Absolute Continuity of the Spectrum of a Schrödinger Operator with a Potential Which is Periodic in Some Directions and Decays in Others

N. Filonov[1] and F. Klopp[2]

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Abstract. We prove that the spectrum of a Schrödinger operator with a potential which is periodic in certain directions and super-exponentially decaying in the others is purely absolutely continuous. Therefore, we reduce the operator using the Bloch-Floquet-Gelfand transform in the periodic variables, and show that, except for at most a set of quasi-momenta of measure zero, the reduced operators satisfies a limiting absorption principle.

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1 Formulation of the result

There are many papers (see, for example, [1, 9]) devoted to the question of the absolute continuity of the spectrum of differential operators with coefficients periodic in the whole space. In the present article, we consider the situation where the coefficients are periodic in some variables and decay very fast (super-exponentially) when the other variables tend to infinity. The corresponding operator describes the scattering of waves on an infinite membrane or filament. Recently, quite a few studies have been devoted to similar problems, for periodic, quasi-periodic or random surface Hamiltonians (see, e.g. [3, 7, 2]).

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Let (x, y) denote the points of the space $\mathbb{R}^{m+d}$. Define $\Omega = \mathbb{R}^m \times (0, 2\pi)^d$ and $\langle x \rangle = \sqrt{x^2 + 1}$. For $a \in \mathbb{R}$, introduce the spaces

$$L_{p,a} = \{f : e^{a\langle x \rangle} f \in L_p(\Omega)\}, \qquad H_a^2 = \{f : e^{a\langle x \rangle} f \in H^2(\Omega)\},$$

where $1 \leq p \leq \infty$ and $H^2(\Omega)$ is the Sobolev space. Our main result is

Theorem 1.1. *Consider in $L_2(\mathbb{R}^{m+d})$ the self-adjoint operator*

$$Hu = -\operatorname{div}(g\nabla u) + Vu \tag{1}$$

and assume that the functions $g : \mathbb{R}^{m+d} \to \mathbb{R}$ and $V : \mathbb{R}^{m+d} \to \mathbb{R}$ satisfy following conditions:

1. *$\forall l \in \mathbb{Z}^d$, $\forall (x, y) \in \mathbb{R}^{m+d}$,*

$$g(x, y + 2\pi l) = g(x, y), \quad V(x, y + 2\pi l) = V(x, y);$$

2. *there exists $g_0 > 0$ such that $(g - g_0)$, Δg, $V \in L_{\infty,a}$ for any $a > 0$;*
3. *there exists $c_0 > 0$ such that $\forall (x, y) \in \mathbb{R}^{m+d}$, $g(x, y) \geq c_0$.*

Then, the spectrum of H is purely absolutely continuous.

Remark 1.1. *Operators with different values of g_0 differ from one another only by multiplication by a constant; so, without loss of generality, we can and, from now on, do assume that $g_0 = 1$.*

Remark 1.2. *If $V \equiv 0$, (1) is the acoustic operator. If $g \equiv 1$, it is the Schrödinger operator with electric potential V.*

The basic philosophy of our proof is the following. To prove the absolute continuity of the spectrum for periodic operators (i.e., periodic with respect to a non degenerate lattice in $\mathbb{R}^d$), one applies the Floquet-Bloch-Gelfand reduction to the operator and one is left with proving that the Bloch-Floquet-Gelfand eigenvalues must vary with the quasi-momentum i.e., that they cannot be constant on sets of positive measure (see e.g. [9]). If one tries to follow the same line in the case of operators that are only periodic with respect to a sub-lattice, the problem one encounters is that, as the resolvent of the Bloch-Floquet-Gelfand reduction of the operator is not compact, its spectrum may contain continuous components and some Bloch-Floquet-Gelfand eigenvalues may be embedded in these continuous components. The perturbation theory of such embedded eigenvalues (needed to control their behavior in the Bloch quasi-momentum) is more complicated than that of isolated eigenvalues. To obtain a control on these eigenvalues, we use an idea of the theory of resonances (see e.g. [13]): if one analytically dilates Bloch-Floquet-Gelfand reduction of the operator, these embedded eigenvalues become isolated eigenvalues, and thus can be controlled in the usual way.

Let us now briefly sketch our proof. We make the Bloch-Floquet-Gelfand transformation with respect to the periodic variables (see section 3) and get a family of operators $H(k)$ in the cylinder Ω. Then, we consider the corresponding resolvent in suitable weighted spaces. It analytically depends on the quasi-momentum k and the spectral (non real) parameter λ. It turns out that we can extend it analytically with respect to λ from the upper half-plane to the lower one (see Theorem 5.1 below) and thus establish the limit absorption principle. This suffices to prove the absolute continuity of the initial operator (see section 7).

Note that an analytic extension of the resolvent of the operator (1) with coefficients g and V which decay in all directions is constructed in the paper [4] (with $m = 3$, $d = 0$; see also [10] for $g \equiv 1$). In the case of a potential decaying in all directions but one (i.e., if $d = 1$), the analytic extension of the resolvent of the whole operator (1) (not only for the operator $H(k)$ (see section 3)) is investigated in [6] when $g \equiv 1$. Note also that our approach has shown to be useful in the investigation of the perturbation of free operator in the half-plane by δ-like potential concentrated on a line (see [5]); the wave operators are also constructed there.

In section 2, we establish some auxiliary inequalities. In section 3, we define the Floquet-Gelfand transformation and construct an analytic extension of the resolvent of free operator in the cylinder Ω. In sections 4 and 5, we prove a limiting absorption principle for the initial operator in the cylinder. An auxiliary fact from theory of functions is established in section 6. Finally, the proof of Theorem 1.1 is completed in section 7.

