A Lambda-Graph System for the Dyck Shift and Its K-Groups

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Abstract. A property of subshifts is described that allows to associate to the subshift a distinguishied presentation by a compact Shannon graph. For subshifts with this property and for the resulting invariantly associated compact Shannon graphs and their λ-graph systems the term 'Cantor horizon' is proposed. The Dyck shifts are Cantor horizon. The C^*-algebras that are obtained from the Cantor horizon λ-graph systems of the Dyck shifts are separable, unital, nuclear, purely infinite and simple with UCT. The K-groups and Bowen-Franks groups of the Cantor horizon λ-graph systems of the Dyck shifts are computed and it is found that the K_0-groups are not finitely generated.

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0. Introduction

Let Σ be a finite alphabet. On the shift space $\Sigma^{\mathbb{Z}}$ one has the left-shift that sends a point $(\sigma_i)_{i\in\mathbb{Z}}$ into the point $(\sigma_{i+1})_{i\in\mathbb{Z}}$. In symbolic dynamics one studies the dynamical systems, called subshifts, that are obtained by restricting the shift to a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$. For an introduction to symbolic dynamics see [Ki] or [LM]. A finite word in the symbols of Σ is said to be admissible for the subshift $X \subset \Sigma^{\mathbb{Z}}$ if it appears somewhere in a point of X. A subshift is uniquely determined by its set of admissible words. Throughout this paper, we denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the set of all nonnegative integers and the set of all positive integers respectively.

A directed graph G whose edges are labeled by symbols in the finite alphabet Σ is called a Shannon graph if for every vertex u of G and for every $\alpha \in \Sigma$, G has at most one edge with initial vertex u and label α. We say that a Shannon

graph G presents a subshift X if every vertex of G has a predecessor and a successor and if the set of admissible words of X coincides with the set of label sequences of finite paths on G. To a Shannon graph G there is associated a topological Markov chain $M(G)$. The state space of $M(G)$ is the set of pairs (u, α), where u is a vertex of G and α is the label of an edge of G with initial vertex u. Here a transition from state (u, α) to state (v, β) is allowed if and only if v is the final vertex of the edge with initial vertex u and label α. For a vertex u of Shannon graph G we denote the forward context of u by $\Gamma^+(u)$. $\Gamma^+(u)$ is the set of sequences in $\Sigma^{\mathbb{N}}$ that are label sequences of infinite paths in G that start at the vertex u. We say that a Shannon graph G is forward separated if vertices of G, that have the same forward context, are identical. The Shannon graphs that we consider in this paper are forward separated. We always identify the vertices of a forward separated Shannon graph G with their forward contexts, and then use on the vertex set of G the topology that is given by the Hausdorff metric on the set of nonempty compact subsets of $\Sigma^{\mathbb{N}}$.
There is a one-to-one correspondence between forward-separated compact Shannon graphs G such that every vertex has a predecessor and a class of λ-graph systems [KM]. We recall that a λ-graph system is a directed labelled Bratteli diagram with an additional structure. We write the vertex set of a λ-graph system as

$$V = \bigcup_{n \in \mathbb{Z}_+} V_{-n}.$$

Every edge with initial vertex in V_{-n}, has its final vertex in $V_{-n+1}, n \in \mathbb{N}$. It is required that every vertex has a predecessor and every vertex except the vertex in V_0 has a successor. In this paper we consider λ-graph-systems that are forward separated Shannon graphs. Their additional structure is given by a mapping

$$\iota : \bigcup_{n \in \mathbb{N}} V_{-n} \to V$$

such that

$$\iota(V_{-n}) = V_{-n+1}, \qquad n \in \mathbb{N}$$

that is compatible with the labeling, that is, if u is the initial vertex of an edge with label α and final vertex v, then $\iota(u)$ is the intial vertex of an edge with label α and final vertex $\iota(v)$.
Given a subshift $X \subset \Sigma^{\mathbb{Z}}$ there is a one-to-one correspondence between the compact forward separated Shannon graphs that present X, and the forward separated Shannon λ-graph systems that present X. To describe this one-to-one correspondence denote for a vertex v of a Shannon graph by v_n the set of initial segments of length n of the sequences in $V, n \in \mathbb{N}$. The λ-graph system that corresponds to the forward separated Shannon graph has as its set V_{-n} the set of $v_n, n \in \mathbb{Z}_+$, v a vertex of G, and if in G there is an edge with initial vertex u and final vertex v and label α then in the corresponding λ-graph system there is an edge with initial vertex u_n, final vertex $v_{n-1}, n \in \mathbb{N}$,

and label α, the mapping ι of the corresponding λ-graph system deleting last symbols.
λ-graph systems can be described by their symbolic matrix systems [Ma]

$$(\mathcal{M}_{-n,-n-1}, I_{-n,-n-1})_{n\in\mathbb{Z}_+}.$$

Here $\mathcal{M}_{-n,-n-1}$ is the symbolic matrix

$$[\mathcal{M}_{-n,-n-1}(u,v)]_{u\in V_{-n}, v\in V_{-n-1}}$$

that is given by setting $\mathcal{M}_{-n,-n-1}(u,v)$ equal to $\alpha_1 + \cdots + \alpha_k$ if in V there is an edge with initial vertex v and final vertex u with label $\alpha_i, i = 1, \dots, k$ and by setting $\mathcal{M}_{-n,-n-1}(u,v)$ equal to zero otherwise. $I_{-n,-n-1}$ is the zero-one matrix

$$[I_{-n,-n-1}(u,v)]_{u\in V_{-n}, v\in V_{-n-1}}$$

that is given by setting $I_{-n,-n-1}(u,v)$ equal to one if $\iota(v) = u$ and by setting $I_{-n,-n-1}(u,v)$ equal to zero otherwise. We remark that the time direction considered here is opposite to the time direction in [Ma]. For symbolic matrix systems there is a notion of strong shift equivalence [Ma] that extends the notion of strong shift equivalence for transition matrices of topological Markov shifts [Wi] and of the symbolic matrices of sofic systems [BK,N].
To a symbolic matrix system there are invariantly associated K-groups and Bowen-Franks groups [Ma]. To describe them, let

