forg} E(\mu) \subset \mathfrak{g}^*$ *is* $E(\mu) \subset \mathrm{T}_\mu \mathfrak{g}^*$ *with base point 'forgotten.' Then for all $\mu \in \mathfrak{g}^*_{\rm reg}$:*

(1) $\operatorname{forg} E(\mu) = [\mathfrak{g}, \mathfrak{g}_\mu]^\circ$.

(2) *$E(\mu)$ is independent of the particular choice of inner product.*

(3) *The restriction $\iota_\mu : \operatorname{forg} E(\mu) \to \mathfrak{g}^*_\mu$ of the natural projection $p_\mu : \mathfrak{g}^* \to \mathfrak{g}^*_\mu$ is an isomorphism.*

(4) *The orthogonal projection $\mathrm{pr}_\mu : \mathfrak{g} \to \mathfrak{g}_\mu$ is independent of the choice of inner product and satisfies the identity*

$$\langle \iota_\mu^{-1}(\nu), \xi \rangle = \langle \nu, \mathrm{pr}_\mu \xi \rangle \qquad (\nu \in \mathfrak{g}^*_\mu,\ \xi \in \mathfrak{g}) \ .$$

(5) *The complementary projection $\mathrm{pr}_\mu^\perp \equiv \mathrm{id} - \mathrm{pr}_\mu$ satisfies the identity*

$$\mathrm{pr}_\mu [\mathrm{pr}_\mu^\perp \xi, \mathrm{pr}_\mu^\perp \eta] = \mathrm{pr}_\mu [\xi, \eta] \qquad (\xi, \eta \in \mathfrak{g}) \ .$$

(6) *There exists a subspace $V \subset \mathfrak{g}^*$ containing μ and an open neighborhood $S \subset V$ of μ such that $\mathrm{T}_s S = E(s)$ for all $s \in S$.*

Remark. One can choose the V in (6) to be G_μ-invariant (see the proof below), so that S (suitably shrunk) is a slice for the co-adjoint action. This is provided, of course, that G has *closed* co-adjoint orbits. Although we do *not* assume that these orbits are closed, the reader may nevertheless find it helpful to think of S as a slice. We do not use (6) until Section 8.

Proof. In fact (3) is true for *any* space $E(\mu)$ complementary to $\mathrm{T}_\mu(G\cdot\mu)$, for this means

$$\mathrm{T}_\mu \mathfrak{g}^* = E(\mu) \oplus \mathrm{T}_\mu(G\cdot\mu) \ , \tag{7}$$

which, on identifying the spaces with subspaces of $\mathfrak{g}^*$, delivers the decomposition

$$\mathfrak{g}^* = \operatorname{forg} E(\mu) \oplus \mathfrak{g}_\mu^\circ \ .$$

Since $\mathfrak{g}_\mu^\circ$ is the kernel of the linear surjection $p_\mu : \mathfrak{g}^* \to \mathfrak{g}_\mu^*$, (3) must be true. The identity in (4) is an immediate corollary.
Because taking annihilator and orthogonalizing are commutable operations, we deduce from the above Lemma the formula $(\mathfrak{g}_\mu^\circ)^\perp = [\mathfrak{g},\mathfrak{g}_\mu]^\circ$. Since $\mathfrak{g}_\mu^\circ = \operatorname{forg} \mathrm{T}_\mu(G\cdot\mu)$, (1) holds. Claim (2) follows.
Regarding (5), we have

$$\begin{aligned}\mathrm{pr}_\mu[\mathrm{pr}_\mu^\perp \xi, \mathrm{pr}_\mu^\perp \eta] &= \mathrm{pr}_\mu[\xi - \mathrm{pr}_\mu \xi, \eta - \mathrm{pr}_\mu \eta] \\ &= \mathrm{pr}_\mu([\xi,\eta] + [\mathrm{pr}_\mu \xi, \mathrm{pr}_\mu \eta] - [\xi, \mathrm{pr}_\mu \eta] + [\eta, \mathrm{pr}_\mu \xi]) \ .\end{aligned}$$

The second term in parentheses vanishes because $\mathfrak{g}_\mu$ is Abelian (since $\mu \in \mathfrak{g}^*_{\mathrm{reg}}$). The third and fourth terms vanish because they lie in $[\mathfrak{g},\mathfrak{g}_\mu]$, which is the kernel of pr_μ, on account of the Lemma. This kernel is evidently independent of the choice of inner product, which proves the first part of (4).
To prove (6), take

$$V \equiv [\mathfrak{g},\mathfrak{g}_\mu]^\circ = \{\nu \in \mathfrak{g}^* \mid \mathfrak{g}_\nu \subset \mathfrak{g}_\mu\} \ ,$$

which clearly contains μ. Since $\dim \mathfrak{g}_\mu = \dim \mathfrak{g}_\nu$ if and only if $\nu \in \mathfrak{g}^*_{\mathrm{reg}}$, we conclude that

$$V \cap \mathfrak{g}^*_{\mathrm{reg}} = \{\nu \in \mathfrak{g}^* \mid \mathfrak{g}_\nu = \mathfrak{g}_\mu\} \ .$$

Since, $\mathfrak{g}^*_{\mathrm{reg}} \subset \mathfrak{g}^*$ is an open set (see Appendix B), it follows that μ has a neighborhood $S \subset V$ of μ such that $S \subset \mathfrak{g}^*_{\mathrm{reg}}$ and $\mathfrak{g}_s = \mathfrak{g}_\mu$ for all $s \in S$. For any $s \in S$ we then have

$$\operatorname{forg}(E(s)) = [\mathfrak{g},\mathfrak{g}_s]^\circ = [\mathfrak{g},\mathfrak{g}_\mu]^\circ = V = \operatorname{forg}(\mathrm{T}_s S) \ , \tag{8}$$

where the first equality follows from (1). Equation (8) implies that $E(s) = \mathrm{T}_s S$, as required. ☐

Henceforth E denotes the connection on the symplectic stratification of $\mathfrak{g}^*$ defined in the above Proposition.

6.2 The general case $Q \neq G$

In general, a connection D on the symplectic stratification of $(\mathrm{T}^*Q)/G \cong \mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q$ is given by

$$D(z \oplus [q,\mu]_G) \equiv \left\{ \frac{d}{dt} z \oplus [q, \mu + t\delta]_G \Big|_{t=0} \ \Big| \ \delta \in \operatorname{forg} E(\mu) \right\} \tag{1}$$
$$\left(z \in \mathrm{T}^*_{\rho(q)}(Q/G),\, q \in Q,\, \mu \in \mathfrak{g}^* \right) \ .$$

If $[q', \mu']_G = [q, \mu]_G$, then the right-hand side of (1) is unchanged by a substitution by primed quantities, because E is G-invariant. This shows that the distribution D is well defined. It is a connection on the symplectic stratification of $\mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q$ because E is a connection on the symplectic stratification of $\mathfrak{g}^*$, and because the symplectic leaf through a point $z \oplus [q, \mu]_G$ is $\mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q$, where $\mathcal{O} \equiv G \cdot \mu$.

6.3 Transverse derivatives.

To determine the transverse derivative operator D_μ determined by D in the special case of cotangent bundles (needed to apply Theorem 2.3), we will need an explicit expression for the isomorphism $L(D, y, \mu) : \mathfrak{g}^*_\mu \to D(y)$ defined in 2.2.

Lemma. *Fix $\mu \in \mathfrak{g}^*_{\mathrm{reg}}$. Then:*

(1) *Each $y \in P_\mu$ is of the form $y = z \oplus [q, \mu]_G$ for some $q \in Q$ and $z \in \mathrm{T}^*_{\rho(q)}(Q/G)$.*

(2) *For each such y one has*

$$L(D, \mu, y)(\nu) = \frac{d}{dt} z \oplus [q, \mu + t\iota_\mu^{-1}(\nu)]_G \Big|_{t=0} \ ,$$

where ι_μ is defined by 6.1(3).