We denote by $B_\delta(k_0)$ a ball in real space

$$B_\delta(k_0) = \{k \in \mathbb{R}^d : |k - k_0| < \delta\}$$

and by k_1 the first coordinate of k, $k = (k_1, k')$. We will use the spaces of function in Ω with periodic boundary conditions,

$$\tilde{H}^2 = \left\{ f \in H^2(\Omega) : f\,|_{y_i=0} = f\,|_{y_i=2\pi},\ \frac{\partial f}{\partial y_i}\,|_{y_i=0} = \frac{\partial f}{\partial y_i}\,|_{y_i=2\pi},\ i = 1, \ldots, d \right\},$$
$$\tilde{H}^2_{loc} = \left\{ f \in H^2_{loc}(\Omega) : f\,|_{y_i=0} = f\,|_{y_i=2\pi}, \frac{\partial f}{\partial y_i}\,|_{y_i=0} = \frac{\partial f}{\partial y_i}\,|_{y_i=2\pi},\ i = 1, \ldots, d \right\}.$$

Finally $B(X, Y)$ is the space of all bounded operators from X to Y, and $B(X) = B(X, X)$, both endowed with their natural topology.

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2 Auxiliary estimations

In this section, we assume that the pair $(k_0, \lambda_0) \in \mathbb{R}^{d+1}$ satisfies

$$(k_0 + n)^2 \neq \lambda_0 \qquad \forall n \in \mathbb{Z}^d. \tag{2}$$

The constants in all the inequalities in this section may depend on (k_0, λ_0). The set

$$J = \{n \in \mathbb{Z}^d : (k_0 + n)^2 < \lambda_0\} \tag{3}$$

is finite. In a neighborhood of (k_0, λ_0), the partition of $\mathbb{Z}^d$ into J and $(\mathbb{Z}^d \setminus J)$ is clearly the same. In other words, there exists $\delta = \delta(k_0, \lambda_0) > 0$ such that

$$\text{if } k \in B_\delta(k_0) \text{ and } \lambda \in B_\delta(\lambda_0), \text{ then } \ (k + n)^2 < \lambda \Leftrightarrow n \in J. \tag{4}$$

Choose $\tilde{k} \in B_\delta(k_0)$ with $\tilde{k}_1 \notin \mathbb{Z}$ and put

$$k(\tau) := (\tilde{k}_1 + i\tau, \tilde{k}') \in \mathbb{C}^d, \quad \tau \in \mathbb{R},$$

and

$$M_1 = M_1(k_0, \lambda_0) := (B_\delta(k_0) \cup \{k(\tau)\}_{\tau \in \mathbb{R}}) \times B_\delta(\lambda_0). \tag{5}$$

Lemma 2.1. *There exists $c > 0$ such that, for all $\zeta \in \mathbb{R}^m$, $(k, \lambda) \in M_1$, $n \in \mathbb{Z}^d \setminus J$ and $\tau \in \mathbb{R}$, we have*

$$|\zeta^2 + (k + n)^2 - \lambda| \geq c,$$
$$|\zeta^2 + (k(\tau) + n)^2 - \lambda| \geq c|\tau|.$$

Proof. By virtue of (4), there exists $c > 0$ such that, for $n \in \mathbb{Z} \setminus J$,

$$\forall k \in B_\delta(k_0), \ \forall \lambda \in B_\delta(\lambda_0), \quad (k + n)^2 - \lambda > c.$$

Hence, for $\zeta \in \mathbb{R}^m$, $n \in \mathbb{Z} \setminus J$,

$$\forall k \in B_\delta(k_0), \ \forall \lambda \in B_\delta(\lambda_0), \quad \zeta^2 + (k + n)^2 - \lambda > c.$$

The second inequality is an immediate corollary of our choice of $\tilde{k}_1$ and the equality

$$\operatorname{Im}(\zeta^2 + (k(\tau) + n)^2 - \lambda) = 2(\tilde{k}_1 + n_1)\tau.$$

This completes the proof of Lemma 2.1. □

In the remaining part of this section, we assume $\lambda_0 > 0$. In this case, we will need to change the integration path in the Fourier transformation; we now describe the contour deformation. Fix $\eta > \sqrt{\lambda_0}$ and, let γ be the contour in the complex plane defined as

$$\gamma = \{-\xi + i\eta\}_{\xi \in [\eta, \infty)} \cup \{\alpha(1 - i)\}_{\alpha \in [-\eta, \eta]} \cup \{\xi - i\eta\}_{\xi \in [\eta, \infty)}. \tag{6}$$

The following two assertions are clear.

Lemma 2.2. *If $g \in L_2(\gamma)$ and $\eta_0 > \eta$ then the function*

$$h(t) = e^{-\eta_0|t|}\int_\gamma e^{itz}g(z)dz$$

belongs to $L_2(\mathbb{R})$.

Lemma 2.3. *Let Γ denote the open set between real axis and γ (it consists of two connected components). Let g be an analytic function in Γ such that $g \in C(\overline{\Gamma})$ and $|g(z)| \leq C(1+|\operatorname{Re} z|)^{-2}$. Then,*

$$\int_{\mathbb{R}} e^{itz}g(z)dz = \int_\gamma e^{itz}g(z)dz \qquad \forall t \in \mathbb{R}.$$

Establish an analogue of Lemma 2.1 for $n \in J$ and $\zeta \in \gamma^m$ i.e., $\zeta = (\zeta_1, \ldots, \zeta_m) \in \mathbb{C}^m$, $\zeta_j \in \gamma$.