$$M_{n,n+1} = [M_{n,n+1}(u,v)]_{u\in V_{-n}, v\in V_{-n-1}}$$

be the nonnegative matrix that is given by setting $M_{n,n+1}(u,v)$ equal to zero if $\mathcal{M}_{-n,-n-1}(u,v)$ is zero, and by setting it equal to the number of the symbols whose sum is $\mathcal{M}_{-n,-n-1}(u,v)$ otherwise. We let $I_{n,n+1}, n \in \mathbb{Z}_+$ be $I_{-n,-n-1}$. Let $m(n)$ be the cardinal number of the vertex set V_{-n}. Also denote by $\bar{I}^t_{n,n+1}, n \in \mathbb{Z}_+$ the homomorphism from $\mathbb{Z}^{m(n)}/(M^t_{n-1,n} - I^t_{n-1,n})\mathbb{Z}^{m(n-1)}$ to $\mathbb{Z}^{m(n+1)}/(M^t_{n,n+1} - I^t_{n,n+1})\mathbb{Z}^{m(n)}$ that is induced by $I^t_{n,n+1}$. Then

$$K_0(M,I) = \varinjlim_n \{\mathbb{Z}^{m(n+1)}/(M^t_{n,n+1} - I^t_{n,n+1})\mathbb{Z}^{m(n)}, \bar{I}^t_{n,n+1}\},$$

$$K_1(M,I) = \varinjlim_n \{\mathrm{Ker}(M^t_{n,n+1} - I^t_{n,n+1}) \text{ in } \mathbb{Z}^{m(n)}, I^t_{n,n+1}\}.$$

Let $\mathbb{Z}_I$ be the group of the projective limit $\varprojlim_n\{\mathbb{Z}^{m(n)}, I_{n,n+1}\}$. The sequence $M_{n,n+1} - I_{n,n+1}, n \in \mathbb{Z}_+$ acts on it as an endomorphism, denoted by $M - I$. The Bowen-Franks groups $BF^I(M,I), i = 0,1$ are defined by

$$BF^0(M,I) = \mathbb{Z}_I/(M-I)\mathbb{Z}_I, \qquad BF^1(M,I) = \mathrm{Ker}(M-I) \quad \text{in} \quad \mathbb{Z}_I.$$

Given a subshift $X \subset \Sigma^{\mathbb{Z}}$ we use for $x = (x_i)_{i\in\mathbb{Z}} \in X$ notation like

$$x_{[j,k]} = (x_i)_{j\leq i\leq k},$$

and we set

$$X_{[j,k]} = \{x_{[j,k]} \mid x \in X\}, \qquad j,k \in \mathbb{Z}, \quad j < k,$$

using similar notations when indices range is infinite intervals. We denote the forward context of a point x^- in $X_{(-\infty,0]}$ by $\Gamma^+(x^-)$,

$$\Gamma^+(x^-) = \{x^+ \in X_{[1,\infty)} \mid (x^-, x^+) \in X\}.$$

The set $G(X) = \{\Gamma^+(x^-) \mid x^- \in X_{(-\infty,0]}\}$ is the vertex set of a forward separated Shannon graph that presents X. The λ-graph system of its closure was introduced in [KM] as the canonical λ-graph system of the subshift X. It is canonically associated to the subshift in the sense that a topological conjugacy of subshifts induces a strong shift equivalence of their canonical λ-graph systems.

For a subshift $X \subset \Sigma^{\mathbb{Z}}$ that is synchronizing [Kr] (or semisynchronizing [Kr]) one has an intrinsically defined shift invariant dense subset $P_s(X)$ of periodic points of X, and one has associated to X the presenting forward separated Shannon graph whose vertex set is the set of forward contexts $\Gamma^+(x)$, where x is left asymptotic to a point in $P_s(X)$. These Shannon graphs are canonically associated to the synchronizing (or semisynchronizing) subshift, in the sense that a topological conjugacy of subshifts induces a block conjugacy of the topological Markov chains of the Shannon graphs and also a strong shift equivalence [Ma] of the λ-graph systems of their closures. Prototype examples of semisynchronizing subshifts are the Dyck shifts that can be defined via the Dyck inverse monoids. The Dyck inverse monoid is the inverse monoid (with zero) with generators $\alpha_n, \beta_n, 1 \leq n \leq N$, and relations

$$\begin{aligned} \alpha_n\beta_n &= 1, && 1 \leq n \leq N, \\ \alpha_n\beta_m &= 0, && 1 \leq n, m \leq N, \quad n \neq m \end{aligned}$$

and the Dyck shift D_N is defined as the subshift $D_N \subset \{\alpha_n, \beta_n \mid 1 \leq n \leq N\}^{\mathbb{Z}}$, whose admissible words $(\gamma_i)_{0\leq i\leq I}$ satisfy the condition

$$\prod_{0\leq i\leq I} \gamma_i \neq 0.$$

In section 1 we introduce another class of subshifts $X \subset \Sigma^{\mathbb{Z}}$ with an intrinsically defined shift invariant dense set $P_{Ch}(X)$ of periodic points. Again the Dyck shifts serve here as prototypes. In the Dyck shift D_N the points in $P_{Ch}(D_N)$ are such that during a period there appears an event that has the potential to influence even the most distant future. In other words, a point $(x_i)_{i\in\mathbb{Z}}$ in

D_N with period p is in $P_{Ch}(D_N)$ if the normal form of the word $(x_i)_{0 \le i < p}$ is a word in the symbols $\alpha_n, 1 \le n \le N$. One can view here the record of an infinite sequence of events as a point in a Cantor discontinuum. With this in mind, we call the subshifts in this class Cantor horizon subshifts. The presenting Shannon graph with vertex set the set of forward contexts $\Gamma^+(y^-)$, where y^- is negatively asymptotic to a point in $P_{Ch}(X)$, is canonically associated to the Cantor horizon subshift $X \subset \Sigma^{\mathbb{Z}}$, and so is the λ-graph system of its closure, that we call the Cantor horizon λ-graph system of X. The Cantor horizon λ-graph system of a Cantor horizon subshift is a sub λ-graph system of its canonical λ-graph system.