Proof. That each $y \in P_\mu$ is of the form given in (1) follows from an argument already given in the proof of Proposition 5.1. Moreover, that proof shows that there exists $x_0 \in \mathbf{J}^{-1}(0)_q$ such that $\rho^\circ(x_0) = z$. We prove (2) by first computing the natural isomorphism $D(y) \xrightarrow{\sim} \mathfrak{g}^*_\mu$ in Lemma 2.2 whose inverse defines $L(D, \mu, y)$. Define $x \equiv M_\mu(x_0)$, where M_μ is the momentum shift defined in 4.3. Then $x \in \mathbf{J}^{-1}(\mu)$. According to (1), an arbitrary vector $v \in D(y)$ is of the form

$$v = \frac{d}{dt} z \oplus [q, \mu + t\delta]_G \Big|_{t=0} \ ,$$

for some $\delta \in \operatorname{forg} E(\mu)$. We claim that the vector

$$w \equiv \frac{d}{dt} M_{\mu + t\delta}(x_0) \Big|_{t=0} \in \mathrm{T}_x P \subset \mathrm{T}_{\mathbf{J}^{-1}(\mu)} P \qquad (P = \mathrm{T}^* Q)$$

is a valid choice for the corresponding vector w in Lemma 2.2. Indeed, one has

$$\begin{aligned}
\mathrm{T}\pi \cdot w = \frac{d}{dt} \pi(M_{\mu+t\delta}(x_0)) \Big|_{t=0} &= \frac{d}{dt} \mathrm{T}^*_{\mathbf{A}}\rho \cdot M_{\mu+t\delta}(x_0) \oplus \mathbf{J}'(M_{\mu+t\delta}(x_0)) \Big|_{t=0} \\
&= \frac{d}{dt} \rho^\circ(x_0) \oplus [\tau^*_Q(M_{\mu+t\delta}(x_0)), \mu + t\delta]_G \Big|_{t=0} \\
&= \frac{d}{dt} z \oplus [q, \mu + t\delta]_G \Big|_{t=0} = v \ ,
\end{aligned}$$

as required. We now compute

$$p_\mu \langle d\mathbf{J}, w\rangle = p_\mu \frac{d}{dt}\, \mathbf{J}(M_{\mu+t\delta}(x_0))\Big|_{t=0} = p_\mu \delta \ .$$

The natural isomorphism $D(y) \xrightarrow{\sim} \mathfrak{g}^*_\mu$ is therefore given by

$$\frac{d}{dt}\, z \oplus [q, \mu + t\delta]_G \Big|_{t=0} \mapsto p_\mu \delta \qquad (\delta \in \operatorname{forg} E(\mu)) \ .$$

Since $L(D, y, \mu)$ is the inverse of this map, this proves (2). □

6.4 The canonical two-form determined by D

We now determine the canonical two-form ω_D determined by D in the cotangent bundle case.

According to Theorem 5.2, the symplectic structure of the leaf $P_\mu = \mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q$ ($\mathcal{O} \equiv G \cdot \mu$) is given by

$$(1) \qquad \omega_\mu = \mathrm{pr}_1^*\, \omega_{Q/G} + \mathrm{pr}_2^* \Big((\omega^-_{\mathcal{O}})_Q - (i_{\mathcal{O}})_Q \wedge \rho^*_{\mathcal{O}} \operatorname{curv} \mathbf{A} \Big) \ ,$$

where $\mathrm{pr}_1 : \mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q \to \mathrm{T}^*(Q/G)$ and $\mathrm{pr}_2 : \mathrm{T}^*(Q/G) \oplus \mathcal{O}_Q \to \mathcal{O}_Q$ are the canonical projections. We claim that the canonical two-form $\omega_D \in \Omega^2(\mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q)$ determined by D (see 2.2) is given by

$$(2) \qquad \omega_D = \mathrm{pr}_1^*\, \omega_{Q/G} + \mathrm{pr}_2^* \Big((\omega_E)_Q - (\mathrm{id}_{\mathfrak{g}^*})_Q \wedge \rho^*_{\mathfrak{g}^*} \operatorname{curv} \mathbf{A} \Big) \ .$$

Here pr_1 and pr_2 denote the canonical projections $\mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q \to \mathrm{T}^*(Q/G)$ and $\mathrm{T}^*(Q/G) \oplus \mathfrak{g}^*_Q \to \mathfrak{g}^*_Q$. The form ω_E denotes the canonical two-form on $\mathfrak{g}^*$ determined by E. The zero-form $(\mathrm{id}_{\mathfrak{g}^*})_Q \in \Omega^0(\mathfrak{g}^*_Q, \rho^*_{\mathfrak{g}^*} \mathfrak{g}^*_Q)$ denotes the form associated with the identity map $\mathrm{id}_{\mathfrak{g}^*} : \mathfrak{g}^* \to \mathfrak{g}^*$, viewed as an element of $\Omega^0(\mathfrak{g}^*, \mathfrak{g}^*)$. (If one makes the identification $\rho^*_{\mathfrak{g}^*} \mathfrak{g}^*_Q \cong \mathfrak{g}^*_Q \oplus \mathfrak{g}^*_Q$, then $(\mathrm{id}_{\mathfrak{g}^*})_Q(\eta) = \eta \oplus \eta$.) Recall that $\rho_{\mathfrak{g}^*} : \mathfrak{g}^*_Q \to Q/G$ denotes associated bundle projection.

The formula in (2) is easily verified by checking that $\omega_D(v, \cdot\,) = 0$ for $v \in D$, and by checking that the restriction of ω_D to a leaf P_μ coincides with the two-form on the right-hand side of (1).

7 The dynamic phase

For general G-invariant Hamiltonians $H : \mathrm{T}^*Q \to \mathbb{R}$ the formula for g_{dyn} in Theorem 3.5 follows from Theorem 2.3, Lemma 6.3, and the definition of D_{μ_0} given in 2.2. In this section we deduce the form taken by this phase in simple mechanical systems, as reported in Theorem 3.4.

7.1 The reduced Hamiltonian

The kinetic energy metric $\langle\!\langle \cdot\, , \cdot \rangle\!\rangle_Q$ induces an isomorphism $\mathrm{T}Q \xrightarrow{\sim} \mathrm{T}^*Q$ sending hor $\equiv \ker \mathbf{A}$ to $\mathbf{J}^{-1}(0)$ and $\ker \mathrm{T}\rho$ to hor° (see 4.2(1) and 4.2(2)). Since $\mathbf{J}^{-1}(0)_q = (\ker \mathrm{T}_q\rho)^\circ$, it is not too difficult to see that

$$(1) \qquad x \in \mathbf{J}^{-1}(0)_q \;\;\Rightarrow\;\; \langle\!\langle x, x \rangle\!\rangle_Q^* = \langle\!\langle \rho^\circ(x), \rho^\circ(x) \rangle\!\rangle_{Q/G}^* \;\; ,$$

where $\langle\!\langle \cdot\, , \cdot \rangle\!\rangle_{Q/G}^*$ is defined in 3.4 and ρ° is defined in 4.1.

If instead, $x \in \mathrm{hor}_q^\circ$, then x is the image under the isomorphism $\mathrm{T}Q \xrightarrow{\sim} \mathrm{T}^*Q$ of $\xi^Q(q)$, for some $\xi \in \mathfrak{g}$. For such ξ, and arbitrary $\eta \in \mathfrak{g}$, we compute

$$\langle \mathbf{J}(x), \eta \rangle = \langle x, \eta^Q(q) \rangle = \langle\!\langle \xi^Q(q), \eta^Q(q) \rangle\!\rangle_Q = \langle \hat{\mathbb{I}}(q)(\xi), \eta \rangle \;\; ,$$

where the first equality follows from 2.4(1). Since $\eta \in \mathfrak{g}$ is arbitrary, it follows that $\xi = \hat{\mathbb{I}}^{-1}(q)(\mathbf{J}(x))$. We now conclude that

$$(2) \qquad x \in \mathrm{hor}_q^\circ \;\;\Rightarrow\;\; \langle\!\langle x, x \rangle\!\rangle_Q^* = \langle\!\langle \xi^Q(q), \xi^Q(q) \rangle\!\rangle_Q = \langle \mathbf{J}(x), \hat{\mathbb{I}}^{-1}(q)(\mathbf{J}(x)) \rangle \;\; .$$

An arbitrary element $x \in \mathrm{T}_q^*Q$ decomposes into unique parts along $\mathbf{J}^{-1}(0)_q$ and hor_q°, the first component being $\mathbf{A}'(x)$. From (1) and (2) one deduces

$$(3) \quad \langle\!\langle x, x \rangle\!\rangle_Q^* = \langle\!\langle \mathrm{T}_{\mathbf{A}}^*\rho \cdot x, \mathrm{T}_{\mathbf{A}}^*\rho \cdot x \rangle\!\rangle_{Q/G}^* + \langle \mathbf{J}(x), \hat{\mathbb{I}}^{-1}(q)(\mathbf{J}(x)) \rangle \qquad (x \in \mathrm{T}_q^*Q) \;\; .$$

Define $h : \mathrm{T}^*(Q/G) \oplus \mathfrak{g}_Q^* \to \mathbb{R}$ by

$$(4) \qquad h(z \oplus [q, \mu]_G) = \frac{1}{2} \langle\!\langle z, z \rangle\!\rangle_{Q/G}^* + \frac{1}{2} \langle \mu, \hat{\mathbb{I}}^{-1}(q)\mu \rangle + V_{Q/G}(\rho(q)) \;\; ,$$

where $V_{Q/G}$ denotes the function on Q/G to which V drops on account of its G-invariance. With the help of (3), one checks that $H = h \circ \pi$, i.e., h is the Poisson-reduced Hamiltonian. Substituting (4) into 3.5(1) delivers the formula

$$(5) \qquad D_{\mu_0} h(z \oplus [q, \mu_0]_G) = \mathrm{pr}_{\mu_0} \hat{\mathbb{I}}^{-1}(q)\mu_0 \;\; ,$$

where $\mathrm{pr}_{\mu_0} : \mathfrak{g} \to \mathfrak{g}_{\mu_0}$ denotes the orthogonal projection.