Lemma 2.4. *Let $\lambda_0 > 0$, $\eta > \sqrt{\lambda_0}$ and γ be defined by (6). There exists $c > 0$ such that, for all $\zeta \in \gamma^m$, $(k,\lambda) \in M_1$, $n \in J$ and $\tau \in \mathbb{R}$, we have*

$$\begin{aligned} |\zeta^2 + (k+n)^2 - \lambda| &\geq c, \\ |\zeta^2 + (k(\tau)+n)^2 - \lambda| &\geq c|\tau|. \end{aligned} \tag{7}$$

Proof. By virtue of (4), there exists $c > 0$ such that, for $n \in J$,

$$\forall k \in B_\delta(k_0),\ \forall \lambda \in B_\delta(\lambda_0), \quad (k+n)^2 - \lambda < -2c.$$

Hence, for $\zeta \in \gamma^m$ such that $|\zeta| \leq \sqrt{c}$, one has

$$\forall k \in B_\delta(k_0),\ \forall \lambda \in B_\delta(\lambda_0), \quad \operatorname{Re}(\zeta^2 + (k+n)^2 - \lambda) < -c.$$

On the other hand, for $\zeta \in \gamma^m$ such that $|\zeta| \geq \sqrt{c}$, one has

$$\forall k \in B_\delta(k_0),\ \forall \lambda \in B_\delta(\lambda_0), \quad \operatorname{Im}(\zeta^2 + (k+n)^2 - \lambda) < -c$$

if one chooses c sufficiently small. Thus, it remains to prove the second inequality. Therefore, we write

$$\zeta^2 = -2i\sum_p \alpha_p^2 + \sum_q (\xi_q - i\eta)^2,$$

where the indexes p correspond to the coordinates of ζ which are in the middle part of γ (i.e., $|\operatorname{Re}\zeta_p| < \eta$) and the indexes q correspond to the extreme parts of γ (i.e., $|\operatorname{Re}\zeta_q| \geq \eta$); it is possible that there are only indexes p or only q. Without loss of generality, we suppose that, for all q, $\xi_q \geq 0$. Thus,

$$\begin{aligned} \zeta^2 + (k(\tau)+n)^2 - \lambda = {} & \sum_q (\xi_q^2 - \eta^2) + (\tilde{k}+n)^2 - \tau^2 - \lambda \\ & + 2i\left(-\sum_p \alpha_p^2 - \sum_q \xi_q\eta + (\tilde{k}_1 + n_1)\tau\right). \end{aligned}$$

Fix some $\sigma \in (\eta^{-1}\sqrt{\lambda_0}, 1)$. If $\sum_q \xi_q \geq \sigma|\tau|$ then,

$$\left|\mathrm{Im}(\zeta^2 + (k(\tau)+n)^2 - \lambda)\right| \geq 2\left(\sigma\eta - |\tilde{k}_1 + n_1|\right)|\tau| > 2(\sigma\eta - \sqrt{\lambda_0})|\tau|,$$

as $(\tilde{k}+n)^2 < \lambda_0$. If $\sum_q \xi_q \leq \sigma|\tau|$ then $\sum_q \xi_q^2 \leq \sigma^2\tau^2$ and

$$\left|\mathrm{Re}(\zeta^2 + (k(\tau)+n)^2 - \lambda)\right| \geq \tau^2 + \lambda - (\tilde{k}+n)^2 - \sigma^2\tau^2 > (1-\sigma^2)\tau^2$$

again by virtue of (4). This completes the proof of Lemma 2.4. $\square$

3 The resolvent of free operator in the cylinder

Let us consider the Floquet-Gelfand transformation

$$(Uf)(k,x,y) = \sum_{l\in\mathbb{Z}^d} e^{i\langle k, y+2\pi l\rangle} f(x, y+2\pi l).$$

It is a unitary operator

$$U : L_2(\mathbb{R}^{m+d}) \to \int_{[0,1)^d}^{\oplus} L_2(\Omega)dk.$$

Introduce the family of operators $(H(k))_{k\in\mathbb{C}^d}$ on the cylinder Ω where for $k \in \mathbb{C}^d$, $\mathrm{Dom}\, H(k) = \tilde{H}^2$ and

$$H(k) = \left(i\nabla - (0,\overline{k})\right)^* g(x,y)\,(i\nabla - (0,k)) + V(x,y). \tag{8}$$

Then, the Schrödinger operator (1) is unitarily equivalent to the direct integral of these operators in Ω:

$$UHU^* = \int_{[0,1)^d}^{\oplus} H(k)dk.$$

In this section, we investigate the free operator

$$A(k) = -\Delta_x + \left(i\nabla_y - \overline{k}\right)^*(i\nabla_y - k) \tag{9}$$

(which corresponds $H(k)$ with $g \equiv 1$, $V \equiv 0$). For $k \in \mathbb{R}^d$ and $\lambda \notin \mathbb{R}$, its resolvent can be expressed as

$$\left((A(k)-\lambda)^{-1}f\right)(x,y) = \sum_{n\in\mathbb{Z}^d}\int_{\mathbb{R}^m} \frac{e^{i\zeta x + iny}(Ff)(\zeta,n)d\zeta}{\zeta^2 + (k+n)^2 - \lambda}, \tag{10}$$

where F denotes the Fourier transformation in the cylinder

$$(Ff)(\zeta,n) = (2\pi)^{-m-d}\int_\Omega e^{-i\zeta x - iny} f(x,y)\,dx\,dy.$$

Let $(k_0, \lambda_0) \in \mathbb{R}^{d+1}$ satisfy (2) and, J and M_1 be defined respectively by formulas (3) and (5) in the previous section.