The K-groups and Bowen-Franks groups of the symbolic matrix system $(\mathcal{M}^{D_N}, I^{D_N})$ for the canonical λ-graph systems of the Dyck shifts $D_N, N \ge 2$ were computed in [Ma2]. These are

$$K_0(\mathcal{M}^{D_N}, I^{D_N}) \cong \sum_{n \in \mathbb{N}} \mathbb{Z}, \qquad K_1(\mathcal{M}^{D_N}, I^{D_N}) \cong 0,$$

$$BF^0(\mathcal{M}^{D_N}, I^{D_N}) \cong 0, \qquad BF^1(\mathcal{M}^{D_N}, I^{D_N}) \cong \prod_{n \in \mathbb{N}} \mathbb{Z}.$$

In section 3 we determine the symbolic matrix system $(\mathcal{M}^{Ch(D_2)}, I^{Ch(D_2)})$ of the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_2)}$ of the Dyck shift D_2, and we compute its K-groups. Denoting the group of all $\mathbb{Z}$-valued continuous functions on the Cantor discontinuum $\mathfrak{C}$ by $C(\mathfrak{C}, \mathbb{Z})$ one has

$$K_0(\mathcal{M}^{Ch(D_2)}, I^{Ch(D_2)}) \cong \mathbb{Z}/2\mathbb{Z} \oplus C(\mathfrak{C}, \mathbb{Z}),$$

and one has

$$K_1(\mathcal{M}^{Ch(D_2)}, I^{Ch(D_2)}) \cong 0.$$

One can construct simple C^*-algebras from irreducible λ-graph systems [Ma3]. A λ-graph system is said to be irreducible if for a sequence $v_{-n} \in V_{-n}, n \in \mathbb{Z}_+$ of vertices with $\iota(v_{-n}) = v_{-n+1}$ and for a vertex u, there exists an $N \in \mathbb{Z}_+$ such that there is a path from v_{-N} to u. It is said to be aperiodic if for a vertex u, there exists an $N \in \mathbb{Z}_+$ such that for all $v \in V_{-N}$ there exist paths from v to u. The Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_2)}$ of the Dyck shift D_2 is irreducible and moreover aperiodic. Hence the resulting C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}$ is simple and purely infinite whose K_0-group and K_1-group are the above groups $K_0(\mathcal{M}^{Ch(D_2)}, I^{Ch(D_2)})$ and $K_1(\mathcal{M}^{Ch(D_2)}, I^{Ch(D_2)})$ respectively (cf. [Ma3]). In section 4, we compute the Bowen-Franks groups of the symbolic matrix system $(\mathcal{M}^{Ch(D_2)}, I^{Ch(D_2)})$.

In section 5, we consider the K-groups and Bowen-Franks groups of the Dyck shifts $D_N, N \ge 2$. Here one has

$$K_0(\mathcal{M}^{Ch(D_N)}, I^{Ch(D_N)}) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{C}, \mathbb{Z}),$$
$$K_1(\mathcal{M}^{Ch(D_N)}, I^{Ch(D_N)}) \cong 0.$$

1. Subshifts with Cantor horizon lambda-graph systems

Denoting for a given subshift $X \subset \Sigma^{\mathbb{Z}}$, the right context of an admissible block $x_{[i,j]}, x \in X, i,j \in \mathbb{Z}, i \le j$, by $\Gamma^+(x_{[i,j]})$,

$$\Gamma^+(x_{[i,j]}) = \{y^+ \in X_{(j,\infty)} \mid (x_{[i,j]}, y^+) \in X_{[i,\infty)}\}$$

and its left context by $\Gamma^-(x_{[i,j]})$,

$$\Gamma^-(x_{[i,j]}) = \{y^- \in X_{(-\infty,i)} \mid (y^-, x_{[i,j]}) \in X_{(-\infty,j]}\},$$

we set

$$\omega^+(x_{[i,j]}) = \bigcap_{y^- \in \Gamma^-(x_{[i,j]})} \{y^+ \in X_{(j,\infty)} \mid (y^-, x_{[i,j]}, y^+) \in X\}.$$

Lemma 1.1. *Let $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}, X \subset \Sigma^{\mathbb{Z}}$ be subshifts and let $\psi : \widetilde{X} \to X$ be a topological conjugacy. Let for some $L \in \mathbb{Z}_+$ ψ be given by a $(2L+1)$-block map Ψ and ψ^{-1} be given by a $(2L+1)$-block map $\widetilde{\Psi}$. Let $\widetilde{N} \in \mathbb{N}$, and let $\tilde{x} \in \widetilde{X}$ be such that*

$$\omega^+(\tilde{x}_{(-L-\widetilde{N},-L]}) = \omega^+(\tilde{x}_{(-L-\tilde{n},-L]}), \qquad \tilde{n} \ge \widetilde{N}. \tag{1.1}$$

Then for $x = \psi(\widetilde{x})$ and $N = \widetilde{N} + 2L$,

$$\omega^+(x_{(-N,0]}) = \omega^+(x_{(-n,0]}), \qquad n \ge N. \tag{1.2}$$

Proof. Let $n \ge N$, and let

$$y^+ \in \omega^+(x_{(-n,0]}). \tag{1.3}$$

Let

$$\tilde{y}^+ = \widetilde{\Psi}(x_{[-L,0]}, y^+).$$

One has

$$\tilde{y}^+ \in \omega^+(\tilde{x}_{(-\widetilde{N}-L,-L]}), \tag{1.4}$$

which implies that

$$y^+ \in \omega^+(x_{(-N,0]}),$$

confirming (1.2). We note that by (1.1) one has that (1.4) follows from

$$\tilde{y}^+ \in \omega^+(\tilde{x}_{(-\tilde{n}-L,-L]}),$$

which in turn follows from (1.3). $\square$

Let $X \subset \Sigma^{\mathbb{Z}}$ be a subshift and $P(X)$ be its set of periodic points. Denote by $P_a(X)$ the set of $x \in P(X)$ such that there is an $N \in \mathbb{N}$ such that

$$\omega^+(x_{(-N,0]}) = \omega^+(x_{(-n,0]}), \qquad n \ge N.$$

Lemma 1.2. *Let $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}, X \subset \Sigma^{\mathbb{Z}}$ be subshifts and let $\psi : \widetilde{X} \to X$ be a topological conjugacy. Then*

$$\psi(P_a(\widetilde{X})) = P_a(X).$$

Proof. Apply Lemma 1.1. □

By Lemma 1.1 the following property of a subshift $X \subset \Sigma^{\mathbb{Z}}$ is invariant under topological conjugacy : For $x \in X$ and $N \in \mathbb{N}$ such that

$$\omega^+(x_{(-N,0]}) = \omega^+(x_{(-n,0]}), \qquad n \geq N$$

there exists an $M \in \mathbb{N}$ such that

$$\omega^+(x_{[-M,0)}) = \omega^+(x_{[-m,0)}), \qquad m \geq M.$$

For a subshift $X \subset \Sigma^{\mathbb{Z}}$ with this property we consider the subgraph $G_{Ch}(X)$ of $G(X)$ with vertices $\Gamma^+(u^-)$ where $u^- \in X_{(-\infty,0]}$ is negatively asymptotic to a point in $P_{Ch}(X) = P(X) \backslash P_a(X)$. If here $G_{Ch}(X)$ presents X then we say that X is a Cantor horizon subshift, and we call the λ-graph system of the closure of $G_{Ch}(X)$ the Cantor horizon λ-graph system of X. By Lemma 1.2 the Cantor horizon property is an invariant of topological conjugacy and the Cantor horizon λ-graph system is invariantly associated to the Cantor horizon subshift.