To establish the formula for g_{dyn} in Theorem 3.4 it remains to show that

$$(6) \qquad \Big((\mathrm{Pr}_{\mu_0})_Q \wedge \xi_{\mathbb{I}} \Big)([q, \mu_0]_G) = \mathrm{pr}_{\mu_0} \hat{\mathbb{I}}^{-1}(q)\mu_0 \;\; ,$$

where $\xi_{\mathbb{I}} \equiv \rho_{\mathcal{O}}^* \mathbb{I}^{-1} \wedge (i_{\mathcal{O}})_Q$. We will be ready to do so after providing the general definition of associated forms alluded to in 3.2.

7.2 Associated forms (general case)

Let V be a real vector space on which G acts linearly and $\mathcal{O}$ an arbitrary manifold on which G acts smoothly. Let λ be a V-valued k-form on $\mathcal{O}$. For the sake of clarity, we will suppose $k = 1$; the extension to general k will be obvious.

Assuming that $\lambda \in \Omega^1(\mathcal{O}, V)$ is equivariant in the sense that

$$\lambda(g \cdot u) = g \cdot \lambda(u) \qquad (g \in G,\, u \in \mathrm{T}\mathcal{O}) \ ,$$

we will construct a *bundle*-valued differential form $\lambda_Q \in \Omega^1(\mathcal{O}_Q, \rho_{\mathcal{O}}^* V_Q)$ called the *associated form*. Recall that $\rho_{\mathcal{O}} : \mathcal{O}_Q \to Q/G$ denotes the projection of the associated bundle $\mathcal{O}_Q \equiv (Q \times \mathcal{O})/G$, and $\rho_{\mathcal{O}}^*$ denotes pullback. As always, we assume $\rho : Q \to Q/G$ is equipped with a connection one-form $\mathbf{A}$.

We begin by noting that an arbitrary vector tangent to $\rho_{\mathcal{O}}^* Q \equiv \mathcal{O}_Q \oplus_{Q/G} Q$ is of the form

$$\frac{d}{dt}\, [q^{\mathrm{hor}}(t), \nu(t)]_G \oplus \exp(t\xi) \cdot q^{\mathrm{hor}}(t) \,\Big|_{t=0} \ , \tag{1}$$

for some $\xi \in \mathfrak{g}$, some $\mathbf{A}$-horizontal curve $t \mapsto q^{\mathrm{hor}}(t) \in Q$, and some curve $t \mapsto \nu(t) \in \mathcal{O}$. Define $\Lambda \in \Omega^1(\rho_{\mathcal{O}}^* Q, V)$ by

$$\Lambda\left(\frac{d}{dt}\, [q^{\mathrm{hor}}(t), \nu(t)]_G \oplus \exp(t\xi) \cdot q^{\mathrm{hor}}(t) \,\Big|_{t=0} \right) \equiv \lambda\left(\frac{d}{dt}\, \nu(t) \,\Big|_{t=0} \right) \ .$$

As the reader is left to verify, the equivariance of λ ensures that Λ is well defined. Now $\rho_{\mathcal{O}}^* Q \equiv \mathcal{O}_Q \oplus_{Q/G} Q$ is a principal G-bundle (G acts according $g \cdot (\eta \oplus q) \equiv \eta \oplus (g \cdot q)$) and we claim that Λ is tensorial.

Proof that Λ is tensorial. The (tangent-lifted) action of G on $\mathrm{T}(\rho_{\mathcal{O}}^* Q)$ is given by

$$\begin{aligned} & g \cdot \frac{d}{dt}\, [q^{\mathrm{hor}}(t), \nu(t)]_G \oplus \exp(t\xi) \cdot q^{\mathrm{hor}}(t) \,\Big|_{t=0} \\ =& \frac{d}{dt}\, [q^{\mathrm{hor}}(t), \nu(t)]_G \oplus g \exp(t\xi) \cdot q^{\mathrm{hor}}(t) \,\Big|_{t=0} \\ =& \frac{d}{dt}\, [g \cdot q^{\mathrm{hor}}(t), g \cdot \nu(t)]_G \oplus \exp(t g \cdot \xi) \cdot (g \cdot q^{\mathrm{hor}}(t)) \,\Big|_{t=0} \ . \end{aligned}$$

Since $t \mapsto g \cdot q^{\mathrm{hor}}(t)$ is $\mathbf{A}$-horizontal, it follows that

$$\begin{aligned} & \Lambda\left(g \cdot \frac{d}{dt}\, [q^{\mathrm{hor}}(t), \nu(t)]_G \oplus \exp(t\xi) \cdot q^{\mathrm{hor}}(t) \,\Big|_{t=0} \right) \\ & \qquad = \lambda\left(\frac{d}{dt}\, g \cdot \nu(t) \,\Big|_{t=0} \right) = g \cdot \lambda\left(\frac{d}{dt}\, \nu(t) \,\Big|_{t=0} \right) \ , \end{aligned}$$

where the second quality follows from the equivariance of λ. What we have just shown is that

$$\Lambda(g \cdot u) = g \cdot \Lambda(u) \qquad (g \in G)$$

for arbitrary $u \in \mathrm{T}(\rho_{\mathcal{O}}^* Q)$, i.e., Λ is equivariant. Also, the generic tangent vector in (1) is vertical (in the principal bundle $\rho_{\mathcal{O}}^* Q \to \mathcal{O}_Q$) if and only if $d/dt\, [q^{\mathrm{hor}}(t), \nu(t)]_G\, |_{t=0} = 0$. This is true if and only if $d/dt\, \nu(t)\, |_{t=0} = 0$. It follows that Λ vanishes on vertical vectors. This fact and the forementioned equivariance establishes that Λ is tensorial. □

Because $\Lambda \in \Omega^1(\rho_{\mathcal{O}}^* Q, V)$ is tensorial, it drops to an element of $\Omega^1(\mathcal{O}_Q, \rho_{\mathcal{O}}^* V_Q)$, which is the sought after associated form λ_Q. By construction one has the implicit formula

$$(2) \quad \lambda_Q \left(\frac{d}{dt} [q^{\mathrm{hor}}(t), \nu(t)]_G \Big|_{t=0} \right) = \left[[q, \nu]_G \oplus q, \lambda \left(\frac{d}{dt}\, \nu(t) \Big|_{t=0} \right) \right]_G \ ,$$

where $q \equiv q^{\mathrm{hor}}(0)$ and $\nu \equiv \nu(0)$.

Formula (2) is for a one-form λ. From the zero-form analogue of (2), one deduces

$$(3) \qquad (\mathrm{Pr}_{\mu_0})_Q([q, \mu_0]_G) = [[q, \mu_0]_G \oplus q, \mathrm{pr}_{\mu_0}]_G$$

$$(4) \qquad (i_{\mathcal{O}})_Q([q, \mu_0]_G) = [[q, \mu_0]_G \oplus q, \mu_0]_G \ .$$

Since

$$(\rho_{\mathcal{O}}^* \mathbb{I}^{-1})([q, \mu_0]_G) = [[q, \mu_0]_G \oplus q, \hat{\mathbb{I}}(q)]_G \ ,$$

we deduce

$$(\mathrm{Pr}_{\mu_0})_Q(\rho_{\mathcal{O}}^* \mathbb{I}^{-1} \wedge (i_{\mathcal{O}})_Q)([q, \mu_0]_G) = \mathrm{pr}_{\mu_0}\, \hat{\mathbb{I}}^{-1}(q) \mu_0 \ ,$$

which proves 7.1(6).

8 The geometric phase

This section derives the formula for g_{geom} reported in Theorem 3.4. We will carry out several computations, some of them somewhat involved. However, our objective throughout is clear: To apply the formula for g_{geom} in 2.3 we must calculate the transverse derivative $D_{\mu_0} \omega_D$ of the leaf symplectic structures $\omega_\mu = \omega_D | P_\mu$. To do so we must first compute $d\omega_D$. Our preference for a coordinate free proof leads us to lift the computation to a bigger space, which we do with the help of the 'slice' S for the co-adjoint action delivered by 6.1(6).

Using the fact that d is an antiderivation, that d commutes with pullbacks, and that $d\omega_{Q/G} = 0$, we obtain from 6.4(2)

$$(1) \quad d\omega_D = \mathrm{pr}_2^* (\, d(\omega_E)_Q - d(\mathrm{id}_{\mathfrak{g}^*})_Q \wedge \rho_{\mathfrak{g}^*}^* \,\mathrm{curv}\, \mathbf{A} - (\mathrm{id}_{\mathfrak{g}^*})_Q \wedge \rho_{\mathfrak{g}^*}^* d \,\mathrm{curv}\, \mathbf{A}\,) \ .$$