Lemma 3.1. *There exists $\mathcal{V}_1$, a neighborhood of the set M_1 in $\mathbb{C}^{d+1}$ such that, for $(k,\lambda) \in \mathcal{V}_1$, the operator $R_1(k,\lambda)$ given by*

$$\left(R_1(k,\lambda)f\right)(x,y) = \sum_{n\in\mathbb{Z}^d\setminus J} \int_{\mathbb{R}^m} \frac{e^{i\zeta x+iny}(Ff)(\zeta,n)d\zeta}{\zeta^2+(k+n)^2-\lambda}$$

is well defined and is bounded from $L_2(\Omega)$ to $H^2(\Omega)$. The $B(L_2(\Omega),H^2(\Omega))$-valued function $(k,\lambda)\mapsto R_1(k,\lambda)$ is analytic in $\mathcal{V}_1$. For $\tau\neq 0$, the estimate

$$\|R_1(k(\tau),\lambda)\|_{B(L_2(\Omega))} \leq C|\tau|^{-1}$$

holds.

Proof. It immediately follows from Lemma 2.1. □

Lemma 3.2. *Let $\lambda_0 > 0$, $\eta > \sqrt{\lambda_0}$, $a > \eta\sqrt{m}$ and the contour γ be defined by (6). Then, there exists a neighborhood of the set M_1, say $\mathcal{V}_2$, such that, for $(k,\lambda)\in\mathcal{V}_2$, the operator $R_2(k,\lambda)$ given by*

$$\left(R_2(k,\lambda)f\right)(x,y) = \sum_{n\in J}\int_\gamma \cdots \int_\gamma \frac{e^{i\zeta x+iny}(Ff)(\zeta,n)}{\zeta^2+(k+n)^2-\lambda}\,d\zeta_1\cdots d\zeta_m \tag{11}$$

is well defined as a bounded operator from $L_{2,a}$ to H^2_{-a}. The $B(L_{2,a},H^2_{-a})$-valued function $(k,\lambda)\mapsto R_2(k,\lambda)$ is analytic in $\mathcal{V}_2$. For $\tau\neq 0$, the estimate

$$\|R_2(k(\tau),\lambda)\|_{B(L_{2,a},\,L_{2,-a})} \leq C|\tau|^{-1}$$

holds.

Proof. If $f\in L_{2,a}$ then the function $(Ff)(\cdot,n)$ is square integrable on γ^m. By Lemma 2.4, the denominator in (11) never vanishes for $(k,\lambda)\in M_1$; therefore, in some neighborhood of M_1. So

$$\left|(\zeta^2+(k+n)^2-\lambda)^{-1}e^{i\zeta x+iny}\right| \leq C|e^{i\zeta x}|$$

where the constant does not depend on $\zeta\in\gamma^m$ and on x; the same is true for the second derivatives of $(\zeta^2+(k+n)^2-\lambda)^{-1}e^{i\zeta x+iny}$ with respect to (x,y). Hence, $R_2(k,\lambda)\in B(L_{2,a},H^2_{-a})$ by virtue of Lemma 2.2. Estimation (7) yields the estimation for the norm of $R_2(k(\tau),\lambda)$. □

Now, we construct an analytic extension of the resolvent of $A(k)$.

Theorem 3.1. *Let $(k_0,\lambda_0)\in\mathbb{R}^{d+1}$ satisfy (2) and the set M_1 be defined in (5). Then, there exists a neighborhood of M_1 in $\mathbb{C}^{d+1}$, say M_0, a real number a and a $B(L_{2,a},H^2_{-a})$-valued function, say $(k,\lambda)\mapsto R_A(k,\lambda)$, defined and analytic in M_0, such that, for $(k,\lambda)\in M_0$, $k\in\mathbb{R}^d$, $\operatorname{Im}\lambda>0$ and $f\in L_{2,a}$, one has*

$$R_A(k,\lambda)f = (A(k)-\lambda)^{-1}f \tag{12}$$

and

$$\|R_A(k(\tau),\lambda)\|_{B(L_{2,a},\,L_{2,-a})} \leq C|\tau|^{-1}. \tag{13}$$

Proof. If $\lambda_0 \leq 0$, we can take $R_A = R_1$ (R_1 is constructed in Lemma 3.1; here, $J = \emptyset$ and $a = 0$).
If $\lambda_0 > 0$ then, we put $R_A = R_1 + R_2$, where R_1, R_2 and a are defined in Lemmas 3.1 and 3.2, and M_0 is the intersection of $\mathcal{V}_1$ and $\mathcal{V}_2$ defined respectively in Lemma 3.1 and Lemma 3.2. If $f \in L_{2,a}$ then $(Ff)(\cdot, n)$ is an analytic function in the domain $\{\zeta : |\operatorname{Im}\zeta| < a\}$ and is uniformly bounded on $\{\zeta : |\operatorname{Im}\zeta| \leq \eta\sqrt{m}\}$. If $\zeta \in \overline{\Gamma}^m$ where Γ is the open set between $\mathbb{R}$ and γ (see Lemma 2.3), then, $\operatorname{Im}\zeta^2 \leq 0$; therefore, the integrand in (11) has no poles when $\operatorname{Im}\lambda > 0$. Hence, the integral in right hand side of (10) for $n \in J$ coincides with the corresponding integral in (11) due to Lemma 2.3, and (12) holds.
The estimate (13) is a simple corollary of the estimations of Lemmas 3.1 and 3.2. □