2. The Dyck shift

We consider the Dyck shift D_2 with alphabet $\Sigma = \Sigma^- \cup \Sigma^+$ where $\Sigma^- = \{\alpha_0, \alpha_1\}, \Sigma^+ = \{\beta_0, \beta_1\}$. A periodic point x of D_2 with period p is not in $P_a(D_2)$ precisely if for some $i \in \mathbb{Z}$ the normal form of the word $(x_{i+q})_{0 \leq q < p}$ is a word in the symbols of Σ^-, in other words, if the multiplier of x in the sense of [HI] is negative. We also note that periodic points with negative multipliers give rise to the same irreducible component of $G_{Ch}(D_2)$ precisely if they have the same multiplier.

We describe the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_2)}$ of D_2: The vertices at level l are given by the words of length l in the symbols of Σ^-. The mapping ι deletes the first symbol of a word. A word $(\alpha_{i(n)})_{1 \leq n \leq l}$ accepts β_i precisely if $i(l) = i, i = 0, 1$, effecting a transition to the word $(\alpha_{i(n)})_{1 \leq n < l}$, and it accepts α_i, effecting a transition to the word $(\alpha_{i(n)})_{2 \leq n \leq l}$. The forward context of the word $a = (\alpha_{i(n)})_{1 \leq n \leq l}$ contains precisely all words $c = (\gamma_n)_{1 \leq n \leq l}$ in symbols of Σ such that (a, c) is admissible for D_2. In describing the Cantor horizon symbolic matrix system $(\mathcal{M}, I)$ of the Dyck shift and the resulting nonnegative matrix system (M, I) we use the reverse lexcographic order on the words in the symbols in Σ^-, that is, we assign to a word $(\alpha_{i(n)})_{1 \leq n \leq l} \in \Sigma^{-[1,l]}$ the number

$$\sum_{1 \leq n \leq l} i(n) 2^{n-1}.$$

One has then

$$\begin{aligned} \mathcal{M}_{0,-1} &= [\beta_0+\alpha_0+\alpha_1,\ \ \beta_1+\alpha_0+\alpha_1] = [\alpha_0+\alpha_1+\beta_0,\ \ \alpha_0+\alpha_1+\beta_1], \\ I_{0,-1} &= [1,\ \ 1]. \end{aligned}$$

For $l \in \mathbb{Z}_+$ and $a \in \{\alpha_0,\alpha_1,\beta_0,\beta_1\}$, let $I_l(a)$ be the $2^l \times 2^l$ diagonal matrix with diagonal entries a, and $\mathcal{S}_l(a)$ be the $2^{l-1} \times 2^{l+1}$ matrix $[\mathcal{S}_l(a)(i,j)]_{1\le i\le 2^{l-1},1\le j\le 2^{l+1}}$ where $\mathcal{S}_l(a)(i,j)$ is a for $j = 4i, 4i-1, 4i-2, 4i-3$, and is otherwise zero.

Proposition 2.1. *For $l = 1,2,\dots,$ the matrix $\mathcal{M}_{-l,-l-1}$ is a $2^l \times 2^{l+1}$ rectangular matrix that is given as the block matrix:*

$$\mathcal{M}_{-l,-l-1} = \begin{bmatrix} \mathcal{S}_l(\alpha_0) \\ \mathcal{S}_l(\alpha_1) \end{bmatrix} + [\, I_l(\beta_0)\ \mid\ I_l(\beta_1)\,] \tag{2.1}$$

and

$$I_{-l,-l-1}(i,j) = \begin{cases} 1 & (j = 2i-1, 2i), \\ 0 & \textit{elsewhere.} \end{cases} \tag{2.2}$$

Proof. The first summand in (2.1) describes the transitions that arise when a vertex accepts a symbol in Σ^+. The second summand arises from the transitions that arise when a vertex accepts a symbol in Σ^-, the arrangement of the components of the matrix as well as (2.2) being a component of the ordering of the vertices at level l and $l-1$. □

We note that the λ-graph systems of the closures of the irreducible components of $G_{Ch}(X)$ are identical.

Proposition 2.2. *The λ-graph system $\mathfrak{L}^{Ch(D_2)}$ for $(\mathcal{M}, I)$ is irreducible and aperiodic.*

Proof. Let $V_{-l}, l \in \mathbb{Z}_+$ be the vertex set of the λ-graph system $\mathfrak{L}^{Ch(D_2)}$. For any vertex u of V_{-l}, there are labeled edges from each of the vertices in V_{-2l} to the vertex u. This implies that $(\mathcal{M}, I)$ is aperiodic. □

3. Computation of the K-groups

Let $M_{l,l+1}$ for $l \in \mathbb{Z}_+$ be the nonnegative matrix obtained from $\mathcal{M}_{-l,-l-1}$ by setting all the symbols of the components of $\mathcal{M}_{-l,-l-1}$ equal to 1. The matrix $I_{l,l+1}$ for $l \in \mathbb{Z}_+$ is defined to be $I_{-l,-l-1}$. For $l > 1$ and $1 \le i \le 2^{l-2}$ let $a_i(l) = [a_i(l)_n]_{n=1}^{2^l}$ be the vector that is given by

$$a_i(l)_n = \begin{cases} 1 & (n = 4i-3, 4i-2, 4i-1, 4i), \\ 0 & \text{elsewhere.} \end{cases}$$

Define for $1 < l \in \mathbb{N}$ E_l as the $2^l \times 2^l$-matrix whose $i-$th column vector and whose $(2^{l-1}+i)-$th column vector are both equal to $a_i(l), 1 \le i \le 2^{l-2}$, the other column vectors being equal to zero vectors.
For $l > 1$ and $1 \le i \le 2^{l-1}$ let $b_i(l) = [b_i(l)_n]_{n=1}^{2^l}$ be the vector that is given by

$$b_i(l)_n = \begin{cases} 1 & (n = 2i-1, 2i), \\ 0 & \text{elsewhere.} \end{cases}$$