Note here that we are using the exterior derivative in the generalized sense of *bundle*-valued forms, as defined with respect to the connection $\mathbf{A}$; see A.5, Appendix A. The last term in parentheses is immediately dispensed with, for one has Bianchi's identity[2]

$$d\,\mathrm{curv}\,\mathbf{A} = 0 \quad . \tag{2}$$

To write down formulas for other terms in (1), it will be convenient to have an appropriate representation for vectors tangent to $\mathfrak{g}^*_Q$. Indeed, as the reader will readily verify, each such vector is of the form

$$\frac{d}{dt}\,[q^{\mathrm{hor}}(t), \mu(t)]_G\,\Big|_{t=0} \ ,$$

for some $\mathbf{A}$-horizontal curve $t \mapsto q^{\mathrm{hor}}(t) \in Q$ and some curve $t \mapsto \mu(t) \in \mathfrak{g}^*$. On occasion, and without loss of generality, we will take $\mu(t)$ to be of the form

$$\mu(t) = \exp(t\xi)\cdot(\mu + tv) \ ,$$

for some $\xi \in \mathfrak{g}$, $\mu \in \mathfrak{g}^*$ and $v \in \mathrm{forg}\,E(\mu)$ (see Proposition 6.1). A straightforward computation gives

$$d(\mathrm{id}_{\mathfrak{g}^*})_Q\left(\frac{d}{dt}\,[q^{\mathrm{hor}}(t), \mu(t)]_G\,\Big|_{t=0}\right) = [[q^{\mathrm{hor}}(0), \mu(0)]_G \oplus q^{\mathrm{hor}}(0), \dot\mu(0)]_G \in \rho^*_{\mathfrak{g}^*}\mathfrak{g}^*_Q,$$

where $\dot\mu(0) \equiv d/dt\,\mu(t)\,|_{t=0} \in \mathfrak{g}^*$. From this follows the formula

$$\begin{aligned}(d(\mathrm{id}_{\mathfrak{g}^*})_Q \wedge \rho^*_{\mathfrak{g}^*}\,\mathrm{curv}\,\mathbf{A})&\left(\frac{d}{dt}\,[q_1^{\mathrm{hor}}(t), \mu_1(t)]_G\,\Big|_{t=0}, \ldots, \frac{d}{dt}\,[q_3^{\mathrm{hor}}(t), \mu_3(t)]_G\,\Big|_{t=0}\right)\\ &= \left\langle\, \dot\mu_1(0),\, \mathbf{DA}(\dot q_2^{\mathrm{hor}}(0), \dot q_3^{\mathrm{hor}}(0))\,\right\rangle\\ &+ \left\langle\, \dot\mu_2(0),\, \mathbf{DA}(\dot q_3^{\mathrm{hor}}(0), \dot q_1^{\mathrm{hor}}(0))\,\right\rangle\\ &+ \left\langle\, \dot\mu_3(0),\, \mathbf{DA}(\dot q_1^{\mathrm{hor}}(0), \dot q_2^{\mathrm{hor}}(0))\,\right\rangle \ ,\end{aligned} \tag{3}$$

where $\mathbf{D}$ denotes exterior covariant derivative and $\dot q_j^{\mathrm{hor}}(0) \equiv d/dt\, q_j^{\mathrm{hor}}(t)\,|_{t=0}$. To compute $d(\omega_E)_Q$ is not so straightforward.[3] The difficulty lies partly in the fact that the co-adjoint orbit symplectic structures, which ω_E 'collects together,' are defined *implicitly* in terms of the infinitesimal generators of the co-adjoint action, and this action is generally not free. We overcome this by pulling $(\omega_E)_Q$ back to a 'bigger' space where we can be explicit. We compute

[2]Perhaps the better known form of this identity is $\mathbf{D}(\mathbf{DA}) = 0$, where $\mathbf{D}$ denotes exterior covariant derivative (see, e.g., [12, Theorem II.5.4]). Since, in the notation of Appendix A, $\mathbf{DA} = (\mathrm{curv}\,\mathbf{A})^\frown$, it follows that $(d\,\mathrm{curv}\,\mathbf{A})^\frown = 0$, which in turn implies (2).

[3]The exterior derivative d does *not* commute with the formation of associated forms!

the derivative in the bigger space and then drop to $\mathfrak{g}^*_Q$. Here is the formula we will derive:

$$
\begin{aligned}
(4)\qquad & d(\omega_E)_Q\biggl(\frac{d}{dt}\,[q_1^{\mathrm{hor}}(t),\exp(t\xi_1)\cdot(\mu+tv_1)]_G\Big|_{t=0},\\
&\qquad \frac{d}{dt}\,[q_2^{\mathrm{hor}}(t),\exp(t\xi_2)\cdot(\mu+tv_2)]_G\Big|_{t=0},\\
&\qquad \frac{d}{dt}\,[q_3^{\mathrm{hor}}(t),\exp(t\xi_3)\cdot(\mu+tv_3)]_G\Big|_{t=0}\biggr)\\
&= -\langle v_1,[\xi_2,\xi_3]\rangle-\langle v_2,[\xi_3,\xi_1]\rangle-\langle v_3,[\xi_1,\xi_2]\rangle\\
&\quad -\langle\mu,[\xi_1,\mathbf{DA}(\dot q_2^{\mathrm{hor}}(0),\dot q_3^{\mathrm{hor}}(0))]_G\rangle\\
&\quad -\langle\mu,[\xi_2,\mathbf{DA}(\dot q_3^{\mathrm{hor}}(0),\dot q_1^{\mathrm{hor}}(0))]_G\rangle\\
&\quad -\langle\mu,[\xi_3,\mathbf{DA}(\dot q_1^{\mathrm{hor}}(0),\dot q_2^{\mathrm{hor}}(0))]_G\rangle\\
&\quad \Bigl(\xi_j\in\mathfrak{g},\ \mu\in\mathfrak{g}^*_{\mathrm{reg}},\ v_j\in\operatorname{forg}E(\mu)\Bigr)\ ,
\end{aligned}
$$

where $q_1^{\mathrm{hor}}(0)=q_2^{\mathrm{hor}}(0)=q_3^{\mathrm{hor}}(0)\equiv q\in Q$. Note that we insist that μ lies in $\mathfrak{g}^*_{\mathrm{reg}}$. In other words, (4) is a formula for $(\omega_E)_Q$ on the open dense set $(\mathfrak{g}^*_{\mathrm{reg}})_Q\subset\mathfrak{g}^*_Q$.

Derivation of (4). With $\mu\in\mathfrak{g}^*_{\mathrm{reg}}$ fixed, let $S\subset V\subset\mathfrak{g}^*$ denote the corresponding 'slice' furnished by Proposition 6.1(6). Define the map

$$
\begin{aligned}
b: Q\times G\times S&\to\mathfrak{g}^*_Q\\
(q,g,s)&\mapsto[q,g\cdot s]_G\ .
\end{aligned}
$$

At each point $(q,g,s)\in Q\times G\times S$ we define, for each $(u,\eta,\xi,v)\in\mathrm{T}_{\rho(q)}(Q/G)\times\mathfrak{g}\times\mathfrak{g}\times V$, the tangent vector

$$
\begin{aligned}
&\langle u,\eta,\xi,v;\,q,g,s\rangle\equiv\\
&\qquad\frac{d}{dt}\Bigl(\exp(t\eta)\cdot q^{\mathrm{hor}}(t),\ \exp(t\xi)\cdot g,\ s+tv\Bigr)\Big|_{t=0}\in\mathrm{T}_{(q,g,s)}(Q\times G\times S)\ ,
\end{aligned}
$$

where $t\mapsto q^{\mathrm{hor}}(t)\in Q$ is any $\mathbf{A}$-horizontal curve satisfying

$$
\begin{aligned}
&q^{\mathrm{hor}}(0)=q\\
\text{and}\qquad&\frac{d}{dt}\rho(q^{\mathrm{hor}}(t))\Big|_{t=0}=u\ .
\end{aligned}
$$

Note that every vector tangent to $Q\times G\times S$ is of the above form, and that

$$
\begin{aligned}
(5)\quad \mathrm{T}b\cdot\langle u,\eta,\xi,v;\,q,g,s\rangle&=\frac{d}{dt}\,[\exp(t\eta)\cdot q^{\mathrm{hor}}(t),\ \exp(t\xi)g\cdot(s+tv)]_G\Big|_{t=0}\\
(6)\qquad\qquad\qquad&=\frac{d}{dt}\,[q^{\mathrm{hor}}(t),\ \exp(-t\eta)\exp(t\xi)g\cdot(s+tv)]_G\Big|_{t=0}\ .
\end{aligned}
$$

From (6) and the definition of associated forms 3.2(1), we obtain

$$(7)\qquad b^*(\omega_E)_Q\Big(\langle u_1,\eta_1,\xi_1,v_1;\,q,g,s\rangle,\,\langle u_2,\eta_2,\xi_2,v_2;\,q,g,s\rangle\Big)$$
$$=\omega_E\left(\frac{d}{dt}\,\exp(-t\eta_1)\exp(t\xi_1)g\cdot(s+tv_1)\Big|_{t=0},\right.$$
$$\left.\frac{d}{dt}\,\exp(-t\eta_2)\exp(t\xi_2)g\cdot(s+tv_2)\Big|_{t=0}\right)\ .$$

Now ω_E is the canonical two-form on $\mathfrak{g}^*$ determined by E and according to 6.1(6), we have

$$\frac{d}{dt}\,s+tv_j\Big|_{t=0}\in \mathrm{T}_sS=E(s)\qquad (j=1,2)\ .$$

It follows from (7) that

$$(8)\quad b^*(\omega_E)_Q\Big(\langle u_1,\eta_1,\xi_1,v_1;\,q,g,s\rangle,\,\langle u_2,\eta_2,\xi_2,v_2;\,q,g,s\rangle\Big)$$
$$=-\Big\langle\, g\cdot s,\,[\xi_1-\eta_1,\xi_2-\eta_2]\,\Big\rangle\qquad (u_j\in \mathrm{T}_{\rho(q)}(Q/G);\,\eta_j,\xi_j\in\mathfrak{g};\,v_j\in V)\ .$$