4 Invertibility of operators of type $(I + WR_A)$

Lemma 4.1. *Let $W \in L_{\infty,b}$ for $b > 2a > 0$. Then, the operator of multiplication by W (we will denote it by the same letter) is*

1. *bounded as an operator from $L_{2,-a}$ to $L_{2,a}$;*
2. *compact as an operator from H^2_{-a} to $L_{2,a}$.*

Proof. The first assertion is evident. In order to prove the second it is enough to introduce functions

$$W_\rho(x,y) = \begin{cases} W(x,y), & |x| < \rho, \\ 0, & |x| \geq \rho, \end{cases}$$

and note that the multiplication by W_ρ is a compact operator from H^2_{-a} to $L_{2,a}$ and that

$$\|W - W_\rho\|_{B(L_{2,-a},L_{2,a})} \to 0$$

when $\rho \to \infty$. □

The next lemma is a well known result from analytic Fredholm theory (see, e.g., [8, 11]).

Lemma 4.2. *Let U be a domain in $\mathbb{C}^p$, $z_0 \in U$. Let $z \mapsto T(z)$ be an analytic function with values in the set of compact operators in some Hilbert space $\mathcal{H}$. Then, there exists a neighborhood U_0 of the point z_0 and an analytic function $h :\ U_0 \to \mathbb{C}$ such that, for $z \in U_0$,*

$$(I + T(z))^{-1} \text{ exists if and only if } h(z) = 0.$$

Now, we can establish the existence of the inverse of $(I + WR_A)$.

Theorem 4.1. *Let (k_0, λ_0) satisfy (2), $R_A(k,\lambda)$ and a be defined as in Theorem 3.1. Pick $b > 2a$, and let $(x,y,\lambda) \mapsto W(x,y,\lambda)$ be a function which belongs to $L_{\infty,b}$ for all λ, and is analytic with respect to λ i.e., $\lambda \mapsto W(\cdot,\cdot,\lambda) \in Hol(\mathbb{C}, L_{\infty,b})$.*
Then, there exists $\varepsilon > 0$, an open set $U \subset \mathbb{C}^{d+1}$ such that $B_\varepsilon(k_0) \times B_\varepsilon(\lambda_0) \subset U$, and an analytic function $h: U \to \mathbb{C}$ such that

$$\forall \lambda \in B_\varepsilon(\lambda_0), \quad \exists k \in B_\varepsilon(k_0) \quad \text{such that} \quad h(k,\lambda) \neq 0, \tag{14}$$

and, for any $(k,\lambda) \in U$, the operator $(I + W(\lambda)R_A(k,\lambda))$ is invertible in $L_{2,a}$ if and only if $h(k,\lambda) \neq 0$.

Proof. Due to Theorem 3.1 and Lemma 4.1, the operator $W(\lambda)R_A(k(\tau),\lambda)$ is compact in $L_{2,a}$ and satisfies the inequality

$$\|W(\lambda)R_A(k(\tau),\lambda)\|_{B(L_{2,a})} \leq C|\tau|^{-1}, \quad \forall \lambda \in B_\varepsilon(\lambda_0).$$

Therefore, for $|\tau|$ large enough, the operator $(I + W(\lambda)R_A(k(\tau),\lambda))^{-1}$ exists and is bounded on $L_{2,a}$. The operator-valued function $\lambda \mapsto W(\lambda)R_A(k,\lambda)$ is analytic in M_0 (defined in Theorem 3.1). The analytic Fredholm alternative yields that, for each $\lambda \in B_\varepsilon(\lambda_0)$, one can find $k \in B_\varepsilon(k_0)$ such that the operator $(I + W(\lambda)R_A(k,\lambda))^{-1}$ exists. Now, applying Lemma 4.2 with $\mathcal{H} = L_{2,a}$, $z = (k,\lambda)$ and $T(z) = WR_A$, completes the proof of Theorem 4.1. □

5 The resolvent of the operator H

We can reduce the general case of operator (1) with a "metric" g to the case of "pure" Schrödinger operator due to the following lemma. This identity (for the totally periodic case) is known (see [1]). We include the proof for the convenience of the reader.

Lemma 5.1. *Let the operators $H(k)$ and $A(k)$ be defined by (8) and (9) respectively, and let the conditions of Theorem 1.1 be fulfilled with $g_0 = 1$. If $u \in \tilde{H}^2$ then,*

$$(H(k) - \lambda)\, g^{-1/2} u = g^{1/2}\,(A(k) + W(\lambda) - \lambda)\, u,$$

where

$$W(\lambda) = \frac{1}{g}\left(\frac{\Delta g}{2} - \frac{|\nabla g|^2}{4g} + V + \lambda(g-1)\right). \tag{15}$$

Remark 5.1. *If $g \equiv 1$ then $W(\lambda) \equiv V$.*

Proof. It is enough to prove the equality

$$\left(i\nabla - (0,\overline{k})\right)^* g\,(i\nabla - (0,k))\,(g^{-1/2}u) = g^{1/2}\left(A(k) + \frac{\Delta g}{2g} - \frac{|\nabla g|^2}{4g^2}\right)u. \tag{16}$$

We have

$$(i\nabla - (0,k))\,(g^{-1/2}u) = ig^{-1/2}\nabla u - \frac{i}{2}g^{-3/2}\nabla g u - (0,k)(g^{-1/2}u).$$