Define for $1 < l \in \mathbb{N}$ F_l as the $2^l \times 2^l$-matrix whose $i-$th column vector is equal to $b_i(l), 1 \le i \le 2^{l-1}$, the other column vectors being equal to zero vectors.
One has

$$I^t_{l,l+1} E_l = E_{l+1} I^t_{l,l+1}, \qquad I^t_{l,l+1} F_l = F_{l+1} I^t_{l,l+1}, \qquad l > 1. \tag{3.1}$$

Let I_l denote the unit matrix of size 2^l. Define a $2^l \times 2^{l-1}$ matrix $H_{l,l-1}$ by setting

$$H_{l,l-1} = \begin{bmatrix} I_{l-1} \\ -I_{l-1} \end{bmatrix}, \qquad l > 1.$$

Lemma 3.1. *For $l > 1$ one has*

$$(E_l - F_l) H_{l,l-1} = -I^t_{l-1,l}.$$

Proof. One has

$$E_l H_{l,l-1} = 0, \qquad F_l H_{l,l-1} = I^t_{l-1,l}.$$

□

Set $y_1(2) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $y_2(2) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $y_3(2) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $y_4(2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and define inductively for $l > 2$ vectors $y_i(l)$, $1 \le i \le 2^l$, where

$$y_i(l) = I^t_{l-1,l} y_i(l-1), \qquad 1 \le i \le 2^{l-1} \tag{3.2}$$

and

$$y_i(l) = H_{l,l-1} y_{i-2^{l-2}}(l-1), \qquad 2^{l-1} < i \le 2^{l-1} + 2^{l-2} \tag{3.3}$$

where one defines the vectors

$$y_i(l) = [y_i(l)_n]_{n=1}^{2^l}, \qquad 2^{l-1} + 2^{l-2} < i \le 2^l$$

by setting
(3.4)

$$y_{2^{l-1}+2^{l-2}+i}(l)_n = \begin{cases} 1 & (n = 4i-1, 4i, 2^{l-1}+2i), \\ -1 & (n = 2^{l-1}+4i-1, 2^{l-1}+4i), \\ 0 & \text{elsewhere,} \end{cases} \qquad 1 \le i \le 2^{l-3}$$

and by setting

$$(3.5)\quad y_{2^{l-1}+2^{l-2}+2^{l-3}+i}(l)_n = \begin{cases} 1 & (n = 2^{l-1}+2^{l-2}+2i), \\ 0 & \textit{elsewhere}, \end{cases} \qquad 1 \le i \le 2^{l-3}.$$

Define for $l > 2$, T_l as the $2^l \times 2^l$-matrix whose column vectors $y_i(l), 1 \le i \le 2^l$. Here $y_1(l)$ has all components equal to 1, that is, $y_1(l)$ is the eigenvector of $E_l - F_l$ for the eigenvalue 1. Also the vectors

$$y_i(l), \qquad 2^{l-1} + 2^{l-2} < i \le 2^l$$

are linearly independent vectors in the kernel of $E_l - F_l$ and one sees from (3.1), (3.2), (3.3), (3.4) and (3.5) that T_l is invertible and that $T_l^{-1}(E_l - F_l)T_l$ is a matrix that is in a normal form. This normal form is a Jordan form in the sense that by conjugation with a suitable permutation matrix, followed by a conjugation with a suitable diagonal matrix whose entries are 1 or -1, the matrix assumes a Jordan form with Jordan blocks arranged along the diagonal. There will be one Jordan block of length 1 for the eigenvalue 1, and there will be 2^{l-1} Jordan blocks for the eigenvalue 0, and if one lists these by decreasing length then the $k-$th Jordan block for the eigenvalue 0 has length $l - \mu(k)$ where $\mu(k)$ be given by $2^{\mu(k)-1} < k \le 2^{\mu(k)}, 1 \le k \le 2^{l-2}$.
By an elementary column operation we will mean the addition or subtraction of one column vector from another or the exchange of two column vectors.

Lemma 3.2. *Let $l \in \mathbb{N}$ and let K be a $2^l \times 2^l$-matrix with column vectors $z_i, 1 \le i \le 2^l$,*

$$z_i = b_i(l), \qquad 1 \le i \le 2^{l-1},$$

and column vectors

$$z_{2^{l-1}+i} = [z_{2^{l-1}+i,n}]_{n=1}^{2^l}, \qquad 1 \le i \le 2^{l-1}$$

such that

$$\begin{aligned} &z_{2^{l-1}+i,2j-1} = z_{2^{l-1}+i,2j} = 0, \qquad 1 \le j < i, \\ &z_{2^{l-1}+i,2i-1} = 0, \qquad z_{2^{l-1}+i,2i} = 1, \\ &z_{2^{l-1}+i,2j-1} = 0, \qquad z_{2^{l-1}+i,2j} \in \{-1,0,1\}, \qquad i < j \le 2^{l-1}. \end{aligned}$$

Then K can be converted into the unit matrix by a sequence of elementary column operations.

Proof. Let the vector $c_j = [c_{j,n}]_{n=1}^{2^l}, 1 \le j \le 2^{l-1}$, be given by

$$c_{j,n} = \begin{cases} 1 & (n = 2j), \\ 0 & \text{elsewhere}. \end{cases}$$

and denote by $K[j], 1 \leq j \leq 2^{l-1}$ the matrix that is obtained by replacing in the matrix K the last j column vectors by the vectors $c_{2^{l-1}-i}, j \geq i \geq 1$. $K[1]$ is equal to K and $K[j], 1 < j \leq 2^{l-1}$, can be obtained from $K[j-1]$ by subtracting from and adding to the $(2^l - j)$-th column appropriate selections of the the $(2^l - i)$-th columns, $1 \leq i \leq j$. $K[2^{l-1}]$ has as its first 2^{l-1} column vectors the vectors $b_i(l), 1 \leq i \leq 2^{l-1}$, and as its last 2^{l-1} column vectors the vectors $c_i(l), 1 \leq i \leq 2^{l-1}$, and can be converted into the unit matrix by elementary column operations. $\square$

Lemma 3.3. *Let $l \in \mathbb{N}$ and let K be a $2^l \times 2^l$-matrix with column vectors $z_i, 1 \leq i \leq 2^l$,*

$$z_i = b_i(l), \qquad 1 \leq i \leq 2^{l-1},$$

and column vectors

$$z_{2^{l-1}+i} = [z_{2^{l-1}+i,n}]_{n=1}^{2^l}, \qquad 1 < i \leq 2^{l-1}$$

such that

$$(z_{2^{l-1}+i,2j-1}, z_{2^{l-1}+i,2j}) \in \{(0,0),(1,1)\}, \qquad 1 \leq j < i,$$

$$z_{2^{l-1}+i,2i-1} = 0, \qquad z_{2^{l-1}+i,2i} = 1,$$

$$(z_{2^{l-1}+i,2j-1}, z_{2^{l-1}+i,2j}) \in \{(-1,-1),(0,0),(1,1),(0,1),(0,-1)\}, \ \ i < j < 2^{l-1}.$$

Then K can be converted into the unit matrix by a sequence of elementary column operations.