It is now that we see the reason for pulling $(\omega_E)_Q$ back to $Q\times G\times S$. For if we define natural projections

$$\pi_Q:Q\times G\times S\to Q:(q,g,s)\mapsto q$$
$$\pi_G:Q\times G\times S\to G:(q,g,s)\mapsto g$$
$$\pi_{\mathfrak{g}^*}:Q\times G\times S\to \mathfrak{g}^*:(q,g,s)\mapsto g\cdot s$$

and denote by $\theta_G\in\Omega^1(G,\mathfrak{g})$ the right-invariant Mauer-Cartan form on G, then (8) may be written intrinsically as

$$b^*(\omega_E)_Q=-\frac{1}{2}\pi_{\mathfrak{g}^*}\wedge\Big((\pi_G^*\theta_G-\pi_Q^*\mathbf{A})\wedge(\pi_G^*\theta_G-\pi_Q^*\mathbf{A})\Big)\ ,$$

where we view $\pi_{\mathfrak{g}^*}:Q\times G\times S\to\mathfrak{g}^*$ as an element of $\Omega^0(Q\times G\times S,\mathfrak{g}^*)$. We can now take d of both sides, obtaining

$$(9)\quad b^*d(\omega_E)_Q=$$
$$-\frac{1}{2}d\pi_{\mathfrak{g}^*}\wedge\Big((\theta'_G-\mathbf{A}'')\wedge(\theta'_G-\mathbf{A}'')\Big)+\pi_{\mathfrak{g}^*}\wedge\Big((\theta'_G-\mathbf{A}'')\wedge d(\theta'_G-\mathbf{A}'')\Big)\ ,$$

where a single prime indicates pullback by π_G, and a double prime indicates pullback by π_Q. We expand and simplify (9) by invoking the following identities:

$$(10)\qquad d\theta'_G=\frac{1}{2}\theta'_G\wedge\theta'_G\ ,$$

$$(11)\qquad d\mathbf{A}''=(\mathbf{DA})''+\frac{1}{2}\mathbf{A}''\wedge\mathbf{A}''\ ,$$

$$(12)\qquad \theta'_G\wedge(\theta'_G\wedge\theta'_G)=0\ ,$$

$$(13)\qquad \mathbf{A}''\wedge(\mathbf{A}''\wedge\mathbf{A}'')=0\ .$$

If the primes are suppressed, then (10) and (11) are the Mauer-Cartan equations for G and the principal bundle Q resp., while (12) and (13) follow from Jacobi's identity. That we may add the primes follows from the fact that d commutes with pullbacks, and that pullbacks distribute over wedge products. After some manipulation, Equation (9) becomes

$$\begin{aligned} b^* d(\omega_E)_Q = &-\frac{1}{2} d\pi_{\mathfrak{g}^*} \wedge (\mathbf{A}'' \wedge \mathbf{A}'') - \frac{1}{2} d\pi_{\mathfrak{g}^*} \wedge (\theta'_G \wedge \theta'_G) \\ &+ \pi_{\mathfrak{g}^*} \wedge \Big(\mathbf{A}'' \wedge (\mathbf{DA})'' \Big) - \pi_{\mathfrak{g}^*} \wedge \Big(\theta'_G \wedge (\mathbf{DA})'' \Big) \\ &- \frac{1}{2} \pi_{\mathfrak{g}^*} \wedge \Big(\mathbf{A}'' \wedge (\theta'_G \wedge \theta'_G) \Big) - \frac{1}{2} \pi_{\mathfrak{g}^*} \wedge \Big(\theta'_G \wedge (\mathbf{A}'' \wedge \mathbf{A}'') \Big) \ . \end{aligned} \tag{14}$$

For future reference, we note here the easily computed formula

$$d\pi_{\mathfrak{g}^*}(\langle u, \eta, \xi, v;\, q, g, s \rangle) = -\operatorname{ad}^*_\xi(g \cdot s) + g \cdot v \ . \tag{15}$$

By (5), we have

$$\frac{d}{dt} \left[q^{\mathrm{hor}}(t), \exp(t\xi) \cdot (\mu + tv) \right]_G \Big|_{t=0} = \mathrm{T}b \cdot \langle u, 0, \xi, v;\, q, \mathrm{id}, \mu \rangle \ ,$$

so that

$$\begin{aligned} d(\omega_E)_Q \bigg(&\frac{d}{dt} \left[q_1^{\mathrm{hor}}(t), \exp(t\xi_1) \cdot (\mu + tv_1) \right]_G \Big|_{t=0}, \\ &\frac{d}{dt} \left[q_2^{\mathrm{hor}}(t), \exp(t\xi_2) \cdot (\mu + tv_2) \right]_G \Big|_{t=0}, \\ &\frac{d}{dt} \left[q_3^{\mathrm{hor}}(t), \exp(t\xi_3) \cdot (\mu + tv_3) \right]_G \Big|_{t=0} \bigg) \\ = b^* d(\omega_E)_Q(&\langle u_1, 0, \xi_1, v_1;\, q, \mathrm{id}, \mu \rangle, \langle u_2, 0, \xi_2, v_2;\, q, \mathrm{id}, \mu \rangle, \langle u_3, 0, \xi_3, v_3;\, q, \mathrm{id}, \mu \rangle), \end{aligned}$$

We now substitute the formula for $b^* d(\omega_E)_Q$ in (14). In fact, since

$$\mathbf{A}''(\langle u_j, 0, \xi_j, v_j;\, q, \mathrm{id}, \mu \rangle) = 0 \qquad (j = 1,\ 2 \text{ or } 3) \ ,$$

the only part on the right-hand side of (14) with a nontrivial contribution is

$$-\frac{1}{2} d\pi_{\mathfrak{g}^*} \wedge (\theta'_G \wedge \theta'_G) - \pi_{\mathfrak{g}^*} \wedge \Big(\theta'_G \wedge (\mathbf{DA})'' \Big)$$

and we obtain, with the help of (15),

$$
\begin{aligned}
d(\omega_E)_Q \Big(& \frac{d}{dt}\, [q_1^{\rm hor}(t), \exp(t\xi_1)\cdot(\mu + tv_1)]_G \Big|_{t=0}, \\
& \frac{d}{dt}\, [q_2^{\rm hor}(t), \exp(t\xi_2)\cdot(\mu + tv_2)]_G \Big|_{t=0}, \\
& \frac{d}{dt}\, [q_3^{\rm hor}(t), \exp(t\xi_3)\cdot(\mu + tv_3)]_G \Big|_{t=0} \Big) \\
= {} & \langle \mathrm{ad}^*_{\xi_1} \mu - v_1, [\xi_2, \xi_3] \rangle \ + \ \text{cyclic terms} \\
& - \ \langle \mu, [\xi_1, \mathbf{DA}(\dot q_2^{\rm hor}(0), \dot q_3^{\rm hor}(0))] \rangle \ - \ \text{cyclic terms} \\
= {} & \langle \mu, [\xi_1, [\xi_2, \xi_3]] \ + \ \text{cyclic terms} \rangle \\
& - \ \langle v_1, [\xi_2, \xi_3] \rangle \ - \ \text{cyclic terms} \\
& - \ \langle \mu, [\xi_1, \mathbf{DA}(\dot q_2^{\rm hor}(0), \dot q_3^{\rm hor}(0))] \rangle \ - \ \text{cyclic terms} \ .
\end{aligned}
$$

The term appearing in the third last row vanishes, by Jacobi's identity, and what is left amounts to Equation (4). ☐

One computes, using Lemma 6.3 and the definition of D_{μ_0} in 2.2,

$$
\begin{aligned}
\Big\langle \nu, D_{\mu_0}\omega_D \Big(& \frac{d}{dt}\, z_1(t) \oplus [q_1^{\rm hor}(t), \exp(t\xi_1)\cdot \mu_0]_G \Big|_{t=0}, \\
& \frac{d}{dt}\, z_2(t) \oplus [q_2^{\rm hor}(t), \exp(t\xi_2)\cdot \mu_0]_G \Big|_{t=0} \Big) \Big\rangle \\
= d\omega_D \Big(& \frac{d}{dt}\, z \oplus [q, \mu_0 + t\iota_{\mu_0}^{-1}(\nu)]_G \Big|_{t=0}, \\
& \frac{d}{dt}\, z_1(t) \oplus [q_1^{\rm hor}(t), \exp(t\xi_1)\cdot \mu_0]_G \Big|_{t=0}, \\
& \frac{d}{dt}\, z_2(t) \oplus [q_2^{\rm hor}(t), \exp(t\xi_2)\cdot \mu_0]_G \Big|_{t=0} \Big) \\
= \quad & - \Big\langle \iota_{\mu_0}^{-1}(\nu),\, [\xi_1, \xi_2] \Big\rangle - \Big\langle \iota_{\mu_0}^{-1}(\nu),\, \mathbf{DA}(\dot q_1^{\rm hor}(0), \dot q_2^{\rm hor}(0)) \Big\rangle \\
= \quad & \Big\langle \nu, -\operatorname{pr}_{\mu_0} \Big([\xi_1, \xi_2] + \mathbf{DA}(\dot q_1^{\rm hor}(0), \dot q_2^{\rm hor}(0) \Big) \Big\rangle \ .
\end{aligned}
$$