Therefore, the left hand side of (16) is equal to

$$\begin{aligned}
&(i\nabla - (0,\overline{k}))^* \left(ig^{1/2}\nabla u - \frac{i}{2} g^{-1/2}\nabla g u - (0,k)(g^{1/2}u)\right) \\
&\quad = -g^{1/2}\Delta u + \frac{1}{2}\operatorname{div}(g^{-1/2}\nabla g)u - i\langle k, \nabla_y(g^{1/2}\overline{u})\rangle_{\mathbb{C}} \\
&\qquad\qquad - ig^{1/2}\langle \nabla_y u, \overline{k}\rangle_{\mathbb{C}} + \frac{i}{2} g^{-1/2}\langle \nabla_y g, \overline{k}\rangle_{\mathbb{C}} u + k^2 g^{1/2} u \\
&\quad = g^{1/2}\left(-\Delta_x u + (i\nabla_y - \overline{k})^*(i\nabla_y - k)u + \frac{1}{2} g^{-1/2}\operatorname{div}(g^{-1/2}\nabla g)u\right).
\end{aligned}$$

This completes the proof of Lemma 5.1. □

In the following theorem, we describe the meromorphic extension of the resolvent of $H(k)$.

Theorem 5.1. *Let the conditions of Theorem 1.1 be fulfilled, the operator $H(k)$ be defined by (8) and $(k_0, \lambda_0) \in \mathbb{R}^{d+1}$ satisfy (2). Then, there exists numbers $a \geq 0$, $\varepsilon > 0$, a neighborhood U of (k_0, λ_0) in $\mathbb{C}^{d+1}$ containing the set $B_\varepsilon(k_0) \times B_\varepsilon(\lambda_0)$, a function $h \in Hol(U)$ satisfying (14) and an operator-valued function $(k,\lambda) \mapsto R_H(k,\lambda)$ having the following properties:*

1. *R_H is defined on the set $\{(k,\lambda) \in U : h(k,\lambda) \neq 0\}$ and is analytic there;*
2. *for $(k,\lambda) \in U$ such that $h(k,\lambda) \neq 0$, one has $R_H(k,\lambda) \in B(L_{2,a}, L_{2,-a})$;*
3. *for $(k,\lambda) \in U$, $k \in \mathbb{R}^d$, $\operatorname{Im}\lambda > 0$, $f \in L_{2,a}$, one has*

$$R_H(k,\lambda)f = (H(k) - \lambda)^{-1} f. \tag{17}$$

Remark 5.2. *It will be seen from the proof that $R_H(k,\lambda) \in B(L_{2,a}, H^2_{-a})$ though we do not need this fact.*

Proof. By the assumptions of Theorem 1.1, for any $b > 0$, $\nabla g \in L_{\infty,b}$. So, if we define $W(\lambda)$ by (15), for any $b > 0$, $W(\lambda) \in L_{\infty,b}$. We can thus apply Theorem 4.1. Let U, h, a and R_A be as in this theorem. On the set where $h(k,\lambda) \neq 0$, we put

$$R_H(k,\lambda) = g^{-1/2} R_A(k,\lambda)\left(I + W(\lambda) R_A(k,\lambda)\right)^{-1} g^{-1/2}.$$

By Theorem 4.1, $R_H(k,\lambda) \in B(L_{2,a}, H^2_{-a})$. Let $f \in L_{2,a}$. Then,

$$\left(I + W(\lambda) R_A(k,\lambda)\right)^{-1} g^{-1/2} f \in L_{2,a} \tag{18}$$

and we can apply Lemma 5.1 to the function

$$u = R_A(k,\lambda)\left(I + W(\lambda) R_A(k,\lambda)\right)^{-1} g^{-1/2} f \in H^2_{-a}, \tag{19}$$

so

$$(H(k)-\lambda)R_H(k,\lambda)f = g^{1/2}\left(A(k)+W(\lambda)-\lambda\right)u. \tag{20}$$

For real k and non real λ, we have by (12) and (18)

$$(A(k)-\lambda)u = \left(I+W(\lambda)R_A(k,\lambda)\right)^{-1} g^{-1/2}f,$$

hence, by (19),

$$\left(A(k)+W(\lambda)-\lambda\right)u = g^{-1/2}f,$$

and, finally, by (20)

$$(H(k)-\lambda)R_H(k,\lambda)f = f. \tag{21}$$

For $\operatorname{Im}\lambda > 0$, the operators $(H(k)-\lambda)^{-1}$ and $(A(k)-\lambda)^{-1}$ are well defined in $L_2(\Omega)$. As $R_H(k,\lambda)f \in L_2(\Omega)$, (21) gives $R_H(k,\lambda)f = (H(k)-\lambda)^{-1}f$. This completes the proof of Theorem 5.1. □

6 One fact from the theory of functions

Lemma 6.1. *Let V be an open subset of $\mathbb{R}^d$. Let f be a real-analytic function in a box $(c,d)\times V$. Let Λ be a subset of V of measure zero,* $\operatorname{mes}\Lambda = 0$. *Then,*

$$\operatorname{mes}\{k\in(c,d) : \exists\lambda\in\Lambda \textit{ s.t. } f(k,\lambda)=0 \textit{ and } \partial_k f(k,\lambda)\neq 0\} = 0. \tag{22}$$