Proof. For all $i, 2^{l-1} < i \leq 2^l$, one subtracts from the i-th column of K and adds to the i-th column of K appropriate selections of the first 2^{l-1} columns of K to obtain a matrix to which Lemma 3.2 applies. $\square$

Proposition 3.4. *The matrix T_l is unimodular.*

Proof. The matrix T_2 can be converted into the unit matrix by elementary column operations. The proof is by induction on l. Assume that the matrix $T_{l-1}, l > 2$ can be converted into the unit matrix by a sequence of elementary column operations. Then by (3.2) the matrix T_l can be converted by a sequence of elementary column operations into a matrix whose first 2^{l-1} column vectors are the vectors $b_i(l), 1 \leq i \leq 2^{l-1}$ and whose last 2^{l-1} column vectors are those of the matrix T_l, and by (3.2),(3.3) and (3.4) Lemma 3.3 is applicable to this matrix. $\square$

Define a $2^l \times 2^l$ matrix L_l by setting

$$L_l = I_l + E_l - F_l, \qquad l > 1.$$

Denote by $0_{k,l}$ the $2^k \times 2^l$ matrix with entries 0's. Also define permutation matrices $P_l(i,j), 1 \leq i,j \leq 2^l, l > 1$, by

$$P_l(i, 2^l - i + 1) = 1, \qquad 1 \leq i \leq 2^l,$$

and set

$$B_{l+1} = \begin{bmatrix} L_l & 0_{l,l} \\ P_l L_l P_l & 0_{l,l} \end{bmatrix}.$$

Lemma 3.5. $B_{l+1} = [\, M^t_{l,l+1} - I^t_{l,l+1} \mid 0_{l+1,l} \,], \quad l > 1.$

Proof. This follows from Proposition 2.1. □

Define $2^{l+1} \times 2^{l+1}$ matrices $J(l+1)$ and U_{l+1} by setting

$$J(l+1) = \begin{bmatrix} T_l^{-1} L_l T_l & 0_{l,l} \\ 0_{l,l} & 0_{l,l} \end{bmatrix}, \qquad U_{l+1} = \begin{bmatrix} T_l & 0_{l,l} \\ P_l T_l & I_l \end{bmatrix}.$$

Lemma 3.6. U_{l+1} *is unimodular and*

$$U_{l+1}^{-1} B_{l+1} U_{l+1} = J(l+1), \qquad l > 1.$$

Proof. One has

$$U_{l+1}^{-1} = \begin{bmatrix} T_l^{-1} & 0_{l,l} \\ -P_l & I_l \end{bmatrix}$$

and further

$$\begin{bmatrix} T_l^{-1} & 0_{l,l} \\ -P_l & I_l \end{bmatrix} \begin{bmatrix} L_l & 0_{l,l} \\ P_l L_l P_l & 0_{l,l} \end{bmatrix} \begin{bmatrix} T_l & 0_{l,l} \\ P_l T_l & I_l \end{bmatrix} = \begin{bmatrix} T_l^{-1} L_l T_l & 0_{l,l} \\ 0_{l,l} & 0_{l,l} \end{bmatrix}.$$

□

Define a $2^{l+1} \times 2^l$ matrix $G_{l+1,l}$ by setting

$$G_{l+1,l} = \begin{bmatrix} I_l \\ 0_{l,l} \end{bmatrix}, \qquad l > 1.$$

One has

$$I^t_{l,l+1} T_l = T_{l+1} G_{l+1,l}, \tag{3.6}$$

$$I^t_{l,l+1} P_l T_l = P_{l+1} T_{l+1} G_{l+1,l}. \tag{3.7}$$

Define a $2^{l+1} \times 2^l$ matrix $J_{l+1,l}$ by setting

$$J_{l+1,l} = \begin{bmatrix} G_{l,l-1} & 0_{l,l-1} \\ 0_{l,l-1} & I^t_{l-1,l} \end{bmatrix}, \qquad l > 1.$$

Lemma 3.7. $I^t_{l,l+1} U_l = U_{l+1} J_{l+1,l}, \quad l > 1.$

Proof. From (3.6) and (3.7), it follows that

$$\begin{aligned} \begin{bmatrix} I^t_{l-1,l} & 0_{l,l-1} \\ 0_{l,l-1} & I^t_{l-1,l} \end{bmatrix} \begin{bmatrix} T_{l-l} & 0_{l,l-1} \\ P_{l-1} T_{l-1} & I_{l-1} \end{bmatrix} &= \begin{bmatrix} T_l G_{l,l-1} & 0_{l,l-1} \\ P_l T_l G_{l,l-1} & I^t_{l-1,l} \end{bmatrix} \\ &= \begin{bmatrix} T_l & 0_{l,l} \\ P_l T_l & I_l \end{bmatrix} \begin{bmatrix} G_{l,l-1} & 0_{l,l-1} \\ 0_{l,l-1} & I^t_{l-1,l} \end{bmatrix}. \end{aligned}$$