The second equality follows from Equations (1)–(4) derived above; the last equality follows from 6.1(4). Since $\nu \in \mathfrak{g}^*_{\mu_0}$ in this computation is arbitrary, we conclude that

$$
\begin{aligned}
(16) \qquad D_{\mu_0}\omega_D \Big(& \frac{d}{dt}\, z_1(t) \oplus [q_1^{\rm hor}(t), \exp(t\xi_1)\cdot \mu_0]_G \Big|_{t=0}, \\
& \frac{d}{dt}\, z_2(t) \oplus [q_2^{\rm hor}(t), \exp(t\xi_2)\cdot \mu_0]_G \Big|_{t=0} \Big) \\
= \qquad & -\operatorname{pr}_{\mu_0}[\xi_1, \xi_2] - \operatorname{pr}_{\mu_0} \mathbf{DA}(\dot q_1^{\rm hor}(0), \dot q_2^{\rm hor}(0)) \ .
\end{aligned}
$$

Proposition 2.8 and the definition 3.2(1) of associated forms delivers the formula

$$
\begin{aligned}
(17)\qquad (\operatorname{curv}\alpha_{\mu_0})_Q\Big(&\frac{d}{dt}\,[q_1^{\mathrm{hor}}(t),\exp(t\xi_1)\cdot\mu_0]_G\,\Big|_{t=0},\\
&\frac{d}{dt}\,[q_2^{\mathrm{hor}}(t),\exp(t\xi_2)\cdot\mu_0]_G\,\Big|_{t=0}\Big)=\mathrm{pr}_{\mu_0}[\xi_1,\xi_2]\ .
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
(\rho_{\mathcal{O}}^*\operatorname{curv}\mathbf{A})\Big(&\frac{d}{dt}\,[q_1^{\mathrm{hor}}(t),\exp(t\xi_1)\cdot\mu_0]_G\,\Big|_{t=0},\\
&\frac{d}{dt}\,[q_2^{\mathrm{hor}}(t),\exp(t\xi_2)\cdot\mu_0]_G\,\Big|_{t=0}\Big)\\
&=[[q,\mu_0]_G\oplus q,\mathbf{DA}(\dot q_1^{\mathrm{hor}}(0),\dot q_2^{\mathrm{hor}}(0))]_G\in\rho_{\mathcal{O}}^*\mathfrak{g}_Q\ .
\end{aligned}
$$

Combining this with 7.2(3) gives

$$
\begin{aligned}
(18)\qquad ((\mathrm{Pr}_{\mu_0})_Q\wedge\rho_{\mathcal{O}}^*\operatorname{curv}\mathbf{A})\Big(&\frac{d}{dt}\,[q_1^{\mathrm{hor}}(t),\exp(t\xi_1)\cdot\mu_0]_G\,\Big|_{t=0},\\
&\frac{d}{dt}\,[q_2^{\mathrm{hor}}(t),\exp(t\xi_2)\cdot\mu_0]_G\,\Big|_{t=0}\Big)\\
&=\mathrm{pr}_{\mu_0}\,\mathbf{DA}(\dot q_1^{\mathrm{hor}}(0),\dot q_2^{\mathrm{hor}}(0))\ .
\end{aligned}
$$

Comparing the right-hand side of (16) with the right-hand sides of (17) and (18), we deduce the intrinsic formula

$$
\begin{aligned}
(19)\qquad D_{\mu_0}\omega_D&=-\,\mathrm{pr}_2^*\Big((\operatorname{curv}\alpha_{\mu_0})_Q+(\mathrm{Pr}_{\mu_0})_Q\wedge\rho_{\mathcal{O}}^*\operatorname{curv}\mathbf{A}\Big)\\
&=-\,\mathrm{pr}_2^*(\Omega_{\mu_0}+(\mathrm{Pr}_{\mu_0})_Q\wedge\Omega_{\mathbf{A}})\ .
\end{aligned}
$$

The curve $t\mapsto\eta_t\in\mathcal{O}_Q$ in Theorem 3.4 is a closed embedded curve because it bounds the surface S. Because $z_t\oplus\eta_t$ and η_t have the same minimal period, it follows that there exists a smooth map $s:\partial S\to\mathrm{T}^*(Q/G)\oplus\mathcal{O}_Q$ such that $s(\eta_t)=z_t\oplus\eta_t$. As $\mathrm{pr}_2:\mathrm{T}^*(Q/G)\oplus\mathcal{O}_Q\to\mathcal{O}_Q$ is a vector bundle, the map s can be extended to a global section $s:\mathcal{O}_Q\to\mathrm{T}^*(Q/G)\oplus\mathcal{O}_Q$ of pr_2. This follows, for example, from [12, Theorem I.5.7]. Define $\Sigma\equiv s(S)$, so that $\mathrm{pr}_2(\Sigma)=S$ and $t\mapsto z_t\oplus\eta_t$ is the boundary of Σ. Appealing to Theorem 2.3 and (19), we obtain

$$
\begin{aligned}
g_{\mathrm{geom}}&=\exp\int_\Sigma D_{\mu_0}\omega_D=\exp\Big(-\int_\Sigma\mathrm{pr}_2^*\,(\Omega_{\mu_0}+(\mathrm{Pr}_{\mu_0})_Q\wedge\Omega_{\mathbf{A}}\,)\Big)\\
&=\exp\Big(-\int_S(\,\Omega_{\mu_0}+(\mathrm{Pr}_{\mu_0})_Q\wedge\Omega_{\mathbf{A}}\,)\Big)\ ,
\end{aligned}
$$

which is the form of g_{geom} given in Theorem 3.4.

A On bundle-valued differential forms

The exterior calculus of differential forms taking values in a vector bundle is ordinarily constructed via Koszul (or 'affine') connections. See, for example, [10, Chap. 9] or [13, Chap. 17]. On the other hand, given an associated vector bundle V_Q (see 3.1 for notation), one can model the exterior calculus of V_Q-valued forms on the exterior covariant calculus of tensorial V-valued forms on Q. In place of a Koszul connection, one prescribes a principal connection on Q. (The corresponding Koszul connection ∇ appears as the $p = 0$ case of Lie derivatives of bundle-valued p-forms; see A.7.) When the vector bundle at hand is realized as an associated bundle, this latter approach, while equivalent to the former, is better suited to explicit computations. As we are unaware of a readily accessible account of it, we outline the basics here.

A.1 Notation

Let $\xi : E \to B$ be a vector bundle with base B and consider the (Abelian) category of real vector bundles over B, restricting attention to morphisms covering the identity on B. Denote by $\mathrm{Alt}^p(\mathrm{T}B, E)$ the bundle over B of all alternating p-linear bundle morphisms from $\mathrm{T}B \oplus \cdots \oplus \mathrm{T}B$ into E . Then an *E-valued differential p-form* is a smooth section of $\mathrm{Alt}^p(\mathrm{T}B, E) \to B$. The space of all such forms is denoted $\Omega^p(B, E)$.

A.2 Bundle-valued forms as tensorial vector-valued forms

Let $\rho : Q \to B$ be a principal G-bundle equipped with a connection one-form $\mathbf{A}$ and let V be a real vector space on which G acts linearly. Let $\Omega^p_{\mathrm{tens}}(Q, V)$ denote the space of tensorial V-valued forms on Q (for a definition of tensorial forms see, e.g., [12, Section II.5]). Here, as elsewhere, all actions are understood to be *left* actions (contrary to the convention adopted in [12]). As is well known, one has an isomorphism

$$\begin{gathered} \lambda \mapsto \hat{\lambda} \\ \Omega^p(B, V_Q) \xrightarrow{\sim} \Omega^p_{\mathrm{tens}}(Q, V) \end{gathered}$$

defined implicitly through the formula

$$\lambda(\mathrm{T}\rho \cdot u_1, \ldots, \mathrm{T}\rho \cdot u_p) = [q, \hat{\lambda}(u_1, \ldots, u_p)]_G \qquad (u_j \in \mathrm{T}_q Q,\ q \in Q) \ . \tag{1}$$

A.3 Pullbacks

If $f : B' \to B$ is a smooth map, then the pullback f^*Q of the principal bundle Q is defined by

$$f^*Q \equiv B' \oplus_B Q$$

(see 4.2 for notation). The manifold f^*Q is itself a principal G-bundle; its base space is B', the bundle projection is $b' \oplus q \to b'$, and G acts according to

$g \cdot (b' \oplus q) \equiv (b' \oplus g \cdot q)$. One defines a map $\hat{f} : f^*Q \to Q$ by $\hat{f}(b' \oplus q) \equiv q$ and has the commutative diagram

$$\begin{array}{ccc} f^*Q & \xrightarrow{\hat{f}} & Q \\ \downarrow & & \downarrow \rho \\ B' & \xrightarrow[f]{} & B \end{array} \quad .$$