Proof. The Implicit Function Theorem implies that, for any point (k_0,λ_0) such that $f(k_0,\lambda_0) = 0 \neq \partial_k f(k_0,\lambda_0)$, we can find rational numbers $\tilde{k}_0$, $\tilde{r}_0 > 0$, a vector $\tilde{\lambda}_0 = (\tilde{\lambda}_0^1,\cdots,\tilde{\lambda}_0^d)$ with rational coordinates, and a cube $C_{\tilde{r}_0}(\tilde{k}_0,\tilde{\lambda}_0)$ where

$$(k_0,\lambda_0)\in C_{\tilde{r}_0}(\tilde{k}_0,\tilde{\lambda}_0) = (\tilde{k}_0-\tilde{r}_0,\tilde{k}_0+\tilde{r}_0)\times C_{\tilde{r}_0}(\tilde{\lambda}_0)\subset (c,d)\times V$$
$$C_{\tilde{r}_0}(\tilde{\lambda}_0) = (\tilde{\lambda}_0^1-\tilde{r}_0,\tilde{\lambda}_0^1+\tilde{r}_0)\times\cdots\times(\tilde{\lambda}_0^d-\tilde{r}_0,\tilde{\lambda}_0^d+\tilde{r}_0)$$

and a real analytic function $\theta:\ C_{\tilde{r}_0}(\tilde{\lambda}_0)\to(\tilde{k}_0-\tilde{r}_0,\tilde{k}_0+\tilde{r}_0)$ such that

1. $\theta(\lambda_0) = k_0$;
2. $f(k,\lambda)=0 \Leftrightarrow \theta(\lambda)=k$ if $(k,\lambda)\in C_{\tilde{r}_0}(\tilde{k}_0,\tilde{\lambda}_0)$.

Therefore,

$$\begin{aligned}\operatorname{mes}\{k : \exists\lambda\in\Lambda \text{ s.t. } (k,\lambda)\in C_{\tilde{r}_0}(\tilde{k}_0,\tilde{\lambda}_0) \text{ and } f(k,\lambda)=0\}&\\ \leq \operatorname{mes}\theta(\Lambda\cap C_{\tilde{r}_0}(\tilde{\lambda}_0)) = 0.&\end{aligned}$$

The set

$$\{(k,\lambda) : f(k,\lambda)=0 \text{ and } \partial_k f(k,\lambda)\neq 0\}$$

can be covered by a countable number of cubes $C_{\tilde r}(\tilde k,\tilde\lambda)$ constructed as above, say $(C_{\tilde r_i}(\tilde k_i,\tilde\lambda_i))_{i\in\mathbb{N}}$; hence, the measure of the set under consideration in (22) is also equal to zero as

$$\begin{aligned}\{k\in(c,d):\exists\lambda\in\Lambda \text{ s.t. } f(k,\lambda)=0 \text{ and } \partial_k f(k,\lambda)\neq 0\}\\ \subset\bigcup_{i\in\mathbb{N}}\operatorname{mes}\{k:\exists\lambda\in\Lambda \text{ s.t. } (k,\lambda)\in C_{\tilde r_i}(\tilde k_i,\tilde\lambda_i) \text{ and } f(k,\lambda)=0\}.\end{aligned}$$

This completes the proof of Lemma 6.1. ☐

Lemma 6.1 has a multidimensional analogue.

Lemma 6.2. *Let U be an open subset of $\mathbb{R}^d$, and V be an open subset of $\mathbb{R}^{d'}$. Let f be a real-analytic function on the set $U\times V$, and pick $\Lambda\subset V$ such that* $\operatorname{mes}\Lambda=0$. *For $k\in U$, we write $k=(k_1,k')$ where k_1 is real and $k'\in\mathbb{R}^{d-1}$. Then,*

$$\operatorname{mes}\{k\in U:\ \exists\lambda\in\Lambda \textit{ s.t. } f(k,\lambda)=0 \textit{ and } \partial_{k_1}f(k,\lambda)\neq 0\}=0. \tag{23}$$

Proof. Cover U with countably many open sets of the form $(a,b)\times\tilde U$ i.e.,

$$U=\bigcup_{i\in\mathbb{N}}(a_i,b_i)\times\tilde U_i.$$

For $i\in\mathbb{N}$, one has

$$\begin{aligned}\{k\in(a_i,b_i)\times\tilde U_i:\ \exists\lambda\in\Lambda \text{ s.t. } f(k,\lambda)=0 \text{ and } \partial_{k_1}f(k,\lambda)\neq 0\}\\ \subset\{k_1\in(a_i,b_i):\ \exists\lambda\in\Lambda \text{ s.t. } f(k_1,k',\lambda)=0 \text{ and}\\ \partial_{k_1}f(k_1,k',\lambda)\neq 0\}\times\tilde U_i.\end{aligned} \tag{24}$$

By Lemma 6.1, the set in the right hand side of equation (24) has measure 0 (as $\tilde U_i\times\Lambda$ has measure zero in $\mathbb{R}^{d+d'-1}$). As

$$\begin{aligned}\{k\in U:\ \exists\lambda\in\Lambda \text{ s.t. } f(k,\lambda)=0 \text{ and } \partial_{k_1}f(k,\lambda)\neq 0\}\\ =\bigcup_{i\in\mathbb{N}}\{k\in(a_i,b_i)\times\tilde U_i:\ \exists\lambda\in\Lambda \text{ s.t. } f(k,\lambda)=0 \text{ and } \partial_{k_1}f(k,\lambda)\neq 0\},\end{aligned}$$

(23) holds, which completes the proof of Lemma 6.2. ☐

Finally, we prove

Theorem 6.1. *Let U be a region in $\mathbb{R}^d$, Λ be a subset of an interval (a,b) such that* $\operatorname{mes}\Lambda=0$. *Let h be a real-analytic function defined on the set $U\times(a,b)$ and suppose that*

$$\forall\lambda\in\Lambda\quad\exists k\in U\quad\textit{such that}\quad h(k,\lambda)\neq 0. \tag{25}$$

Then,

$$\operatorname{mes}\{k\in U:\ \exists\lambda\in\Lambda \textit{ s.t. } h(k,\lambda)=0\}=0.$$