□

Lemma 3.8.
$$J_{l+1,l}J(l) = J(l+1)J_{l+1,l}, \qquad l > 1.$$

Proof. By (3.1)
$$I^t_{l-1,l}L_{l-1} = L_l I^t_{l-1,l}$$
and by (3.6)
$$T^{-1}_{l+1}I^t_{l,l+1}T_l = G_{l+1,l}T^{-1}_l.$$
Therefore by (3.6)
$$\begin{aligned} G_{l,l-1}T^{-1}_{l-1}L_{l-1}T_{l-1} &= T^{-1}_l I^t_{l-1,l}L_{l-1}T_{l-1} \\ &= T^{-1}_l L_l I^t_{l-1,l}T_{l-1} \\ &= T^{-1}_l L_l T_l G_{l,l-1} \end{aligned}$$
and therefore
$$\begin{aligned} &\begin{bmatrix} G_{l,l-1} & 0_{l,l-1} \\ 0_{l,l-1} & I^t_{l-1,l} \end{bmatrix} \begin{bmatrix} T^{-1}_{l-1}L_{l-1}T_{l-1} & 0_{l,l-1} \\ 0_{l,l-1} & 0_{l,l-1} \end{bmatrix} \\ &= \begin{bmatrix} G_{l,l-1}T^{-1}_{l-1}L_{l-1}T_{l-1} & 0_{l,l-1} \\ 0_{l,l-1} & 0_{l,l-1} \end{bmatrix} \\ &= \begin{bmatrix} T^{-1}_l L_l T_l G_{l,l-1} & 0_{l,l-1} \\ 0_{l,l-1} & 0_{l,l-1} \end{bmatrix} \\ &= \begin{bmatrix} T^{-1}_l L_l T_l & 0_{l,l-1} \\ 0_{l,l-1} & 0_{l,l-1} \end{bmatrix} \begin{bmatrix} G_{l,l-1} & 0_{l,l-1} \\ 0_{l,l-1} & I^t_{l-1,l} \end{bmatrix}. \end{aligned}$$

□

By the preceding lemma, the matrix $J_{l+1,l}$ induces a homomorphism
$$\bar{J}_{l+1,l} : \mathbb{Z}^{2^l}/J(l)\mathbb{Z}^{2^l} \to \mathbb{Z}^{2^{l+1}}/J(l+1)\mathbb{Z}^{2^{l+1}}$$
and by Proposition 3.4 and Lemma 3.6 the matrix U_l, as $B_l\mathbb{Z}^{2^l} = (M^t_{l-1,l} - I^t_{l-1,l})\mathbb{Z}^{2^{l-1}}$, induces a homomorphism
$$\bar{U}_l : \mathbb{Z}^{2^l}/J(l)\mathbb{Z}^{2^l} \to \mathbb{Z}^{2^l}/B_l\mathbb{Z}^{2^l}.$$

Lemma 3.9. *The diagram :*

$$\begin{array}{ccc} \mathbb{Z}^{2^l}/(M^t_{l-1,l} - I^t_{l-1,l})\mathbb{Z}^{2^{l-1}} & \xrightarrow{\bar{I}^t_{l,l+1}} & \mathbb{Z}^{2^{l+1}}/(M^t_{l,l+1} - I^t_{l,l+1})\mathbb{Z}^{2^l} \\ \bar{U}_l \uparrow & & \bar{U}_{l+1} \uparrow \\ \mathbb{Z}^{2^l}/J(l)\mathbb{Z}^{2^l} & \xrightarrow{\bar{J}_{l+1,l}} & \mathbb{Z}^{2^{l+1}}/J(l+1)\mathbb{Z}^{2^{l+1}} \end{array}$$

is commutative.

Proof. Apply Lemma 3.7. □

Define a diagonal matrix $D(l)$ by setting

$$D(l) = \operatorname{diag}(2, \overbrace{1, 1, \dots, 1}^{2^l - 1}), \qquad l \in \mathbb{N}.$$

As $J(l+1)\mathbb{Z}^{2^{l+1}} = D(l)\mathbb{Z}^{2^l} \oplus (\overbrace{0, \dots, 0}^{2^l})$ and hence $\mathbb{Z}^{2^{l+1}}/J(l+1)\mathbb{Z}^{2^{l+1}} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{2^l}$, through the map $\varphi_l \oplus \mathrm{id}$ where $\varphi_l : \mathbb{Z}^{2^l}/D(l)\mathbb{Z}^{2^l} \to \mathbb{Z}/2\mathbb{Z}$ is defined by $\varphi_l([(x_i)_{i=1}^{2^l}]) = [x_1] \pmod 2$, we have

PROPOSITION 3.10. *The following diagram is commutative:*

$$\begin{array}{ccc}
\mathbb{Z}^{2^l}/J(l)\mathbb{Z}^{2^l} & \xrightarrow{\bar{J}_{l+1,l}} & \mathbb{Z}^{2^{l+1}}/J(l+1)\mathbb{Z}^{2^{l+1}} \\
{\scriptstyle \varphi_{l-1}\oplus\mathrm{id}}\Big\downarrow & & {\scriptstyle \varphi_l\oplus\mathrm{id}}\Big\downarrow \\
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{2^{l-1}} & \xrightarrow{\mathrm{id}\oplus I^t_{l-1,l}} & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}^{2^l}.
\end{array}$$

COROLLARY 3.11.

$$\varinjlim_l \{\mathbb{Z}^{2^{l+1}}/(M^t_{l,l+1} - I^t_{l,l+1})\mathbb{Z}^{2^l}, \bar{I}^t_{l,l+1}\} \cong \mathbb{Z}/2\mathbb{Z} \oplus C(\mathfrak{C}, \mathbb{Z}).$$

Proof. As the group of the inductive limit: $\varinjlim_l \{I^t_{l,l+1} : \mathbb{Z}^{2^l} \to \mathbb{Z}^{2^{l+1}}\}$ is isomorphic to $C(\mathfrak{C}, \mathbb{Z})$, we get the assertion. □

THEOREM 3.12.

$$K_0(M, I) \cong \mathbb{Z}/2\mathbb{Z} \oplus C(\mathfrak{C}, \mathbb{Z}), \qquad K_1(M, I) \cong 0.$$

By Proposition 2.2, the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_2)}$ of D_2 is aperiodic so that the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}$ associated with the λ-graph system $\mathfrak{L}^{Ch(D_2)}$ is simple and purely infinite ([Ma3;Proposition 4.9]). It satisfies the UCT by [Ma3;Proposition 5.6] (cf.[Bro],[RS]). By [Ma3;Theorem 5.5], the K-groups $K_i(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}})$ are isomorphic to the K-groups $K_i(M, I)$ so that we get

COROLLARY 3.13. *The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}$ associated with the λ-graph system $\mathfrak{L}^{Ch(D_2)}$ is separable, unital, nuclear, simple, purely infinite with UCT such that*

$$K_0(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}) \cong \mathbb{Z}/2\mathbb{Z} \oplus C(\mathfrak{C}, \mathbb{Z}), \qquad K_1(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}) \cong 0.$$

4. Computation of the Bowen-Franks groups

We compute next the Bowen-Franks groups $BF^0(M,I)$ and $BF^0(M,I)$.

Lemma 4.1. $\mathrm{Ext}^1_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}) \cong 0$.