A connection one-form for $f^*Q \to B'$ is $\hat{f}^*\mathbf{A}$.
If $g : B'' \to B'$ is a second map, then a natural isomorphism $(f \circ g)^*Q \cong g^*(f^*Q)$ is given by

$$\begin{gathered} B'' \oplus_B Q \xrightarrow{\sim} B'' \oplus_{B'} (B' \oplus_B Q) \\ b'' \oplus q \to b'' \oplus (g(b'') \oplus q) \ . \end{gathered}$$

The pullback f^*V_Q of an associated vector bundle V_Q can be defined analogously but we will define it in a way making the pullback itself an associated bundle:

$$f^*V_Q \equiv V_{f^*Q} \ .$$

By the above we have $(f \circ g)^*V_Q \cong g^*(f^*V_Q)$. This definition of f^*V_Q is equivalent to the forementioned alternative, for we have an isomorphism

$$\begin{gathered} f^*V_Q \xrightarrow{\sim} B' \oplus_B V_Q \\ [b' \oplus q, v]_G \mapsto b' \oplus [q, v]_G \ . \end{gathered}$$

The map $f : B' \to B$ defines a pullback operator on forms $f^* : \Omega^p(B, V_Q) \to \Omega^p(B', f^*V_Q)$ defined through

$$(f^*\lambda)\widehat{} = \hat{f}^*\hat{\lambda} \ ,$$

where the pullback on the right-hand side is the usual one for vector-valued forms. Making the identification $(f \circ g)^*V_Q \cong g^*(f^*V_Q)$ indicated above, we have $(f \circ g)^* = g^* \circ f^*$.

A.4 Wedge products

The wedge product $\lambda \wedge \mu \in \Omega^{p+q}(B, (U \otimes V)_Q)$ of forms $\lambda \in \Omega^p(B, U_Q)$ and $\mu \in \Omega^q(B, V_Q)$ is defined through

$$(\lambda \wedge \mu)\widehat{} = \hat{\lambda} \wedge \hat{\mu} \ .$$

Suppose there is a natural, bilinear pairing $(u, v) \mapsto \langle u, v \rangle : U \times V \to W$ that is equivariant in the sense that $\langle g \cdot u, g \cdot v \rangle = g \cdot \langle u, v \rangle$. Then there is a G-invariant homomorphism $U \otimes V \to W$ allowing one to identify $\hat{\lambda} \wedge \hat{\mu}$ with an element of $\Omega^{p+q}_{\text{tens}}(Q, W)$; $\lambda \wedge \mu$ is correspondingly identified with an element of $\Omega^{p+q}(B, W_Q)$. In the special case that G acts trivially on W (e.g., $W = \mathbb{R}$), one has $W_Q \cong W \times Q$ and there is a further identification $\Omega^{p+q}(B, W_Q) \cong \Omega^{p+q}(B, W)$.

A.5 Exterior derivatives

The exterior derivative $d\lambda \in \Omega^{p+1}(B, V_Q)$ of a form $\lambda \in \Omega^p(B, V_Q)$ is defined through

$$(d\lambda)\widehat{\ }= \mathbf{D}\hat{\lambda} \ ,$$

where $\mathbf{D}$ denotes exterior covariant derivative with respect to the connection $\mathbf{A}$ (see [12]).

A.6 Curvature

We next define the *curvature form* $\mathbf{B}_V$, which measures the degree to which Poincaré's identity $d^2 = 0$ fails for V_Q-valued differential forms.

By its equivariance, a tensorial zero-form $F \in \Omega^0_{\text{tens}}(Q, V)$ satisfies the identity

$$dF\left(\frac{d}{dt}\exp(t\xi)\cdot q\Big|_{t=0}\right) = \operatorname{ad}^V_\xi F(q) \qquad (\xi \in \mathfrak{g},\, q \in Q) \ ,$$

where ad^V_ξ denotes the infinitesimal generator of the linear action of G on V along ξ, viewed as an element of $\operatorname{Hom}(V, V)$. From the definition of exterior covariant derivative, one deduces the identity $\mathbf{D}F = dF - \mathbf{A}_V \wedge F$, where $\mathbf{A}_V \in \Omega^1(Q, \operatorname{Hom}(V, V))$ is defined by

$$\mathbf{A}_V(u) \equiv \operatorname{ad}^V_{\mathbf{A}(u)} \qquad (u \in \mathrm{T}Q) \ .$$

It follows that $\mathbf{D}^2 F = -\mathbf{D}\mathbf{A}_V \wedge F$. Note that by the linearity of $\xi \mapsto \operatorname{ad}^V_\xi$, we have

$$\mathbf{D}\mathbf{A}_V(u_1, u_2) = \operatorname{ad}^V_{\mathbf{D}\mathbf{A}(u_1,u_2)} \qquad (u_1, u_2 \in \mathrm{T}_q Q,\, q \in Q) \ .$$

The two-form $\mathbf{D}\mathbf{A}_V$ is tensorial (with G acting on $\operatorname{Hom}(V, V)$ by conjugation), and so defines a two-form $\mathbf{B}_V \in \Omega^2(B, \operatorname{Hom}(V, V)_Q)$ through[4]

$$\hat{\mathbf{B}}_V = -\mathbf{D}\mathbf{A}_V \ ,$$

allowing us to write $\mathbf{D}^2 F = \hat{\mathbf{B}}_V \wedge F$. Moreover, one can show that F in this identity can be replaced by an arbitrary, tensorial, V-valued p-form. One does so using the fact that such a form is an $\mathbb{R}$-linear combination of products of the form $\omega \wedge F$, for some $\omega \in \Omega^p_{\text{tens}}(Q, \mathbb{R})$ and $F \in \Omega^0_{\text{tens}}(Q, V)$. In particular, replacing F by $\hat{\lambda}$ ($\lambda \in \Omega^p(B, V_Q)$), one deduces the important identity

(1) $$d^2\lambda = \mathbf{B}_V \wedge \lambda \qquad (\lambda \in \Omega^p(B, V_Q)) \ .$$

[4]We have inserted a minus sign in the formula defining $\mathbf{B}_V$ to ensure that the identity (1) conforms with the case of *right* principal bundles, as well as the theory as developed via Koszul connections.

Notice that $d^2 = 0$ if and only if $\mathbf{B}_V = 0$, which is true if and only if G acts trivially on V or $\mathbf{A}$ is a flat connection.

Note. The two-form $\operatorname{curv}\mathbf{A} \in \Omega^2(B, \mathfrak{g}_Q)$ defined through $(\operatorname{curv}\mathbf{A})\widehat{\ } = \mathbf{DA}$ is known as the *curvature of* $\mathbf{A}$. It is related to $\mathbf{B}_\mathfrak{g}$ in the following way: $\mathbf{B}_\mathfrak{g}$ is the image of $\operatorname{curv}\mathbf{A}$ under the natural map $\Omega^2(B, \mathfrak{g}_Q) \to \Omega^2(B, \operatorname{Hom}(\mathfrak{g}, \mathfrak{g})_Q)$ induced by $\xi \mapsto -\operatorname{ad}_\xi : \mathfrak{g} \to \operatorname{Hom}(\mathfrak{g}, \mathfrak{g})$.

A.7 Lie derivatives and interior products

If X is a vector field on B and X^h denotes its $\mathbf{A}$-horizontal lift to a vector field on Q, then the (covariant) Lie derivative $\mathcal{L}_X : \Omega^p(B, V_Q) \to \Omega^p(B, V_Q)$ (often denoted ∇_X) is defined through

$$(\mathcal{L}_X \lambda)\widehat{\ } = \mathcal{L}_{X^h} \hat{\lambda} \qquad (\lambda \in \Omega^p(B, V_Q)) \ ,$$

where $\mathcal{L}_{X^h} : \Omega^p(Q, V) \to \Omega^p(Q, V)$ is the standard Lie derivative along X^h. The interior product (contraction) $X \lrcorner \lambda$ of a vector field X on B with a V_Q-valued p-form λ satisfies the identity

$$(X \lrcorner \lambda)\widehat{\ } = X^h \lrcorner \hat{\lambda} \ .$$

A.8

Many familiar identities generalize to the vector bundle case, despite the fact that $d^2 \neq 0$. These are derived by simply dropping the appropriate identity for tensorial forms. For example, one has the well known identity

$$\frac{d}{dt} \exp\Big(t\mathbf{DA}(X^h(q), Y^h(q))\Big) \cdot q\,\Big|_{t=0} = [X, Y]^h(q) - [X^h, Y^h](q) \qquad (q \in Q) \ ,$$

which, for an arbitrary zero-form $F \in \Omega^0(B, V_Q)$, implies

$$\mathbf{DA}_V(X^h, Y^h) \wedge \hat{F} = \mathcal{L}_{[X,Y]^h} \hat{F} - \mathcal{L}_{X^h} \mathcal{L}_{Y^h} \hat{F} + \mathcal{L}_{Y^h} \mathcal{L}_{X^h} \hat{F} \ .$$

Dropping to B, we obtain

$$\mathbf{B}_V(X, Y) \wedge F = \mathcal{L}_X \mathcal{L}_Y F - \mathcal{L}_Y \mathcal{L}_X F - \mathcal{L}_{[X,Y]} F \ ,$$

which characterizes $\mathbf{B}_V$ in terms of (covariant) Lie derivatives.
Similarly, just as $\mathbf{D}$ is an antiderivation on tensorial vector-valued forms, d is an antiderivation on bundle-valued forms, i.e.,

$$d(\lambda \wedge \mu) = d\lambda \wedge \mu + (-1)^p \lambda \wedge d\mu \qquad (\lambda \in \Omega^p(B, V_Q),\ \mu \in \Omega^q(B, V_Q)) \ .$$

In particular, applying d to both sides of A.6(1) gives

$$d^2(d\lambda) = d\mathbf{B}_V \wedge \lambda + \mathbf{B}_V \wedge d\lambda \ .$$

Replacing λ in A.6(1) by $d\lambda$ and substituting into the above equation yields

$$\begin{aligned} &\mathbf{B}_V \wedge d\lambda = d\mathbf{B}_V \wedge \lambda + \mathbf{B}_V \wedge d\lambda \\ \Rightarrow \quad &d\mathbf{B}_V \wedge \lambda = 0 \ . \end{aligned}$$

Since $\lambda \in \Omega^p(B, V_Q)$ is arbitrary, we conclude that $d\mathbf{B}_V = 0$ (Bianchi's identity).