Proof. For any $k \in U$ and $\lambda \in \Lambda$, by assumption (25), there exists a multi-index $\alpha \in \mathbb{Z}_+^d$ such that $\partial_k^\alpha h(k,\lambda) \neq 0$. Therefore,

$$\{k \in U : h(k,\lambda) = 0 \text{ for some } \lambda \in \Lambda\} \\ \subset \bigcup_{j=1}^{d} \bigcup_{\alpha \in \mathbb{Z}_+^d} \left\{k \in U : \partial_k^\alpha h(k,\lambda) = 0, \partial_{k_j}\partial_k^\alpha h(k,\lambda) \neq 0 \text{ for some } \lambda \in \Lambda\right\}.$$

Reference to Lemma 6.2 then completes the proof of Theorem 6.1. □

7 The proof of Theorem 1.1

The following lemma is well known (see for example [12]).

Lemma 7.1. *Fix $b > 0$. Let B be a self-adjoint operator in $L_2(\Omega)$. Suppose that R_B is an analytic function defined in a complex neighborhood of an interval $[\alpha, \beta]$ except at a finite number of points $\{\mu_1, \dots, \mu_N\}$, that the values of R_B are in $B(L_{2,b}, L_{2,-b})$ and that*

$$R_B(\lambda)\varphi = (B-\lambda)^{-1}\varphi \quad \textit{if } \operatorname{Im}\lambda > 0, \varphi \in L_{2,b}.$$

Then, the spectrum of B in the set $[\alpha,\beta] \setminus \{\mu_1,\dots,\mu_N\}$ is absolutely continuous. If $\Lambda \subset [\alpha,\beta]$, $\operatorname{mes}\Lambda = 0$ and $\mu_j \notin \Lambda$, $j = 1,\dots,N$, then $E_B(\Lambda) = 0$, where E_B is the spectral projector of B.

Proof of Theorem 1.1. By Theorem 5.1, the set of all points $(k,\lambda) \in \mathbb{R}^{d+1}$ satisfying (2) can be represented as the following union

$$\{(k,\lambda) \in \mathbb{R}^{d+1} \text{ s.t. (2) be satisfied}\} = \bigcup_{j=1}^{\infty} B_{\varepsilon_j}(k_j) \times B_{\varepsilon_j}(\lambda_j), \tag{26}$$

where, for every j, there exists

- a number $a_j \geq 0$,
- an analytic scalar function h_j defined in a complex neighborhood of $\overline{B_{\varepsilon_j}(k_j) \times B_{\varepsilon_j}(\lambda_j)}$ with the property
$$\forall \lambda \in B_{\varepsilon_j}(\lambda_j) \quad \exists k \in B_{\varepsilon_j}(k_j) \quad \text{such that} \quad h_j(k,\lambda) \neq 0,$$
- an analytic $B(L_{2,a_j}, L_{2,-a_j})$-valued function $R_H^{(j)}$ defined on the set where $h_j(k,\lambda) \neq 0$ and satisfying (17).

Now, pick $\Lambda \subset \mathbb{R}$ such that $\operatorname{mes}\Lambda = 0$. Set

$$K_0 = \{k \in [0,1]^d : (k+n)^2 = \lambda \text{ for some } n \in \mathbb{Z}^d, \lambda \in \Lambda\},$$
$$K_1 = \{k \in [0,1]^d : h_j(k,\lambda) = 0 \text{ for some } j \in \mathbb{N}, \lambda \in \Lambda\}.$$

Thanks to Theorem 6.1, we know

$$\operatorname{mes} K_0 = \operatorname{mes} K_1 = 0. \tag{27}$$

For $k \notin K_0$, denote

$$\Lambda_j(k) = \{\lambda \in \Lambda : (k, \lambda) \in B_{\varepsilon_j}(k_j) \times B_{\varepsilon_j}(\lambda_j)\}.$$

It is clear that $\Lambda_j(k) \subset (\lambda_j - \varepsilon_j, \lambda_j + \varepsilon_j)$, $\operatorname{mes} \Lambda_j(k) = 0$, and, by (26),

$$\Lambda = \bigcup_{j=1}^{\infty} \Lambda_j(k) \quad \forall k \notin K_0. \tag{28}$$

If $k \notin (K_0 \cup K_1)$ and $\Lambda_j(k) \neq \emptyset$ then $h_j(k, \lambda) \neq 0$ for $\lambda \in \Lambda_j(k)$ and $\lambda \mapsto h_j(k, \lambda)$ has at most a finite number of zeros in $[\lambda_j - \varepsilon_j, \lambda_j + \varepsilon_j]$. So we can apply Lemma 7.1; therefore,

$$E_{H(k)}(\Lambda_j(k)) = 0 \quad \forall j.$$

This and (28) implies that

$$E_{H(k)}(\Lambda) = 0.$$

Finally, one computes

$$E_H(\Lambda) = \int_{[0,1]^d} E_{H(k)}(\Lambda)\, dk = \int_{[0,1]^d \setminus K_0 \setminus K_1} E_{H(k)}(\Lambda)\, dk = 0$$

by virtue of (27). So, we proved that the spectral resolution of H vanishes on any set of Lebesgue measure 0, which means, by definition, that the spectrum of the operator H is purely absolutely continuous.

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N. Filonov
Department of
Mathematical Physics
St Petersburg
State University
198504 St Petersburg-
Petrodvorets
Russia
filonov@mph.phys.spbu.ru

F. Klopp
LAGA, Institut Galilée
Université Paris-Nord
F-93430 Villetaneuse
France
klopp@math.univ-paris13.fr