Proof. From [Ro] (cf.[Sch;Theorem 1.3]) one has that for an inductive sequence $\{G_i\}$ of abelian groups there exists a natural short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(G_i,\mathbb{Z}) \to \mathrm{Ext}^1_{\mathbb{Z}}(\varinjlim G_i,\mathbb{Z}) \to \varprojlim \mathrm{Ext}^1_{\mathbb{Z}}(G_i,\mathbb{Z}) \to 0.$$

The lemma follows therefore from $C(\mathfrak{C},\mathbb{Z}) = \varinjlim_l \{I^t_{l,l+1} : \mathbb{Z}^{2^l} \to \mathbb{Z}^{2^{l+1}}\}$ and $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}^{2^l},\mathbb{Z}) = \varprojlim{}^1 \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{2^l},\mathbb{Z}) = 0$. $\square$

As in [Ma;Theorem 9.6], one has the following lemma that provides a universal coefficient type theorem.

Lemma 4.2. *For $i = 0,1$ there exists an exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K_i(M,I),\mathbb{Z}) \to BF^i(M,I) \to \mathrm{Hom}_{\mathbb{Z}}(K_{i+1}(M,I),\mathbb{Z}) \to 0.$$

Theorem 4.3.

$$BF^0(M,I) \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. By Theorem 3.12 and by Lemma 4.2 one has

$$BF^0(M,I) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

As $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, the theorem follows from Lemma 4.1. $\square$

Theorem 4.4.

$$BF^1(M,I) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

Proof. $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z})$ is trivial. Therefore by Theorem 3.12

$$\mathrm{Hom}_{\mathbb{Z}}(K_0(M,I)) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

Since the group $K_1(M,I)$ is trivial, by Lemma 4.2, one gets

$$BF^1(M,I) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

$\square$

As the Bowen-Franks groups $BF^0(M,I)$ and $BF^1(M,I)$ are isomorphic to the Ext-groups $\mathrm{Ext}^1(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}})(= \mathrm{Ext}(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}))$ and $\mathrm{Ext}^0(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}})(= \mathrm{Ext}(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}} \otimes C_0(\mathbb{R}))$ for the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}$ (cf. [Ma3]), we obtain

Corollary 4.5.

$$\mathrm{Ext}^1(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}) \cong \mathbb{Z}/2\mathbb{Z}, \qquad \mathrm{Ext}^0(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

5. General Dyck shifts

One can extend the preceding results for the Dyck shift D_2 to the general Dyck shifts D_N with $2N$ symbols $\alpha_n, \beta_n, 1 \le n \le N$, for $N > 2$, generalizing the previous discussions for the case of $N = 2$. We will briefly explain this. We consider the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_N)}$ of D_N as in the previous case and write its symbolic matrix system $(\mathcal{M}^{Ch(D_N)}_{-l,-l-1}, I^{Ch(D_N)}_{-l,-l-1})$ as $(\mathcal{M}_{l,l+1}, I_{l,l+1})$. We define the nonnegative matrices $M_{l,l+1}, I_{l,l+1}, l \in \mathbb{Z}_+$ in a similar way. The size of the matrices $M_{l,l+1}, I_{l,l+1}$ is $N^l \times N^{l+1}$. Let $I_l^{(N)}$ be the unit matrix with size N^l. For $l > 1$ and $1 \le i \le N^{l-2}$ let $a_i^{(N)}(l) = [a_i^{(N)}(l)_n]_{n=1}^{N^l}$ be the vector that is given by

$$a_i^{(N)}(l)_n = \begin{cases} 1 & (N^2(i-1)+1 \le n \le N^2 i), \\ 0 & \text{elsewhere.} \end{cases}$$

Define for $1 < l \in \mathbb{N}$ $E_l^{(N)}$ as the $N^l \times N^l$-matrix whose $i-$th, $(N^{l-1}+i)-$th, $(2N^{l-1}+i)-$th, $\dots$, $((N-1)N^{l-1}+i)-$th column vectors are all equal to $a_i^{(N)}(l), 1 \le i \le N^{l-2}$, the other column vectors being equal to zero vectors. Let $b_i^{(N)}(l) = [b_i^{(N)}(l)_n]_{n=1}^{N^l}$ be the vector that is given by

$$b_i^{(N)}(l)_n = \begin{cases} 1 & (N(i-1)+1 \le n \le Ni), \\ 0 & \text{elsewhere.} \end{cases}$$

Define for $1 < l \in \mathbb{N}$ $F_l^{(N)}$ as the $N^l \times N^l$-matrix whose $i-$th column vector is equal to $b_i^{(N)}(l), 1 \le i \le N^{l-1}$, the other column vectors being equal to zero vectors.

Define a $N^l \times N^l$ matrix by

$$L_l^{(N)} = I_l^{(N)} + E_l^{(N)} - F_l^{(N)}, \qquad l \in \mathbb{N}.$$

Also define permutation matrices $P_l^{(N),k}, k = 1, 2, \dots, N-1$, by

$$P_l^{(N),k}(i, N^l - i + 1 - (k-1)) = 1, \qquad 1 \le i \le N^l,$$

where $1 \le N^l - i + 1 - (k-1) \le N^l$ is taken mod N^l. Denote by $0_{k,l}$ the $N^k \times N^l$ matrix with entries 0's.

We define an $N^{l+1} \times N^{l+1}$ matrix $B_{l+1}^{(N)}$ by

$$B_{l+1}^{(N)} = [M_{l,l+1}^t - I_{l,l+1}^t \mid 0_{l+1,l}],$$

that is written as the block matrix

$$\begin{bmatrix} L_l^{(N)} & 0_{l,l} \\ P_l^{(N),1} L_l^{(N)} P_l^{(N),1} & 0_{l,l} \\ \vdots & \vdots \\ P_l^{(N),N-1} L_l^{(N)} P_l^{(N),N-1} & 0_{l,l} \end{bmatrix}.$$

By an argument that is similar to the one of the previous sections, one can conclude then

Theorem 5.1.

$$K_0(M,I) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{C},\mathbb{Z}), \qquad K_1(M,I) \cong 0,$$
$$BF^0(M,I) \cong \mathbb{Z}/N\mathbb{Z}, \qquad BF^1(M,I) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

Corollary 5.2. *The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ associated with the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_N)}$ of D_N is separable, unital, nuclear, simple, purely infinite such that*

$$K_0(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{C},\mathbb{Z}), \qquad K_1(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong 0,$$
$$\mathrm{Ext}^1(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong \mathbb{Z}/N\mathbb{Z}, \qquad \mathrm{Ext}^0(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathfrak{C},\mathbb{Z}),\mathbb{Z}).$$

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