B On regular points of the co-adjoint action

This appendix is devoted to the proof of Lemma 6.1. While this could be done using standard structure theory, we opt for a 'direct' proof based on the following well known fact:[5]

Theorem (Duflo-Vergne [11]). *Let G be any finite-dimensional Lie group. Then $\mathfrak{g}^*_{\mathrm{reg}}$ is open and dense in $\mathfrak{g}^*$. Furthermore, for all $\mu \in \mathfrak{g}^*_{\mathrm{reg}}$, $\mathfrak{g}_\mu$ is Abelian.*

Fix an Ad-invariant inner product on $\mathfrak{g}$ and equip $\mathfrak{g}^*$ with the corresponding Ad*-invariant inner product. The product on $\mathfrak{g}$ defines an equivariant isomorphism $\rho : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$, establishing an equivalence between the adjoint and co-adjoint representations. We therefore begin by examining the adjoint action, where computations are easier.
Put $\lambda \equiv \rho^{-1}(\mu)$ ($\mu \in \mathfrak{g}^*_{\mathrm{reg}}$). By invariance of the inner product on $\mathfrak{g}$, the map $\xi \mapsto [\xi, \lambda]$ is skew-symmetric. Its kernel $\mathfrak{g}_\lambda$ and image $[\mathfrak{g}, \lambda]$ are therefore orthogonal:

$$\mathfrak{g}_\lambda^\perp = [\mathfrak{g}, \lambda] \ . \tag{1}$$

In addition, we claim that

$$[\mathfrak{g}, \lambda] = [\mathfrak{g}, \mathfrak{g}_\lambda] \ . \tag{2}$$

Proof of (2). Equation (1) implies that $\mathfrak{g} = \mathfrak{g}_\lambda + [\mathfrak{g}, \lambda]$ so that for an arbitrary $\eta \in \mathfrak{g}_\lambda$ we have

$$[\mathfrak{g}, \eta] = [\mathfrak{g}_\lambda + [\mathfrak{g}, \lambda], \eta] = [[\mathfrak{g}, \lambda], \eta] \ . \tag{3}$$

The second equality holds because $\mathfrak{g}_\lambda$ is Abelian (by the Theorem and the equivalence of the adjoint and co-adjoint representations). It follows from (3) that an arbitrary element of $[\mathfrak{g}, \eta]$ is of the form $[[\xi, \lambda], \eta]$, for some $\xi \in \mathfrak{g}$. But, by Jacobi's identity, we have

$$\begin{aligned} &[[\xi, \lambda], \eta] = -[[\eta, \xi], \lambda] - [[\lambda, \eta], \xi] \\ &\qquad\qquad = -[[\eta, \xi], \lambda] \ , \quad \text{since } [\lambda, \eta] = 0 \ (\mathfrak{g}_\lambda \text{ is Abelian}) \\ \Rightarrow \quad &[[\xi, \lambda], \eta] \in [\mathfrak{g}, \lambda] \ . \end{aligned}$$

So $[\mathfrak{g}, \eta] \subset [\mathfrak{g}, \lambda]$. Since $\eta \in \mathfrak{g}_\lambda$ was arbitrary, we conclude that $[\mathfrak{g}, \mathfrak{g}_\lambda] \subset [\mathfrak{g}, \lambda]$. As the reverse containment is obvious, Equation (2) is established. □

[5] Proofs of this theorem in English are given in [18] and [25].

Together (1) and (2) imply that $\mathfrak{g}_\lambda^\perp = [\mathfrak{g}, \mathfrak{g}_\lambda]$. By the equivalence of the adjoint and co-adjoint representations, we have $\mathfrak{g}_\mu^\perp = [\mathfrak{g}, \mathfrak{g}_\mu]$, as claimed.

References

[1] R. Abraham and J. E. Marsden. *Foundations of Mechanics*. Addison-Wesley Publishing Co., Reading, Massachusetts, 2nd edition, 1978.

[2] M. S. Alber, G. G. Luther, J. E. Marsden, and J. M. Robbins. Geometric phases, reduction and Lie-Poisson structure for the resonant three-wave interaction. *Phys. D*, 123:271–290, 1998.

[3] M. S. Alber and J. E. Marsden. On geometric phases for soliton equations. *Comm. Math. Phys*, 149:217–240, 1992.

[4] V. I. Arnold. *Mathematical Methods of Classical Mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer, 2nd edition, 1989.

[5] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt. *Dynamical Systems III.* Encycl. Math. Sci. Springer, New York, 1988.

[6] A. D. Blaom. Reconstruction phases via Poisson reduction. *Differential Geom. Appl.*, 12(3):231–252, 2000.

[7] A. D. Blaom. A geometric setting for Hamiltonian perturbation theory. *Mem. Amer. Math. Soc.*, 153(727):1–112, 2001.

[8] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and G. Sánchez de Alvarez. Stabilization of rigid body dynamics by internal and external torques. *Automatica J. IFAC*, 28:745–756, 1992.

[9] H. Cendra, D. D. Holm, J. E. Marsden, and T. S. Ratiu. Lagrangian reduction, the Euler-Poincaré equations, and semidirect products. *Amer. Math. Soc. Transl. Ser. 2*, 186:1–25, 1998.

[10] R. W. R. Darling. *Differential Forms and Connections.* Cambridge University Press, 1994.

[11] M. Duflo and M. Vergne. Une proprieté de la représentation coadjointe d'une algébre de Lie. *C. R. Acad. Sci. Paris Sér. A-B*, 268:583–585, 1969.

[12] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry, Volume I.* Wiley, New York, 1963.

[13] I. Madsen and J. Tornehave. *From Calculus to Cohomology.* Cambridge University Press, 1997.

[14] J. E. Marsden. *Lectures on Mechanics*, volume 174 of *London Mathematical Society Lecture Notes Series.* Cambridge University Press, 1992.

[15] J. E. Marsden. Park City lectures on mechanics, dynamics, and symmetry. In *Symplectic geometry and topology (Park City, UT, 1997)*, pages 335–430. Amer. Math. Soc., Providence, RI, 1999.

[16] J. E. Marsden, R. Montgomery, and T. Ratiu. Reduction, symmetry, and phases in mechanics. *Mem. Amer. Math. Soc.*, 88(436):1–110, 1990.

[17] J. E. Marsden and J. Ostrowski. Symmetries in motion: Geometric foundations of motion control. *Nonlinear Sci. Today*, January, 1998.

[18] J. E. Marsden and T. S. Ratiu. *Introduction to Mechanics and Symmetry*, volume 17 of *Texts in Applied Mathematics*. Springer, 1994.

[19] J. E. Marsden, T. S. Ratiu, and J. Scheurle. Reduction theory and the Lagrange-Routh Equations. *J. Math. Phys.*, 41:3379–3429, 2000.

[20] J. E. Marsden and A. Weinstein. Reduction of symplectic manifolds with symmetry. *Rep. Math. Phys.*, 5:121–130, 1974.

[21] K. R. Meyer. Symmetries and Integrals in Mechanics. In M. Peixoto, editor, *Dynamical Systems*, pages 259–273. Academic Press, New York, 1973.

[22] R. Montgomery. *The Bundle Picture in Mechanics*. PhD thesis, University of California, Berkeley, 1986.

[23] R. Montgomery. How much does a rigid body rotate? A Berry's phase from the 18th century. *Amer. J. Phys.*, 59:394–398, 1991.

[24] M. Perlmutter. *Symplectic Reduction by Stages*. PhD thesis, University of California, Berkeley, 1999.

[25] A. Weinstein. The local structure of Poisson manifolds. *J. Differential Geom.*, 18:523–557, 1983.

Anthony D. Blaom
Washington, D. C. and
Princeton, New Jersey
Current address:
Department of Mathematics
University of Auckland
Private Bag 92019
Auckland, New Zealand
a.blaom@auckland.ac.nz
www.math.auckland.ac.nz/∼blaom